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Application of Ujlayan-Dixit fractional chi-square probability distribution

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Abstract In this study, we take into account the Ujlayan-Dixit (UD) fractional derivative in order to introduce the fractional probability density function for the Chi-Square distribution (CSD), and to establish certain new applications for this distribution through the use of fractional concepts in probability theory, such as cumulative distribution, survival and hazard functions. Furthermore, other ideas and applications for continuous random variables are developed using the UD fractional analogs of statistical measures wiche is expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Lastly, we provide the UD fractional entropy measures including Shannon, Tsallis and Rényi entropy.

Keywords Chi-Square Distribution (CSD), Continuous Random Variables, Probability Distribution, UD Fractional Derivative, Entropy

AMS 2010 subject classifications 26A33, 62E10

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1. Introduction

Fractional calculus is an extension of classical calculus that generalizes integrals and derivatives to non-integer orders. It has recently gained prominence as a research field focused on modeling phenomena and solving practical problems [1, 2, 3, 4, 5]. Many definition of fractional integrals and derivatives have been suggested, including Riemann-Liouville [6], Caputo [7], Caputo-Hadamard [8], and Conformable derivative [9]. Fractional calculus has several applications in scientific and engineering areas, and plays an essential role in various disciplines, like physics, chemistry, biology, computer science and economics; for details, see the references [10, 11, 12, 13, 14, 15, 16].

In 2020, Dixit and Ujlayan [17, 18] introduced a new fractional derivative called "Ujlayan-Dixit (UD) fractional derivative" which converts the fractional derivative into a convex combination of the function and its ordinary derivative. The UD fractional derivative aims to provide certain specific computational or physical advantages, depending on its use in fractional order systems and fractional differential equations.

Probability distributions are a key concept in probability theory and statistics, and describe how to assign probabilities to the results of a random variable. It provides a systematic method for modeling and simulating volatility and uncertainty in real-world phenomena (see [19, 20]). Among the interesting distributions is the Chi-square distribution (CSD), which is a probability distribution widely used in statistics, primarily for hypothesis testing and analysing categorical data. It is derived as the distribution of the sum of the squares of the parameter k independent standard normal random variables. The chi-square distribution (CSD) is essential in fields such as the

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social sciences, biology, and engineering, due to its flexibility and explanatory power that make it a cornerstone of modern statistical inference, which has led to the interest of scholars in it; for example, see the references [21, 22].

In actuality, the relationship between fractional calculus and probability theory has been the focus of many researchers, as the intersection of these two fields represents an effective means of modeling complex phenomena characterized by randomness and long memory. Especially in probability distributions, fractional derivatives play an important role in modeling stochastic processes, as they generalize classical stochastic models. This makes fractional calculus an essential tool for understanding probabilistic systems; see the references [23, 24]. The authors have recently applied fractional derivatives to probability distributions, obtaining interesting results, particularly with regard to the conformable fractional derivative, where they have developed the majority of applications to conformable fractional probability distributions; further details can be found in the sources [25].

In [26], the authors introduced properties and applications of conformable fractional Chi-square probability distribution (CSD), namely density, cumulative distribution, survival and hazard functions. In addition, it developed conformable fractional analogues, and it presented conformable fractional analogues to some entropy measures. Recently, Alhribat *et al.* established the UD fractional probability distribution functions for the exponential, Pareto, Levy, and Lomax distributions in [27], by using the UD fractional differential equations and based on previously existing probability distributions.

The choice of the Ujlayan-Dixit (UD) fractional derivative over more traditional operators such as the Caputo or Riemann–Liouville derivatives is motivated by its simpler structure, absence of singular kernels, and better analytical tractability. Unlike the Caputo and Riemann–Liouville derivatives, which often involve complex integral forms and memory-dependent behavior, the UD derivative maintains a linear and non-singular form that enables the derivation of closed-form expressions for probability density functions and related statistical measures. This makes it especially useful in constructing generalized distributions while preserving analytical simplicity.

Based on the above, in this study we are interested to present the UD fractional probability density function (FPDF) for the Chi-Square distribution (CSD) by using the UD fractional derivative, and to provide some applications for this distribution like cumulative distribution, survival and hazard functions. Then, we develop new notions of the UD fractional analogues, including expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Finally, we will establish the UD fractional entropy measures such as Shannon, Tsallis, and Rényi entropy.

2. Basic Concepts

In this section, we will review the basic concepts and properties of the UD fractional derivative and the UD fractional integral; for further details, refer to these references [17, 18].

