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The Survival Power Weibull Distribution With Application

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Abstract The primary objective of this research is to investigate a novel lifetime distribution characterized by three parameters, which is constructed through the amalgamation of the Weibull distribution and the Survival Powe-G family. The recently introduced model is referred to as the SPW distribution. The newly formulated distribution possesses the advantage of effectively modeling various data types, thus proving to be instrumental in the domains of reliability and lifespan statistics. Several statistical properties pertinent to the SPW distribution are examined in this study. The recommended estimation approach is the maximum likelihood method. Empirical tests of the SPW distribution are presented by using two real datasets. Furthermore, SPW distribution demonstrates a good fit, backed by comparisons with Weibull-based models and other alternative distributions using several goodness-of-fit assessments.

Keywords Weibull distribution; Maximum likelihood estimation; Entropy; Monte Carlo simulation; Family of distributions.

AMS 2010 subject classifications 62E15, 62G30, 62F10, 60E05, 62P05.

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1. Introduction

The Weibull distribution constitutes a highly adaptable and extensively employed probability distribution within the domains of reliability engineering [1], life data analysis, and failure time modeling. Its adaptability arises from its capacity to effectively model a diverse array of data types via its shape parameter, which facilitates the modification of the distribution's structure to align with varying datasets. The principal generalizations derived from the Weibull distribution encompass its ability to represent increasing, constant, or decreasing failure rates, thereby rendering it suitable for a wide spectrum of real-world phenomena.

The prominent upgrade of the Weibull distribution is the addition of a new parameter, generally labelled as the scale parameter. By integrating this parameter, we boost the goodness-of-fit for empirical data and supply a better into the foundational processes that drive the phenomena observed, ultimately better decision-making. Moreover, the scale parameter proves to be particularly advantageous in reliability analysis. While this paper does not delineate specific studies or references, it is possible to summarize several notable contributions to the generalization of the Weibull distribution, where there have been various notable research explorations in recent times, including the following: Barnard et al. introduced the Linearly Decreasing Stress Weibull, it shares similarities with the Weibull distribution but offers advantages in certain parameter configurations, providing competitive fit quality in specific applications [2]. Teamah et al. introduced the Right Truncated Fréchet-Weibull Distribution, this distribution is tailored for datasets with truncation, and it is beneficial when the data is incomplete or censored [3]. Almazah introduced the Compound Weibull Distribution by combining three Weibull distributions,

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offering enhanced flexibility for modelling some datasets [4]. L. Benkhelifa introduced the Weibull Birnbaum-Saunders distribution as an extension of the flexible Weibull distribution, it demonstrates its potential over some other important distributions [5]. Jia et al. introduced the q-Weibull Distribution as a generalization of the Weibull distribution, the q-Weibull is supported by robust confidence intervals derived from bootstrap methods [6]. Nwezza et al. introduced the Gumbel Marshall-Olkin-Weibull Distribution, which is generated by combining the Gumbel and Weibull distributions, providing a flexible tool for modelling some complex datasets [7]. Hamed introduced the Mixture Weibull-Generalized Gamma Distribution, which is a mixture model that combines Weibull and generalized gamma distributions. It is designed to handle inhomogeneous populations and is particularly useful in life testing and reliability analysis [8]. Ishaq and Abiodun introduced the Maxwell-Weibull Distribution, as a generalization that combines the Maxwell and Weibull distributions, and It is particularly effective in modelling some lifetime data, as exchange rates and material strengths compared to some other distributions [9]. Hassan et al. introduced a new family of distributions based on the Weibull Lindley distribution and cumulative distribution function (CDF), it is called the Weibull Lindley general family of distributions [10]. Oluyede et al. introduced the Log Generalized Lindley-Weibull distribution, which has comprehensive statistical properties and is applied to real data to show its effectiveness [11]. Alizadeh et al. a groundbreaking distribution is introduced that features four parameters and is referred to as the new generalized modified Weibull distribution, which represents a generalization of the modified Weibull distribution. Moreover, this distribution can be expressed as an infinite linear combination of modified Weibull distributions, thereby offering a diverse array of shapes with varying degrees of skewness, differing tail weights, contingent upon its additional parameters [12]. Nwezza et al. introduced the Generalized Transmuted Weibull Distribution, it offers additional flexibility by incorporating transmutation parameters, which allow for better fitting of real-life data sets [14]. Adnan et al. introduced the Weibull-Lindley Rayleigh distribution as another extension that combines features of the Weibull, Lindley, and Rayleigh distributions. This new distribution is shown to fit certain datasets better than the individual distributions it encompasses [15]. Aljumaily and Saieed introduced the Gull Alpha Power transform Weibull distribution introducing additional flexibility to the standard Weibull distribution, making it applicable in fields like cancer modeling and engineering [16]. Anabike et al. introduced the New Weibull Exponential Distribution extends the Weibull distribution by adding a shape parameter, improving its fit and applicability in reliability studies, such as aeroplane windshield failure times [17].

This study shows a new distribution as an adaptation of the two-parameter Weibull distribution, named the Survival Power Weibull (SPW) distribution, by integrating an extra parameter, thus augmenting its flexibility. The novel PSW distribution is contingent upon the family of Survival Power-G (SP-G) distributions, which Kalt.[18] proposed, this family represents a new approach to incorporate an additional parameter to the baseline distribution to provide enhanced adaptability in modelling some real data. The new distribution provides a widerange of shapes with varying skewness, varied tail weights and shifting modes based on its additional parameters. Let's delve into the survival function linked to the baseline distribution marked as $S(x;\theta)$, along with the probability density function (PDF) noted as $g(x;\theta)$ with the vector of parameters θ. The cumulative distribution function (CDF) and the PDF of the SP-G family are delineated by

$$F_{SPG}(x;\alpha,\theta) = \alpha^{S(x;\theta)} - \alpha S(x;\theta), \quad 0 < \alpha,\ x \in \mathbb{R} \tag{1}$$

and

$$f_{SPG}(x;\alpha,\theta) = g(x;\theta)\left(\alpha - \alpha^{S(x;\theta)}\log(\alpha)\right) \tag{2}$$

where α is a shape parameter. Conversely, the Weibull distribution has found extensive application across various fields including actuarial science, reliability analysis, agricultural science, and health sciences. The survival function and PDF linked to the Weibull distribution are

$$S_W(x;\beta,\kappa) = \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}, \quad x \geq 0, \beta > 0, \kappa > 0 \tag{3}$$

and

$$g_W(x;\beta,\kappa)=\frac{k}{\beta}\left(\frac{x}{\beta}\right)^{\kappa-1}\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}} \tag{4}$$

The structure of this paper is as follows: In Section 2, the most important functions of the new SPW distribution are defined as CDF and PDF, this section also delves into the study of other important statistical functions of the SPW distribution, such as the hazard function, inverse hazard function, quantile function, and moment functions, as well as the derivation of the maximum likelihood estimator. Section 3 introduces the simulation study of maximum likelihood estimators using the mean square error (MSE) criterion and the average bias of the estimators for the three parameters. The section 4 introduces a test of the application of the new SPW distribution through an analysis of two real datasets.

2. Survival Power Weibull (SPW) Distribution

In this section, we have proposed the Survival Power Weibull (SPW) distribution by using the survival function in Equation (3) and the PDF in Equation (4) of the Weibull distribution in the Equations (1) and (2). The CDF of the proposed (SPW) distribution is

$$F_{SPW}(x;\alpha,\beta,\kappa)=\alpha^{\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}}-\alpha\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}},\quad x\geq 0,\ \alpha,\beta,\kappa>0 \tag{5}$$

The PDF of SPW distribution is

$$f_{SPW}(x;\alpha,\beta,\kappa)=\beta^{-\kappa}\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}x^{\kappa-1}\kappa\left(\alpha-\alpha^{\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}}\log(\alpha)\right) \tag{6}$$

where α and β scale parameters and κ is shape parameter. The SPW distribution provides some specific distributions as a special case, for example, for $\alpha=1$, the distribution reduces to a Weibull distribution, and for $\kappa=1$, it reduces to the survival power exponential (SPE) distribution [18]. The plots for the CDF and the PDF of the SPW distribution are sketched in Figure 1 and Figure 2 respectively.

