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Robust Numerical Approach to CRR Model under Self-financing assumption

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Abstract
The accurate pricing of options is crucial for minimizing financial risks and making informed investment decisions in dynamic markets. Traditional models like the Black-Scholes often fail to account for the early exercise feature of American options and the self-financing replicating portfolio concept, leading to less realistic pricing. This study address these gaps by employing various metaheuristic algorithms, including Particle Swarm Optimization, Differential Evolution, Grey Wolf Optimization, and Simulated Annealing Algorithm, to estimate the parameters for a modified Cox-Ross-Rubinstein model. We derive a Brownian motion model incorporating upward and downward factors and use the Euler-Maruyama method to simulate stock price paths. By comparing these simulated paths with real stock data, we evaluate the effectiveness of the estimated parameters. Additionally, we improve the numerical method for estimating American option prices via the CRR model by integrating the self-financing replicating portfolio concept. The results demonstrate that Particle Swarm and Grey Wolf optimization algorithms provide parameter estimates that yield simulated paths closely matching the real stock data, thereby offering computationally realistic prices for American options. This study highlights the potential of integrating metaheuristic algorithms with traditional models to enhance the accuracy and reliability of option pricing.

Keywords Option Pricing; Metaheuristic Algorithms; Cox-Ross-Rubinstein Model; American Options

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1. Introduction

Pricing options in a manner that accurately reflects various market and investment conditions can minimize losses and ensure informed decisions on long or short positions in the market. Such conditions include early exercise, as seen in American options, and the concept of a self-financing replicating portfolio. The Cox-Ross-Rubinstein (CRR) model has been found to be more suitable for American options compared to the Black-Scholes model, as it accommodates early exercise for option pricing [5]. Despite significant advancements in modeling underlying stocks, particularly with the Geometric Brownian Motion (GBM) model [19], the existing literature has yet to adequately address the limitations in estimating jump factors within the CRR model under the self-financing replicating portfolio framework. This research aims to bridge these gaps by proposing a robust numerical approach that enhances the accuracy of jump estimation without deviating from the full solution of Brownian Motion, thereby providing a more reliable method for pricing American options.

In the literature, some authors have focused on modeling the underlying stock using different models, such as the Geometric Brownian Motion (GBM) model [10, 13, 17]. This model is more easily incorporated into the

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Black-Scholes model than the CRR model [28]. However, to use the CRR model, some authors estimate jump factors using a maximum likelihood algorithm criterion that counts the number of times a stock has jumped against the number of times it has not, expressing this as a percentage [21]. The parametric Geometric Brownian Motion, which is known for its desirable results in fitting data, has been sidelined for a long time. This led authors, especially those working under the self-financing replicating portfolio framework, to manipulate the solution of the Brownian Motion to estimate average daily upward and downward jump factors [11]. The common approach involves removing Brownian Motion noise from the solution and then varying the signs of the volatility components to estimate jumps [15, 22]. This results in estimates that somewhat deviate from the full solution of the Brownian Motion [24].

Numerical methods for the CRR model mainly apply the technique of removing noise in the solution of Brownian Motion to estimate jumps, in order to achieve a self-financing replicating portfolio according to the European options. Most authors estimate volatility from returns to ensure that the factors accurately reflect jumps [2]. The concept of a self-financing replicating portfolio is rarely used in trees for European options. [17] and [12] assume that a tree is inherently a self-financing framework. Many numerical methods have been developed without this concept, as seen in [4] and [20] for the CRR model.

In recent years, research has presented numerous metaheuristic algorithms that have been applied in various scenarios, typically discrete, such as optimizing positions to minimize losses in electric power systems, transportation expenses, and even mitigating denial-of-service attacks [9]. Few methods, like the Simulated Annealing Algorithm (SAA), Particle Swarm Optimization (PSO), Differential Evolution (DE) algorithm, and Grey Wolf Optimization (GWO), have been tried for continuous cases, usually in terms of optimization for discrete factors [1, 18].

Furthermore, advancements in computational power have enabled more sophisticated approaches to option pricing models, incorporating machine learning and artificial intelligence techniques. These methods aim to refine the estimation of jump factors and volatility, providing more accurate and robust models for both European and American options. Studies such as [16] and [25] have demonstrated the potential of these advanced algorithms in enhancing the predictive power and efficiency of option pricing models.