Definition 2.1
[18] For a function $f : [0, +\infty) \to \mathbb{R}$, the UD fractional derivative of order $\alpha \in [0, 1]$ is defined as:

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} f\left(x e^{\frac{\varepsilon \alpha}{x}}\right) - f(x)}{\varepsilon}, \tag{1}$$

if limit exists. Also, if f is UD differentiable in the interval $(0, x)$ and for $x > 0$ and $\alpha \in [0, 1]$ such that $\lim_{x \to 0^+} f^\alpha(x)$ exist, then

$$f^\alpha(0) = \lim_{x \to 0^+} f^\alpha(x),$$

we notice that,

$$D^\alpha f(x) = \frac{d^\alpha f}{dx^\alpha}.$$

Theorem 2.2
[17] Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function. Then, f is UD differentiable, and

$$D^\alpha f(x) = (1 - \alpha) f(x) + \alpha f'(x), \quad \alpha \in [0, 1]. \tag{2}$$

In particular cases, if $\alpha = 0$ we have $D^0 f(x) = f(x)$, and if $\alpha = 1$ we have $D^1 f(x) = f'(x)$.

Lemma 2.3
[18] Let $a, b \in \mathbb{R},\ x \geq 0$ and for $\alpha \in [0,1]$. The UD derivatives of some elementary real-valued differentiable are given by:

- $D^\alpha(c) = (1-\alpha)c.$
- $D^\alpha((ax+b)^n) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}.$
- $D^\alpha(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}.$
- $D^\alpha(e^{ax+b}) = ((1-\alpha)+a\alpha)e^{ax+b}.$

Properties 2.4
[17] Let $f_1, f_2 : [0,+\infty) \to \mathbb{R}$ be two differentiable functions and for $0 \leq \alpha, \beta \leq 1$. The UD fractional derivative has the following properties:

- The UD derivative is a linear operator, such that for all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have:
$$D^\alpha(\lambda_1 f_1(x) + \lambda_2 f_2(x)) = \lambda_1 D^\alpha f_1(x) + \lambda_2 D^\alpha f_2(x).$$
- The UD derivative satisfies the product rule, i.e:
$$D^\alpha(f_1(x).f_2(x)) = (D^\alpha f_1(x))f_2(x) + \alpha(D^\alpha f_2(x))f_1(x).$$
Hence, The UD derivative does not satisfy the Leibnitz's rule, i.e:
$$D^\alpha(f_1(x).f_2(x)) \neq f_2(x)D^\alpha f_1(x) + f_1(x)D^\alpha f_2(x).$$
- The UD derivative satisfies the quotient rule, i.e:
$$D^\alpha(f_1(x).f_2(x)) = \frac{(D^\alpha f_1(x))f_2(x) - \alpha(D^\alpha f_2(x))f_1(x)}{(f_2(x))^2}, \quad \text{with } \ f_2(x) \neq 0.$$
- The UD derivative is a commutative operator, i.e:
$$D^\alpha(D^\beta f_1(x)) = D^\beta(D^\alpha f_1(x)).$$
- The UD derivative does not satisfy the semi-group property, i.e:
$$D^\alpha(D^\beta f_1(x)) \neq D^{\alpha+\beta} f_1(x).$$

Definition 2.5
[17] For a continuous function $f : [a,b] \to \mathbb{R}$, the UD fractional integral of order $\alpha \in (0,1]$ is defined as:
$$I_a^\alpha f(x) = \frac{1}{\alpha}\int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f(t)dt.$$

Properties 2.6
[17] Let $f_1, f_2 : [0,+\infty) \to \mathbb{R}$ be two continuous functions and for $0 \leq \alpha, \beta \leq 1$. The UD fractional integral has the following properties:

- The UD integral is a linear operator, such that for all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have:
$$I^\alpha(\lambda_1 f_1(x) + \lambda_2 f_2(x)) = \lambda_1 I^\alpha f_1(x) + \lambda_2 I^\alpha f_2(x).$$
- The UD integral is a commutative operator, i.e:
$$I^\alpha(I^\beta f_1(x)) = I^\beta(I^\alpha f_1(x)).$$
- The UD integral does not satisfy the semi-group property, i.e.:
$$I^\alpha(I^\alpha f_1(x)) \neq I^{2\alpha} f_1(x).$$

3. Main Results

This section focuses on presenting the main results related to the fractional probability density function of the Chi-Square distribution (CSD) by using the UD derivative, and establishing some new applications for this distribution.

3.1. The UD Fractional Chi-Square Distribution (UDFCSD)

The probability density function (PDF) of a continuous random variable X for the Chi-Square distribution (CSD) with k degrees of freedom is given by (see [28]):

$$f(x,k) = \frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}, \quad x \geq 0, \tag{3}$$

where $k > 0$ and $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$ is called the Gamma function. If X has Chi-Square distribution (CSD), then we denote it by $X \sim \chi^2(k)$.
Firstly, let's take $y = \frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}$. Then, we give the first derivative of y as:

$$y' = \frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}\left(\frac{k-2}{2x} - \frac{1}{2}\right) = \left(\frac{k-2}{2x} - \frac{1}{2}\right)y.$$

