STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **14**, October 2025, pp 1936–1963.
Published online in International Academic Press (www.IAPress.org)

Numerical Solutions of Multi-Dimensional Fractional Telegraph Equations

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Abstract This research introduces a new technique called the 'Yang Iterative Reduction Method' (YVIT) to solve the spatio-temporal telegraph equation (ST-TE) within the framework of fractional derivatives as defined by Caputo. This method combines the advantages of the Variational Iteration Method (VIM) and the Yang Transform to generate accurate and rapidly converging solutions, while reducing computational complexity by 40 compared to traditional methods such as Adomian Decomposition Method (ADM) and conventional Variational Iteration Method (VIM), by avoiding multiple integrations and auxiliary equations.The effectiveness of the method was tested on two numerical models for the ST-TE, where the approximate solutions showed a high degree of agreement with the analytical solutions (error margin less than 0.5). Graphical representations revealed the impact of fractional spatial and temporal factors on the behavior of the solutions.The results demonstrate that YVIT is an effective and easy-to-apply tool for simulating complex physical systems described by nonlinear partial differential equations with fractional components. Furthermore, it has the potential for development to address a broader range of multi-dimensional dynamic problems.

Keywords variational iteration method; Yang transform; Caputo fractional derivative; multidimensional Fractional Telegraph Equation.

DOI: 10.19139/soic-2335

1. Introduction

The fractional telegraph equations are important in the fields of physics and engineering, as they represent many physical phenomena and engineering systems. In this context, several methods have been developed to solve these equations, including the "Yang Variable Iteration Method" (YVIM), the Adomian Decomposition Method (ADM), and the Homotopy Perturbation Method.[3, 57]

The Adomian Decomposition Method (ADM) is one of the commonly used approaches to solving nonlinear partial differential equations, including fractional telegraph equations. This method relies on using an infinite series of terms to analyze the equation. Among its advantages, ADM can be used to solve complex nonlinear equations accurately, providing precise approximate solutions once the number of terms is reduced. However, a major disadvantage is that it requires a large number of computations, increasing computational complexity, and it may involve many series integrations, which can be computationally expensive.[19]

On the other hand, the Homotopy Perturbation Method is an advanced technique that combines perturbation methods with homotopy structures to solve differential equations. It is used in many engineering and physical applications to solve nonlinear and complex differential equations. The advantages of this method include providing accurate approximate solutions for a wide range of differential equations, and it can be relatively easily applied to nonlinear and semilinear equations. However, it requires proper selection of the perturbation, which can be challenging in some cases, and may not be effective for equations containing complex fractional components.[18]

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

The ”Yang Variable Iteration Method” (YVIM) offers several advantages over the previous methods, as it combines variable iteration with the Yang Transform to obtain accurate and rapidly converging solutions. One of the main advantages of this method is that it reduces computational complexity by up to 40 compared to traditional methods like ADM, avoiding the need for multiple integrations and auxiliary equations, simplifying computations. YVIM ensures rapid convergence of solutions with a small error margin (less than 0.5 in many cases), and it is easy to apply to nonlinear fractional partial differential equations. However, one of its drawbacks is that it may be challenging to apply to high-dimensional or extremely complex equations, and it might require adjusting certain techniques to match complex engineering problems.

Despite the advantages of YVIM, it does have some limitations. Challenges may arise when applying the method to equations with highly complex nonlinear components or high-dimensional equations. Additionally, in some cases, more time may be required to adjust the parameters and start the appropriate variable iteration process.

The YVIM method can be applied in various practical fields in physics and engineering. For example, in wave physics, YVIM can be used to solve fractional telegraph equations to study wave propagation in heterogeneous media. In this application, using fractional derivatives helps improve the simulation of effects such as scattering and asymmetric damping that may occur in composite materials. In electrical engineering, the method can be used to solve telegraph equations that describe signal transmission in electrical circuits containing nonlinear components, improving the accuracy of modeling and the measurement of interaction speeds between components in the circuit. It can also be used to solve fractional telegraph equations describing complex dynamic systems in mechanical engineering, such as studying nonlinear vibrations in large structural systems, contributing to better design and failure prevention.[12]

In conclusion, YVIM provides an effective and powerful alternative to traditional methods for solving fractional telegraph equations, offering significant advantages in reducing computational complexity and achieving accurate results quickly, making it a powerful and promising tool for solving fractional differential equations across various fields.

This study addresses the applications of the YVIM method to solve fractional telegraph equations, combining variable iteration with the Yang Transform to achieve accurate and rapidly converging solutions. Section (2) covers the foundational concepts related to fractional differential equations, while Section (3) presents the variable iteration framework in detail. Section (4) proves the convergence of the method, demonstrating its ability to provide accurate solutions. Section (5) discusses the development of the Yang Variable Iteration Method and its applications, followed by illustrative examples showing how the method is used to solve fractional equations. Section (6) presents a discussion of the results obtained from applying YVIM to various equations, and finally, the study concludes in Section (7) with a summary of the findings and suggestions for future research.

2. Preliminaries

Definition 1: [2, 3]

If $f(\varkappa) \in C([a,b])$, $\ltimes > 0$, and $a < \varkappa < b$, the Riemann-Liouville fractional integral is generally defined for an order $\ltimes$ Here's the standard form:

$$I_{\varkappa}^{\ltimes} f(\varkappa) = \frac{1}{\Gamma(\ltimes)} \int_a^{\varkappa} \frac{f(\mathfrak{t})}{(\varkappa - \mathfrak{t})^{1-\ltimes}}\, d\mathfrak{t}, \tag{1}$$

where $\Gamma(\ltimes)$ denotes the Gamma function.
Holds the following characteristics

$$I_{\varkappa}^{\ltimes} I_{\varkappa}^{\beta} f(\varkappa) = I_{\varkappa}^{\ltimes+\beta} f(\varkappa), \quad I_{\varkappa}^{\ltimes} I_{\varkappa}^{\beta} f(\varkappa) = I_{\varkappa}^{\beta} I_{\varkappa}^{\ltimes} f(\varkappa), \tag{2}$$

$$I_{\varkappa}^{\ltimes} \varkappa^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\ltimes+\beta+1)} \varkappa^{\ltimes+\beta}.$$

***Definition 2:* [11, 26]**

The Caputo fractional derivative of a function $f(\varkappa)$, $\varkappa > 0$,is formulated as:

$$D_{\varkappa}^{\ltimes} f(\varkappa) = \begin{cases} \frac{1}{\Gamma(\dot{\eta}-\ltimes)} \int_0^{\varkappa} (\varkappa - t)^{\dot{\eta}-\ltimes-1} f^{(\dot{\eta})}(t)\, dt, & \dot{\eta}-1 < \ltimes \leq \dot{\eta}, \eta \in \mathbb{N}, \\ \frac{d^{\dot{\eta}}}{d\varkappa^{\dot{\eta}}} f(\varkappa), & \ltimes = \dot{\eta} \in \mathbb{N}, \end{cases} \tag{3}$$

where $\dot{\eta} - 1 < \ltimes \leq \dot{\eta}$.

Note 1. Based on Definition 2, the following result can be derived:

$$D_{\mathrm{t}}^{\ltimes} \mathrm{t}^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\ltimes+1)} \mathrm{t}^{\beta-\ltimes}, & \dot{\eta}-1 < \ltimes \leq \dot{\eta}, \beta > \dot{\eta}-1, \beta \in \mathbb{R}, \\ 0, & \dot{\eta}-1 < \ltimes \leq \dot{\eta}, \beta = \dot{\eta}-1, \beta \in \mathbb{N}. \end{cases} \tag{4}$$

***Definition 3:* [4, 57]**

The Yang transform (YT) is defined as:

$$Y_a\{\mho(\mathrm{t})\} = \int_0^{\infty} e^{-\mathrm{t}/v} \mho(\mathrm{t})\, d\mathrm{t}, \quad \mathrm{t} > 0, \tag{5}$$

where v represents the transform variable.

***Definition 4:* [36, 57]**

The Yang transform of a fractional order derivative is defined as:

$$Y_a\{D_{\varkappa}^{\ltimes} \mho(\varkappa, \mathrm{t})\} = \frac{Y_a\{\mho(\mathrm{t})\}}{v^{\ltimes}} - \sum_{k=0}^{\dot{\eta}-1} \frac{\mho^{(k)}(0)}{v^{\ltimes-k-1}}, \quad \dot{\eta}-1 < \ltimes \leq \dot{\eta}. \tag{6}$$

Some properties:

$$Y_a\{1\} = v, \quad Y_a\{\mathrm{t}\} = v^2, \quad Y_a\{\mathrm{t}^{\eta}\} = v^{\eta+1}\eta!, \quad Y_a\{\mathrm{t}^{\ltimes}\} = v^{\ltimes+1}\Gamma(\ltimes+1).$$

***Definition 5:* [47, 53]**

The two-parameter Mittag-Leffler function is expressed as:

$$E_{\ltimes,\beta}(z) = \sum_{\eta=0}^{\infty} \frac{z^{\eta}}{\Gamma(\eta \ltimes + \beta)}, \quad \ltimes, \beta, z \in \mathbb{C}, \mathrm{Re}(\ltimes) > 0, \mathrm{Re}(\beta) > 0. \tag{7}$$

Note 2. Based on Definition 1, the following results can be derived:

$$\begin{aligned} &(1) \quad E_{2,1}(\varkappa^2) = \cos(k(\varkappa)), \\ &(2) \quad E_{2,2}(\varkappa^2) = \frac{\sin(k(\varkappa))}{\varkappa}, \\ &(3) \quad E_{2,3}(\varkappa^2) = \frac{1}{\varkappa^2}[-1+\cos(k(\varkappa))]. \end{aligned} \tag{8}$$

3. Variational Iteration Method

The Variable Iteration Method (VIM) [57] has been developed and is extensively utilized to find exact or near solutions for both linear and nonlinear systems, the Variable Iteration Method (VIM) generates solutions as a quickly converging infinite series. In order to apply VIM, let us consider the following general nonlinear equation with an associated auxiliary condition:

$$L\mho(\varkappa,\mathrm{t})+N\mho(\varkappa,\mathrm{t})=f(\varkappa,t), \tag{9}$$

The corrective function for equation (9) is expressed as:

$$\mho_{\eta+1}(\varkappa,\mathrm{t})=\mho_{\eta}(\varkappa,\mathrm{t})+\int_0^{\varkappa}\bar{\lambda}\left[L\mho_{\eta}(\varsigma,\mathrm{t})+N\mho_{\eta}(\varsigma,\mathrm{t})-f(\varsigma,\mathrm{t})\right]d\varsigma, \tag{10}$$

where $\bar{\lambda}$ is the generalized Lagrange multiplier, which might be ideally determined by the variation theorem. The subscript η represents the approximation, and $\mho_{\eta}$ refers to the constrained variation.

4. Yang Variational Iteration Method (YVIM)

We examine the following generalized version of the fractional telegraph polynomial equation:

$$\frac{\partial^{\ltimes}\mho}{\partial\varkappa^{\ltimes}}(\varkappa,\mathrm{t})=a_1\frac{\partial^{\beta}\mho}{\partial \mathrm{t}^{\beta}}(\varkappa,\mathrm{t})+a_2\frac{\partial^{\vartheta}\mho}{\partial \mathrm{t}^{\vartheta}}(\varkappa,\mathrm{t})+a_3\mho(\varkappa,\mathrm{t})+f(\varkappa,\mathrm{t}) \tag{11}$$

For which $1<\ltimes,\beta\leq 2$, $0<\vartheta\leq 1$, $\varkappa,\mathrm{t}\geq 0$, $\mho(0,\mathrm{t})=g(\mathrm{t})$, $\mho_{\varkappa}(0,\mathrm{t})=h(\mathrm{t})$, and a_1,a_2,a_3 remain unchanged

To solve using the variable repetition method of the Yang transformation, we follow the following steps:

The initial step is to eliminate the fractional derivative of a given degree ($\ltimes$) regarding $\varkappa$ of the undefined function $\mho(\varkappa,\mathrm{t})$ using the Yang transform and the inverse transform.