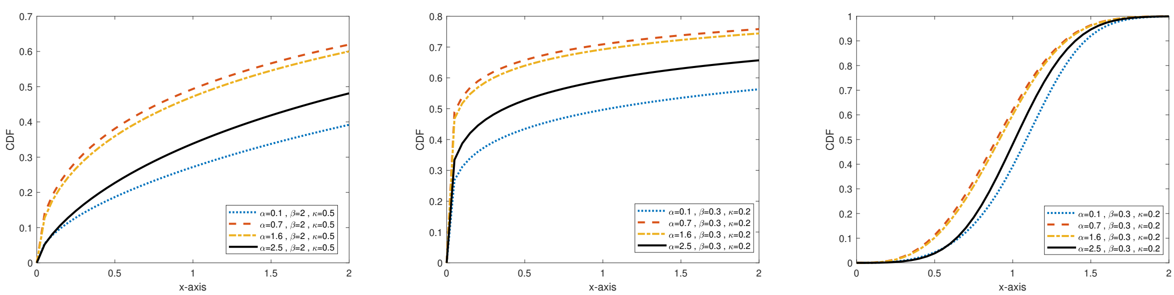

Figure 1. The plot of CDFs to SPW distribution.

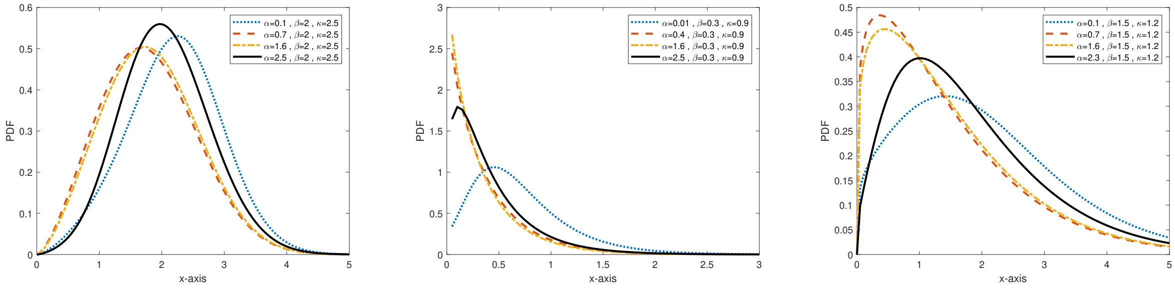

Figure 2. The plot of PDFs to SPW distribution.

2.1. *Reliability function*

The Reliability function $R_{SPW}(x;\alpha,\beta,\kappa)$ of the SPW distribution is

$$R_{SPW}(x;\alpha,\beta,\kappa) = 1 - F_{SPW}(x;\alpha,\beta,\kappa) = 1 - \alpha^{\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}} + \alpha \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}} \tag{7}$$

The plots for the Reliability function of the SPW distribution are sketched in Figure 3.

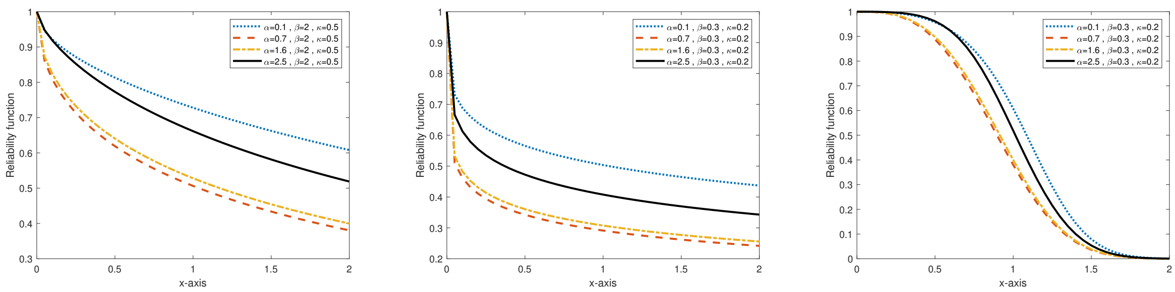

Figure 3. The plot of Reliability function to SPW distribution.

2.2. *Hazard and reversed hazard rate functions*

The hazard rate $h_{SPW}(x;\alpha,\beta,\kappa)$ and the reverse hazard rate $r_{SPW}(x;\alpha,\beta,\kappa)$ of the SPW distribution are

$$h_{SPW}(x;\alpha,\beta,\kappa) = \frac{f_{SPW}(x;\alpha,\beta,\kappa)}{1 - F_{SPW}(x;\alpha,\beta,\kappa)} = \frac{\beta^{-k}\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}} x^{\kappa-1}\kappa \left(\alpha - \alpha^{\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}} \log(\alpha)\right)}{1 - \alpha^{\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}} + \alpha \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}} \tag{8}$$

and

$$r_{SPW}(x;\alpha,\beta,\kappa) = \frac{f_{SPW}(x;\alpha,\beta,\kappa)}{F_{SPW}(x;\alpha,\beta,\kappa)} = \frac{\beta^{-k}\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}} x^{\kappa-1}\kappa \left(\alpha - \alpha^{\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}} \log(\alpha)\right)}{\alpha^{\mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}} - \alpha \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{\kappa}}} \tag{9}$$

2.3. *Quantile function*

Through using the inverse of the CDF of SPW distribution (5), we can find the quantile function of the SPW distribution as follows

$$Q_{SPW}(v) = \beta \left(\log(\alpha) + \log(\log(\alpha)) - \log\left(-v\log(\alpha) - \mathrm{W}\left(-\log(\alpha)\alpha^{\frac{-\alpha-v}{\alpha}}\right)\alpha\right)\right)^{\frac{1}{\kappa}} \tag{10}$$

where W is the function of Lambert-W and $0 < v < 1$. The quantile function can be used to find the median of the new SPW distribution when v=0.5 in Equation (11) as follows.

$$Median = Q_{SPW}(0.5) = \beta\left(\log(\alpha) + \log(\log(\alpha)) - \log\left(-0.5\log(\alpha) - \mathrm{W}\left(-\log(\alpha)\alpha^{\frac{-\alpha-0.5}{\alpha}}\right)\alpha\right)\right)^{\frac{1}{\kappa}} \quad (11)$$

the Bowley skewness is one of the measures to the skewness [19], which can be found by using the quantile function as follows.

$$SK_{SPW} = \frac{Q_{SPW}\left(\frac{3}{4}\right) + Q_{SPW}\left(\frac{1}{4}\right) - 2Q_{SPW}\left(\frac{1}{2}\right)}{Q_{SPW}\left(\frac{3}{4}\right) - Q_{SPW}\left(\frac{1}{4}\right)} \quad (12)$$

also, the kurtosis based on quantiles is expressed as the following [20]

$$KU_{SPW} = \frac{Q_{SPW}\left(\frac{7}{8}\right) - Q_{SPW}\left(\frac{5}{8}\right) + Q_{SPW}\left(\frac{3}{8}\right) - Q_{SPW}\left(\frac{1}{8}\right)}{Q_{SPW}\left(\frac{6}{8}\right) - Q_{SPW}\left(\frac{2}{8}\right)} \quad (13)$$

Where $Q_{SPW}(.)$ is the quantile function of SPW distribution. Since the metrics of SK_{SPW} and KU_{SPW} exhibit reduced action to outliers and apply to the SPW distribution.