Additionally, the integration of metaheuristic algorithms with traditional models has shown promising results in handling large datasets and complex market conditions. For instance, [27] and [26] utilized a hybrid approach combining PSO with GBM to capture market anomalies and improve pricing accuracy. This hybrid methodology not only enhances the flexibility of the models but also provides a robust framework for future research in financial modeling and option pricing.

The existing literature has yet to address several critical questions: 1) How can jumps be estimated in a way that does not violate the principles of GBM for use in the CRR model, particularly for American option exercises? 2) If it is possible to minimize the violation of GBM, how can metaheuristic algorithms be utilized to estimate jumps while ensuring a self-financing replicating portfolio scenario? 3) Furthermore, once this issue is resolved, how can the self-financing replicating portfolio be integrated into the CRR model to compute realistic prices for American options?

This paper proposes a robust numerical approach to the CRR model under the self-financing assumption. Specifically, we derive parameters from a modified Brownian motion to enhance the accuracy of jump estimation.

In summary, this work makes the following contributions:

(i) Integration of a self-financing replicating portfolio into the numerical method for the American option exercise under the CRR model. The combination of this approach with metaheuristic algorithms results in computationally realistic prices for American options.

(i) Development of a Brownian motion model that includes both upward and downward factors. Using the Euler-Maruyama method and a sampling correction procedure, a fitness function is constructed that converges effectively to provide accurate estimates of jumps using PSO, DE, SAA, and GWO.

The rest of the paper is organized as follows: Section 2 describes the methodology employed to achieve the study's objectives. Section 3 presents the results, illustrating the effectiveness of the methods used. Finally, Section 4 concludes the paper by discussing the findings and their implications.

2. Methodology

This section presents the methodology used to achieve the goals of this paper. First, it covers the concepts of self-financing and replicating portfolios. It then delves into the use of these concepts to improve numerical methods for calculating both American and European options. This work proposes using metaheuristic techniques instead of traditional likelihood-based approaches to enhance the estimation process. Specifically, it explains the application of Simulated Annealing (SAA), Differential Evolution (DE), Particle Swarm Optimization (PSO), and Grey Wolf Optimization (GWO) algorithms, which necessitate the use of sufficient data.

2.1. Utilization of Self-Financing and Replicating Portfolio Principles

A self-financing portfolio is one where the changes in the portfolio's value over time are solely due to the changes in the prices of the assets it holds. That is, no additional funds are added or withdrawn from the portfolio; any changes in the value are the result of gains or losses from the assets' price movements. Mathematically, the following definition [14] can be adopted.

Definition 1
Consider a market with N assets $\{S_j\}_{j=1}^N$. Let $\omega_t = \{\omega_{j,t}\}_{j=1}^N$ be a vector of weights representing the proportion of wealth invested in asset S_j at time t. Let V_t be the value of the portfolio at time t. The portfolio ω_t is called self-financing if the change in the portfolio value over the time interval $[t, t+\Delta t]$ is given by:

$$V_{t+\Delta t} - V_t = \sum_{j=1}^{N} \omega_{j,t} \Delta S_j, \tag{1}$$

where $\Delta S_j = S_{j,t+\Delta t} - S_{j,t}$ represents the change in the price of asset S_j over the time interval $[t, t+\Delta t]$.

A replicating portfolio is one that matches the payoff of a derivative at maturity. The portfolio is constructed by holding positions in the underlying assets and adjusting these positions over time to ensure that the portfolio value replicates the value of the derivative. The following definition [12] can be adopted:

Definition 2
The value of the replicating portfolio V_t at any time t is given by:

$$V_t = \Delta_t S_t + \Psi_t, \tag{2}$$

where Δ_t represents the number of shares of the underlying asset held in the portfolio, and Ψ_t represents the amount of cash or risk-free asset held. The cash or risk-free asset component Ψ_t is estimated as:

$$\Psi_t = e^{-r\Delta t}(V_{t+\Delta t} - \Delta_t S_{t+\Delta t}), \tag{3}$$

where $V_{t+\Delta t}$ is the portfolio value at time $t+\Delta t$, $S_{t+\Delta t}$ is the asset price at time $t+\Delta t$, and r is the risk-free rate.

The value of an American option can be represented as the value of a self-financing replicating portfolio in the CRR model.