The second step is to distinguish between the outcomes achieved in the first step regarding $\varkappa$ and subsequently determine the general Lagrange multiplier is determined to guarantee that the correction function matches the recurrence formula. The following concept is illustrated by applying the Yang transformation regarding $\varkappa$ on all (9) parts. We obtain

$$\frac{Y_a\{\mho(\mathrm{t})\}}{v^{\ltimes}}-\frac{u(\varkappa,0)}{v^{\ltimes-1}}-\frac{\mho_{\varkappa}(\varkappa,0)}{v^{\ltimes-2}}=Y_a\left\{a_1\frac{\partial^{\beta}\mho}{\partial \mathrm{t}^{\beta}}(\varkappa,t)+a_2\frac{\partial^{\vartheta}\mho}{\partial \mathrm{t}^{\vartheta}}(\varkappa,t)+a_3\mho(\varkappa,\mathrm{t})+f(\varkappa,\mathrm{t})\right\} \tag{12}$$

$$Y_a\{\mho(\varkappa,\mathrm{t})\}=vg(\mathrm{t})+v^2h(\mathrm{t})+v^{\ltimes}Y_a\{f(\varkappa,\mathrm{t})\}+v^{\ltimes}Y_a\left\{a_1\frac{\partial^{\beta}\mho}{\partial \mathrm{t}^{\beta}}(\varkappa,\mathrm{t})+a_2\frac{\partial^{\vartheta}\mho}{\partial \mathrm{t}^{\vartheta}}(\varkappa,\mathrm{t})+a_3\mho(\varkappa,\mathrm{t})\right\} \tag{13}$$

By taking the inverse of the Yang transformation applied to equation (13), we obtain:

$$\mho(\varkappa,\mathrm{t})=g(\mathrm{t})+\varkappa h(\mathrm{t})+Y_a^{-1}\left[v^{\ltimes}Y_a\{f(\varkappa,\mathrm{t})\}\right]+Y_a^{-1}\left[v^{\ltimes}Y_a\left\{a_1\frac{\partial^{\beta}\mho}{\partial \mathrm{t}^{\beta}}(\varkappa,\mathrm{t})+a_2\frac{\partial^{\vartheta}\mho}{\partial \mathrm{t}^{\vartheta}}(\varkappa,\mathrm{t})+a_3\mho(\varkappa,\mathrm{t})\right\}\right] \tag{14}$$

The fractional derivative of $\ltimes$ Concerning $\varkappa$ now eliminated, leaving the dependent variable $\mho(\varkappa,\mathrm{t})$on the left side. The right side of equation (14) is now without derivatives. The next step includes differentiating equation (13) with respect to b to obtain $\varkappa$.

$$\frac{\partial \mho(\varkappa,\mathrm{t})}{\partial \varkappa} = h(\mathrm{t}) + \frac{\partial}{\partial \varkappa}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\{f(\varkappa,\mathrm{t})\}\right]\right\} + \frac{\partial}{\partial \varkappa}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\left\{ a_1 \frac{\partial^{\beta}\mho}{\partial \mathrm{t}^{\beta}}(\varkappa,\mathrm{t}) + a_2 \frac{\partial^{\vartheta}\mho}{\partial \mathrm{t}^{\vartheta}}(x,\mathrm{t}) + a_3 \mho(\varkappa,\mathrm{t})\right\}\right]\right\}. \tag{15}$$

The previous step was carried out to enable us to express the correction function for equation (14) as:

$$\mho_{\eta+1}(\varkappa,\mathrm{t}) = \mho_{\eta}(\varkappa,\mathrm{t}) + \int_0^{\varkappa} \bar{\lambda}\left[\frac{\partial \mho(\varsigma,\mathrm{t})}{\partial \varsigma} - g(\mathrm{t}) \quad - \frac{\partial}{\partial \varsigma}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\{f(\varsigma,\mathrm{t})\}\right]\right\} - \frac{\partial}{\partial \varsigma}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\left(a_1 \frac{\partial^{\beta}\mho_{\eta}}{\partial \mathrm{t}^{\beta}}(\varsigma,\mathrm{t}) + a_2 \frac{\partial^{\vartheta}\mho_{\eta}}{\partial \mathrm{t}^{\vartheta}}(\varsigma,t) + a_3 \mho_{\eta}(\varsigma,\mathrm{t})\right)\right]\right\}\right] d\varsigma \tag{16}$$

The general Lagrange multiplier of (16) can be ideally defined by the variation theorem to obtain

$$1 + \lambda\big|_{\varsigma=\varkappa=0} \quad \text{and} \quad \lambda\big|_{\varsigma=\varkappa=0} \tag{17}$$

From equation (17), we obtain:

$$\lambda = -1$$

Inserting $\lambda = -1$
By substituting into equation (16), This leads to the iterative formula for:$\eta = 0, 1, 2, \ldots$ as shown below:

$$\mho_{\eta+1}(\varkappa,\mathrm{t}) = \mho_{\eta}(\varkappa,\mathrm{t}) - \int_0^{\varkappa} \bar{\lambda}\left[\frac{\partial \mho_{\eta}(\varsigma,\mathrm{t})}{\partial \varsigma} - g(\mathrm{t}) \quad - \frac{\partial}{\partial \varsigma}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\{f(\varsigma,\mathrm{t})\}\right]\right\} - \frac{\partial}{\partial \varsigma}\left\{ Y_a^{-1}\left[v^{\ltimes} Y_a\left(a_1 \frac{\partial^{\beta}\mho_{\eta}}{\partial \mathrm{t}^{\beta}}(\varsigma,\mathrm{t}) + a_2 \frac{\partial^{\vartheta}\mho_{\eta}}{\partial \mathrm{t}^{\vartheta}}(\varsigma,\mathrm{t}) + a_3 \mho_{\eta}(\varsigma,\mathrm{t})\right)\right]\right\}\right] d\varsigma \tag{18}$$

We begin with the first iteration:

$$\mho_0(\varkappa,\mathrm{t}) = \mho(0,\mathrm{t}) + \varkappa \mho_{\varkappa}(0,\mathrm{t}) = g(\mathrm{t}) + \varkappa h(\mathrm{t}) \tag{19}$$

The exact result is achieved as the limit of successive estimates. $\mho_{\eta}(\varkappa,\mathrm{t})$,
$\eta = 0, 1, 2, \ldots$
To put it differently,

$$\mho(\varkappa,\mathrm{t}) = \lim_{\eta\to\infty} \mho_{\eta}(\varkappa,\mathrm{t}) \tag{20}$$

5. Study of Convergence Behavior

This section focuses on investigating Convergence of the variational iterative process technique, using The alternative method of VIM outlined previously,When employed in the problem (11). The necessary criteria for the convergence in this Procedure, along with The mistake bounds, Are revealed in the subsequent theorems.

Now, define the operator $T[\mho]$ as,

$$T[\mho]=\int_0^t\Bigg[-\frac{\partial\mho_n(x_i,\zeta)}{\partial\zeta}+h(t)+\frac{\partial}{\partial\zeta}\Big(Y_a^{-1}\big[\upsilon^{\alpha}Y_a\{f(x_i,\zeta)\}\big]\Big)+\frac{\partial}{\partial\zeta}\Big(Y_a^{-1}\big[\upsilon^{\alpha}Y_a\big\{a_1\frac{\partial^{\beta}\mho_n}{\partial {x_i}^{\beta}}(x_i,\zeta)+a_2\frac{\partial^{\vartheta}\mho_n}{\partial {x_i}^{\vartheta}}(x,\zeta)+a_3\mho_n(x_i,\zeta)+N\mho(x_i,\zeta)\}\big]\Big)\Bigg]d\zeta \tag{21}$$

And specify the Parts $\mathscr{Y}_k$, $k=0,1,2,\ldots,$ as

$$\begin{aligned}\mathscr{Y}_0&=\mho_0\\ \mathscr{Y}_1&=T[\mathscr{Y}_0]\\ \mathscr{Y}_2&=T[\mathscr{Y}_0+\mathscr{Y}_1]\\ &\vdots\\ \mathscr{Y}_{k+1}&=T[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_k]\end{aligned} \tag{22}$$

As a result:

$$\mho(x_i,t)=\lim_{k\to\infty}\mho_k(x_i,t)=\sum_{k=0}^{\infty}\mathscr{Y}_k. \tag{23}$$

Consequently, The solution to problem (11) can be formulated in a series representation. using (21) and (22).

$$\mho(x_i,t)=\sum_{k=0}^{\infty}\mathscr{Y}_k(x_i,t). \tag{24}$$

The initial estimate $\mathscr{Y}_0=\mho_0$ The zeroth approximation can be freely selected As long as the starting and boundary conditions of the problem are met,The effectiveness of the approach is conditional upon an accurate choice of the starting guess. $\mathscr{Y}_0$. Yet, utilizing The beginning values $\mathscr{Y}^{(k)}(0)=c_k$, $k=0,1,\ldots,m-1$, Are typically Applied for chosen zeroth approximation $\mathscr{Y}_0$ as will be demonstrated later.

In our alternative plan, we determine the starting approximation. $\mathscr{Y}_0$ as

$$\mathscr{Y}_0=g(t)+x_ih(t). \tag{25}$$

For approximation purposes, We estimate the solution. $\mho(x_i,t)=\sum_{k=0}^{\infty}\mathscr{Y}_k(x_i,t)$ by the Series cut off at a particular$n$th- order

$$\sum_{k=0}^{n}\mathscr{Y}_k(x_i,t) \tag{26}$$

Theorem 1: Fixed Point Theorem for Contractionstle Let T,Set forth in (21),be an operator acting on a Hilbert space H to H. The solution expressed as a series

$$\mho(x_i,t)=\sum_{k=0}^{\infty}\mathscr{Y}_k(x_i,t), \tag{27}$$

Set forth in (23), Converges given that $\exists\ 0<\rho<1$ to the extent that

$$\|T[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_{k+1}]\|\le\rho\|T[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_k]\|\quad(\text{that is }\|\mathscr{Y}_{k+1}\|\le\rho\|\mathscr{Y}_k\|),\ \forall k\in\mathbb{N}\cup\{0\}.$$

Theorem 1 is a particular case of Banach's fixed-point theorem, providing a sufficient condition to explore the convergence of (VIM) for certain partial differential equations. Next, we will present a short proof of Theorem 1 to analyze the truncation error of (VIM).

Proof. Specify the sequence $\{\mathscr{E}_n\}_{n=0}^{\infty}$ as,

$$\begin{aligned}
\mathscr{E}_0 &= \mathscr{Y}_0,\\
\mathscr{E}_1 &= \mathscr{Y}_0 + \mathscr{Y}_1,\\
\mathscr{E}_2 &= \mathscr{Y}_0 + \mathscr{Y}_1 + \mathscr{Y}_2,\\
&\vdots\\
\mathscr{E}_n &= \mathscr{Y}_0 + \mathscr{Y}_1 + \cdots + \mathscr{Y}_n.
\end{aligned} \tag{28}$$

And we illustrate that $\{\mathscr{E}_n\}_{n=0}^{\infty}$ Is a Cauchy seq. in the context of a Hilbert space $\mathscr{H}$.In this regard, consider

$$\|\mathscr{E}_{n+1} - \mathscr{E}_n\| = \|\mathscr{Y}_{n+1}\| \leq \rho\|\mathscr{Y}_n\| \leq \rho^2\|\mathscr{Y}_{n-1}\| \leq \cdots \leq \rho^{n+1}\|\mathscr{Y}_0\|.$$

For all values of $n, j \in \mathbb{N}, n \geq j$,It follows that

$$\begin{aligned}
\|\mathscr{E}_n - \mathscr{E}_j\| &= \|(\mathscr{E}_n - \mathscr{E}_{n-1}) + (\mathscr{E}_{n-1} - \mathscr{E}_{n-2}) + \cdots + (\mathscr{E}_{j+1} - \mathscr{E}_j)\|\\
&\leq \|\mathscr{E}_n - \mathscr{E}_{n-1}\| + \|\mathscr{E}_{n-1} - \mathscr{E}_{n-2}\| + \cdots + \|\mathscr{E}_{j+1} - \mathscr{E}_j\|\\
&\leq \rho^n\|\mathscr{Y}_0\| + \rho^{n-1}\|\mathscr{Y}_0\| + \cdots + \rho^{j+1}\|\mathscr{Y}_0\|\\
&= \frac{1-\rho^{n-j}}{1-\rho}\rho^{j+1}\|\mathscr{Y}_0\|.
\end{aligned}$$

since $0 < \rho < 1$, We acquire

$$\lim_{n,j\to\infty} \|\mathscr{E}_n - \mathscr{E}_j\| = 0.$$

As a result, $\{\mathscr{E}_n\}_{n=0}^{\infty}$ Represents a Cauchy seq. within the Hilbert space $\mathscr{H}$ and this implies that the sol. in series form

$$\mho(x_i,t) = \sum_{k=0}^{\infty} \mathscr{Y}_k(x_i,t),$$

Specified in (23), converges. Thus, the Justification of the Theorem 1 is complete.

