STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, April 2025, pp 1690–1713.

Dynamics of a Fractional Order Harvested Predator-Prey Model Incorporating Fear Effect and Refuge

Siti Nurul Afiyah [1,2], Fatmawati [3,*], Windarto [3], Afeez Abidemi [4]

[1]*Doctoral Program of Mathematics and Natural Science, Faculty of Science and Technology,*
Universitas Airlangga, Surabaya 60115, Indonesia
[2] *Informatics Department, Faculty of Technology and Design, Institut Teknologi dan Bisnis Asia, Malang, 65113, Indonesia*
[3]*Department of Mathematics, Faculty of Science and Technology, Universitas Airlangga, Surabaya 60115, Indonesia*
[4]*Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State, PMB 704, Nigeria*

Abstract This study presents a fractional-order predator-prey dynamics model that considers the impact of fear, refuge, and harvesting on the population, respectively. The proposed model uses the Caputo fractional derivative to successfully obtain the memory effects of this interaction between predators and preys. We prove the existence and uniqueness of solution to ensure the non-negativity and boundedness of the system, which is indispensable for maintaining biologically feasible populations. The stability analysis is conducted on the equilibrium points at local and global levels, explaining the conditions that guarantee these points are stable or lead to periodic dynamics through Hopf bifurcation. To support the analytical results, numerical simulations are provided, which demonstrate the essential roles played by fear, refuge, and harvesting in the survival of prey and the overall dynamics of the system.

Keywords Caputo fractional derivative, stability analysis, harvesting, fear effect, prey refuge.

AMS 2010 subject classifications 92B05, 92D50, 37N25

DOI: 10.19139/soic-2310-5070-2341

1. Introduction

In complex wildlife, the predators and prey interactions have become the subject of in-depth and exciting research. How these creatures interact in ecosystems, affecting each other's growth and survival, has been the focus of natural scientists' attention for centuries. One of the difficulties in understanding this relationship is that predators and prey always influence each other in an ever-changing environment. In an effort to simplify and explain these complex dynamics, the predator-prey model has become an important tool in ecology [1].

The predator-prey model is a mathematical tool that allows us to describe and understand changes in predator and prey populations over time. This model, first developed by Lotka and Volterra in the early twentieth century, has helped us solve ecological mysteries and uncover several unexpected ecological impacts. Moreover, numerous mathematical models have been developed using biologically plausible assumptions. A multitude of researchers have developed predator-prey models by adding some modifications including harvesting [2, 3, 4], refuge [5, 6, 7, 8, 9, 10], cannibalism [10, 11, 12, 13, 14], fear effects [15, 16, 17, 18, 19, 20, 21], hunting cooperation [15, 22], migration [23], and more.

On predator-prey interactions, research on the effects of fear is one of the interesting topics to discuss. Fear effects are physiological changes related to behavior and stress due to the presence of predators, prey remains

---

[*] Correspondence to: Fatmawati (Email: fatmawati@fst.unair.ac.id). Department of Mathematics, Faculty of Science and Technology, Universitas Airlangga, Surabaya 60115, Indonesia.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

vigilant at all times, anticipating potential assaults from predators. [24]. In 2017, Elliott et al. [25] conducted a field study on the effects of fear on the fitness of populations of Drasophila melanogaster (prey) and mantids (predators) in relation to species density. The authors have shown that the reproductive success of drasophila decreases when they come into contact with mantids, especially when there are few mantids around. This decline occurs during both the breeding and non-breeding seasons. In recent years, several researchers have studied the modified predator-prey model with the inclusion of fear effects, one of the studies conducted by Pal et al. [26] in 2019. Pal et al. [26] conducted a study on the dynamic analysis of Leslie-Gower models. They specifically investigated the impact of fear on prey and how it affects the cooperation between predators. The study's findings demonstrated that the fear-effect parameter exerted an influence on the prey's condition, leading to a reduction in the prey population that could ultimately result in extinction.