2.4. Moments

This subsection shows the computation of the rth moment r about the origin point of the SPW distribution. By the definition of the rth moment, we have

$$\mu'_r = \int_0^\infty x^r f_{SPW}(x;\alpha,\beta,\kappa)dx = \int_0^\infty \beta^{-\kappa}\mathrm{e}^{-x^\kappa\beta^{-\kappa}}x^{r+\kappa-1}\kappa\left(\alpha - \alpha^{\mathrm{e}^{-x^\kappa\beta^{-\kappa}}}\log(\alpha)\right)dx \quad (14)$$

We can factor out the constants from the integral

$$\mu'_r = \kappa\beta^{-\kappa}\alpha\int_0^\infty \mathrm{e}^{-x^\kappa\beta^{-\kappa}}x^{r+\kappa-1}dx - \kappa\beta^{-\kappa}\log(\alpha)\int_0^\infty \mathrm{e}^{-x^\kappa\beta^{-\kappa}}x^{r+\kappa-1}\alpha^{\mathrm{e}^{-x^\kappa\beta^{-\kappa}}}dx \quad (15)$$

The first integral of Equation (15) involves only a standard gamma-like integral. Substituting $u = x^k\beta^{-\kappa} \Rightarrow x = \beta u^{\frac{1}{\kappa}}$ and $dx = \frac{1}{\kappa}\beta u^{\frac{1}{\kappa}-1}du$ this becomes:

$$I_1 = \int_0^\infty \mathrm{e}^{-x^\kappa\beta^{-\kappa}}x^{r+\kappa-1}dx = \frac{1}{\kappa}\beta^{r+\kappa}\int_0^\infty e^{-u}u^{\frac{r}{\kappa}}du = \frac{1}{\kappa}\beta^{r+\kappa}\Gamma\left(\frac{r}{\kappa}+1\right) \quad (16)$$

to solve the second integral of Equation (15)

$$I_2 = \int_0^\infty \mathrm{e}^{-x^\kappa\beta^{-\kappa}}x^{r+\kappa-1}\alpha^{\mathrm{e}^{-x^\kappa\beta^{-\kappa}}}dx \quad (17)$$

Let's perform the substitution: $u = x^k\beta^{-\kappa}$ so that $x = \beta u^{\frac{1}{\kappa}}$ and $dx = \frac{1}{\kappa}\beta u^{\frac{1}{\kappa}-1}du$. Thus, the integral transforms into:

$$I_2 = \int_0^\infty \mathrm{e}^{-u}\beta^{r+\kappa-1}u^{\frac{r+\kappa-1}{\kappa}}\alpha^{\mathrm{e}^{-u}}\frac{1}{\kappa}\beta u^{\frac{1}{\kappa}-1}du = \frac{1}{\kappa}\beta^{r+\kappa}\int_0^\infty \mathrm{e}^{-u}u^{\frac{r}{\kappa}}\alpha^{\mathrm{e}^{-u}}du \quad (18)$$

Using the series expansion:

$$\alpha^{\mathrm{e}^{-u}} = \sum_{m=0}^{\infty}\frac{(\log(\alpha))^m\mathrm{e}^{-mu}}{m!} \quad (19)$$

Substitute this into the integral in Equation (18)

$$I_2 = \frac{1}{\kappa}\beta^{r+\kappa}\int_0^\infty \mathrm{e}^{-u}u^{\frac{r}{k}}\left(\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m\mathrm{e}^{-mu}}{m!}\right)du \tag{20}$$

Distribute the sum inside the integral

$$I_2 = \frac{1}{\kappa}\beta^{r+\kappa}\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!}\int_0^\infty \mathrm{e}^{-(m+1)u}u^{\frac{r}{\kappa}}du \tag{21}$$

so we have

$$\int_0^\infty \mathrm{e}^{-(m+1)u}u^{\frac{r}{k}}du = \frac{\Gamma\left(\frac{r}{\kappa}+1\right)}{(m+1)^{\frac{r}{\kappa}+1}} \tag{22}$$

Substitute this result back into the sum

$$I_2 = \frac{1}{\kappa}\beta^{r+\kappa}\Gamma\left(\frac{r}{\kappa}+1\right)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{r}{\kappa}+1}} \tag{23}$$

By substituting both I_1 and I_2 in Equation (15), then the rth moment is

$$\mu_r' = k\beta^{-\kappa}\alpha\left(\frac{1}{\kappa}\beta^{r+\kappa}\Gamma\left(\frac{r}{\kappa}+1\right)\right) - \kappa\beta^{-\kappa}\log(\alpha)\left(\frac{1}{\kappa}\beta^{r+\kappa}\Gamma\left(\frac{r}{\kappa}+1\right)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{r}{\kappa}+1}}\right) \tag{24}$$

$$\mu_r' = \beta^r\Gamma\left(\frac{r}{\kappa}+1\right)\left(\alpha - \log(\alpha)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{r}{\kappa}+1}}\right) \tag{25}$$

According to the results given in Equation (25), the mean and the variance of the SPW distribution are $\mu = \mu_1'$ and $Var(X) = \mu_2' - \mu^2$ in Equations (26) and (27) respectively, as follows.

$$\mu = E(X) = \beta\Gamma\left(\frac{1}{\kappa}+1\right)\left(\alpha - \log(\alpha)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{1}{\kappa}+1}}\right) \tag{26}$$

$$Var(X) = \beta^2\Gamma\left(\frac{2}{\kappa}+1\right)\left(\alpha - \log(\alpha)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{2}{\kappa}+1}}\right) - \left(\beta\Gamma\left(\frac{1}{\kappa}+1\right)\left(\alpha - \log(\alpha)\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m}{m!(m+1)^{\frac{1}{\kappa}+1}}\right)\right)^2 \tag{27}$$

2.5. *Moment generating function*

The moment generating function (mgf) of the SPW distribution about the zero ,which denoted by $M_X(t)$, can derive as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}f_{SPW}(x;\alpha,\beta,\kappa)dx \tag{28}$$

By using Taylor series to the function e^{tx}.

$$e^{tx} = \sum_{r=0}^{\infty}\frac{t^r x^r}{r!} \tag{29}$$

By substituting Equation (29) into Equation (28), then

$$M_X(t) = \int_0^\infty \sum_{r=0}^{\infty}\frac{t^r x^r}{r!}f_{SPW}(x;\alpha,\beta,\kappa)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r' \tag{30}$$

By substituting Equation (25) in Equation (30), we get the mgf of the SPW distribution.

2.6. Entropy

The Renyi entropy to the variable X serves as an indicator of the variability of uncertainty and has found application across numerous disciplines. According to Renyi (1961) [27], the Renyi entropy of a random variable has the SPW distribution calculated using the following theorem.

Theorem: For X is a random variable that has the SPW distribution, the Renyi entropy is given by

$$H(w)=\frac{1}{1-w}\log\left(\alpha^{w}\sum_{n=0}^{\infty}\binom{w}{n}\left(\frac{-\log(\alpha)}{\alpha}\right)^{n}\frac{1}{n^{w}}\gamma(w,n\ln\alpha)\right) \tag{31}$$

Proof
The Rennie entropy of the SPW distribution is calculated as follows

$$H(w)=\frac{1}{1-w}\log\int_{0}^{\infty}\left(\beta^{-\kappa}\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}x^{\kappa-1}\kappa\left(\alpha-\alpha^{\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}}\log(\alpha)\right)\right)^{w}\mathrm{d}x, w>0 \text{ and } w\neq 1 \tag{32}$$

First we will solve the integral.

$$I=\int_{0}^{\infty}\left(\beta^{-\kappa}\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}x^{\kappa-1}\kappa\left(\alpha-\alpha^{\mathrm{e}^{-x^{\kappa}\beta^{-\kappa}}}\log(\alpha)\right)\right)^{w}dx \tag{33}$$

A substitution may simplify the integrand $u=x^{\kappa}\beta^{-k}\implies du=\kappa x^{\kappa-1}\beta^{-\kappa}dx$ The limits of integration remain 0 to ∞, and this substitution simplifies part of the integrand

$$I=\int_{0}^{\infty}\left(e^{-u}\left(\alpha-\alpha^{e^{-u}}\log(\alpha)\right)\right)^{w}du \tag{34}$$