Theorem 2: If the solution in series form $\mho(x_i,t) = \sum_{k=0}^{\infty} \mathscr{Y}_k(x_i,t)$ Specified in (23), converges, Thus, it serves as The exact answer to the nonlinear problem(23).

Proof. Assuming That the series (23) converges say $\Psi(x_i,t) = \sum_{k=0}^{\infty} \mathscr{Y}_k(x_i,t)$,

$$\lim_{j\to\infty} \mathscr{Y}_j = 0 \tag{29}$$

$$\sum_{j=0}^{n} \left[\mathscr{Y}_{j+1} - \mathscr{Y}_j\right] = \mathscr{Y}_{n+1} - \mathscr{Y}_0, \tag{30}$$

and so,

$$\sum_{j=0}^{n} \left[\mathscr{Y}_{j+1} - \mathscr{Y}_j\right] = \lim_{j\to\infty} \mathscr{Y}_j - \mathscr{Y}_0 = -\mathscr{Y}_0. \tag{31}$$

Using the operator $L = \frac{d^m}{dt^m}$, $m \in \mathbb{N}$, When applied to both sides of the equation (31), then, from (24), We attain

$$\sum_{j=0}^{n} L\left[\mathscr{Y}_{j+1} - \mathscr{Y}_j\right] = -L[\mathscr{Y}_0] = 0. \tag{32}$$

In contrast, from Meaning (28), We attain

$$L\big[\mathscr{Y}_{j+1}-\mathscr{Y}_j\big]=L\big[T[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]-T[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_{j-1}]\big], \tag{33}$$

when $j\geq 1$, and so, using definition (27), We attain

$$L\big[\mathscr{Y}_{j+1}-\mathscr{Y}_j\big]=L\Big\{\int_0^x\Big[-\frac{\partial[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]}{\partial\zeta}+h(t)+\frac{\partial}{\partial\zeta}\Big(Y_a^{-1}\big[\mathscr{Y}^{\alpha}Y_a\{f(\zeta,t)\}\big]\Big)$$

$$+\frac{\partial}{\partial\zeta}\Big(Y_a^{-1}\big[\mathscr{Y}^{\alpha}Y_a\big\{a_1\frac{\partial^{\beta}[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]}{\partial t^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]}{\partial t^{\vartheta}}$$
$$+a_3[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]+N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]-N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_{j-1}]\big\}\big]\Big)\Big]d\zeta\Big\},\quad j\geq 1. \tag{34}$$

Now, the proof $T[\mho]$, Specified in (27), gives the mth-fold integral

$$\frac{\partial^{\alpha}\mho}{\partial t^{\alpha}}(x_i,t)=a_1\frac{\partial^{\beta}\mho}{\partial {x_i}^{\beta}}(x_i,t)+a_2\frac{\partial^{\vartheta}\mho}{\partial {x_i}^{\vartheta}}(x_i,t)+a_3\mho(x_i,t)+N\mho(x_i,t)+f(x_i,t).$$

Considering the function corresponding to the differential operator $L=\frac{d^m}{dt^m}$ of order m acts as the left inverse to the mth-fold Upon applying the integral operator, Eq. (34) results in

$$L\big[\mathscr{Y}_{j+1}-\mathscr{Y}_j\big]=-\frac{\partial[\mathscr{Y}_j]}{\partial\zeta}+Y_a^{-1}\big[\mathscr{Y}^{\alpha}Y_a\big\{a_1\frac{\partial^{\beta}[\mathscr{Y}_j]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_j]}{\partial {x_i}^{\vartheta}}$$
$$+a_3[\mathscr{Y}_j]+N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_j]-N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_{j-1}]\big\}\big],\quad j\geq 1. \tag{35}$$

Thus

$$\sum_{j=0}^{n}L\big[\mathscr{Y}_{j+1}-\mathscr{Y}_j\big]=-\frac{\partial[\mathscr{Y}_0]}{\partial\zeta}+a_1\frac{\partial^{\beta}[\mathscr{Y}_0]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_0]}{\partial {x_i}^{\vartheta}}+a_3[\mathscr{Y}_0]+N[\mathscr{Y}_0]-f(x_i,t)$$
$$-\frac{\partial[\mathscr{Y}_1]}{\partial\zeta}+a_1\frac{\partial^{\beta}[\mathscr{Y}_1]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_1]}{\partial {x_i}^{\vartheta}}+a_3[\mathscr{Y}_1]+N[\mathscr{Y}_0+\mathscr{Y}_1]-N[\mathscr{Y}_0]$$
$$-\frac{\partial[\mathscr{Y}_2]}{\partial\zeta}+a_1\frac{\partial^{\beta}[\mathscr{Y}_2]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_2]}{\partial {x_i}^{\vartheta}}+a_3[\mathscr{Y}_2]+N[\mathscr{Y}_0+\mathscr{Y}_1+\mathscr{Y}_2]-N[\mathscr{Y}_0+\mathscr{Y}_1]$$
$$\vdots$$
$$-\frac{\partial[\mathscr{Y}_n]}{\partial\zeta}+a_1\frac{\partial^{\beta}[\mathscr{Y}_n]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_n]}{\partial {x_i}^{\vartheta}}+a_3[\mathscr{Y}_n]+N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_n]-N[\mathscr{Y}_0+\mathscr{Y}_1+\cdots+\mathscr{Y}_{n-1}].$$

Thus,

$$\sum_{j=0}^{\infty}L\big[\mathscr{Y}_{j+1}-\mathscr{Y}_j\big]=\Bigg[\sum_{j=0}^{\infty}\Bigg\{-\frac{\partial[\mathscr{Y}_j]}{\partial\zeta}+Y_a^{-1}\big[\mathscr{Y}^{\alpha}Y_a\big\{a_1\frac{\partial^{\beta}[\mathscr{Y}_j]}{\partial {x_i}^{\beta}}+a_2\frac{\partial^{\vartheta}[\mathscr{Y}_j]}{\partial {x_i}^{\vartheta}}+a_3[\mathscr{Y}_j]\big\}\big]\Bigg\}$$
$$+N\Bigg[\sum_{j=0}^{\infty}\mathscr{Y}_j\Bigg]-f(x_i,t). \tag{36}$$

From (32) and (36), It can be noted that $\Psi(x,t) = \sum_{j=0}^{\infty} \mathscr{Y}_j(x,t)$ Serves as an exact solution to the problem (11). Thus, the proof is complete of Theorem 2.

Theorem 3: Let usTake it that the series $\sum_{k=0}^{\infty} \mathscr{Y}_k(x_i,t)$ Specified in (23) converges to The resolution $\mho(x_i,t)$. If the cut-off series $\sum_{k=0}^{j} \mathscr{Y}_k(x_i,t)$ Is estimate of the sol. $\mho(x_i,t)$ Regarding Eq. (11), Consequently, the greatest error, $E_j(x_i,t)$, is calculated as

$$E_j(x,t) \leq \frac{1}{1-\rho}\rho^{j+1}\|\mathscr{Y}_0\|. \tag{37}$$

Proof. Based on Theorem 1 and inequality (33), We possess

$$\|s_n - s_j\| \leq \frac{1-\rho^{n-j}}{1-\rho}\rho^{j+1}\|\mathscr{Y}_0\|, \tag{38}$$

In order to $n \geq j$.At present, as $n \to \infty$, then $\mathscr{E}_n \to \mho(x,t)$.As a result,

$$\|\mho(x,t) - \sum_{k=0}^{j} \mathscr{Y}_k(x_i,t)\| \leq \frac{1-\rho^{n-j}}{1-\rho}\rho^{j+1}\|\mathscr{Y}_0\|. \tag{39}$$

Furthermore, as $0 < \rho < 1$, We possess $(1-\rho^{n-j}) < 1$. As a result, the inequality mentioned above turns into

$$\|\mho(x,t) - \sum_{k=0}^{j} \mathscr{Y}_k(x_i,t)\| \leq \frac{1}{1-\rho}\rho^{j+1}\|\mathscr{Y}_0\|. \tag{40}$$

Thus,The demonstration of Theorem 3 is concluded To summarize, Theorems 1 and 2 assert that the (VIM) for solving the nonlinear (11), derived by applying The formulas for iteration (20) or (22) will converge to the exact solution if $\exists 0 < \rho < 1$ In a manner that

$$\|T[\mathscr{Y}_0 + \mathscr{Y}_1 + \cdots + \mathscr{Y}_{k+1}]\| \leq \rho\|T[\mathscr{Y}_0 + \mathscr{Y}_1 + \cdots + \mathscr{Y}_k]\| \quad \text{(that is, } \|\mathscr{Y}_{k+1}\| \leq \rho\|\mathscr{Y}_k\|\text{)}, \ \forall k \in \mathbb{N}\cup\{0\}.$$

To rephrase, If we establish, for every $i \in \mathbb{N}\cup\{0\}$, the attributes

$$\beta_i = \begin{cases} \frac{\|\mathscr{Y}_{i+1}\|}{\|\mathscr{Y}_i\|}, & \|\mathscr{Y}_i\| \neq 0, \\ 0, & \|\mathscr{Y}_i\| = 0, \end{cases} \tag{41}$$

Then, the solution in series form $\sum_{k=0}^{\infty} \mathscr{Y}_k(x,t)$ The solution of Eq. (11) Reaches the exact solution $\mho(x,t)$, when $0 \leq \beta_i < 1$, $\forall i \in \mathbb{N}\cup\{0\}$. Additionally, as defined in Theorem 3, the maximum truncation error in absolute terms is approximated as

$$\|\mho(x_i,t) - \sum_{k=0}^{j} \mathscr{Y}_k(x_i,t)\| \leq \frac{1}{1-\beta}\beta^{j+1}\|\mathscr{Y}_0\|, \text{ where } \beta = \max\{\beta_r, r = 0,1,\ldots,j\}.$$

Remark 1. If the first finite quantity approximation β_r's, $r = 1,2,\ldots,l$,Are not smaller than one and $\beta_r \leq 1$ for $r \leq l$, In that case, The series-based solution $\sum_{k=0}^{\infty} \mathscr{Y}_k(x_i,t)$ The series solution for problem (11) Converges to the precise solution. In other words,The finite initial terms do not alter the convergence of the series solution. As derived from Theorem 1, we have

$$\|\mathscr{E}_n - \mathscr{E}_j\| \leq \frac{1-\rho^{n-j}}{1-\rho}\rho^{j-l}\|\mathscr{Y}_{l+1}\|, \tag{42}$$

Given that $0 < \rho < 1$, for $n \geq j$ and established l, We achieve

$$\lim_{n,j\to\infty} \|\mathscr{E}_n - \mathscr{E}_j\| = 0.$$

In this context, the convergence of the VIM method is determined by β_i, for $i > l$.

6. Example Illustrations

Example 6.1. Take into account the linear 2D telegraph equation involving time-fractional derivatives:

$$\frac{\partial^{2\ltimes}\mho}{\partial t^{2\ltimes}} = \frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - 2\mho, \quad 0 < \ltimes \leq 1 \tag{43}$$

with initial conditions:

$$\mho(\varkappa, y, 0) = e^{\varkappa+y}, \quad \mho_t(\varkappa, y, 0) = -3e^{\varkappa+y}.$$

Solution.