Theorem 1
Let V_t be the value of an American option at time t in the Cox-Ross-Rubinstein (CRR) model. Then

$$V_t = \Delta_t S_t + \Psi_t, \tag{4}$$

where Δ_t and Ψ_t are determined such that the portfolio replicates the payoff of the American option at each time step and allows for early exercise, satisfying:

$$V_t = \max\left(\text{payoff}(S_t), e^{-r\Delta t}(pV^u_{t+\Delta t} + (1-p)V^d_{t+\Delta t})\right), \tag{5}$$

with $p = \frac{e^{r\Delta t}-d}{u-d}$, and $V^u_{t+\Delta t}$ and $V^d_{t+\Delta t}$ representing the option values at the next time step for the up and down states, respectively.

Proof
See [6] and [23]. □

Using the concepts defined in the two definitions and the theorem, we propose an improvement to the algorithm in [21] to incorporate self-replicating concepts and American options. This is presented in the following algorithm:

Algorithm 1 American Option Valuation Using Self-Financing Replicating Portfolio

Require: $S_0, K, r, u, d, n, optionType, earlyExerciseThreshold$
  $\Delta t \leftarrow \frac{1}{n}$
  $p \leftarrow \frac{e^{r\Delta t}-d}{u-d}$
  Initialize matrices V, S, Δ, and Θ of size $(n+1)\times(n+1)$ with zeros
  for $i \leftarrow 0$ to n **do**
    for $j \leftarrow 0$ to i **do**
      $S[i,j] \leftarrow S_0 \cdot u^j \cdot d^{i-j}$
  for $j \leftarrow 0$ to n **do**
    if $optionType =$ ”call” **then**
      $V[n,j] \leftarrow \max(0, S[n,j]-K)$
    else
      $V[n,j] \leftarrow \max(0, K - S[n,j])$
  for $i \leftarrow n-1$ down to 0 **do**
    for $j \leftarrow 0$ to i **do**
      $continuationValue \leftarrow e^{-r\Delta t}\cdot(p\cdot V[i+1,j+1] + (1-p)\cdot V[i+1,j])$
      if $optionType =$ ”call” **then**
        $exerciseValue \leftarrow S[i,j]-K$
        $V[i,j] \leftarrow \max(earlyExerciseThreshold \cdot exerciseValue, continuationValue)$
      else
        $exerciseValue \leftarrow K - S[i,j]$
        $V[i,j] \leftarrow \max(earlyExerciseThreshold \cdot exerciseValue, continuationValue)$
      $\Delta[i,j] \leftarrow \frac{V[i+1,j+1]-V[i+1,j]}{S[i+1,j+1]-S[i+1,j]}$
      $\Psi[i,j] \leftarrow e^{-r\Delta t}\cdot(V[i+1,j] - \Delta[i,j]\cdot S[i+1,j])$
      $V[i,j] \leftarrow \Delta[i,j]\cdot S[i,j] + \Psi[i,j]$
  return $V[0,0]$

In the CRR model, any European call or put option can be perfectly replicated by a self-financing portfolio consisting of shares of the underlying stock and a bond. This is derived from the fact that if the early exercise value is excluded, as described in Theorem 1, we are left with the characteristics of a European option.

2.2. Optimal CRR Option Prices Using Metaheuristic Algorithms

We assume that the closing stock price, X_t, follows a Geometric Brownian Motion (GBM) as characterized by the parameters σ (volatility) and μ (drift), under the condition that $2\mu \neq \sigma^2$. The GBM is a widely used model in financial mathematics to describe the stochastic behavior of asset prices, where

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{6}$$

represents the differential equation governing the price evolution [8]. This assumption underpins our approach to option pricing within the CRR model, as it allows for a realistic representation of stock price movements over time.

Suppose we adopt a self-financing replicating portfolio framework for derivatives in this study. We can express σ as:

$$\sigma = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{u}{d}\right), \tag{7}$$

where $u = e^{\mu+\sigma\sqrt{\Delta t}}$ and $d = e^{\mu-\sigma\sqrt{\Delta t}}$. Similarly, μ can be expressed as:

$$\mu = \frac{1}{2}\ln(ud). \tag{8}$$

Consequently, we propose the following problem to integrate metaheuristic methods for estimating options under the self-financing replicating portfolio conditions.