Since e^{-u} appears in both terms, we might attempt a substitution to simplify this further. Let's make the substitution $v=e^{-u}$, which gives $dv=-e^{-u}du$ or equivalently $du=-\frac{dv}{v}$. Under this substitution, when $u=0$ then $v=1$ and when $u\to\infty$ then $v\to 0$. Thus, the integral transforms into:

$$I=\int_{1}^{0}\left(v\left(\alpha-\alpha^{v}\log(\alpha)\right)\right)^{w}\cdot\left(-\frac{dv}{v}\right)=\int_{0}^{1}v^{w-1}\left(\alpha-\alpha^{v}\log(\alpha)\right)^{w}dv \tag{35}$$

Since w is not a natural number, we can use the generalized binomial series for $(1+x)^{w}$, which applies to real or complex w :

$$(1+x)^{w}=\sum_{n=0}^{\infty}\binom{w}{n}x^{n} \tag{36}$$

In our case, let $x=\frac{-\alpha^{v}\log(\alpha)}{\alpha}$. Now, applying the binomial series expansion:

$$I=\int_{0}^{1}v^{w-1}(\alpha)^{w}\left(1+\frac{-\alpha^{v}\log(\alpha)}{\alpha}\right)^{w}dv=\alpha^{w}\int_{0}^{1}v^{w-1}\sum_{n=0}^{\infty}\binom{w}{n}\left(\frac{-\alpha^{v}\log(\alpha)}{\alpha}\right)^{n}dv \tag{37}$$

Now we simplify each term in the sum. The integral becomes:

$$I=\alpha^{w}\sum_{n=0}^{\infty}\binom{w}{n}\left(\frac{-\log(\alpha)}{\alpha}\right)^{n}\int_{0}^{1}v^{w-1}\alpha^{nv}dv \tag{38}$$

To solve the last integral, we can use the substitution $u=nv$, which implies $v=\frac{u}{n}$ and $dv=\frac{du}{n}$. Rewrite the limits, When $v=0,u=0$, and When $v=1,u=n$. The integral becomes:

$$\int_0^1 v^{w-1} e^{\beta v} dv = \int_0^n \left(\frac{u}{n}\right)^{w-1} \alpha^u \frac{du}{n} = \frac{1}{n^w} \int_0^n u^{w-1} \alpha^u du \tag{39}$$

This is a special case of the incomplete gamma function $\gamma(s,x)$ and is not solvable in terms of elementary functions for general α and w. However, if you want a solution in terms of special functions, it can be expressed as:

$$\int_0^1 v^{w-1} e^{\beta v} dv = \frac{1}{n^w} \gamma(w, n \ln \alpha) \tag{40}$$

where $\gamma(s,x)$ is the lower incomplete gamma function.

So the integral I becomes:

$$I = \alpha^w \sum_{n=0}^{\infty} \binom{w}{n} \left(\frac{-\log(\alpha)}{\alpha}\right)^n \frac{1}{n^w} \gamma(w, n \ln \alpha) \tag{41}$$

This gives a general solution in terms of a series involving the gamma function. So, the Renyi entropy is

$$H(w) = \frac{1}{1-w} \log \left(\alpha^w \sum_{n=0}^{\infty} \binom{w}{n} \left(\frac{-\log(\alpha)}{\alpha}\right)^n \frac{1}{n^w} \gamma(w, n \ln \alpha) \right) \tag{42}$$

□

2.7. Maximum likelihood estimation

A multitude of methodologies have been proposed to ascertain the estimations of unknown parameters with the objective of deriving predictive outcomes. The preeminent technique among these methodologies is the maximum likelihood estimation (MLE) methodology in terms of both Bayesian and non-Bayesian approaches. [29, 23, 24, 22, 28]. The estimators obtained through this methodology demonstrate favourable properties and so can be utilized to construct confidence intervals and other statistical evaluations. The approximation of the MLEs can be adeptly handled either via numerical calculations or analytical methodologies. More profound elucidations regarding MLEs can be found. In this specific section, we select to utilize the MLE methodology to estimate the three parameters of the SPW distribution. We can express $x_1, x_2, \ldots, x_n$ as a randomly chosen sample from the SPW distribution.

The likelihood function that aligns with Equation (6) is

$$L(\alpha, \beta, \kappa; x) = \prod_{i=1}^{n} f_{SPW}(x_i; \alpha, \beta, \kappa) \tag{43}$$

$$L(\alpha, \beta, \kappa; x) = \left(-\beta^{-\kappa}\kappa\right)^n \left(\prod_{i=1}^{n} \mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}} x_i^{\kappa-1} \left(\alpha^{\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}} \log(\alpha) - \alpha \right) \right) \tag{44}$$

The logarithm of Equation (44) is

$$\begin{aligned} \ell(\alpha, \beta, \kappa; x) = & -\kappa n \log(\beta) + n \log(\kappa) - \beta^{-\kappa} \left(\sum_{i=1}^{n} x_i^{\kappa} \right) + \\ & \log \left(\prod_{i=1}^{n} x_i^{\kappa-1} \right) + \sum_{i=1}^{n} \log \left(\alpha - \alpha^{\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}} \log(\alpha) \right) \end{aligned} \tag{45}$$

In order to ascertain the values of the three parameters that maximize the Likelihood function, the formulations for the partial derivatives concerning the three parameters of the Equation (45) are presented, as follows:

$$\frac{\partial \ell(\alpha,\beta,\kappa;x)}{\partial \alpha} = \sum_{i=1}^{n} \frac{1+\alpha^{\mathrm{e}^{-x_i^k\beta^{-k}}} - 1\mathrm{e}^{-x_i^k\beta^{-k}}\log(\alpha) + \alpha^{\mathrm{e}^{-x_i^k\beta^{-k}}-1}}{\alpha - \alpha^{-x_i^k\beta^{-k}}\log(\alpha)} \tag{46}$$

$$\frac{\partial \ell(\alpha,\beta,\kappa;x)}{\partial \beta} = -\frac{\kappa n}{\beta} + \frac{\kappa\left(\sum_{i=1}^{n} x_i^{\kappa}\right)}{\beta^{\kappa+1}} + \sum_{i=1}^{n}\left(-\frac{\kappa x_i^{\kappa}\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}\alpha^{\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}}\log(\alpha)^2}{\beta^{\kappa+1}\left(\alpha - \alpha^{\mathrm{e}^{-x_i^{k}\beta^{-\kappa}}}\log(\alpha)\right)}\right) \tag{47}$$

$$\begin{aligned}\frac{\partial \ell(\alpha,\beta,\kappa;x)}{\partial \kappa} =& -n\log(\beta) + \frac{n}{\kappa} + \beta^{-\kappa}\log(\beta)\left(\sum_{i=1}^{n} x_i^{\kappa}\right) - \beta^{-\kappa}\left(\sum_{i=1}^{n} x_i^{k}\log\left(x_i\right)\right)\\ &+ \frac{\sum_{i=1}^{n}\left(\prod_{il\sim=1}^{i-1} x_{il\sim}^{\kappa-1}\right)\left(\prod_{il\sim=i+1}^{n} x_{il\sim}^{\kappa-1}\right)x_i^{\kappa-1}\log\left(x_i\right)}{\prod_{i=1}^{n} x_i^{\kappa-1}}\\ &+ \sum_{i=1}^{n}\left(-\frac{\alpha^{\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}}\left(-x_i^{\kappa}\log\left(x_i\right)\beta^{-\kappa} + x_i^{\kappa}\beta^{-\kappa}\log(\beta)\right)\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}\log(\alpha)^2}{\alpha - \alpha^{\mathrm{e}^{-x_i^{\kappa}\beta^{-\kappa}}}\log(\alpha)}\right)\end{aligned} \tag{48}$$

Deriving the MLEs for the parameters α, β and κ involves equating the nonlinear Equation (46), (47) and (48), to zero and resolving them simultaneously. These Equations are a system of nonlinear equations and cannot be solved simultaneously, so we can solve them by optimized Newton-Raphson's iterative method facilitated by computational software such as R or Matlab.