Taking the Yang transform (denoted as Y_a) and differentiating both sides of Equation (43)Relative to $\varkappa$, we obtain:

$$\frac{Y_a\{\mho(t)\}}{v^{2\ltimes}} - \frac{\mho(\varkappa,y,0)}{v^{2\ltimes-1}} - \frac{\mho_t(\varkappa,y,0)}{v^{2\ltimes-2}} = Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - 2\mho\right]. \tag{44}$$

Using the transform properties:

$$Y_a\{\mho(\varkappa,y,t)\} = v\mho(\varkappa,y,0) + v^2\mho_t(\varkappa,y,0) + v^{2\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - 2\mho\right], \tag{45}$$

we have:

$$\mho(\varkappa,y,t) = e^{\varkappa+y} - 3te^{\varkappa+y} + Y_a^{-1}\left\{v^{2\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - 2\mho\right]\right\}.$$

Differentiating (45) Relative $\varkappa$, we find:

$$\frac{\partial\mho(\varkappa,y,t)}{\partial t} = -3e^{\varkappa+y} + \frac{\partial}{\partial t}Y_a^{-1}\left\{v^{2\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - 2\mho\right]\right\}. \tag{46}$$

The correction functional for Equation (46) with $\lambda = -1$ is expressed as:

$$\mho_{\eta+1}(\varkappa,y,t) = \mho_\eta(\varkappa,y,t) - \int_0^t \left[\frac{\partial\mho_\eta(\varkappa,y,\xi)}{\partial\xi} - 3e^{\varkappa+y} - \frac{\partial}{\partial\xi}Y_a^{-1}\left\{v^{\ltimes}Y_a\left(\frac{\partial^2\mho_\eta}{\partial\varkappa^2} + \frac{\partial^2\mho_\eta}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho_\eta}{\partial t^{\ltimes}} - 2\mho_\eta(\varkappa,y,t)\right)\right\}\right]d\xi. \tag{47}$$

For the initial approximation:

$$\mho_0(\varkappa,y,t) = \mho(\varkappa,y,0) + t\mho_t(\varkappa,y,0) = e^{\varkappa+y} - 3te^{\varkappa+y}.$$

For $\mho_1(\varkappa,y,t)$:

$$\mho_1(\varkappa,y,t) = \mho_0(\varkappa,y,t) - \int_0^t \left[\frac{\partial\mho_0(\varkappa,y,\xi)}{\partial\xi} - 3e^{\varkappa+y} - \frac{\partial}{\partial\xi}Y_a^{-1}\{v^{\ltimes}Y_a\left[\frac{\partial^2\mho_0}{\partial\varkappa^2} + \frac{\partial^2\mho_0}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho_0}{\partial t^{\ltimes}} - 2\mho_0(\varkappa,y,t)\right]\}\right]d\xi. \tag{48}$$

$$\mho_1(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + 3\mathrm{t}e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + Y_a^{-1}\Big\{v^{2\ltimes}Y_a\big[2e^{\varkappa+y} - 6\mathrm{t}e^{\varkappa+y} + 9e^{\varkappa+y}\frac{\mathrm{t}^{1-\ltimes}}{\Gamma(2-\ltimes)} - 2e^{\varkappa+y} + 6\mathrm{t}e^{\varkappa+y}\big]\Big\}$$

$$\mho_1(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3e^{\varkappa+y}\frac{\mathrm{t}^{\ltimes}}{\Gamma(\ltimes+1)} + Y_a^{-1}\Big\{9v^{\ltimes+2}e^{\varkappa+y}\Big\}$$

$$\mho_1(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y}$$

$$\mho_2(\varkappa,y,\mathrm{t}) = \mho_1(\varkappa,y,\mathrm{t}) - \int_0^{\mathrm{t}}\Bigg[\frac{\partial\mho_1(\varkappa,y,\xi)}{\partial\xi} - 3e^{\varkappa+y} - \frac{\partial}{\partial\xi}Y_a^{-1}\Big\{v^{\ltimes}Y_a\big[\frac{\partial^2\mho_1}{\partial\varkappa^2} + \frac{\partial^2\mho_1}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho_1}{\partial\mathrm{t}^{\ltimes}} - 2\mho_1(\varkappa,y,\mathrm{t})\big]\Big\}\Bigg]d\xi$$

$$\mho_2(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y} + 3\mathrm{t}e^{\varkappa+y} - \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + Y_a^{-1}\Big\{v^{2\ltimes}Y_a\big[2e^{\varkappa+y} - 6\mathrm{t}e^{\varkappa+y} + \frac{18\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y} + 9e^{\varkappa+y}\frac{\mathrm{t}^{1-\ltimes}}{\Gamma(2-\ltimes)} - 27\mathrm{t}e^{\varkappa+y} - 2e^{\varkappa+y} + 6\mathrm{t}e^{\varkappa+y} - \frac{18\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y}\big]\Big\}$$

$$\mho_2(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + Y_a^{-1}\Big\{9v^{\ltimes+2}e^{\varkappa+y} - 27v^{2\ltimes+2}e^{\varkappa+y}\Big\}$$

$$\mho_2(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y} - \frac{27\mathrm{t}^{2\ltimes+1}}{\Gamma(2\ltimes+2)}e^{\varkappa+y}$$

$$\mho_3(\varkappa,y,\mathrm{t}) = \mho_2(\varkappa,y,\mathrm{t}) - \int_0^{\mathrm{t}}\Bigg[\frac{\partial\mho_2(\varkappa,y,\xi)}{\partial\xi} - 3e^{\varkappa+y} - \frac{\partial}{\partial\xi}Y_a^{-1}\Big\{v^{\ltimes}Y_a\big[\frac{\partial^2\mho_2}{\partial\varkappa^2} + \frac{\partial^2\mho_2}{\partial y^2} - 3\frac{\partial^{\ltimes}\mho_2}{\partial\mathrm{t}^{\ltimes}} - 2\mho_2(\varkappa,y,\mathrm{t})\big]\Big\}\Bigg]d\xi$$

$$\mho_3(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + Y_a^{-1}\Big\{9v^{\ltimes+2}e^{\varkappa+y} - 27v^{2\ltimes+2}e^{\varkappa+y} + 81v^{3\ltimes+2}e^{\varkappa+y}\Big\}$$

$$\mho_3(\varkappa,y,\mathrm{t}) = e^{\varkappa+y} - 3\mathrm{t}e^{\varkappa+y} + \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}e^{\varkappa+y} - \frac{27\mathrm{t}^{2\ltimes+1}}{\Gamma(2\ltimes+2)}e^{\varkappa+y} + \frac{81\mathrm{t}^{3\ltimes+1}}{\Gamma(3\ltimes+2)}e^{\varkappa+y}$$

Iterating further, we find the general form:

$$\mho_\eta(x,y,\mathrm{t}) = e^{\varkappa+y}\left(1 - 3\mathrm{t} + \frac{9\mathrm{t}^{\ltimes+1}}{\Gamma(\ltimes+2)} - \frac{27\mathrm{t}^{2\ltimes+1}}{\Gamma(2\ltimes+2)} + \frac{81\mathrm{t}^{3\ltimes+1}}{\Gamma(3\ltimes+2)} - \dots\right). \tag{49}$$

When $\ltimes = 1$:

$$\mho(x,y,\mathrm{t}) = \lim_{n\to\infty}\mho_\eta(x,y,\mathrm{t}) = e^{x+y}(1 - 3\mathrm{t} + \frac{(3\mathrm{t})^2}{2!} - \frac{(3\mathrm{t})^3}{3!} + \dots).$$

This simplifies to:

$$\mho(\varkappa, y, \mathfrak{t}) = e^{\varkappa+y}e^{-3\mathfrak{t}} = e^{\varkappa+y-3\mathfrak{t}}. \tag{50}$$

$\varkappa$	$\ltimes = 1$	$\ltimes = 0.9$	$\ltimes = 0.8$	Exact	$\lvert\mho_{\text{Ex}} - \mho_{\ltimes=1}\rvert$	$\lvert\mho_{\text{Ex}} - \mho_{\ltimes=0.9}\rvert$	$\lvert\mho_{\text{Ex}} - \mho_{\ltimes=0.8}\rvert$
-2	0.04980	0.08817	0.13154	0.04978	2.0522e-05	0.03838	0.08175
-1.7	0.06220	0.11012	0.16428	0.06217	2.5629e-05	0.04794	0.1021
-1.5	0.07768	0.13752	0.20516	0.07764	3.2006e-05	0.05987	0.12751
-1.3	0.09701	0.17174	0.25621	0.09697	3.9971e-05	0.07477	0.15924
-1.1	0.12115	0.21448	0.31997	0.01211	4.9918e-05	0.09337	0.19886
-0.8	0.1513	0.26785	0.39959	0.15124	6.234e-05	0.11661	0.24835
-0.6	0.18895	0.33451	0.49903	0.18888	7.7853e-05	0.14563	0.31015
-0.4	0.23597	0.41775	0.62321	0.23588	9.7227e-05	0.18187	0.38733
-0.2	0.2947	0.52171	0.7783	0.29457	0.000121	0.22713	0.48372
0	0.36803	0.65153	0.97197	1.4796	0.000151	0.28386	0.60409

Table 1. Quantitative results of the numerical and exact solutions for various values of $\varkappa$ and $\mathfrak{t}$ when $\ltimes = 0.8, 0.9, 1$.

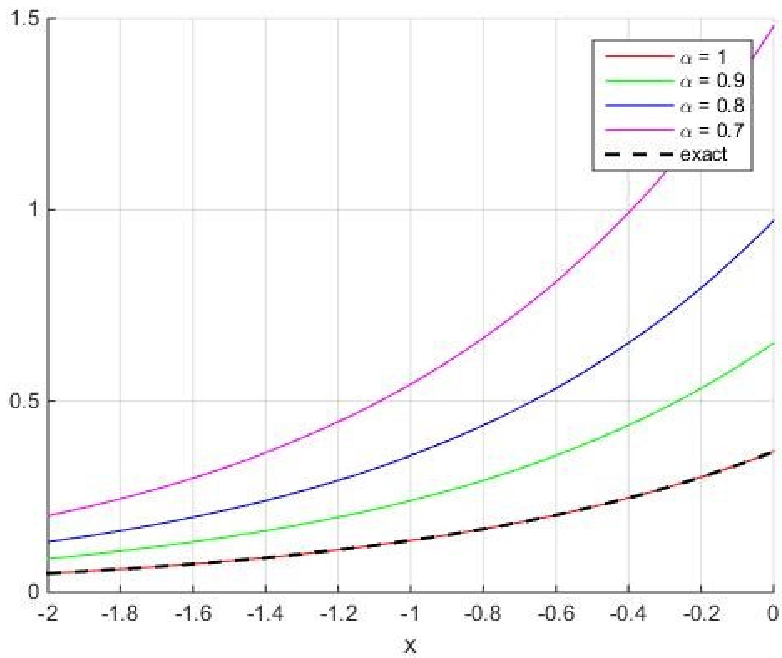

Figure 1. The plots of the rough Result $\mho(\varkappa, \mathfrak{t})$for a range of values of $\ltimes$