Protection is an additional factor that influences the interaction between predators and prey. In the process of predation, predator populations constantly chew prey populations so that the prey population continues to decrease and is threatened with extinction. To avoid extinction, prey protection factors are needed. In the wild, predator and prey instincts for shelter are factors that need to be taken into account in modeling. Prey will get out of his shelter only when he feels safe from predators. The protective effect reduces the likelihood of prey being preyed, thus avoiding extinction [27]. Zhang et al. [28] investigated a predator-prey model that incorporates the influence of fear and refuge on prey.

$$\begin{aligned}\frac{dP}{dt} &= \frac{r_1 P}{1+fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P},\\ \frac{dQ}{dt} &= -r_2 Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P},\end{aligned} \tag{1}$$

where all the parameters are non-negative. $P(t)$ and $Q(t)$ are prey and predator densities at time t respectively.r_1 is the intrinsic growth rate of prey,f is level of fear, ν is the coefficient of predation, c is the efficiency Food conversion from prey to predator, δ is the coefficient of refuge on prey, r_2 is the predator natural death rate, and b is the intrinsic growth rate per carrying capacity of prey.

In expansion to the impacts of fear and refuge on prey, there are other variables that impact the dynamics of predator and prey populaces, specifically harvesting. Harvesting of populations is another vital calculate in predator-prey elements [3]. Chakraborty et al. [29] have investigated the interaction between predator and prey with addition harvesting, they considered a basic predator–prey interaction where predator population is subjected to harvesting. They watched comes about have been tried with the parameter values of Paramecium Aurelia as prey and Didinium Nasutum as predator.

Moreover, the system of integer order differential equations indicates that the rate of population growth is not solely determined by the current conditions. Nevertheless, it is an established fact that the rate of population change is influenced by all preceding conditions, a phenomenon referred to as the memory effect. This memory effect is referred to as fractional order. A system is considered to be memoryless if its output at any given time t is solely determined by the input at the same time t, without any dependence on previous inputs. Conversely, if the system needs to retain past input values in order to determine the present output value, it is referred to as a memory system [30]. By incorporating memory effects, fractional-order models allow us to represent ecological processes more realistically, particularly in systems where delayed responses or cumulative interactions significantly influence population dynamics. For instance, in predator-prey systems, the memory effect can reflect how past predation events affect prey behavior, reproduction, and refuge-seeking strategies, or how predators adjust their hunting efficiency based on historical prey availability. These capabilities highlight the advantages of fractional calculus in ecological modeling compared to traditional integer-order approaches.

This fractional differential equation model provides a more accurate description than the first-order dynamic system for complex natural dynamics [31]. The development of the predator-prey model of fractional orders has increased the interest of researchers in analyzing the dynamic system of fractions of orders [32, 33, 34, 35, 36, 37, 38]. Nevertheless, fractional calculus also implement in different fields such as physics, engineering, and more [39, 40, 41]. The use of fractional calculus thus enables researchers to capture these nuanced dynamics, offering insights into long-term system behavior, stability conditions, and the impact of historical interactions. This makes fractional-order models particularly useful for exploring ecological challenges that involve delayed feedbacks or cumulative environmental impacts, which are difficult to capture using conventional methods.

The fractional approaches are Riemann-Liouville, Caputo, and Grunwald-Letnikov. Among some of these approaches, Caputo's approach is more frequently used by researchers, including the research carried out by Themairi and Alqudah [42], Das and Samanta [43], Higazy et al. [44] and Jafari et al. [45]. Caputo's approach has an advantage over the Riemann-Liouville approach. In Caputo's approach, there is no need to define the initial conditions of fractional orders. Furthermore, one of the important advantages in using a Caputo-type fractions is that the Caputo derivative with a constant is zero. In Caputo's approach, the initial value can use the same starting value as in an integer order. Based on that, this research will use the Caputo approach.

The previous model (1) was designed for predator prey model with fear effect and prey refuge only and the harvesting is not yet implemented. However, this research investigated new model by complementing the harvesting on the model (1). Beside, the model (1) was built by using a couple of first-order derivative nonlinear differential equations. Then, this research considered a fractional-order predator prey model with fear effect, prey refuge and harvesting on both population via Caputo derivative. Therefore, the novelties of this research article is a new model combination between fear effect, prey refuge and harvesting in both population and memory effect.

The subsequent sections of this paper are structured in the following manner: In Section 2, we present a fractional order harvested predator-prey model that includes the fear effect and refuge. In Section 3, the fundamental principles of fractional calculus were examined. In the following section, Section 4, we will discuss the primary outcome, which includes the existence, uniqueness, non-negativity, and boundedness of equilibrium points, as well as the existence of equilibrium points, local stability, global stability of equilibrium points, and Hopf-bifurcation. In Section 5, we presented a numerical simulation to validate the analytical findings and discuss the outcomes of our research. In the final section, we provided conclusions derived from our study.