3. Simulation study

In this section, a comprehensive evaluation is conducted regarding the effectiveness of the MLEs pertaining to the three parameters for the SPW distribution. The examination emphasizes the application of this novel distribution as a particularized distribution for simulation purposes. The simulation methodology encompasses a systematic series of steps, wherein $N = 1000$ iterations of the samples of varying sizes ($n = 10, 25, 50, 75, 100, 250, 500$) are generated from the SPW distribution by employing the inversion technique for four distinct sets of parameter values: ($\alpha = 0.9, \beta = 1.8, \kappa = 0.7$), ($\alpha = 0.3, \beta = 2, \kappa = 0.5$), ($\alpha = 1.8, \beta = 0.5, \kappa = 1.5$) and ($\alpha = 1.8, \beta = 1.5, \kappa = 2.5$). The mean square error (MSE) also average bias (AB) of the estimates of the three parameters associated with the new distribution are computed, with repetitions conducted for each sample size n, as delineated below.

$$\mathrm{MSE}(\zeta) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\zeta}_j - \zeta\right)^2 \text{ and } \mathrm{AB}(\zeta) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\zeta}_j - \zeta\right) \tag{49}$$

where ζ is either α, β or κ. The simulation results are systematically summarized in Tables 1, 2, 3 and 4, accompanied by the graphical representation in Figure (4) depicting MSE and Figure (5) illustrating AB. The information presented in this table and associated figures elucidates the advantageous efficacy of the estimations pertaining to the parameters of the SPW distribution, revealing negligible bias and credible MSEs across all evaluated scenarios, thereby indicating the dependability and accuracy of these estimations in relation to the true parameter values. Moreover, the biases approach zero as the sample size escalates, signifying that the estimations have propertis as unbiased estimators. In addition, the MSEs demonstrate a decline with the augmentation of sample size, implying the robustness of these three estimators in accurately estimating the three parameters of the SPW distribution.

Table 1. The MSE and AB of simulation for SPW distribution when $\alpha = 0.9$, $\beta = 1.8$, $\kappa = 0.7$.

n	$MSE(\hat{\alpha})$	$AB(\hat{\alpha})$	$MSE(\hat{\beta})$	$AB(\hat{\beta})$	$MSE(\hat{\kappa})$	$AB(\hat{\kappa})$
10	0.20708	0.24368	0.65256	-0.02645	0.06523	0.13443
25	0.23901	0.19703	0.29953	-0.16576	0.01094	0.03224
50	1.04133	0.23548	0.20716	-0.19125	0.00567	-0.00469
75	0.13723	0.19438	0.10618	-0.10228	0.00401	0.00606
100	0.16207	0.15221	0.08933	-0.15077	0.00241	0.00537
250	0.10012	0.02531	0.05668	-0.12942	0.00124	0.00687
500	0.05252	0.01181	0.02861	-0.08073	0.00071	0.01194

Table 2. The MSE and AB of simulation for SPW distribution when $\alpha = 0.3$, $\beta = 2$, $\kappa = 0.5$.

n	$MSE(\hat{\alpha})$	$AB(\hat{\alpha})$	$MSE(\hat{\beta})$	$AB(\hat{\beta})$	$MSE(\hat{\kappa})$	$AB(\hat{\kappa})$
10	5.31366	1.08112	1.52116	0.30006	0.08571	0.18821
25	4.22198	0.89818	0.76493	0.13977	0.02929	0.12575
50	3.12962	0.64112	0.43244	0.12475	0.02291	0.12081
75	0.30755	0.40561	0.27989	0.22597	0.01866	0.12659
100	0.33265	0.44457	0.17557	0.20331	0.01868	0.12831
250	1.11513	0.40089	0.16143	0.17637	0.01697	0.12138
500	1.04141	0.32411	0.11895	0.14043	0.01591	0.11824

Table 3. The MSE and AB of simulation for SPW distribution when $\alpha = 1.8$, $\beta = 0.5$, $\kappa = 1.5$.

n	$MSE(\hat{\alpha})$	$AB(\hat{\alpha})$	$MSE(\hat{\beta})$	$AB(\hat{\beta})$	$MSE(\hat{\kappa})$	$AB(\hat{\kappa})$
10	11.0787	0.92833	0.02244	-0.03367	0.28434	0.16901
25	3.60732	0.17591	0.00864	-0.00501	0.12801	0.06672
50	2.90411	0.26255	0.00748	-0.03044	0.06681	-0.04805
75	1.60433	-0.00217	0.00619	-0.01408	0.04633	-0.03659
100	1.56389	0.12921	0.00551	-0.02828	0.04399	-0.06535
250	0.15149	0.03781	0.00208	-0.02198	0.01493	-0.07044
500	0.11193	0.11329	0.00186	-0.02992	0.01353	-0.08301

Table 4. The MSE and AB of simulation for SPW distribution when $\alpha = 1.8$, $\beta = 1.5$, $\kappa = 2.5$.

n	$MSE(\hat{\alpha})$	$AB(\hat{\alpha})$	$MSE(\hat{\beta})$	$AB(\hat{\beta})$	$MSE(\hat{\kappa})$	$AB(\hat{\kappa})$
10	5.67313	0.26245	0.05021	-0.05105	0.69141	0.26067
25	9.70291	0.90915	0.05918	-0.06487	0.34125	-0.01824
50	5.66512	0.45063	0.05093	-0.06217	0.25099	-0.07574
75	3.00272	0.14012	0.03083	-0.03431	0.13757	-0.05784
100	4.37931	0.35324	0.04581	-0.05818	0.21647	-0.10668
250	0.31097	-0.09631	0.00985	-0.02568	0.06528	-0.04585
500	0.21648	-0.03368	0.00651	-0.02057	0.03329	-0.05214

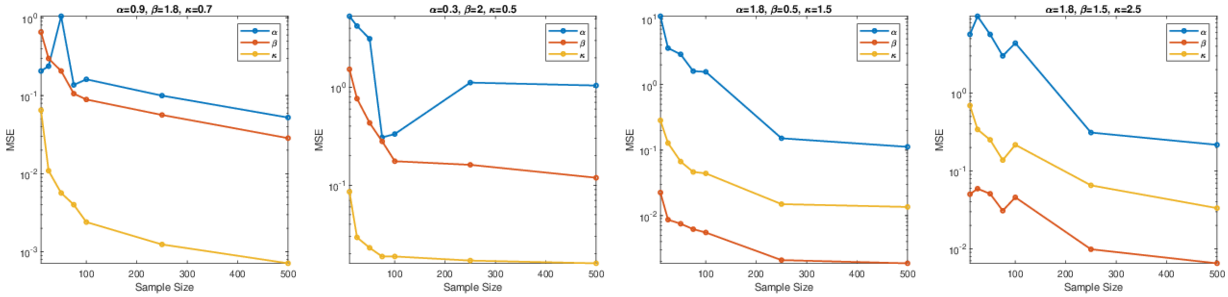

Figure 4. The Mean Squared Errors of the simulation.

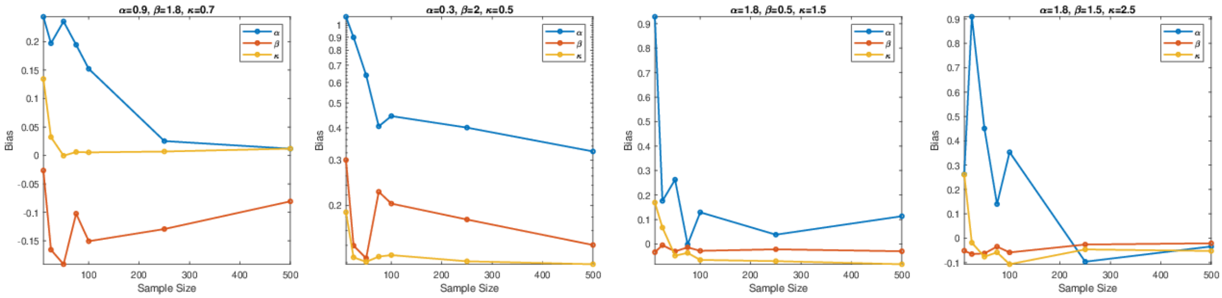

Figure 5. The Average Bias of the simulation.