Therefore, it can be written by:

$$y' - \left(\frac{k-2}{2x} - \frac{1}{2}\right)y = 0. \tag{4}$$

So, the equation (4) is a first-order ordinary differential equation.
Secondly, we consider the α-order differential equation with respect to the UD derivative as follows:

$$\begin{aligned}
y^{(\alpha)} - \left(\frac{k-2}{2x} - \frac{1}{2}\right)y &= 0,\\
(1-\alpha)y + \alpha y' - \left(\frac{k-2}{2x} - \frac{1}{2}\right)y &= 0,\\
\alpha y' + \left(1 - \alpha - \frac{k-2}{2x} + \frac{1}{2}\right)y &= 0,\\
y' + \left(\frac{3-2\alpha}{2\alpha} - \frac{k-2}{2\alpha x}\right)y &= 0.
\end{aligned} \tag{5}$$

Hence, the equation (5) is a linear first-order differential equation with an integrating factor

$$\begin{aligned}
\varphi(x) &= e^{\int\left(\frac{3-2\alpha}{2\alpha} - \frac{k-2}{2\alpha x}\right)dx},\\
&= e^{\frac{3-2\alpha}{2\alpha}x - \frac{k-2}{2\alpha}ln x},\\
&= x^{-\frac{k-2}{2\alpha}}e^{\frac{3-2\alpha}{2\alpha}x}.
\end{aligned}$$

Thus, we express the general solution to the equation (5) as:

$$\begin{aligned}
y &= \frac{\mathcal{C}}{\varphi(x)},\\
&= \mathcal{C}x^{\frac{k-2}{2\alpha}}e^{-\frac{3-2\alpha}{2\alpha}x}.
\end{aligned}$$

As a result, the new probability distribution is given by:

$$f_\alpha(x) = \mathcal{C}x^{\frac{k-2}{2\alpha}}e^{-\frac{3-2\alpha}{2\alpha}x}. \tag{6}$$

In order to determine the normalizing constant $\mathcal{C}$, we solve the following equation:

$$\int_0^{+\infty} f_\alpha(x)dx = 1.$$

This means that,

$$\begin{aligned}\int_0^{+\infty} \mathcal{C}\, x^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x} dx &= 1,\\ \mathcal{C}\,\frac{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}} &= 1.\end{aligned}$$

Consequently, we find the normalizing constant $\mathcal{C}$ as:

$$\mathcal{C} = \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}.$$

Finally, The UD fractional probability density function (UDFPDF) of the α-Chi-square distribution (α-CSD) can be expressed as follows:

$$f_\alpha(x) = \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)} x^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x}, \quad x>0,\ k>0,\ 0<\alpha<1. \tag{7}$$

Notice that,

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} = f(x,k). \tag{8}$$

Fig. 1 shows the comparison between the classical case of the probability density function (PDF) for the chi-square distribution (CSD) and the UD fractional probability density function (UDFPDF) for the α-chi-siquar distribution (α-CSD) for $\alpha = 1$ according to $k = 3$. Fig. 2 also shows the UD fractional probability density function (UDFPDF) for the α-chi-square distribution (α-CSD) by taking various values of α according to $k = 3$.

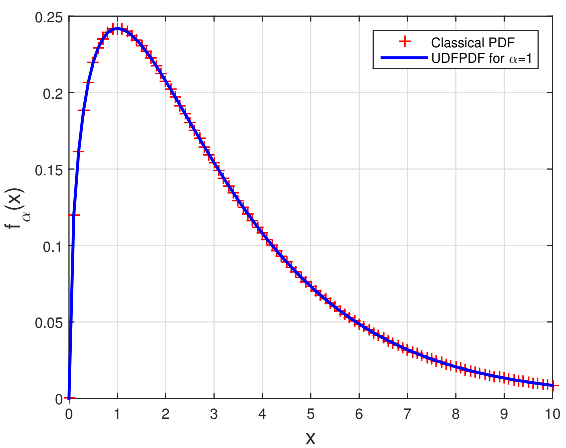

Figure 1. Comparison of the classical PDF of CSD with the UDFPDF of α-CSD for $\alpha = 1$ and $k = 3$.

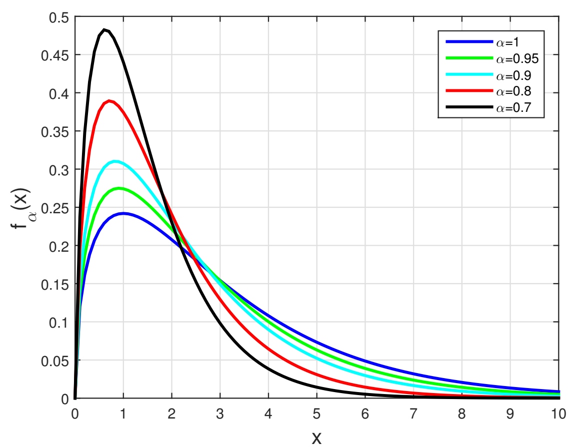

Figure 2. the UDFPDF of α-CSD for various values of α according to $k = 3$.