Proposition 1
To estimate u and d for a self-financing replicating portfolio, one must solve the optimization problem:

$$\arg\min_{u,d} J(u,d) \tag{9}$$

subject to the constraint:

$$4\mu\Delta t + \log(d) > \log(u), \tag{10}$$

where $J(u,d)$ is defined as:

$$J(u,d) = \sqrt{\frac{1}{N}\sum_{j=0}^{N-1}\left(\Delta X_j - 0.5\log(ud)X_j\Delta t - \frac{1}{2\Delta t}\sqrt{N}\log\left(\frac{u}{d}\right)X_j\Delta W_t\right)^2}. \tag{11}$$

The objective function $J(u,d)$ can be derived from the stochastic differential equation (SDE) of the form:

$$dX_t = 0.5\log(ud)X_t dt + \frac{1}{2\Delta t}\log\left(\frac{u}{d}\right)X_t dW_t. \tag{12}$$

Equation (12) can be discretized using the Euler-Maruyama method, resulting in the following expression:

$$X_{j+1} - X_j = 0.5\log(ud)X_j\Delta t + \frac{1}{2\Delta t}\sqrt{N}\log\left(\frac{u}{d}\right)X_j\Delta W_t, \quad j = 0,1,2,\ldots,N-1. \tag{13}$$

2.3. Methodological Approach

The metaheuristic algorithms selected for this study include Simulated Annealing (SAA), Grey Wolf Optimization (GWO), Particle Swarm Optimization (PSO), Differential Evolution (DE), and a hybrid of PSO and a neural network (PSO-NN) found in [3]. These methods are systematically applied to the objective function, as detailed in Equation (7). The estimates of u and d are then used in Algorithm 1 to estimate prices.

To adapt these algorithms for effective operation within the stochastic environment of the objective function, a sampling correction factor of $\frac{1}{\sqrt{N}}$ was integrated. This modification enhances the algorithms' performance, addressing the challenge that, while these techniques fundamentally rely on random number generation, they are conventionally more suited to deterministic scenarios. This strategic adjustment is crucial for leveraging their capabilities in stochastic contexts, as detailed by [7]. These five methods were specifically chosen to reflect a range of global search strategies—SAA for probabilistic annealing, GWO for swarm-based exploration inspired by natural leadership hierarchies, PSO and DE for efficient population-based optimization, and PSO-NN for its demonstrated ability to combine PSO's global search with the adaptive learning power of neural networks [2].

3. Results of Numerical Experiments

Using the scheme in Equation (13), stock data are simulated with parameters $T = N \in \{21, 50, 100, 120\}$, $u = 1.15$, and $d = 0.95$. A simple binomial tree framework generates possible stock paths based on the specified up (u) and down (d) factors. The results show that the accuracy of SAA improves with larger sample sizes, while the accuracy of GWO, PSO, DE, and PSO-NN generally decreases as N increases (Table 1). Notably, PSO, DE, and PSO-NN still achieve nearly exact parameter estimates despite this trend. The initial score values tend to vary depending on the choice of parameters. For example, for $K = 120$, $u = 1.12$, and $d = 0.85$, the PSO-NN method produces relatively high scores during the initial epochs. However, after about 8 epochs, the convergence trend becomes similar across all cases (Figure 2).

Table 1. Estimates for different values of N using SAA, GWO, PSO, DE, and PSO-NN

N	Parameter	SAA	GWO	PSO	DE	PSO-NN
21	u	1.154	1.149	1.150	1.150	1.160
	d	0.949	0.950	0.950	0.950	0.960
	MSE for u	9.15×10^{-4}	1.7×10^{-7}	5.6×10^{-41}	0.000	4.5×10^{-8}
	MSE for d	1.12×10^{-3}	5.5×10^{-8}	6.5×10^{-31}	0.000	3.2×10^{-8}
	Score	4×10^{-1}	3.4×10^{-3}	5×10^{-15}	2.99×10^{-15}	1.2×10^{-15}
50	u	1.162	1.151	1.150	1.150	1.165
	d	0.940	0.949	0.950	0.949	0.961
	MSE for u	1.4×10^{-4}	3×10^{-7}	3.6×10^{-41}	4.93×10^{-32}	2.3×10^{-7}
	MSE for d	9.2×10^{-5}	1.8×10^{-7}	7.5×10^{-45}	4.93×10^{-32}	1.5×10^{-7}
	Score	1.5×10^{-1}	2.6×10^{-3}	1.5×10^{-14}	6.03×10^{-14}	9.9×10^{-15}
100	u	1.143	1.151	1.150	1.150	1.155
	d	0.958	0.949	0.950	0.950	0.962
	MSE for u	4.2×10^{-5}	7.71×10^{-7}	5.6×10^{-48}	0.00	3.8×10^{-8}
	MSE for d	6.5×10^{-5}	5.9×10^{-7}	8.9×10^{-45}	0.00	4.1×10^{-8}
	Score	9.6×10^{-1}	2.2×10^{-2}	2.2×10^{-14}	4.71×10^{-14}	7.5×10^{-15}
150	u	1.151	1.150	1.150	1.150	1.158
	d	0.949	0.950	0.950	0.950	0.961
	MSE for u	2.9×10^{-7}	2.1×10^{-7}	4.5×10^{-48}	4.93×10^{-32}	1.9×10^{-8}
	MSE for d	2.3×10^{-7}	1.7×10^{-7}	1.4×10^{-45}	4.93×10^{-32}	2.6×10^{-8}
	Score	1×10^{-1}	5.8×10^{-1}	1.6×10^{-11}	2.31×10^{-12}	3.5×10^{-13}