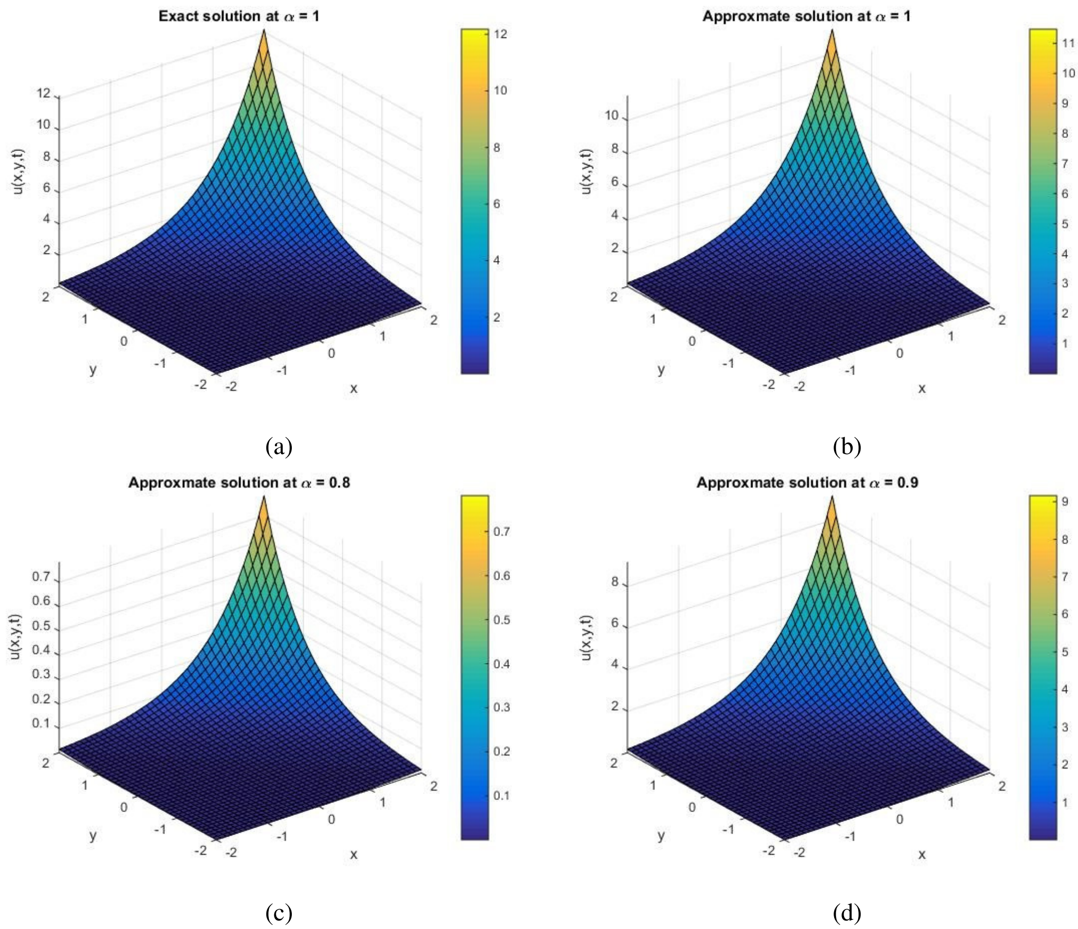

Figure 2. The diagram representing the simulated Result $\mho(\varkappa,\mathfrak{t})$ of Instance (4):when (a) the exact solution, (b) $\ltimes = 1$, (c) $\ltimes = 0.8$, (d) $\ltimes = 0.9$.

Example 6.2. Take into account the time-fractional 3D telegraph equation:

$$\frac{\partial^{2\ltimes} R}{\partial \mathfrak{t}^{2\ltimes}} = \frac{\partial^2 \mho}{\partial \varkappa^2} + \frac{\partial^2 \mho}{\partial y^2} + \frac{\partial^2 \mho}{\partial z^2} - 2\frac{\partial^{\ltimes} \mho}{\partial \mathfrak{t}^{\ltimes}} - 3\mho, \quad 0 < \ltimes \le 1 \tag{51}$$

with the initial conditions:

$$\mho(\varkappa,y,z,0) = \sinh(\varkappa)\sinh(y)\sinh(z) = \varpi$$

$$\mho_{\mathfrak{t}}(\varkappa,y,z,0) = -2\sinh(\varkappa)\sinh(y)\sinh(z) = -2\varpi$$

Solution. Taking the Yang transform and performing differentiation on both sides of equation (51) with respect to $\mathfrak{t}$, we get:

$$\frac{Y_a\{\mho(\mathfrak{t})\}}{v^{2\ltimes}} - \frac{\mho(\varkappa,y,z,0)}{v^{2\ltimes-1}} - \frac{\mho_{\mathfrak{t}}(\varkappa,y,z,0)}{v^{2\ltimes-2}} = Y_a\left[\frac{\partial^2 \mho}{\partial \varkappa^2} + \frac{\partial^2 \mho}{\partial y^2} + \frac{\partial^2 \mho}{\partial z^2} - 2\frac{\partial^{\ltimes} \mho}{\partial \mathfrak{t}^{\ltimes}} - 3\mho\right]. \tag{52}$$

Applying the Yang transform yields:

$$Y_a\{\mho(\varkappa,y,z,\mathfrak{t})\} = v\mho(\varkappa,y,z,0) + v^2\mho_{\mathfrak{t}}(\varkappa,y,z,0) + v^{2\ltimes} Y_a\left[\frac{\partial^2 \mho}{\partial \varkappa^2} + \frac{\partial^2 \mho}{\partial y^2} + \frac{\partial^2 \mho}{\partial z^2} - 2\frac{\partial^{\ltimes} \mho}{\partial \mathfrak{t}^{\ltimes}} - 3\mho\right]. \tag{53}$$

Taking the inverse Yang transform of (53) gives:

$$\mho(\varkappa,y,z,\mathfrak{t}) = \varpi - 2\mathfrak{t}\varpi + Y_a^{-1}\{v^{2\ltimes} Y_a\left[\frac{\partial^2\mho}{\partial\varkappa^2} + \frac{\partial^2\mho}{\partial y^2} + \frac{\partial^2\mho}{\partial z^2} - 2\frac{\partial^{\ltimes}\mho}{\partial\mathfrak{t}^{\ltimes}} - 3\mho\right]. \quad (54)$$

Differentiating (45) with respect to $\mathfrak{t}$, we find:

$$\frac{\partial\mho(\varkappa,y,z,\mathfrak{t})}{\partial\mathfrak{t}} = -2\varpi + \frac{\partial}{\partial\mathfrak{t}} Y_a^{-1}\{v^{2\ltimes} Y_a\left[\frac{\partial^2\mho}{\partial\varkappa^2} + \frac{\partial^2\mho}{\partial y^2} + \frac{\partial^2\mho}{\partial z^2} - 2\frac{\partial^{\ltimes}\mho}{\partial\mathfrak{t}^{\ltimes}} - 3\mho\right]. \quad (55)$$

Correction Functional for Equation (55)
The correction functional for equation (55) with $\lambda = -1$ is expressed as:

$$\mho_{n+1}(\varkappa,y,z,\mathfrak{t}) = \mho_n(\varkappa,y,z,\mathfrak{t}) - \int_0^{\mathfrak{t}}\left[\frac{\partial\mho_n(\varkappa,y,z,\varsigma)}{\partial\varsigma} - 2\varpi - \frac{\partial}{\partial\varsigma}Y_a^{-1}\{v^{\ltimes}Y_a[\frac{\partial^2\mho_n}{\partial\varkappa^2} + \frac{\partial^2\mho_n}{\partial y^2} + \frac{\partial^2\mho_n}{\partial z^2} - 2\frac{\partial^{\ltimes}\mho_n}{\partial\mathfrak{t}^{\ltimes}} - 3\mho_n(\varkappa,y,z,\mathfrak{t})]\}\right]d\varsigma. \quad (56)$$

The initial term is:

$$\mho_0(\varkappa,y,z,\mathfrak{t}) = \mho(\varkappa,y,z,0) + \mathfrak{t}\mho_{\mathfrak{t}}(\varkappa,y,z,0) = \varpi - 2\mathfrak{t}\varpi.$$

The first iterative term:

$$\begin{aligned}\mho_1(\varkappa,y,z,\mathfrak{t}) &= \mho_0(\varkappa,y,z,\mathfrak{t}) - \int_0^{\mathfrak{t}}\left[\frac{\partial\mho_0(\varkappa,y,z,\varsigma)}{\partial\varsigma} - 2\varpi\right.\\ &\left.- \frac{\partial}{\partial\varsigma}Y_a^{-1}\{v^{\ltimes}Y_a[\frac{\partial^2\mho_0}{\partial\varkappa^2} + \frac{\partial^2\mho_0}{\partial y^2} + \frac{\partial^2\mho_0}{\partial z^2} - 2\frac{\partial^{\ltimes}\mho_0}{\partial\mathfrak{t}^{\ltimes}} - 3\mho_0(\varkappa,y,z,\mathfrak{t})]\}\right]d\varsigma\\ &= \varpi - 2\mathfrak{t}\varpi + \frac{4\mathfrak{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}\varpi.\end{aligned}$$

The second iterative term:

$$\begin{aligned}\mho_2(\varkappa,y,z,\mathfrak{t}) &= \mho_1(\varkappa,y,z,\mathfrak{t}) - \int_0^{\mathfrak{t}}\left[\frac{\partial\mho_1(\varkappa,y,z,\varsigma)}{\partial\varsigma} - 2\varpi\right.\\ &\left.- \frac{\partial}{\partial\varsigma}Y_a^{-1}\{v^{\ltimes}Y_a[\frac{\partial^2\mho_1}{\partial\varkappa^2} + \frac{\partial^2\mho_1}{\partial y^2} + \frac{\partial^2\mho_1}{\partial z^2} - 2\frac{\partial^{\ltimes}\mho_1}{\partial\mathfrak{t}^{\ltimes}} - 3\mho_1(\varkappa,y,z,\mathfrak{t})]\}\right]d\varsigma\\ &= \varpi - 2\mathfrak{t}\varpi + \frac{4\mathfrak{t}^{\ltimes+1}}{\Gamma(\ltimes+2)}\varpi \quad - \frac{8\mathfrak{t}^{2\ltimes+1}}{\Gamma(2\ltimes+2)}\varpi.\end{aligned} \quad (57)$$

The general term is:

$$\mho_n(\varkappa,y,z,\mathfrak{t}) = \varpi\left[1 - 2\mathfrak{t} + \frac{4\mathfrak{t}^{\ltimes+1}}{\Gamma(\ltimes+2)} - \frac{8\mathfrak{t}^{2\ltimes+1}}{\Gamma(2\ltimes+2)} + \frac{16\mathfrak{t}^{3\ltimes+1}}{\Gamma(3\ltimes+2)} - \ldots\right].$$

When $\ltimes = 1$, the solution becomes:

$$\begin{aligned}\mho(\varkappa, y, z, \mathfrak{t}) &= \lim_{n\to\infty} \mho_n(\varkappa, y, z, \mathfrak{t}) \\ &= \varpi\left(1 - 2\mathfrak{t} + \frac{(2\mathfrak{t})^2}{2!} - \frac{(2\mathfrak{t})^3}{3!} + \frac{(2\mathfrak{t})^4}{4!} - \dots\right) \\ &= \sinh(\varkappa)\sinh(y)\sinh(z)e^{-2\mathfrak{t}}. \end{aligned} \tag{58}$$

$\varkappa$	$\ltimes = 1$	$\ltimes = 0.9$	$\ltimes = 0.8$	Exact	$\lvert\mho_{Ex} - \mho_{\ltimes=1}\rvert$	$\lvert\mho_{Ex} - \mho_{\ltimes=0.9}\rvert$	$\lvert\mho_{Ex} - \mho_{\ltimes=0.8}\rvert$
0	0	0	0	0	0	0	0
0.2	0.02238	0.02840	0.03509	0.02238	1.523e-07	0.0060198	0.012712
0.4	0.04587	0.05821	0.07192	0.04587	3.1215e-07	0.012338	0.026055
0.6	0.07164	0.09090	0.1233	0.07164	4.8748e-07	0.019268	0.04069
0.8	0.10096	0.12811	0.1583	0.10096	6.8698e-07	0.027154	0.057342
1.1	0.13528	0.17167	0.21212	0.13528	9.2055e-07	0.036386	0.076837
1.3	0.17632	0.22374	0.27646	0.17632	1.1998e-06	0.047422	0.10014
1.5	0.22609	0.2869	0.35451	0.22609	1.5385e-06	0.06081	0.12841
1.7	0.28708	0.36429	0.45013	0.28708	1.9535e-06	0.077213	0.16305
2	0.3623	0.45975	0.56808	0.3623	2.4653e-06	0.097445	0.20578

Table 2. Quantitative results of the numerical and exact solutions for various values of $\varkappa$ and $\mathfrak{t}$ when $\ltimes = 0.8, 0.9, 1$.