2. Mathematical Modelling

The predator-prey model with prey refuge, fear effect, and harvesting in a first-order derivative non-linear differential equation is as follows:

$$\begin{aligned}\frac{dP}{dt} &= \frac{r_1 P}{1+fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_1(1-\delta)P,\\ \frac{dQ}{dt} &= -r_2 Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_2 Q,\end{aligned} \tag{2}$$

where z_1 and z_2 represents the prey and predator harvesting rate are non-negative.

In this paper we consider the fractional predator prey order model with effects of fear, refuge on prey and harvesting in both populations.

$$\begin{aligned}{}^{C}D^{\alpha}P(t) &= \frac{r_1 P}{1+fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_1(1-\delta)P,\\ {}^{C}D^{\alpha}Q(t) &= -r_2 Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_2 Q,\end{aligned} \tag{3}$$

where condition $P(0) > 0, Q(0) > 0$ and $P(t)$ and $Q(t)$ represent prey and predator population densities at time t defined on domain $\Omega = \{(P,Q) \in R^2\}$.

3. Preliminary

The preliminary section outlines the mathematical foundations required for fractional-order analysis. Key definitions and theorems are provided to support the theoretical framework, laying the groundwork for the subsequent dynamic analysis.

Definition 3.1
[46, 47, 48] The Riemann-Liouville fractional integral operator of order α applied to a continuous function $f \in L_1[0, a]$, for $t \in [0, a]$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^a (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

Here, $\Gamma(\alpha)$ denotes the Euler Gamma function.

Definition 3.2
[46, 47, 48] The Caputo fractional derivative of order α for a continuous function $f(x) \in C^n([t_0, +\infty])$ is defined by

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau.$$

Here, $\Gamma(.)$ denotes the Euler's Gamma function, and the operator ${}^C D^\alpha$ represents the Caputo differential operator of order α, where $t \geq 0$ and n is a positive integer satisfying $n-1 < \alpha < n$, with $n \in \mathbb{N}$.
Particularly, when $0 < \alpha < 1$,

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f'(\tau)}{(t-\tau)^\alpha}, d\tau.$$

If $\alpha > 0, n-1 < \alpha \leq n (n \in N)$, then for $\alpha = n \in N$ the Caputo derivative ${}^C D^\alpha f$ coincide with the usual derivative of order n :

$${}^C D^\alpha f(t) = (D^n f)(t)(n \in N; D = \frac{d}{dx}).$$

4. Main Result

4.1. Existence and Uniqueness

Theorem 4.1
The system (3) for any non-negative initial conditions (P_0, Q_0), has a unique solution $H(t) \in \Omega$.

Proof
Define $\Omega := \{(P, Q) \in R^2 : P, Q \geq 0, max|P|, |Q| \leq M\}$ with $M > 0$. It will be shown that the model always has a single solution in $[t_0, \inf) \times \Omega$ if the initial value is given in Ω for every $t \geq t_0$.
Consider $H = (P, Q)$ and $\bar{H} = (\bar{P}, \bar{Q})$ are the solution of the model (3).
Let $B(H) = (B_1(H), B_2(H))$, and

$$\begin{aligned} B_1(H) &= \frac{r_1 P}{1 + fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_1(1-\delta)P, \\ B_2(H) &= -r_2 Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_2 Q, \end{aligned}$$