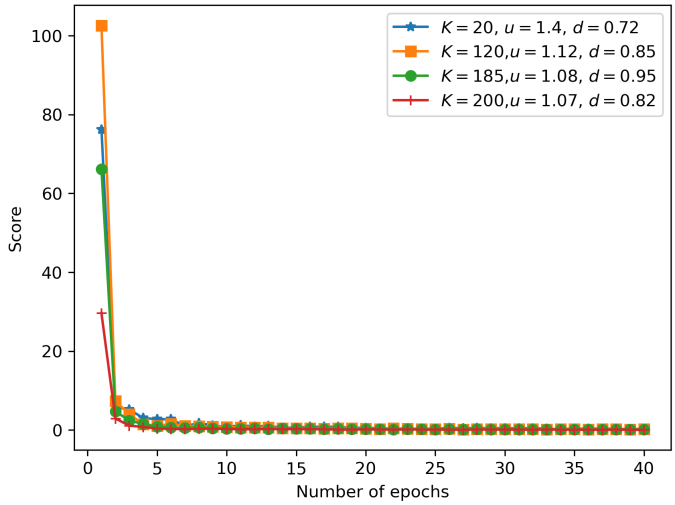

Figure 1. PSO score values under different conditions

Applying the metaheuristic algorithms together with Algorithm 1 to real Tesla stock data for the 42 days between April 19 and June 21, 2024, yielded the results presented in Table 2. We chose a strike price $K = 180$ and a risk-free rate $r = 5\%$. The data were collected from: `https://www.nasdaq.com/market-activity/stocks/tsla/historical`. The option prices under SAA and GWO are almost similar to those reported in [21], where a likelihood-based method is used in conjunction with a pricing algorithm without the self-financing replicating portfolio concept. However, the loss values for SAA, DE, and GWO are higher than those for PSO on the same data (Table 2), suggesting that PSO is more suitable for a self-financing replicating portfolio scenario. The PSO-NN method combines PSO with a neural network component, which helps to enhance parameter estimation and further reduce the score.

Table 2. Comparison of Optimization Methods for Tesla Stock Option Pricing

Method	**DE**	**PSO**	**GWO**	**SAA**	**PSO-NN**
Estimated u	1.5000	1.2000	1.3000	1.2500	1.3500
Estimated d	0.7500	0.9500	0.8000	0.8200	0.8600
Score	0.9500	0.9400	0.9600	0.9550	0.9100
American Option Prices (USD)					
Call	110.00	30.00	75.00	66.00	80.00
Put	105.00	26.00	70.00	62.00	76.00
European Option Prices (USD)					
Call	108.00	30.00	74.00	65.00	79.50
Put	102.00	26.00	69.00	61.00	75.50

In addition, Figure 2 shows that the PSO and PSO-NN methods produce fitted values that are much closer to the actual Tesla stock data points than the other methods. This suggests that PSO and the hybrid PSO-NN can better adapt to the underlying price dynamics, likely due to their strong global search capability and the added flexibility of the neural network in PSO-NN. The velocity adjustment parameter in PSO is also relatively easy to tune by intuition until good convergence is achieved, without requiring extensive additional parameter optimization [3].