The four models for this simulation were randomly selected to achieve the differences in their values in order to observe the superiority of the estimators of these models with the change in sample size. We notice from these results that there is no constant behavior for the estimators resulting from these simulation models with the change in sample sizes in an absolute way, but approximately the estimators of the parameters are better with the increase in the random sample sizes.

4. Applications of the SPW distribution on real data

In this segment, a thorough analysis and interpretation of five genuine datasets will be conducted to clarify the benefits associated with the implementation of the SPW distribution. The evaluation of the model's relevance involved the discernment of multiple information criteria. Identifying a suitable model usually demands a meticulous appraisal of a collection of information benchmarks that incorporate the Akaike Information Criterion (AIC), the Consistent Akaike Information Criterion (CAIC), the Baysian Information Criterion (BIC), the Hannan-Quinn Information Criterion (HQIC), along with the Kolmgorov-Smirenov Criterions (K-S). It is imperative to highlight that a reduction in the values of the goodness-of-fit Criterions metrics signifies a more favorable alignment of the data. In the following sections, we will present the actual data to which the novel distribution methodology was applied. The abundance and diversity of data may create obstacles for researchers in identifying the most appropriate data that accurately represents the new distribution. Therefore, during the data selection phase, alternative distributions were selected for comparative analysis based on prior studies related to the relevant data, rather than opting for the distributions themselves for comparative purposes across the five datasets, which will be elaborated upon below.

The first dataset has been sourced from Bjerkedal (1960) [21] and encapsulates the survival durations (measured in days) of 72 guinea pigs subjected to infection by virulent tubercle bacilli. This authentic dataset is subjected to analysis in order to elucidate the advantages of the SPW distribution in comparison to certain sub-models; specifically, Exponentiated Exponential Weibull (EEW), Exponentiated Weibull (EW), exponentiated Weibull exponential (EWE), Weibull exponential (WE), exponential-exponential (EE), Rayleigh exponential (RE), inverse Weibull distribution (IWD), Marshall Olkin Weibull (MOW) and Weibull (W) distributions. The dataset is delineated as follows:"0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55". The outcomes of the comparative analysis employing the goodness-of-fit statistics for the initial dataset are presented in Table 5.

Table 5. The goodness of fit comparison criterias of the first data

Dist.	AIC	CAIC	BIC	HQIC	K-S
SPW	192.08	201.91	198.91	194.81	0.04
EEW	196.16	209.27	205.27	199.79	0.041
EW	194.16	203.99	200.99	196.88	0.04
EWE	225.04	226.64	224.47	228.66	0.09
WE	298.65	299.01	298.23	301.37	0.13
EE	308.55	308.72	308.26	310.36	0.28
RE	289.02	289.19	288.74	290.83	0.13
IWD	238.65	238.82	243.20	240.46	0.19
MOW	220.94	223.66	227.77	224.36	0.09
W	226.24	228.05	230.79	228.52	0.08

The second data set delineates the remission intervals (measured in months) of a random sample comprising 128 bladder cancer patients, as examined by Lee and Wang (2003) [25] and subsequently analyzed by Al-Zahrani et al. [26]. We exemplify the adaptability of the SPW distribution through the application of this empirical data set. The SPW distribution is juxtaposed against alternative models, including the Exponentiated Exponential Weibull (EEW), the Exponentiated Weibull (EW), the (MAPTIW) distribution, the inverse Lomax (IL) distribution, the alpha power inverse Lomax (APILom) distribution, the inverse Weibull (IW) distribution, the alpha power inverse Weibull (APIW) distribution, the alpha power inverse Lindley (APILin) distribution, and the Weibull (W) distribution. The data are enumerated as follows:"0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52,4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47,14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77,32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66,15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63,17.12, 46.12, 1.26, 2.83, 4.33,5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79,18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69".The outcomes of the comparative analysis employing the goodness-of-fit statistics for the initial dataset are presented in Table 6.

Table 6. The goodness of fit comparison criterias of the second data

Dist.	AIC	CAIC	BIC	HQIC	K-S
SPW	825.56	837.12	834.12	837.12	0.0156
EEW	829.36	844.76	840.76	833.99	0.015
EW	827.36	838.91	835.91	830.83	0.015
MAPTIW	829.74	829.93	838.29	833.21	0.0398
IL	853.35	853.44	859.05	855.66	0.1184
APTILom	868.78	868.97	877.33	872.25	0.1017
IW	892.00	892.09	897.70	894.31	0.1407
APIW	860.26	860.45	868.81	863.73	0.0957
APILin	866.94	867.04	872.64	869.26	0.0924
W	832.17	832.26	837.87	834.49	0.0699

The third dataset is displayed by Murthy (2004) [30] about the times between failures for 30 which are repairable items. This real data sets are analyzed to illustrate the merit of SPW distribution compared with some other models; namely, the Exponentiated Exponential Weibull (EEW), the Exponentiated Weibull (EW), the beta modified Weibull distribution (BMWD), the Weibull power function distribution (WPFD), the power function distribution (PFD), the Kumerswmay power function distribution (KPFD), the Exponentiated Weibull-Power Function Distribution (EWPF) and the transmitted power function distribution(TPFD). The third dataset is: ”1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17”. The outcomes of the comparative analysis employing the goodness-of-fit statistics for the initial dataset are presented in Table 7.

Table 7. The goodness of fit comparison criterias of the third data

Dist.	AIC	CAIC	BIC	HQIC	K-S
SPW	85.820	93.024	90.024	87.165	0.0666
EEW	87.226	96.831	92.831	89.019	0.0666
EW	85.22	92.429	89.429	86.57	0.0666
BMW	94.406	96.906	101.412	96.647	0.8666
WPF	92.324	93.924	97.929	94.117	0.0770
PF	445.571	446.015	448.373	446.46	0.2881
KPF	109.747	111.347	115.352	111.54	0.2629
EWPF	89.248	91.748	96.254	91.489	0.0732
TPF	101.458	102.381	105.662	102.80	0.1391

The fourth dataset is displayed by Tahir, et al.(2015) [31] about the failure times of 84 aircraft windshields. These real datasets are analyzed to illustrate the merit of SPW distribution compared with some other models; namely, the New Generalization of the Inverse Generalized Weibull (NEGIGW), the Generalized Inverse Generalized Weibull Distribution (GIGW), The Exponential Fréchet (NEXF), The Exponentiated Generalized Inverse Weibull (EGIW), The Exponentiated Weibull Exponential (EWE), The Inverse Weibull (IW). The fourth dataset is: ”0.040 1.866 2.385 3.443 0.301 1.876 2.481 3.467 0.309 1.899 2.610 3.478 0.557 1.911 2.625 3.578 0.943 1.912 2.632 3.595 1.070 1.914 2.646 3.699 1.124 1.981 2.661 3.779 1.248 2.010 2.688 3.924 1.281 2.038 2.823 4.035 1.281 2.085 2.890 4.121 1.303 2.089 2.902 4.167 1.432 2.097 2.934 4.240 1.480 2.135 2.962 4.255 1.505 2.154 2.964 4.278 1.506 2.190 3.000 4.305 1.568 2.194 3.103 4.376 1.615 2.223 3.114 4.449 1.619 2.224 3.117 4.485 1.652 2.229 3.166 4.570 1.652 2.300 3.344 4.602 1.757 2.324 3.376 4.663”. The outcomes of the comparative analysis employing the goodness-of-fit statistics for the initial dataset are presented in Table 8.