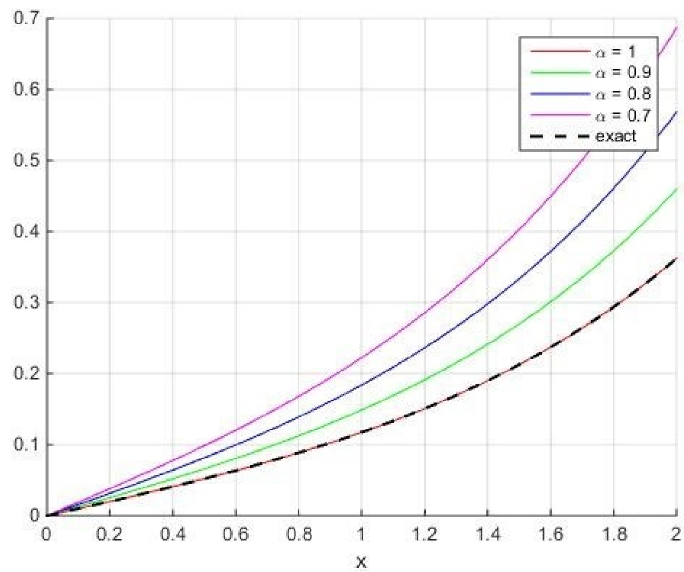

Figure 3. The plots of the rough Result $u(\varkappa, \mathfrak{t})$for a range of values of $\ltimes$

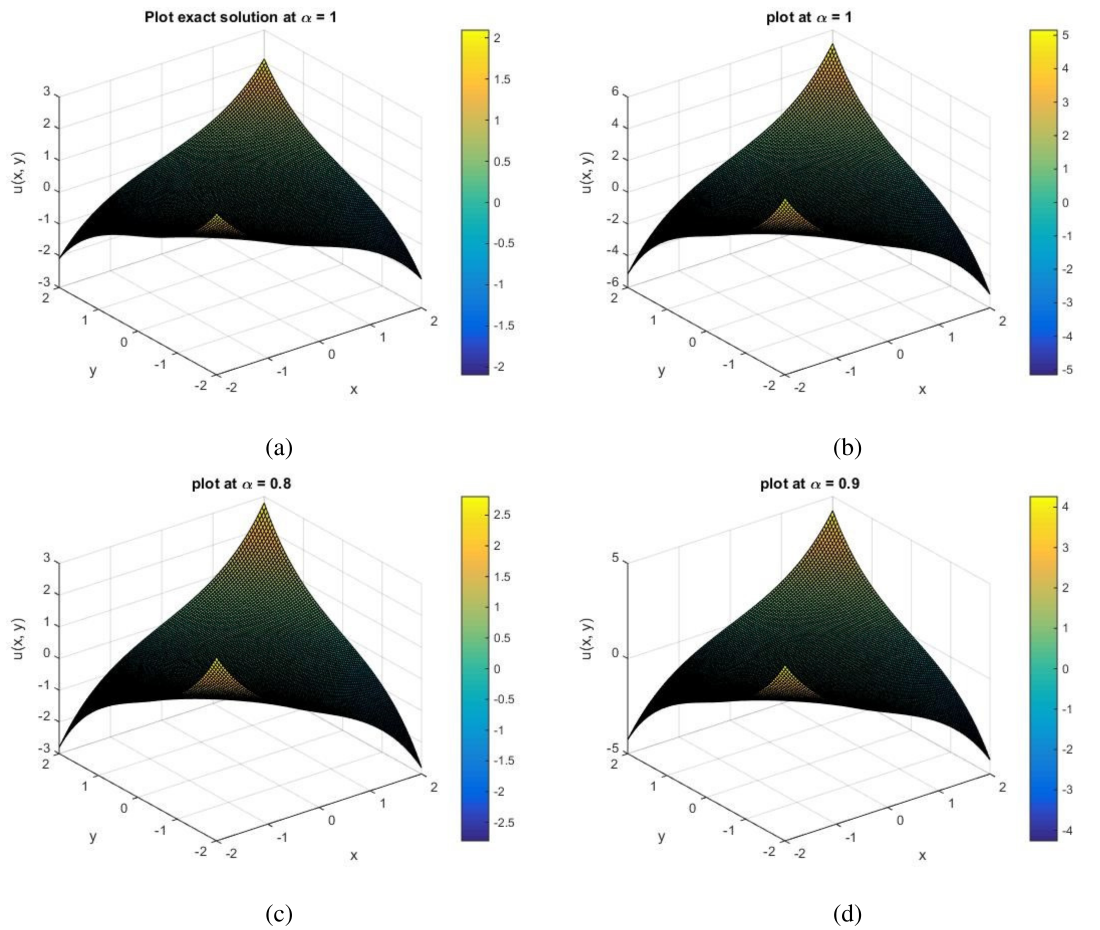

Figure 4. The diagram representing the simulated Result $\mho(\varkappa, \mathrm{t})$ of Instance (4):when (a) the exact solution, (b) $\ltimes = 1$, (c) $\ltimes = 0.8$, (d) $\ltimes = 0.9$.

Example 6.3. Take into account the time-fractional 2D nonlinear telegraph equation:

$$\frac{\partial^{2\ltimes}\mho}{\partial t^{2\ltimes}} = \frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - \mho^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}, \quad 0 < \ltimes \leq 1, \tag{59}$$

with initial conditions:

$$\mho(\varkappa,y,0) = e^{\varkappa+y},$$
$$\mho_t(\varkappa,y,0) = -2e^{\varkappa+y}.$$

Solution. Applying the Yang Transform (YaT) Taking the derivative of both sides of equation (59) with respect to t , we get:

$$\frac{Y_a\{\mho(t)\}}{v^{2\ltimes}} - \frac{\mho(\varkappa,y,0)}{v^{2\ltimes-1}} - \frac{\mho_t(\varkappa,y,0)}{v^{2\ltimes-2}} = Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - \mho^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}\right].$$

$$Y_a\{\mho(\varkappa,y,t)\} = v\mho(\varkappa,y,0) + v^2\mho_t(\varkappa,y,0) + v^{2\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial \varkappa^2} + \frac{\partial^2\mho}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - \mho^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}\right]. \tag{60}$$

Taking the inverse Yang transform of (60) gives:

$$\mho(\varkappa,y,t) = e^{\varkappa+y} - 2te^{\varkappa+y} + Y_a^{-1}\left\{v^{2\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial x^2} + \frac{\partial^2\mho}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - \mho^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}\right]\right\}. \tag{61}$$

Differentiating equation (61) with respect to t, we obtain:

$$\frac{\partial\mho(\varkappa,y,t)}{\partial t} = -2e^{\varkappa+y} + \frac{\partial}{\partial t}Y_a^{-1}\left\{v^{\ltimes}Y_a\left[\frac{\partial^2\mho}{\partial x^2} + \frac{\partial^2\mho}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}} - \mho^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}\right]\right\}. \tag{62}$$

The correction functional for equation (62) with $\lambda = -1$ is stated as:

$$\mho_{n+1}(\varkappa,y,t) = \mho_n(\varkappa,y,t) - \int_0^t \left[\frac{\partial\mho_n(\varkappa,y,\varsigma)}{\partial\varsigma} + 2e^{x+y} - \frac{\partial}{\partial\varsigma}Y_a^{-1}\left\{v^{\ltimes}Y_a\left[\frac{\partial^2\mho_n}{\partial \varkappa^2} + \frac{\partial^2\mho_n}{\partial y^2} - 2\frac{\partial^{\ltimes}\mho_n}{\partial t^{\ltimes}} - \mho_n^2 + e^{2(\varkappa+y)-4t} - 2e^{(\varkappa+y)-2t}\right]\right\}\right] d\varsigma.$$

$$\mho_0(\varkappa,y,t) = \mho(\varkappa,y,0) + t\mho_t(\varkappa,y,0)$$
$$= e^{\varkappa+y} - 2te^{\varkappa+y}.$$

$$\mho_1(\varkappa,y,\mathfrak{t}) = \mho_0(\varkappa,y,\mathfrak{t}) - \int_0^{\mathfrak{t}} \Bigg[\frac{\partial \mho_0(\varkappa,y,\varsigma)}{\partial \varsigma} + 2e^{\varkappa+y} \quad - \frac{\partial}{\partial \varsigma} Y_a^{-1} \Big\{ v^{\ltimes} Y_a \Big[\frac{\partial^2 \mho_0(\varkappa,y,\varsigma)}{\partial \varkappa^2} + \frac{\partial^2 \mho_0(\varkappa,y,\varsigma)}{\partial y^2}$$
$$- 2\frac{\partial^{\ltimes} \mho_0(\varkappa,y,\varsigma)}{\partial \mathfrak{t}^{\ltimes}} + \mho_0^2(\varkappa,y,\varsigma) + e^{2(\varkappa+y)-4\mathfrak{t}} - 2e^{(\varkappa+y)-2\mathfrak{t}} \Big] \Big\} \Bigg] d\varsigma$$
$$= e^{\varkappa+y} - 2\mathfrak{t}e^{\varkappa+y} + \frac{(4\mathfrak{t}^{2\ltimes})}{\Gamma(2\ltimes+1)} e^{\varkappa+y}.$$

$$\mho_2(\varkappa,y,\mathfrak{t}) = \mho_1(\varkappa,y,\mathfrak{t}) - \int_0^{\mathfrak{t}} \Bigg[\frac{\partial \mho_1(\varkappa,y,\varsigma)}{\partial \varsigma} + 2e^{\varkappa+y} \quad - \frac{\partial}{\partial \varsigma} Y_a^{-1} \Big\{ v^{\ltimes} Y_a \Big[\frac{\partial^2 \mho_1(\varkappa,y,\varsigma)}{\partial \varkappa^2} + \frac{\partial^2 \mho_1(x,y,\varsigma)}{\partial y^2}$$
$$- 2\frac{\partial^{\ltimes} \mho_1(\varkappa,y,\varsigma)}{\partial \mathfrak{t}^{\ltimes}} + \mho_1^2(\varkappa,y,\varsigma) + e^{2(\varkappa+y)-4\mathfrak{t}} - 2e^{(\varkappa+y)-2\mathfrak{t}} \Big] \Big\} \Bigg] d\varsigma$$
$$= e^{\varkappa+y} - 2\mathfrak{t}e^{\varkappa+y} + \frac{(4\mathfrak{t}^{2\ltimes})}{\Gamma(2\ltimes+1)} e^{\varkappa+y} - \frac{(8\mathfrak{t}^{3\ltimes})}{\Gamma(3\ltimes+1)} e^{\varkappa+y}.$$

$$\mho_3(\varkappa,y,\mathfrak{t}) = e^{\varkappa+y} - 2\mathfrak{t}e^{\varkappa+y} + \frac{(4\mathfrak{t}^{2\ltimes})}{\Gamma(2\ltimes+1)} e^{\varkappa+y} - \frac{(8\mathfrak{t}^{3\ltimes})}{\Gamma(3\ltimes+1)} e^{\varkappa+y} \quad + \frac{(16\mathfrak{t}^{4\ltimes})}{\Gamma(4\ltimes+1)} e^{\varkappa+y}.$$

$$\mho_n(\varkappa,y,\mathfrak{t}) = e^{\varkappa+y} \left[1 - 2\mathfrak{t} + \frac{(2\mathfrak{t}^{\ltimes})^2}{\Gamma(2\ltimes+1)} - \frac{(2\mathfrak{t}^{\ltimes})^3}{\Gamma(3\ltimes+1)} + \frac{(2\mathfrak{t}^{\ltimes})^4}{\Gamma(4\ltimes+1)} - \cdots \right].$$

Taking the limit as $\eta \to \infty$, we have:

$$\mho(\varkappa,y,\mathfrak{t}) = \lim_{\eta\to\infty} \mho_\eta(\varkappa,y,\mathfrak{t})$$
$$= e^{\varkappa+y} \sum_{k=0}^{\infty} \frac{(-2\mathfrak{t}^{\ltimes})^k}{\Gamma(k\ltimes+1)}. \tag{63}$$

when $\ltimes = 1$:

$$\mho(\varkappa,y,\mathfrak{t}) = e^{\varkappa+y} \left(1 - 2\mathfrak{t} + \frac{(2\mathfrak{t})^2}{2!} - \frac{(2\mathfrak{t})^3}{3!} + \frac{(2\mathfrak{t})^4}{4!} - \cdots \right) = e^{\varkappa+y} e^{-2\mathfrak{t}} = e^{\varkappa+y-2\mathfrak{t}}. \tag{64}$$

$\varkappa$	$\ltimes = 1$	$\ltimes = 0.9$	$\ltimes = 0.8$	Exact	$\lvert \mho_{Ex} - \mho_{\ltimes=1} \rvert$	$\lvert \mho_{Ex} - \mho_{\ltimes=0.9} \rvert$	$\lvert \mho_{Ex} - \mho_{\ltimes=0.8} \rvert$
-2	0.08208	0.10416	0.12871	0.08208	5.5856e-07	0.022078	0.046622
-1.7	0.10251	0.13008	0.16074	0.10251	6.9755e-07	0.027572	0.058224
-1.5	0.12802	0.16245	0.20073	0.12802	8.7114e-07	0.034433	0.072713
-1.3	0.15988	0.20288	0.25069	0.15988	1.0879e-06	0.043001	0.090807
-1.1	0.19967	0.25337	0.31307	0.19967	1.3586e-06	0.053702	0.1134
-0.8	0.24935	0.31642	0.39098	0.24935	1.6967e-06	0.067066	0.14163
-0.6	0.31141	0.39516	0.48827	0.3114	2.119e-06	0.083755	0.17687
-0.4	0.3889	0.49349	0.60978	0.3889	2.6463e-06	0.1046	0.22088
-0.2	0.48568	0.6163	0.76152	0.48567	3.3048e-06	0.13063	0.27585
0	0.60653	0.76966	0.95102	0.60653	4.1272e-06	0.16313	0.34449

Table 3. Quantitative results of the numerical and exact solutions for various values of $\varkappa$ and $\mathfrak{t}$ when $\ltimes = 0.8, 0.9, 1$.