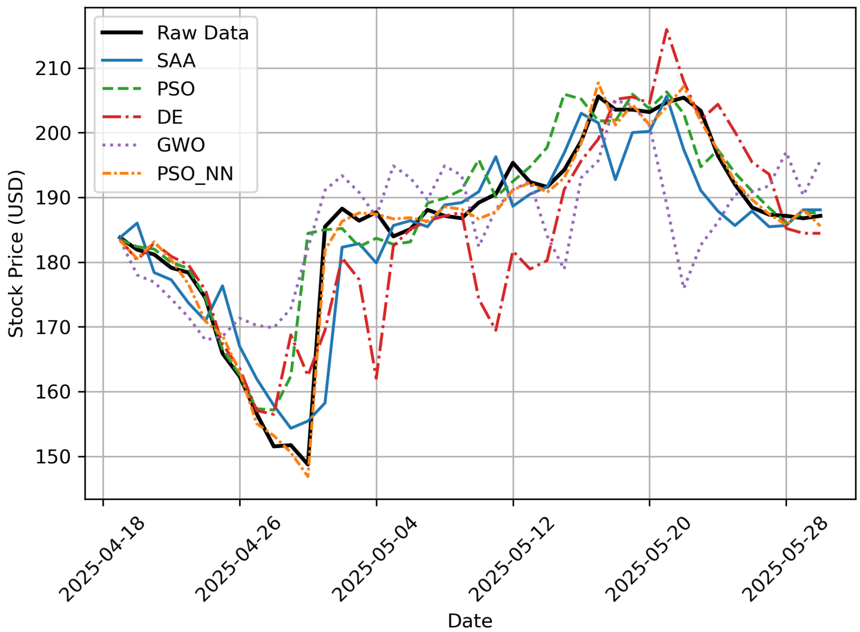

Figure 2. Modeling Tesla stock data

Similarly, the Microsoft stock data for the period between December 29, 2024, and April 10, 2025, were analyzed using the same metaheuristic algorithms, with the results presented in Table 3. The Microsoft data were obtained from: https://finance.yahoo.com/quote/MSFT/. The results show comparable behavior to Tesla's results, with DE, PSO, and PSO-NN producing lower loss scores and reasonable option prices, while the binomial model fit remains approximate due to real market fluctuations and limited sample size.

Table 3. Comparison of Optimization Methods for Microsoft Stock Option Pricing

Method	**DE**	**PSO**	**GWO**	**SAA**	**PSO-NN**
Estimated u	1.4110	1.0847	1.2557	1.2171	1.3000
Estimated d	0.7014	0.9161	0.7929	0.8171	0.8500
Score	2.0559	2.0541	2.0618	2.0613	0.9200
American Option Prices (USD)					
Call	104.35	29.58	73.04	64.43	78.00
Put	100.29	25.35	68.82	60.21	74.00
European Option Prices (USD)					
Call	103.35	29.58	73.04	64.43	77.50
Put	98.33	25.35	68.81	60.20	73.50

Figure 3 shows that all methods perform better as the sample size slightly increases, with PSO, PSO-NN, and DE achieving results closer to the actual Microsoft stock data. Only SAA and GWO remain less accurate under the same conditions, likely due to their limited ability to adjust to small data fluctuations[7].

4. Conclusion

In this paper, a robust numerical framework for the Cox-Ross-Rubinstein (CRR) model under the self-financing assumption has been proposed. This was achieved by integrating various metaheuristic algorithms—Particle