Table 8. The goodness of fit comparison criterias of the fourth data

Dist.	AIC	CAIC	BIC	HQIC	K-S
SPW	262.889	273.181	270.181	265.82	0.0238
NEGIGW	286.495	287.265	298.650	291.381	0.1067
GIGW	300.618	301.124	310.341	304.527	0.1455
NEXF	307.723	308.023	315.016	310.655	0.1616
EGIW	359.862	360.368	369.585	363.770	0.2334
EWE	335.685	336.192	345.409	339.594	0.1812
IW	393.073	393.221	397.935	395.027	0.3127

The fifth dataset is displayed by Abdul-Moniem and Seham (2015) [32] about the data represent the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 % stress level until all had failed. These real datasets are analyzed to illustrate the merit of SPW distribution compared with some other models; namely, the New Generalization of the Inverse Generalized Weibull (NEGIGW), the Generalized Inverse Generalized Weibull Distribution (GIGW), The Exponential Fréchet (NEXF), The Exponentiated Generalized Inverse Weibull (EGIW), The Exponentiated Weibull Exponential (EWE), The Inverse Weibull (IW). The fifth dataset is: "0.0251 0.0886 0.0891 0.2501 0.3113 0.3451 0.4763 0.5650 0.5671 0.6566 0.6748 0.6751 0.6753 0.7696 0.8375 0.8391 0.8425 0.8645 0.8851 0.9113 0.9120 0.9836 1.0483 1.0596 1.0773 1.1733 1.2570 1.2766 1.2985 1.3211 1.3503 1.3551 1.4595 1.4880 1.5728 1.5733 1.7083 1.7263 1.7460 1.7630 1.7746 1.8275 1.8375 1.8503 1.8808 1.8878 1.8881 1.9316 1.9558 2.0048 2.0408 2.0903 2.1093 2.1330 2.2100 2.2460 2.2878 2.3203 2.3470 2.3513 2.4951 2.5260 2.9911 3.0256 3.2678 3.4045 3.4846 3.7433 3.7455 3.9143 4.8073 5.4005 5.4435 5.5295 6.5541 9.0960". The outcomes of the comparative analysis employing the goodness-of-fit statistics for the initial dataset are presented in Table 9.

Table 9. The goodness of fit comparison criterias of the fifth data

Dist.	AIC	CAIC	BIC	HQIC	K-S
SPW	247.229	257.221	254.221	250.024	0.0131
NEGIGW	257.287	258.144	268.940	261.944	0.0845
GIGW	262.855	263.418	272.178	266.581	0.1044
NEXF	269.187	269.521	276.179	271.982	0.1310
EGIW	263.461	264.024	272.784	267.187	0.1177
EWE	260.961	261.525	270.284	264.687	0.0934
IW	311.078	311.242	315.739	312.941	0.1893

It is evident from the analysis presented in Tables 5, 6, 7, 8 and 9 that the Survival Power Weibull (SPW) Distribution demonstrates a significantly superior fit in comparison to the various other competitive modeling alternatives that were subjected to evaluation in this study. This conclusion is further substantiated by the observation that the SPW exhibits the lowest values for the statistic (K-S), as well as for the criterion (AIC), the criterion (CAIC), the criterion (BIC), and the criterion (HQIC) among all the models that have been meticulously considered within the scope of this research. In addition, The degree to which the SPW distribution aligns with these five data sets is further depicted in Figures 6, 7, 8, 9 and 10.

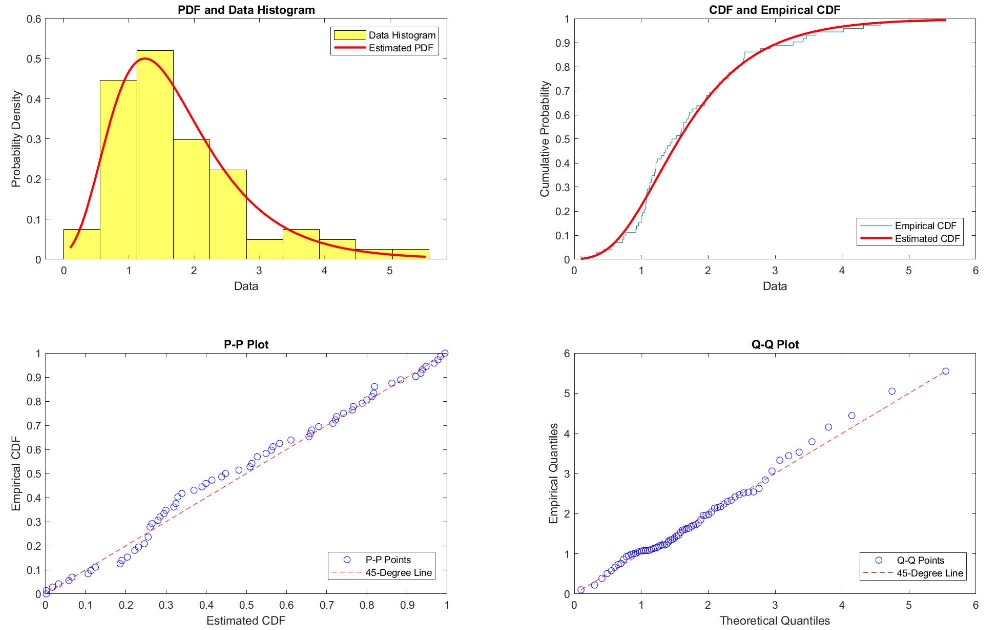

Figure 6. Comparison results for applying the SPW distribution to the first data

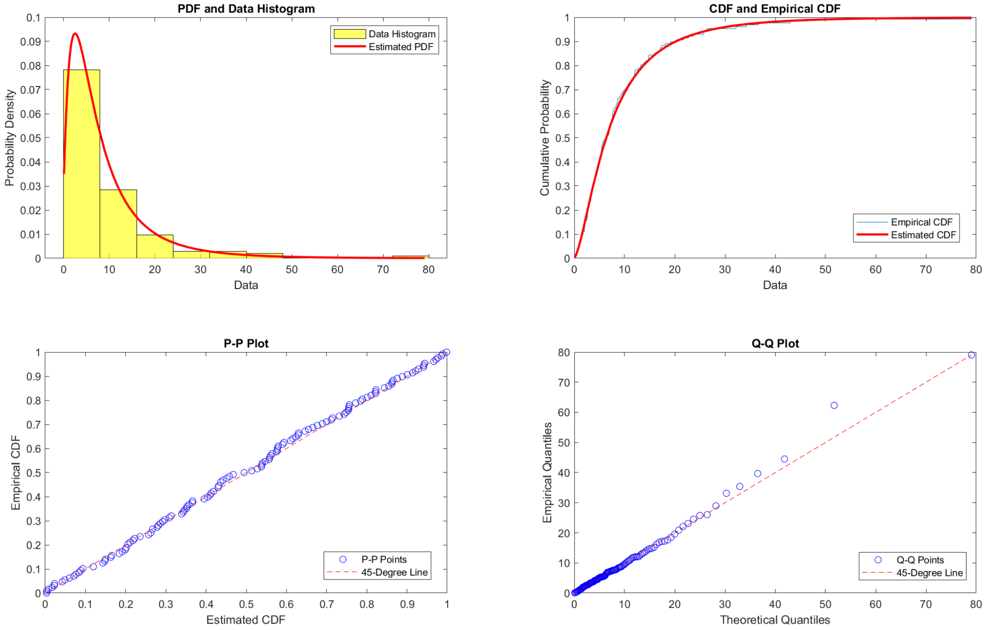

Figure 7. Comparison results for applying the SPW distribution to the second data