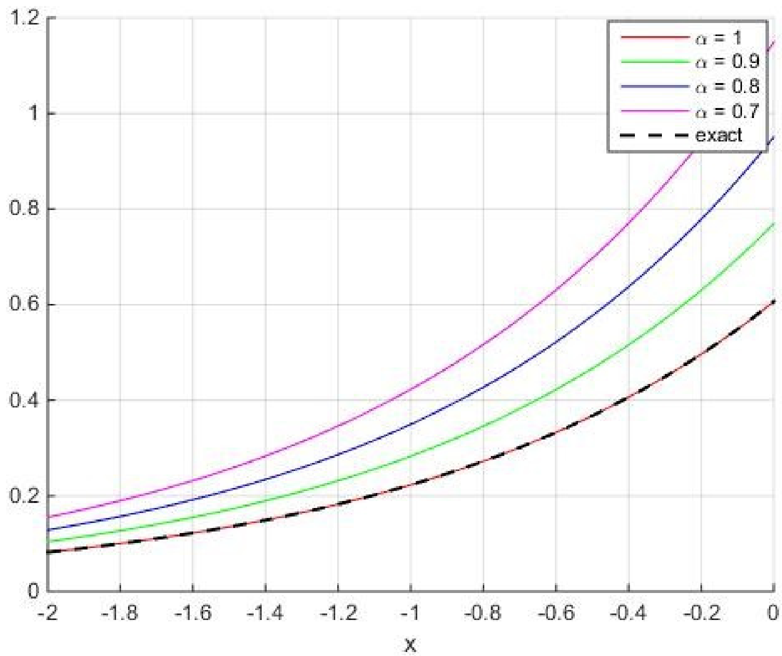

Figure 5. The plots of the rough Result $\mho(\varkappa, \mathfrak{t})$for a range of values of $\ltimes$

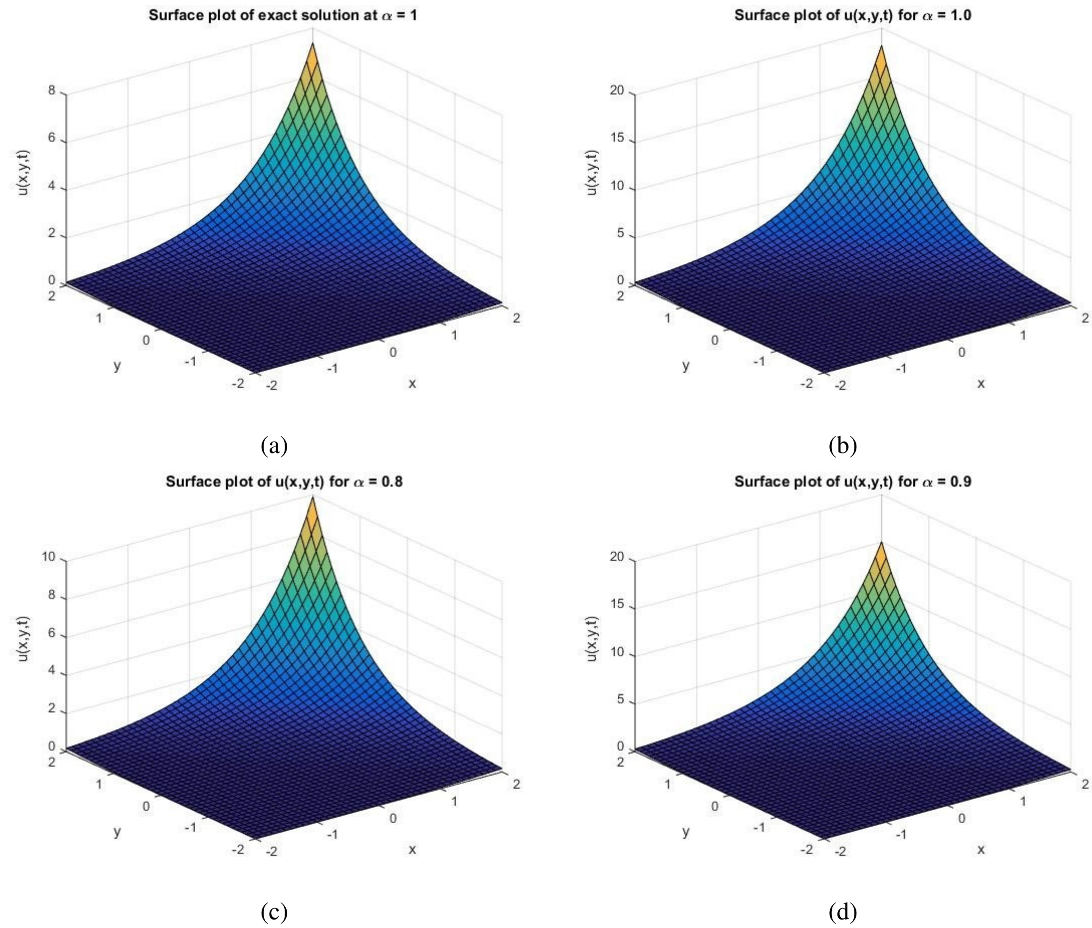

Figure 6. The diagram representing the simulated Result $\mho(\varkappa, \mathrm{t})$ of Instance (4):when (a) the exact solution, (b) $\ltimes = 1$, (c) $\ltimes = 0.8$, (d) $\ltimes = 0.9$.

Example 6.4. Take into account the 3D time-fractional nonlinear telegraph equation:

$$\frac{\partial^{2\ltimes}\mho}{\partial t^{2\ltimes}}=\frac{\partial^2\mho}{\partial\varkappa^2}+\frac{\partial^2\mho}{\partial y^2}+\frac{\partial^2\mho}{\partial z^2}+\frac{\partial^{\ltimes}\mho}{\partial t^{\ltimes}}-\mho^2+e^{2(\varkappa-y-z)-2t}-e^{(\varkappa-y-z)-t},\quad 0<\ltimes\leq 1, \tag{65}$$

with initial conditions:

$$\mho(\varkappa,y,z,0)=e^{\varkappa-y-z},\quad \mho_t(\varkappa,y,z,0)=-e^{\varkappa-y-z}.$$

Solution.

Taking the Yang transform of both sides of equation (65) with respect to t, we get:

$$\frac{\mathrm{Ya}[\mho(t)]}{v^{2\ltimes}}-\frac{\mho(\varkappa,y,z,0)}{v^{2\ltimes-1}}-\frac{\mho_t(\varkappa,y,z,0)}{v^{2\ltimes-2}}=\mathrm{Ya}\left[\frac{\partial^2\mho}{\partial\varkappa^2}+\frac{\partial^2\mho}{\partial y^2}+\frac{\partial^2\mho}{\partial z^2}+\frac{\partial^{\ltimes}\mho}{\partial \mathfrak{t}^{\ltimes}}-\mho^2+e^{2(\varkappa-y-z)-2\mathfrak{t}}-e^{(\varkappa-y-z)-\mathfrak{t}}\right].$$

$$\mathrm{Ya}[\mho(\varkappa,y,z,\mathfrak{t})]=v\mho(\varkappa,y,z,0)+v^2\mho_{\mathfrak{t}}(\varkappa,y,z,0)+v^{2\ltimes}\mathrm{Ya}\left[\frac{\partial^2\mho}{\partial\varkappa^2}+\frac{\partial^2\mho}{\partial y^2}+\frac{\partial^2\mho}{\partial z^2}-\mho^2+e^{2(\varkappa-y-z)-2t}-e^{(\varkappa-y-z)-t}\right]. \tag{66}$$

Taking the inverse (YaT) of eq. (66), we arrive at:

$$\mho(\varkappa,y,z,\mathfrak{t})=e^{\varkappa-y-z}-\mathfrak{t}e^{\varkappa-y-z}\quad+\mathrm{Ya}^{-1}\left\{v^{2\ltimes}\mathrm{Ya}\left[\frac{\partial^2\mho}{\partial\varkappa^2}+\frac{\partial^2\mho}{\partial y^2}+\frac{\partial^2\mho}{\partial z^2}+\frac{\partial^{\ltimes}\mho}{\partial \mathfrak{t}^{\ltimes}}-\mho^2+e^{2(\varkappa-y-z)-2\mathfrak{t}}-e^{(\varkappa-y-z)-\mathfrak{t}}\right]\right\}. \tag{67}$$

Differentiating equation (67) with respect to $\mathfrak{t}$:

$$\frac{\partial\mho(\varkappa,y,z,\mathfrak{t})}{\partial\mathfrak{t}}=-e^{\varkappa-y-z}+\frac{\partial}{\partial t}\mathrm{Ya}^{-1}\left\{v^{2\ltimes}\mathrm{Ya}\left[\frac{\partial^2\mho}{\partial\varkappa^2}+\frac{\partial^2\mho}{\partial y^2}+\frac{\partial^2\mho}{\partial z^2}+\frac{\partial^{\ltimes}\mho}{\partial \mathfrak{t}^{\ltimes}}-\mho^2+e^{2(\varkappa-y-z)-2\mathfrak{t}}-e^{(\varkappa-y-z)-\mathfrak{t}}\right]\right\}. \tag{68}$$