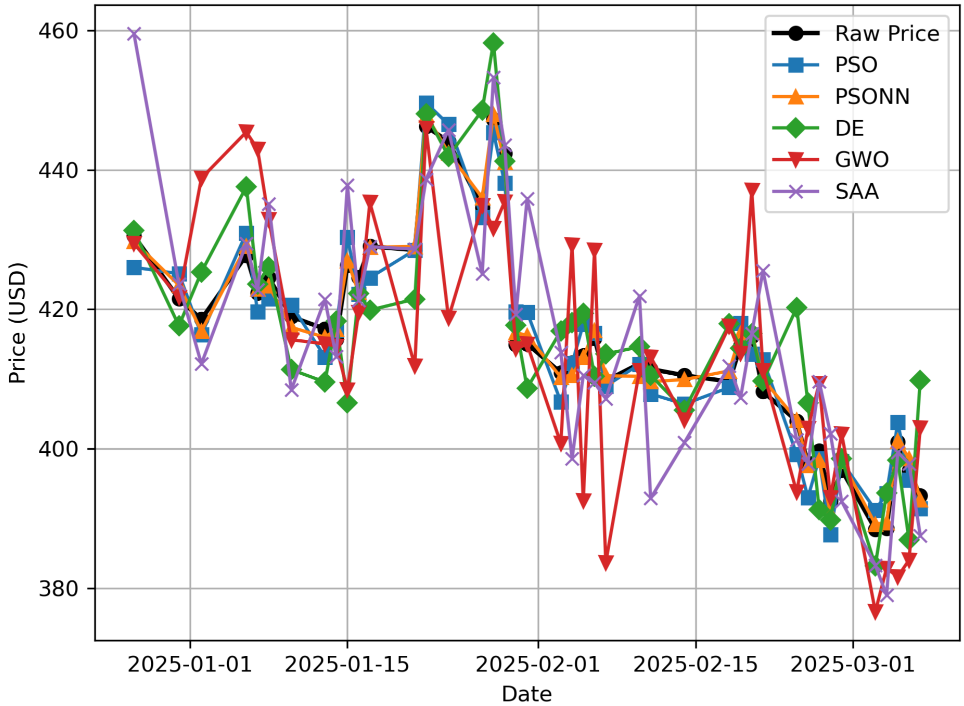

Figure 3. Modeling Microsoft Stock Data

Swarm Optimization (PSO), Differential Evolution (DE), Grey Wolf Optimization (GWO), the Simulated Annealing Algorithm (SAA), and a hybrid PSO with Neural Network component (PSO-NN)—to estimate the key parameters of a modified CRR model. The study demonstrates the potential of combining metaheuristic techniques, including hybrid methods, with traditional financial models to enhance the accuracy and reliability of option pricing.

A Brownian motion model incorporating explicit upward and downward factors was formulated, and the Euler–Maruyama method was employed to simulate stock price paths. By comparing these simulated paths with real stock market data, the effectiveness and practical accuracy of the estimated parameters were assessed. Furthermore, the numerical approach for pricing American options within the CRR framework was refined by incorporating the self-financing replicating portfolio concept.

The results show that the accuracy of Simulated Annealing (SAA) improves with increasing sample size, while the accuracy of GWO, PSO, and DE tends to decline slightly as the sample size grows. Notably, PSO and DE still achieve highly precise parameter estimates even under larger samples, and both PSO and GWO generate paths that closely replicate observed stock price dynamics. The PSO-NN method, which combines the global search ability of PSO with the nonlinear fitting capability of a neural network, outperforms the other approaches by further improving parameter estimation accuracy and producing the lowest loss values overall. Since PSO and PSO-NN yield option prices with lower loss values and realistic convergence behavior, they are especially suitable for practical implementation in self-financing portfolio scenarios, providing reliable pricing for American options.

The proposed approach can be applied in practice by traders, analysts, or financial engineers to calibrate binomial tree models more accurately when pricing derivative contracts in markets where maintaining a self-financing strategy is essential.