3.2. Applications To The UD Fractional Chi-Square Distribution (α-CSD)

In this part, we devlop new applications of the UD fractional probability for the α-chi-square distribution (α-CSD) on probabilistic random variables; based on these references [26, 29].

3.2.1. The UD fractional cumulative distribution function The UD fractional cumulative distribution function (UDFCDF) for the α-CSD can be expressed as:

$$\mathcal{F}_\alpha(x) = \frac{(3-2\alpha)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-2}{2\alpha}+1, \frac{x}{2\alpha}\right), \tag{9}$$

where $\gamma(x,y) = \int_0^y t^{x-1}e^{-t}dt$ is called lower incomplete Gamma function. In effect,

$$\begin{aligned}
\mathcal{F}_\alpha(x) &= \mathcal{P}_\alpha(X \le x), \\
&= I_0^\alpha f(x), \\
&= \frac{1}{\alpha}\int_0^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f_\alpha(t)dt, \\
&= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} \int_0^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} t^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}t}dt, \\
&= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x} \int_0^x t^{\frac{k-2}{2\alpha}} e^{-\frac{t}{2\alpha}}dt,
\end{aligned}$$

Using the change of variable $u = \frac{t}{2\alpha}$, we get:

$$\begin{aligned}
\mathcal{F}_\alpha(x) &= \frac{(3-2\alpha)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1)\right)} e^{\frac{(\alpha-1)}{\alpha}x} \int_0^{\frac{x}{2\alpha}} u^{\frac{k-2}{2\alpha}} e^{-u}du, \\
&= \frac{(3-2\alpha)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-2}{2\alpha}+1, \frac{x}{2\alpha}\right).
\end{aligned}$$

If $\alpha \to 1^-$ in the formula (9), then we get the classical cumulative distribution function (CDF) for CSD, i.e:

$$\lim_{\alpha\to 1^-} \mathcal{F}_\alpha(x) = \frac{\gamma\left(\frac{k}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \mathcal{F}(x). \tag{10}$$

Actually, we can explain the graphical comparison between the UD fractional cumulative distribution function (UDFCDF) of α-chi-square distribution (α-CSD) for $\alpha = 1$ and the classical case of the cumulative distribution function (CDF) of chi-square distribution (CSD) according to $k = 3$ in Fig. 3. Also, we can plot the UD fractional cumulative distribution function (UDFCDF) of α-chi-square distribution (α-CSD) for different values of α according to $k = 3$ as shown in Fig. 4.

3.2.2. The UD fractional survival distribution function For α-CSD, the UD fractional survival distribution function (UDFSDF) of X is given by:

$$\begin{aligned}
\mathcal{S}_\alpha(x) &= 1 - \mathcal{F}_\alpha(x), \\
&= 1 - \frac{(3-2\alpha)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\gamma\left(\frac{k-2}{2\alpha}+1, \frac{x}{2\alpha}\right).
\end{aligned} \tag{11}$$

Notice that, for $\alpha \to 1^{-}$, we have the classical survival distribution function (SDF) for CSD, i.e :

$$\lim_{\alpha\to 1^{-}} S_\alpha(x) = 1 - \frac{\gamma\left(\frac{k}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = S(x). \tag{12}$$

In Fig. 5, we compare the classical case of the survival distribution function (SDF) for chi-square distribution (CSD) with the UD fractional survival distribution function (UDFCDF) of α-chi-square distribution (α-CSD) for $\alpha = 1$ and $k = 3$. In Fig. 6, we display the UD fractional survival distribution function (UDFSDF) for α-chi-square distribution (α-CSD) under different value of α according to $k = 3$.

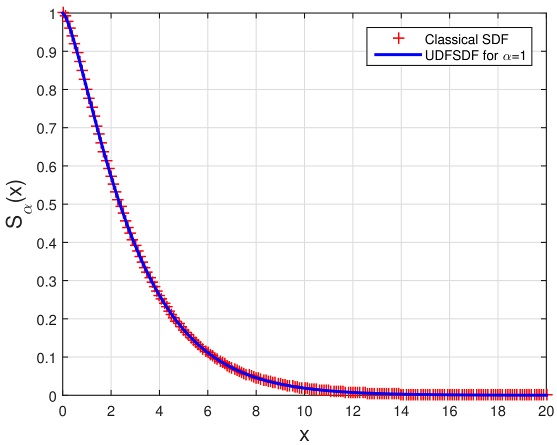

Figure 5. Comparison of the classical SDF of CSD with the UDFSDF of α-CSD for $\alpha = 1$ and $k = 3$.

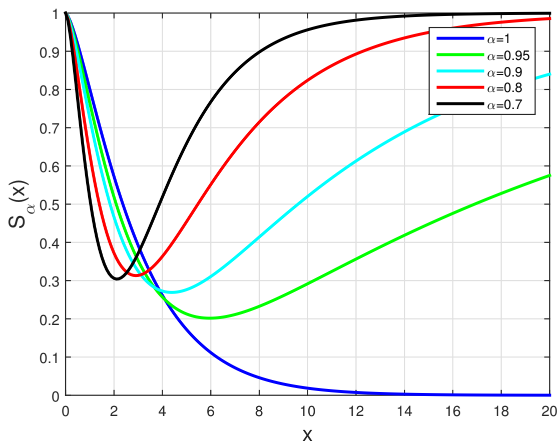

Figure 6. the UDFSDF of α-CSD for different values of α according to $k = 3$.