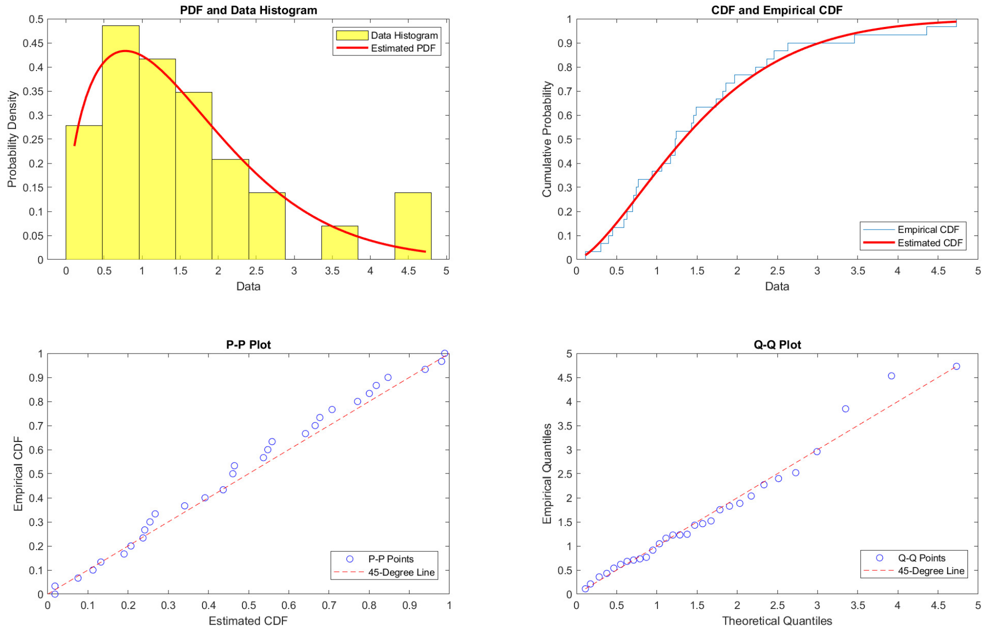

Figure 8. Comparison results for applying the SPW distribution to the third data

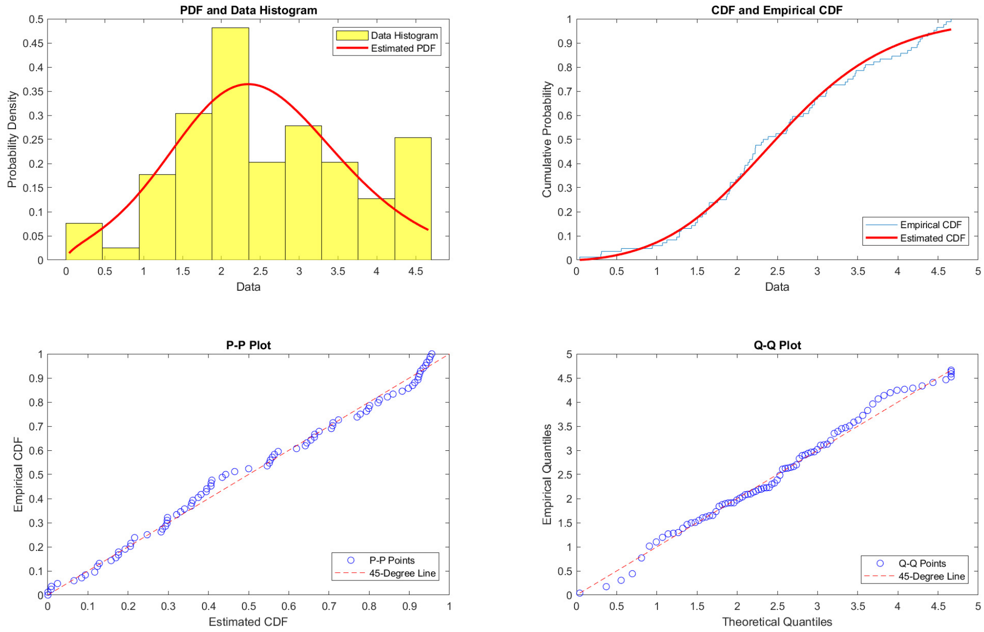

Figure 9. Comparison results for applying the SPW distribution to the fourth data

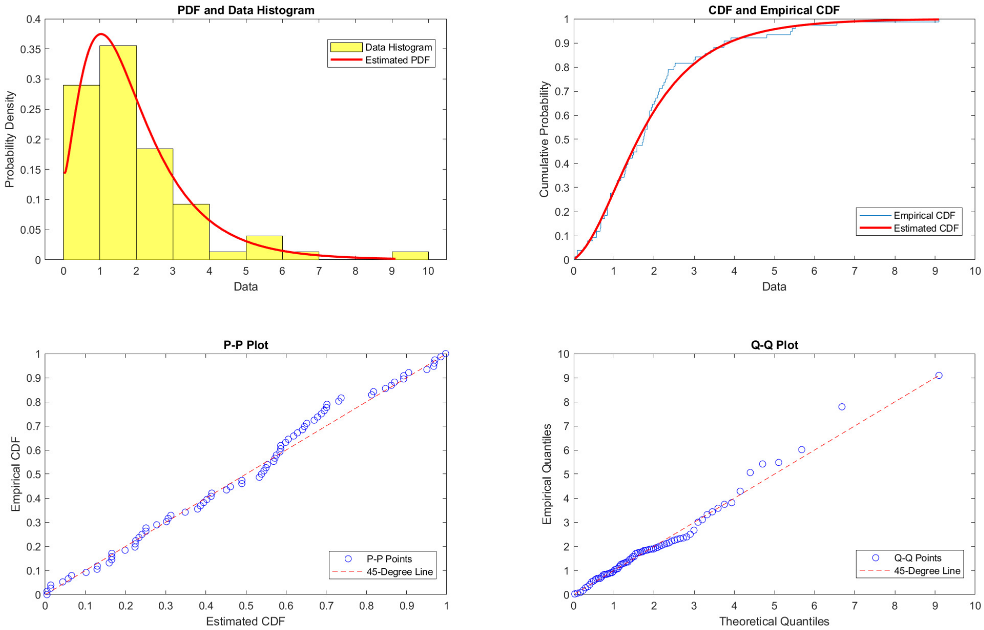

Figure 10. Comparison results for applying the SPW distribution to the fifth data

The five disparate data sets underwent rigorous analysis employing the innovative SPW distribution. We computed the mean and variance by using first and second order moment function, furthermore, by leveraging in the quantile function, we determined the Skewness, Kurtosis and Median, as illustrated in Table 10.

Table 10. The analyzes of descriptive of the five datasets

Dataset	Mean	Variance	Skewness	Median	Kurtosis	Mini	Maxi
1	1.768194	1.070291	1.34187	1.495	4.991056	0.10	5.55
2	9.365625	110.42497	3.28657	6.395	18.48308	0.08	79.05
3	1.542667	1.271675	1.295462	1.235	4.31917	0.11	4.73
4	2.557452	1.251768	0.099494	2.3545	2.34768	0.04	4.663
5	1.959241	2.477415	1.979558	1.73615	8.16079	0.0251	9.096

5. Conclusion

Multiple reworks to the Weibull distribution have been established to enhance their versatility to fit some of the data sets. This manuscript introduces the novel SPW distribution, which includes the addition of a third parameter to the two-parameter Weibull distribution. A distinctive parameter that represents the power function of the survival function in a pioneering way within well-established continuous distributions. The unique features of this new model relative to the SP-G family were studied by choosing the Weibull distribution, which was considered suitable for the SP-G family of continuous distributions. The SPW distribution, characterized by three principal parameters, was advanced, and a plethora of mathematical functions and properties were scrutinized, with MLE computed for the trio of parameters. At the same time, Monte Carlo simulations were conducted to demonstrate the efficacy of these estimators. The efficiency of this distribution has been proven experimentally through the application of two real datasets, Confirming its superiorityr fit for the chosen data in comparison to alternative models. Herein, we underscore this family for prospective research trajectories aimed at modifying other probability distributions to enhance their applicability to specific continuous data types. The Weibull distribution and its generalized distributions are also known for their suitability for failure tests and data related to failure or death rates. Hence, our new distribution will be studied in the future in terms of other methods of estimating parameters as well as reliability or finding entropy, especially since this distribution contains three parameters, which expands its suitability for real data.

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