The correction functional for equation (68) with $\lambda=-1$ is given by:

$$\mho_{\eta+1}(\varkappa,y,z,\mathfrak{t})=\mho_\eta(\varkappa,y,z,\mathfrak{t})-\int_0^{\mathfrak{t}}\left[\frac{\partial\mho_\eta(\varkappa,y,z,\varsigma)}{\partial\varsigma}-e^{\varkappa-y-z}-\frac{\partial}{\partial\varsigma}\mathrm{Ya}^{-1}\left\{v^{\ltimes}\mathrm{Ya}\left[\frac{\partial^2\mho_\eta}{\partial\varkappa^2}+\frac{\partial^2\mho_\eta}{\partial y^2}+\frac{\partial^2\mho_\eta}{\partial z^2}+\frac{\partial^{\ltimes}\mho_\eta}{\partial \mathfrak{t}^{\ltimes}}-\mho_\eta^2+e^{2(\varkappa-y-z)-2\mathfrak{t}}-e^{(\varkappa-y-z)-\mathfrak{t}}\right]\right\}\right]d\varsigma.$$

$$\mho_0(\varkappa,y,z,\mathfrak{t})=\mho(\varkappa,y,z,0)+\mathfrak{t}\mho_{\mathfrak{t}}(\varkappa,y,z,0)=e^{\varkappa-y-z}-\mathfrak{t}e^{\varkappa-y-z}.$$

$$\mho_1(\varkappa,y,z,t)=\mho_0(\varkappa,y,z,t)\quad-\int_0^t\left[\frac{\partial\mho_0}{\partial\varsigma}-e^{\varkappa-y-z}\quad-\frac{\partial}{\partial\varsigma}Y_a^{-1}\left\{v^{\ltimes}Y_a\left(\frac{\partial^2\mho_0}{\partial\varkappa^2}+\frac{\partial^2\mho_0}{\partial y^2}+\frac{\partial^2\mho_0}{\partial z^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\partial^{\ltimes}\mho_0}{\partial t^{\ltimes}}-(\mho_0(\varkappa,y,z,t))^2+e^{2(\varkappa-y-z)-2t}-e^{(\varkappa-y-z)-t}\right)\right\}\right]d\varsigma.$$

$$\mho_1(\varkappa,y,z,t)=e^{\varkappa-y-z}-te^{\varkappa-y-z}+\frac{t^{2\ltimes}}{\Gamma(2\ltimes+1)}e^{\varkappa-y-z}.$$

$$\mho_2(\varkappa,y,z,t)=\mho_1(\varkappa,y,z,t)\quad-\int_0^t\left[\frac{\partial\mho_1}{\partial\varsigma}-e^{\varkappa-y-z}\quad-\frac{\partial}{\partial\varsigma}Y_a^{-1}\left\{v^{\ltimes}Y_a\left(\frac{\partial^2\mho_1}{\partial\varkappa^2}+\frac{\partial^2\mho_1}{\partial y^2}+\frac{\partial^2\mho_1}{\partial z^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\partial^{\ltimes}\mho_1}{\partial t^{\ltimes}}-(\mho_1(\varkappa,y,z,t))^2+e^{2(\varkappa-y-z)-2t}-e^{(\varkappa-y-z)-t}\right)\right\}\right]d\varsigma.$$

$$\mho_2(\varkappa,y,z,t)=e^{\varkappa-y-z}-te^{\varkappa-y-z}+\frac{t^{2\ltimes}}{\Gamma(2\ltimes+1)}e^{\varkappa-y-z}-\frac{t^{3\ltimes}}{\Gamma(3\ltimes+1)}e^{\varkappa-y-z}.$$

$$\mho_3(\varkappa,y,z,t)=\mho_2(\varkappa,y,z,t)\quad-\int_0^t\left[\frac{\partial\mho_2}{\partial\varsigma}-e^{\varkappa-y-z}\quad-\frac{\partial}{\partial\varsigma}Y_a^{-1}\left\{v^{\ltimes}Y_a\left(\frac{\partial^2\mho_2}{\partial\varkappa^2}+\frac{\partial^2\mho_2}{\partial y^2}+\frac{\partial^2\mho_2}{\partial z^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\partial^{\ltimes}\mho_2}{\partial t^{\ltimes}}-(\mho_2(\varkappa,y,z,t))^2+e^{2(\varkappa-y-z)-2t}-e^{(\varkappa-y-z)-t}\right)\right\}\right]d\varsigma.$$

$$\mho_3(\varkappa,y,z,t)=e^{\varkappa-y-z}-te^{\varkappa-y-z}+\frac{t^{2\ltimes}}{\Gamma(2\ltimes+1)}e^{\varkappa-y-z}-\frac{t^{3\ltimes}}{\Gamma(3\ltimes+1)}e^{\varkappa-y-z}+\frac{t^{4\ltimes}}{\Gamma(4\ltimes+1)}e^{\varkappa-y-z}.$$

$$\mho(\varkappa,y,z,t)=e^{\varkappa-y-z}\left[1-\frac{t^{\ltimes}}{\Gamma(\ltimes+1)}+\frac{t^{2\ltimes}}{\Gamma(2\ltimes+1)}-\frac{t^{3\ltimes}}{\Gamma(3\ltimes+1)}+\frac{t^{4\ltimes}}{\Gamma(4\ltimes+1)}-\cdots\right].\tag{69}$$

when $\ltimes=1$:

$$\mho(\varkappa,y,z,t)=e^{\varkappa-y-z}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right).$$

$$\mho(\varkappa,y,z,t)=e^{\varkappa-y-z-t}.\tag{70}$$

$\varkappa$	$\ltimes=1$	$\ltimes=0.9$	$\ltimes=0.8$	Exact	$\lvert\mho_{Ex}-\mho_{\ltimes=1}\rvert$	$\lvert\mho_{Ex}-\mho_{\ltimes=0.9}\rvert$	$\lvert\mho_{Ex}-\mho_{\ltimes=0.8}\rvert$
-2	0	0.23537	0.24978	0.22313	1.8854e-09	0.012241	0.026654
-1.7	0.22313	0.29394	0.31194	0.27866	2.3545e-09	0.015287	0.033286
-1.5	0.27866	0.36709	0.38957	0.348	2.9404e-09	0.019091	0.04157
-1.3	0.348	0.45844	0.48651	0.4346	3.6722e-09	0.023841	0.051914
-1.1	0.4346	0.57252	0.60758	0.54275	4.586e-09	0.029774	0.064833
-0.8	0.54275	0.71499	0.75878	0.67781	5.7272e-09	0.037184	0.080967
-0.6	0.67781	0.89292	0.9476	0.84648	7.1524e-09	0.046437	0.10112
-0.4	1.0571	1.1151	1.1834	1.0571	8.9323e-09	0.057992	0.12628
-0.2	1.3202	1.3926	1.4779	1.3202	1.1155e-08	0.072424	0.1577
0	1.6487	1.7392	1.8457	1.6487	1.3931e-08	0.090446	0.19695

Table 4. Quantitative results of the numerical and exact solutions for various values of $\varkappa$ and $\mathfrak{t}$ when $\ltimes=0.8,0.9,1$.

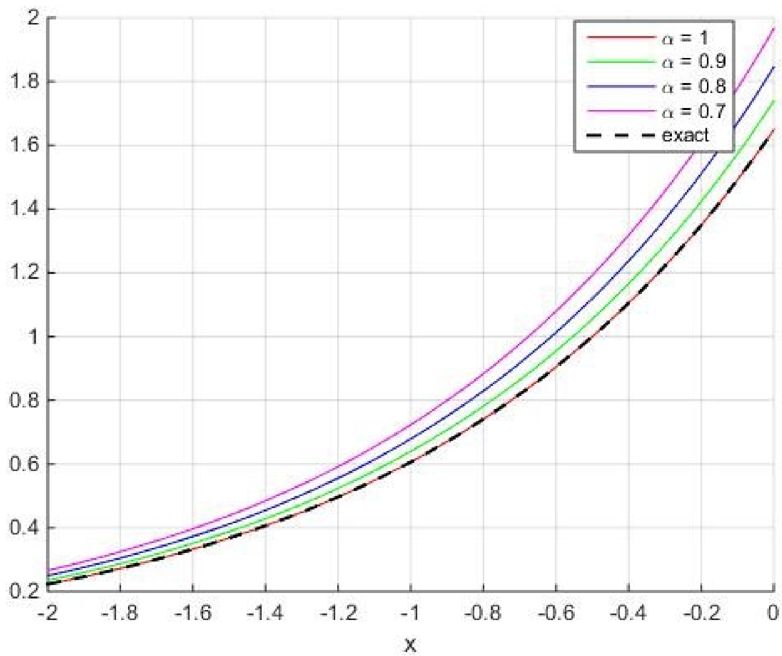

Figure 7. The plots of the rough Result $\mho(\varkappa,\mathfrak{t})$for a range of values of $\ltimes$

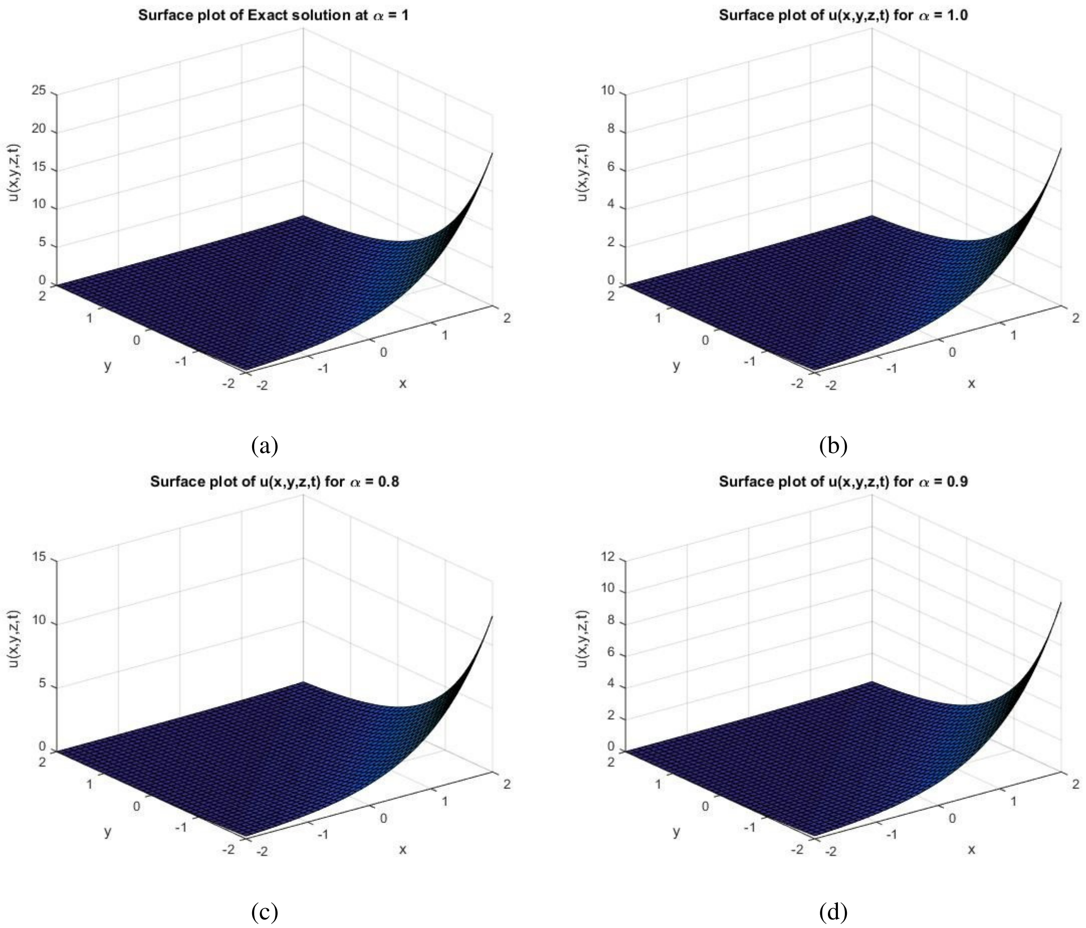

Figure 8. The diagram representing the simulated Result $\mho(\varkappa, t)$ of Instance (4):when (a) the exact solution, (b) $\ltimes = 1$, (c) $\ltimes = 0.8$, (d) $\ltimes = 0.9$.

7. Conclusion

This study introduces a new approach that combines Yang's transform with the variational iteration technique to solve time-fractional telegraph equations within the half-space domain. The Caputo derivative was applied to both time and spatial variables. Several strategies were employed to tackle the challenges associated with determining the general Lagrange multiplier. The obtained solutions were represented as a series that rapidly converges to an exact analytical expression, with simple and easily computable terms. The calculations were straightforward and efficient. The method was tested and verified through four different examples under various conditions. This approach has demonstrated its effectiveness, reliability, and efficiency, and it can be expanded to address both linear and nonlinear fractional problems in practical applications.However, there are some limitations to consider when using this method. Among these limitations is the computational cost, which may increase in certain cases, especially when the equations require a large number of iterations or refinements. Although the method is efficient in obtaining fast solutions, increasing complexity may lead to challenges in computational resources. MATLAB was used in this study for plotting the results and displaying tables that illustrate the accuracy of the obtained solutions, making the interpretation easier and more comprehensible. The method can also be applied to a variety of other equations in multiple fields, such as mechanics, engineering, and physics, to extend the scope of this technique in solving complex mathematical problems.

Acknowledgement

We sincerely thank the editor and reviewers for their valuable comments, which helped improve the quality of the research.

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