3.2.3. The UD fractional hazard distribution function For α-CSD, the UD fractional hazard distribution function (UDFHDF) of X is defined as:

$$\begin{aligned} h_\alpha(x) &= \frac{f_\alpha(x)}{S_\alpha(x)}, \\ &= \frac{\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)} x^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x}}{1 - \frac{(3-2\alpha)^{\frac{k-2}{2\alpha}+1}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x} \gamma\left(\frac{k-2}{2\alpha}+1, \frac{x}{2\alpha}\right)}, \\ &= \frac{\alpha\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1} x^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x}}{\alpha\Gamma\left(\frac{k-2}{2\alpha}+1\right) - (3-2\alpha)^{\frac{k-2}{2\alpha}+1} e^{\frac{(\alpha-1)}{\alpha}x} \gamma\left(\frac{k-2}{2\alpha}+1, \frac{x}{2\alpha}\right)}. \end{aligned} \tag{13}$$

In particular case,

$$\lim_{\alpha\to 1^{-}} h_\alpha(x) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\left(\Gamma\left(\frac{k}{2}\right) - \gamma\left(\frac{k}{2}, \frac{x}{2}\right)\right)} = h(x), \tag{14}$$

where h is the classical hazard distribution function (HDF) for CSD.
A comparison between the classical case of the hazard distribution function (HDF) for the chi-square distribution (CSD) and the UD fractional hazard distribution function (UDFHDF) of α-chi-square distribution (α-CSD) for $\alpha = 1$ and $k = 3$ can be illustrated in Fig. 7. Then, the UD fractional hazard distribution function (UDFHDF) for the α-chi-square distribution (α-CSD) can be plotted by taking different values of α according to $k = 3$ in Fig. 8.

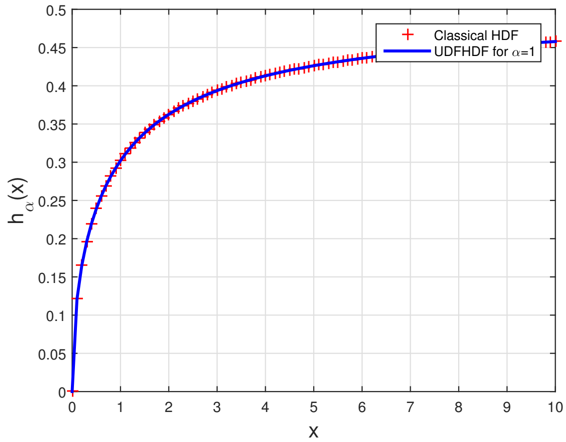

Figure 7. Comparison of the classical HDF of CSD with the UDFHDF of α-CSD for $\alpha = 1$ and $k = 3$.

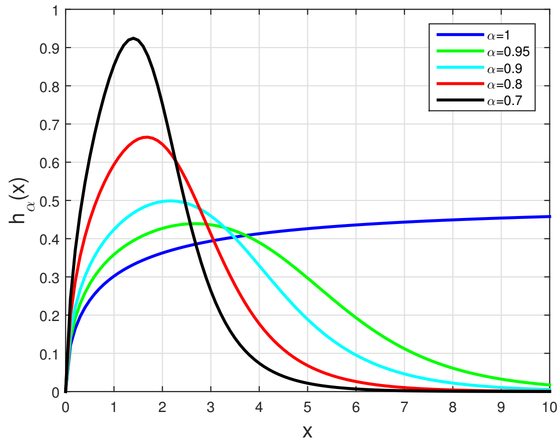

Figure 8. the UDFHDF of α-CSD for different values of α according to $k = 3$.

3.2.4. The UD fractional expectation

• *The r^{th} UD fractional moment $E_\alpha[X^r]$* The UD fractional moment of order r denote by $E_\alpha[X^r]$ of continuous random variable X whose UDFPDF $f_\alpha(x)$ is given as:

$$\begin{aligned} E_\alpha[X^r] &= \int_0^{+\infty} x^r f_\alpha(x)dx, \\ &= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)} \int_0^{+\infty} x^r x^{\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x}, \\ &= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)} \int_0^{+\infty} x^{r+\frac{k-2}{2\alpha}} e^{-\frac{3-2\alpha}{2\alpha}x}, \\ &= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)} \frac{\Gamma\left(\frac{k-2}{2\alpha}+r+1\right)}{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+r+1}}, \\ &= \frac{\left(\frac{2\alpha}{3-2\alpha}\right)^r \Gamma\left(\frac{k-2}{2\alpha}+r+1\right)}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}. \end{aligned} \tag{15}$$