we get,

$$\begin{aligned}
\|B(H)-B(\bar H\| &= |B_1(H)-B_1(\bar H)| + |B_2(H)-B_2(\bar H)| \\
&= |\frac{r_1P}{1+fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_1(1-\delta)P - \frac{r_1\bar P}{1+f\bar Q} + b\bar P^2 + \frac{\nu(1-\delta)\bar P\bar Q}{1+\eta(1-\delta)\bar P} + z_1(1-\delta)\bar P| \\
&+ | - r_2Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_2Q + r_2\bar Q - \frac{c\nu(1-\delta)\bar P\bar Q}{1+\eta(1-\delta)\bar P} + z_2\bar Q|, \\
&= |\frac{r_1P}{1+fQ} - \frac{r_1\bar P}{1+f\bar Q} - bP^2 + b\bar P^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} \\
&+ \frac{\nu(1-\delta)\bar P\bar Q}{1+\eta(1-\delta)\bar P} - z_1(1-\delta)P + z_1(1-\delta)\bar P| \\
&+ \left|-r_2P + r_2\bar Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - \frac{c\nu(1-\delta)\bar P\bar Q}{1+\eta(1-\delta)\bar P} - z_2\bar Q + z_2\bar Q\right|, \\
&\le \left[r_1 + z_1(1-\delta) + b(P+\bar P + r_1f\bar Q + \frac{(1+c)\nu(1-\delta)Q}{(1+\eta(1-\delta)P)(1+\eta(1-\delta)\bar P)}\right]|P-\bar P| \\
&+ \left[r_1f\bar P + (r_2+z_2) + \frac{(1+c)\nu(1-\delta)\bar P}{(1+\eta(1-\delta)\bar P)}\right]|Q-\bar Q|, \\
&\le [r_1 + z_1(1-\delta) + 2bM + r_1fM + (1+c)\nu(1-\delta)M]\,|P-\bar P| \\
&+ [r_1fM + (r_2+z_2) + (1+c)\nu(1-\delta)M]\,|Q-\bar Q|, \\
&= L_1|P-\bar P| + L_2|Q-\bar Q|,
\end{aligned}$$

where,

$$\begin{aligned}
L_1 &= r_1 + z_1(1-\delta) + 2bM + r_1fM + (1+c)\nu(1-\delta)M, \\
L_2 &= r_1fM + (r_2+z_2) + (1+c)\nu(1-\delta)M.
\end{aligned}$$

By choosing a positive $L = max\{L_1, L_2\}$, we get

$$\|B(H)-B(\bar H\| \le L\|H-\bar H\|.$$

The function $B(H)$ satisfies the Lipschitz condition. We can conclude that there exist a unique solution $H(t)$ of the system (3). □

4.2. *Non-Negativity and Boundedness*

Theorem 4.2
All solutions of system (3) with initial values $(P(0), Q(0)) \in R^+$ are non-negative.

Proof
To prove this theorem, we use Theorem 6 in Cresson and Szafranka [49]. For $\alpha = 1$, system (3) can be written to equation (2) From the first equation,

$$\begin{aligned}
\frac{dP}{dt} &= \frac{r_1P}{1+fQ} - bP^2 - \frac{\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_1(1-\delta)P, \\
\frac{dP}{P} &= \left(\frac{r_1}{1+fQ} - bP - \frac{\nu(1-\delta)Q}{1+\eta(1-\delta)P} - z_1(1-\delta)\right)dt, \\
\ln(P) &= \int_0^t \left(\frac{r_1}{1+fQ(\tau)} - bP(\tau) - \frac{\nu(1-\delta)Q(\tau)}{1+\eta(1-\delta)P(\tau)} - z_1(1-\delta)\right)d\tau + \ln P(0), \\
P(t) &= P(0)\exp\left\{\int_0^t \left(\frac{r_1}{1+fQ(\tau)} - bP(\tau) - \frac{\nu(1-\delta)Q(\tau)}{1+\eta(1-\delta)P(\tau)} - z_1(1-\delta)\right)d\tau\right\} \ge 0.
\end{aligned}$$

From the second equation, by applying the same work as in the first equation (see Appendix A), we get

$$Q(t) = Q(0)\exp\left\{\int_0^t \left(-r_2 + \frac{c\nu(1-\delta)P(\tau)}{1+\eta(1-\delta)P(\tau)} - z_2\right) d\tau\right\} \geq 0.$$

Since $\overrightarrow{f}(\overrightarrow{x}(t))$ satisfies Local Lipschitz, $P(t) \geq 0$ and $Q(t) \geq 0$ for $\alpha = 1$ and non-negative initial value, then according Theorem 6 by Cresson and Szafranka [49], model (3) has non-negative solution. ☐

Theorem 4.3
All solutions of system (3) with initial values $(P(0), Q(0)) \in R^+$ are uniformly bounded.