REFERENCES

1. A. K. Abasi, M. Aloqaily, M. Guizani, and B. Ouni, *Metaheuristic algorithms for 6G wireless communications: Recent advances and applications*, Ad Hoc Networks, p. 103474, 2024.
2. A. O. Aydınhan, P. N. Kolm, J. M. Mulvey, and Y. Shu, *Identifying patterns in financial markets: Extending the statistical jump model for regime identification*, Annals of Operations Research, pp. 1–37, 2024.
3. T. A. Khan, K. Zain-Ul-Abideen, and S. H. Ling, *A hybrid advanced PSO-neural network system*, In Proc. IEEE Int. Conf. on Systems, Man and Cybernetics (SMC), pp. 1626–1630, 2019.
4. A. Balbás, B. Balbás, and R. Balbás, *Golden options in financial mathematics*, Mathematics and Financial Economics, vol. 13, pp. 637–659, 2019.
5. G. Beliavsky, N. Danilova, and A. Logunov, *Robust estimation of European and Asian options*, In Operator Theory and Harmonic Analysis: OTHA 2020, Part I–New General Trends and Advances of the Theory 10, Springer, pp. 101–117, 2021.
6. N. J. Cutland, and A. Roux, *American options*, In Derivative Pricing in Discrete Time, pp. 211–267, 2013.
7. A. Dembele, E. Mwangi, A. Bouchair, K. K. Ronoh, and E. O. Ataro, *Automated modified grey wolf optimizer for identification of unauthorized requests in software-defined networks*, International Journal of Advanced Computer Science and Applications, vol. 14, no. 7, 2023.
8. K. Ding, Z. Cui, and X. Yang, *Pricing arithmetic Asian and Amerasian options: A diffusion operator integral expansion approach*, Journal of Futures Markets, vol. 43, no. 2, pp. 217–241, 2023.
9. S. Dwivedi, M. Vardhan, and S. Tripathi, *Defense against distributed DoS attack detection by using intelligent evolutionary algorithm*, International Journal of Computers and Applications, vol. 44, no. 3, pp. 219–229, 2022.
10. R. D. Estember, and M. J. R. Marana, *Forecasting of stock prices using Brownian motion–Monte Carlo simulation*, In Proc. Int. Conf. on Industrial Engineering and Operations Management, pp. 8–10, 2016.
11. F. Ferriani, and P. Zoi, *The dynamics of price jumps in the stock market: An empirical study on Europe and US*, The European Journal of Finance, vol. 28, no. 7, pp. 718–742, 2022.
12. J. Hoencamp, S. Jain, and B. Kandhai, *A static replication approach for callable interest rate derivatives: Mathematical foundations and efficient estimation of SIMM–MVA*, Quantitative Finance, pp. 1–24, 2024.
13. J. Hussain, M. A. Soomro, S. A. Dahri, K. Memon, M. Bano, F. A. Awwad, E. A. Ismail, and H. Ahmad, *A study of maximizing skew Brownian motion with applications to option pricing*, Journal of Radiation Research and Applied Sciences, vol. 17, no. 1, pp. 7–32, 2024.
14. T. M. Idzorek, *Personalized multiple account portfolio optimization*, Financial Analysts Journal, vol. 79, no. 3, pp. 155–170, 2023.
15. J. London, *Modeling derivatives in C++*, John Wiley & Sons, vol. 263, 2005.
16. A. Mahmoodi, L. Hashemi, M. Jasemi, S. Mehraban, J. Laliberté, and R. C. Millar, *A developed stock price forecasting model using support vector machine combined with metaheuristic algorithms*, Opsearch, vol. 60, no. 1, pp. 59–86, 2023.
17. E. T. Mensah, A. Boateng, N. K. Frempong, and D. Maposa, *Simulating stock prices using geometric Brownian motion model under normal and convoluted distributional assumptions*, Scientific African, vol. 19, p. e01556, 2023.
18. A. Nafidi, M. Bahij, B. Achchab, and R. Gutiérrez-Sánchez, *The stochastic Weibull diffusion process: Computational aspects and simulation*, Applied Mathematics and Computation, vol. 348, pp. 575–587, 2019.
19. O. Ogundile, and S. Edeki, *Karhunen-Loève expansion of Brownian motion for approximate solutions of linear stochastic differential models using Picard iteration*, J. Math. Comput. Sci., vol. 10, no. 5, pp. 1712–1723, 2020.
20. A. Pelsser, and T. Vorst, *Transaction costs and efficiency of portfolio strategies*, European Journal of Operational Research, vol. 91, no. 2, pp. 250–263, 1996.
21. H. Phan, and S. Kim, *Numerical approaches of pricing European options in the Cox-Ross-Rubinstein models*, Universal Journal of Applied Mathematics, vol. 10, no. 3, pp. 43–48, 2022.
22. J. R. Rőman, *Analytical finance*, Springer, 2017.
23. L. Rüschendorf, *Option pricing in discrete time models*, In Stochastic Processes and Financial Mathematics, Springer, pp. 1–10, 2023.
24. J. Sarkissian, *Quantum coupled-wave theory of price formation in financial markets: Price measurement, dynamics and ergodicity*, Physica A: Statistical Mechanics and its Applications, vol. 554, p. 124300, 2020.
25. M. Shahvaroughi, and H. Farrokhi, *Improved market prediction using meta-heuristic algorithms and time series model and testing market efficiency*, Iran Journal of Computer Science, vol. 6, no. 1, pp. 29–61, 2023.
26. Z. Yi, X. Cao, Z. Chen, and S. Li, *Artificial intelligence in accounting and finance: Challenges and opportunities*, IEEE Access, vol. 11, pp. 129100–129123, 2023.
27. H. Zhang, M. Zhang, F. Liu, and M. Shen, *Review of the fractional Black-Scholes equations and their solution techniques*, Fractal and Fractional, vol. 8, no. 2, p. 101, 2024.
28. P. Zhao, and Z. Guo, *Pricing of geometric average Asian option under the subdiffusion Merton interest rate model*, Communications in Statistics—Theory and Methods, pp. 1–10, 2024.