For $r = 1$, we find the UD fractional expectation $E_\alpha[X]$:

$$\begin{aligned} E_\alpha[X] &= \frac{\left(\frac{2\alpha}{3-2\alpha}\right)\Gamma\left(\frac{k-2}{2\alpha}+2\right)}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}, \\ &= \frac{2\alpha+k-2}{3-2\alpha}. \end{aligned} \tag{16}$$

For $r = 2$, we have:

$$\begin{aligned} E_\alpha[X^2] &= \frac{\left(\frac{2\alpha}{3-2\alpha}\right)^2\Gamma\left(\frac{k-2}{2\alpha}+3\right)}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}, \\ &= \frac{(4\alpha+k-2)(2\alpha+k-2)}{(3-2\alpha)^2}. \end{aligned} \tag{17}$$

Note that, if $\alpha \to 1^-$ in the formula (15), then we get the classical the r^{th} moment $E[X^r]$, i.e:

$$\lim_{\alpha \to 1^-} E_\alpha[X^r] = \frac{2^r \Gamma\left(\frac{k}{2} + r\right)}{\Gamma\left(\frac{k}{2}\right)} = E[X^r]. \tag{18}$$

• *The r^{th} UD fractional central moment $E_\alpha(X-\mu)^r$* Firstly, we take:

$$\mu = E_\alpha[X] = \frac{(2\alpha + k - 2)}{(3 - 2\alpha)}. \tag{19}$$

Then, the r^{th} UD fractional central moment $E_\alpha(X-\mu)^r$ of X is defined as:

$$E_\alpha(X-\mu)^r = \int_0^{+\infty} (x-\mu)^r f_\alpha(x) dx. \tag{20}$$

According to the formula (20), we can obtain the first and second of central moments:

- First central moment:

$$E_\alpha(X-\mu) = 0. \tag{21}$$

- Second central moment:

$$E_\alpha(X-\mu)^2 = \frac{2\alpha(2\alpha + k - 2)}{(3-2\alpha)^2}. \tag{22}$$

• *The UD fractional variance Var_α* For α-CSD, we give the UD fractional variance Var_α of X as:

$$\begin{aligned} Var_\alpha(X) &= E_\alpha(X^2) - (E_\alpha(X))^2, \\ &= \frac{(4\alpha + k - 2)(2\alpha + k - 2)}{(3-2\alpha)^2} - \left(\frac{2\alpha + k - 2}{3 - 2\alpha}\right)^2, \\ &= \frac{2\alpha(2\alpha + k - 2)}{(3-2\alpha)^2}. \end{aligned} \tag{23}$$

Note that,

$$Var_\alpha(X) = 2k = Var(X). \tag{24}$$

where Var is the classical variance of X.

• *The UD fractional standard deviation σ_α* For α-CSD, the UD fractional standard deviation σ_α of X is square root of the variance and given as:

$$\begin{aligned} \sigma_\alpha &= \sqrt{Var_\alpha(X)}, \\ &= \sqrt{\frac{2\alpha(2\alpha + k - 2)}{(3-2\alpha)^2}}, \\ &= \frac{\sqrt{2\alpha(2\alpha + k - 2)}}{(3-2\alpha)^2}. \end{aligned} \tag{25}$$

In particular case,

$$\lim_{\alpha \to 1^-} \sigma_\alpha = \sqrt{2k} = \sigma. \tag{26}$$

where σ is the classical standard deviation of X.

3.2.5. The UD fractional Entropy Measures

• *The UD fractional Shannon entropy $\alpha\mathcal{H}$* For α-CSD, the UD fractional Shannon entropy $\alpha\mathcal{H}$ of a random variable X is defined by:

$$\alpha\mathcal{H}(X) \quad = \quad -\int_0^{+\infty} f_\alpha(x)\log(f_\alpha(x))dx. \tag{27}$$

Now, we calculate the following quantity:

$$\begin{aligned}\log(f_\alpha(x)) &= \log\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}x^{\frac{k-2}{2\alpha}}e^{-\frac{3-2\alpha}{2\alpha}x}\right),\\ &= \log\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)+\frac{k-2}{2\alpha}\log x-\frac{3-2\alpha}{2\alpha}x.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha\mathcal{H}(X) &= -\int_0^{+\infty} f_\alpha(x)\log(f_\alpha(x))dx,\\ &= -\int_0^{+\infty} f_\alpha(x)\left[\log\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)+\frac{k-2}{2\alpha}\log x-\frac{3-2\alpha}{2\alpha}x\right]dx,\\ &= \frac{3-2\alpha}{2\alpha}\int_0^{+\infty} x\ f_\alpha(x)dx-\log\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)\int_0^{+\infty} f_\alpha(x)dx-\frac{k-2}{2\alpha}\int_0^{+\infty}\log x\ f_\alpha(x)dx,\end{aligned}$$

Next, we simplify the following terms:

$$\int_0^{+\infty} f_\alpha(x)dx \quad = \quad 1,$$

$$\int_0^{+\infty} x\ f_\alpha(x)dx = E[X] = \frac{(2\alpha+k-2)}{(3-2\alpha)},$$

and

$$\begin{aligned}\int_0^{+\infty}\log x\ f_\alpha(x)dx &= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\int_0^{+\infty}\log x\ x^{\frac{k-2}{2\alpha}}e^{-\frac{3-2\alpha}{2\alpha}x}dx,\\ &= \frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\frac{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}\left[\psi\left(\frac{k-2}{2\alpha}+1\right)-\log\left(\frac{3-2\alpha}{2\alpha}\right)\right],\\ &= \psi\left(\frac{k-2}{2\alpha}+1\right)-\log\left(\frac{3-2\alpha}{2\alpha}\right),\end{aligned}$$

where $\psi(x+1)=\frac{\Gamma'(x+1)}{\Gamma(x+1)})$ is called Digamma function.
Lastly, we find the UD fractional Shannon entropy $\alpha\mathcal{H}$ as:

$$\alpha\mathcal{H}(X) \quad = \quad \frac{(2\alpha+k-2)}{2\alpha}+\log\left(\left(\frac{2\alpha}{3-2\alpha}\right)\Gamma\left(\frac{k-2}{2\alpha}+1\right)\right)-\frac{k-2}{2\alpha}\psi\left(\frac{k-2}{2\alpha}+1\right). \tag{28}$$

For $\alpha\to 1^-$ in the formula (28), we notice that it gives us the classical Shannon entropy $\mathcal{H}$ of X for CSD, i.e:

$$\lim_{\alpha\to 1^-}\alpha\mathcal{H}(X)=\frac{k}{2}+\log\left(2\Gamma\left(\frac{k}{2}\right)\right)+\left(1-\frac{k}{2}\right)\psi\left(\frac{k}{2}\right)=\mathcal{H}(X). \tag{29}$$

• *The UD fractional Tsallis entropy* $\alpha\mathcal{T}_q$ For α-CSD, the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$ of a random variable X is defined by:

$$\alpha\mathcal{T}_q(X) = \frac{1}{q-1}\left[1-\int_0^{+\infty}(f_\alpha(x))^q\,dx\right]. \tag{30}$$

Firstly, we calculate the integral $\int_0^{+\infty}(f_\alpha(x))^q\,dx$, we have:

$$\begin{aligned}\int_0^{+\infty}(f_\alpha(x))^q\,dx &= \int_0^{+\infty}\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}x^{\frac{k-2}{2\alpha}}e^{-\frac{3-2\alpha}{2\alpha}x}\right)^q dx,\\ &= \left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)^q\int_0^{+\infty}x^{\frac{k-2}{2\alpha}q}e^{-\frac{3-2\alpha}{2\alpha}qx},\\ &= \left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)^q\frac{\Gamma\left(q\frac{k-2}{2\alpha}+1\right)}{\left(q\frac{3-2\alpha}{2\alpha}\right)^{q\frac{k-2}{2\alpha}+1}}.\end{aligned}$$

Then, we get the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$ as:

$$\begin{aligned}\alpha\mathcal{T}_q(X) &= \frac{1}{q-1}\left[1-\int_0^{+\infty}(f_\alpha(x))^q\,dx\right],\\ &= \frac{1}{q-1}\left[1-\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)^q\frac{\Gamma\left(q\frac{k-2}{2\alpha}+1\right)}{\left(q\frac{3-2\alpha}{2\alpha}\right)^{q\frac{k-2}{2\alpha}+1}}\right].\end{aligned} \tag{31}$$

As $q\to 1$, the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$ reduces to the UD fractional Shannon entropy $\alpha\mathcal{H}$, i.e:

$$\lim_{q\to 1}\alpha\mathcal{T}_q(X) = \alpha\mathcal{H}(X). \tag{32}$$

Notice that,

$$\lim_{\alpha\to 1^-}\alpha\mathcal{T}_q(X) = \frac{1}{q-1}\left[1-\frac{\left(\frac{2}{q}\right)^{q\left(\frac{k}{2}-1\right)+1}\Gamma\left(q\left(\frac{k}{2}-1\right)+1\right)}{\left(2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\right)^q}\right] = \mathcal{T}_q(X), \tag{33}$$

where $\mathcal{T}_q$ is the classical Tsallis entropy of X for CSD.

• *The UD fractional Rényi entropy* $\alpha\mathcal{R}_q$ For α-CSD, the UD fractional Rényi entropy $\alpha\mathcal{R}_q$ of a random variable X is defined by:

$$\alpha\mathcal{R}_q(X) = \frac{1}{1-q}\log\left(\int_0^{+\infty}[f_\alpha(x)]^q dx\right). \tag{34}$$

We know that,

$$\int_0^{+\infty}(f_\alpha(x))^q\,dx = \left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)^q\frac{\Gamma\left(q\frac{k-2}{2\alpha}+1\right)}{\left(q\frac{3-2\alpha}{2\alpha}\right)^{q\frac{k-2}{2\alpha}+1}}.$$

Thus, we obtain the UD fractional Rényi entropy $\alpha\mathcal{R}_q$ as:

$$\alpha\mathcal{R}_q(X) \quad = \quad \frac{1}{1-q}\log\left(\left(\frac{\left(\frac{3-2\alpha}{2\alpha}\right)^{\frac{k-2}{2\alpha}+1}}{\Gamma\left(\frac{k-2}{2\alpha}+1\right)}\right)^q \frac{\Gamma\left(q\frac{k-2}{2\alpha}+1\right)}{\left(q\frac{3-2\alpha}{2\alpha}\right)^{q\frac{k-2}{2\alpha}+1}}\right). \tag{35}$$

As $q \to 1$, the UD fractional Rényi entropy $\alpha\mathcal{R}_q$ reduces to the UD fractional Shannon entropy $\mathcal{H}_q$, i.e:

$$\lim_{q\to 1}\alpha\mathcal{R}_q(X) = \alpha\mathcal{H}(X). \tag{36}$$

Note that, if $\alpha \to 1^-$ in the formula (35), then we get the classical Rényi entropy $\mathcal{R}_q$ of X for CSD, i.e:

$$\lim_{\alpha\to 1^-}\alpha\mathcal{R}_q(X) = \frac{1}{1-q}\log\left(\frac{\left(\frac{2}{q}\right)^{q\left(\frac{k}{2}-1\right)+1}\Gamma\left(q\left(\frac{k}{2}-1\right)+1\right)}{\left(2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\right)^q}\right) = \mathcal{R}_q(X). \tag{37}$$

It is worth mentioning that as the parameter α increases toward 1, all fractional entropy measures (Shannon, Tsallis, and Rényi) converge to their classical counterparts, reflecting a reduction in uncertainty and a return to standard behavior. In contrast, as α decreases, the entropies increase, indicating greater dispersion and uncertainty due to the dominance of the memory component inherent in the UD fractional derivative. This shows that the fractional order α acts as a tunable parameter that controls the spread and uncertainty of the distribution, offering flexibility in modeling complex systems.

3.3. *Assumptions and Limitations*

Although the UD fractional derivative does not satisfy the semi-group property, this limitation is not critical in the current statistical context, where the focus is on modeling rather than solving multi-step dynamic systems. On the other hand, throughout this work, we assume that the fractional order parameter satisfies $0 < \alpha < 1$ and the degrees of freedom $\nu > 0$. These conditions ensure the convergence of the integral definitions and preserve the probabilistic validity of the model. Additionally, a current limitation of the UDFPDF approach is the absence of empirical validation, which will be addressed in future work. Additionally, the model's behavior depends heavily on the selection of α, and determining its optimal value from data remains an open question. Finally, while the UDFPDF generalizes the classical chi-square distribution, the computational complexity of its analytical form may limit its use in closed-form statistical inference, motivating the development of numerical or approximate techniques.

3.4. *Asymptotic Behavior of the UDFPDF*

We now briefly examine the behavior of the UDFPDF in the limiting cases $\alpha \to 1$ and $\alpha \to 0$.

Case 1: $\alpha \to 1$. In this limit, the UD fractional derivative reduces to the classical first-order derivative. Consequently, the UDFPDF converges to the standard chi-square distribution with ν degrees of freedom. This demonstrates that the proposed model is a true generalization of the classical case.

Case 2: $\alpha \to 0$. As $\alpha \to 0$, the weight of the fractional component dominates, and the density function becomes highly diffused and loses its probabilistic meaning due to divergence or flattening. This indicates that the model is not meaningful in this extreme and supports restricting $\alpha \in (0, 1)$.

These asymptotic observations confirm the continuity of the model with respect to α and its compatibility with classical theory.

4. Conclusion

In this study, we have present some novel results for the Chi-square distribution (CSD) by applying the UD fractional derivative so that we create the fractional probability density function (FPDF) for the Chi-square distribution (CSD). Some of the applications of the α-Chi-square distribution (α-CUD) are developed, including cumulative distribution, survival and hazard functions with certain graphical representations of these applications. Also, we establish new concepts of the UD fractional analogues such as expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Finally, the UD fractional entropy measures which is Shannon, Tsallis and Rényi entropy are provided.

The proposed UDFPDF model may find useful applications in various fields where non-local behavior and memory effects are significant. In statistics, it can be used to model skewed or heavy-tailed data more flexibly than the classical chi-square distribution. In reliability engineering, the fractional behavior can capture wear-out processes that evolve over time. Similarly, in economics and finance, where long-range dependencies and irregular fluctuations are common, the UDFPDF may provide improved modeling accuracy. These potential applications motivate further study, including empirical fitting and model validation with real datasets.

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