Proof
Define function $K(t) = P(t) + \frac{1}{c}Q(t)$, then we have

$$^C D^\alpha K(t) = {}^C D^\alpha P(t) + \frac{1}{c}{}^C D^\alpha Q(t).$$

For any positive constant ρ, we have

$$\begin{aligned} ^C D^\alpha K(t) + \rho K(t) &= \frac{r_1 P}{1+fQ} - bP^2 - z_1(1-\delta)P + \rho P + (\rho - r_2 - z_2)\frac{1}{c}Q, \\ &\leq -b\left[P - \frac{\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho}{2b}\right]^2 + \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho)}{4b}, \\ &\leq \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho)^2}{4b}. \end{aligned}$$

By employing the Fractional Comparison Principle in Li et al [50], we have

$$K(t) \leq \left(K(0) - \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho)^2}{4b}\right) E_\alpha(-\rho(t)^\alpha) + \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho)^2}{4b}.$$

Since $t \to \infty$ then $E_\alpha(-\rho(t)^\alpha) \to 0$, so that $K(t) \to \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta)+\rho)^2}{4b}, t \to \infty$. That indicates that each solution of the system (3) that initiates R^+ is contained in the region σ, where

$$\sigma = \left((P,Q) \in R^+ : P + \frac{1}{c}Q \leq \frac{(\frac{r_1 P}{1+fQ} - z_1(1-\delta) + \rho)^2}{4b} + \epsilon, \epsilon > 0\right).$$

☐

4.3. Equilibrium Points

The system (3) contains three equilibrium points, detailed as follows:

1. The trivial equilibrium point $E_0 = (0,0)$, which always exists.
2. The predator free equilibrium point $E_1 = (\frac{r_1 - z_1(1-\delta)}{b}, 0)$, which is exists under the condition $r_1 > z_1(1-\delta)$.
3. The interior equilibrium point $E_2 = (P^*, Q^*)$, where $P^* = \frac{r_2 + z_2}{(1-\delta)(c\nu - \eta(r_2+z_2))}$ and Q^* are to be obtained from quadratic equation as follows:

$$a_0(Q^*)^2 + a_1 Q^* + a_2 = 0, \tag{4}$$

where,

$$\begin{aligned} a_0 &= \nu f(1-\delta), \\ a_1 &= \eta b f(1-\delta)(P^*)^2 + bfP^* + \eta z_1 f(1-\delta)^2 P^* + \nu(1-\delta) + z_1(1-\delta)f, \\ a_2 &= \eta b(1-\delta)(P^*)^2 + bP^* - \eta r_1(1-\delta)P^* - r_1 + z_1(1-\delta) + \eta z_1(1-\delta)^2. \end{aligned}$$

We can see that $a_0 > 0$ and $a_1 > 0$, with using quadratic equation root properties, the equation (4) has a positive root if $a_2 < 0$, that is

$$r_1(1+\eta(1-\delta)P^*) > bP^*(1+\eta(1-\delta)P^*) + z_1(1-\delta)(1+\eta(1-\delta)),$$

with $Q^* = \frac{-a_1+\sqrt{a_1^2-4a_0a_2}}{2a_0}$. Furthermore, the interior equilibrium point $E_2 = (P^*, Q^*)$ exist under condition $c\nu > \eta(r_2+z_2)$ and $r_1(1+\eta(1-\delta)P^*) > bP^*(1+\eta(1-\delta)P^*) + z_1(1-\delta)(1+\eta(1-\delta))$.

4.4. Local Stability

The linearization technique is used to analyze the local stability of the fractional-order predator-prey model that incorporates fear, refuge, and harvesting. The first step is to calculate the Jacobian matrix of the system at the equilibrium points. This matrix provides insight into how small perturbations around these points affect the system, allowing us to determine whether the system will return to equilibrium (stable) or diverge away from it (unstable). The Jacobian matrix of the system (3) at an equilibrium point $(P*, Q*)$ is given by

$$J(P,Q) = \begin{bmatrix} \frac{r_1}{1+fQ} - 2bP - \frac{\nu(1-\delta)Q}{1+\eta(1-\delta)P} + \frac{\eta\nu(1-\delta)^2PQ}{(1+\eta(1-\delta)P)^2} - z_1(1-\delta) & -\frac{r_1fP}{(1+fQ)^2} - \frac{\nu(1-\delta)P}{1+\eta(1-\delta)P} \\ \frac{c\nu(1-\delta)Q}{(1+\eta(1-\delta)P)^2} & -r_2 + \frac{c\nu(1-\delta)P}{1+\eta(1-\delta)P} - z_2 \end{bmatrix}.$$

Theorem 4.4
The trivial equilibrium point $E_0 = (0,0)$ is stable if $r_1 < z_1(1-\delta)$, otherwise it is unstable (saddle).

Proof
The Jacobian matrix at $E_0 = (0,0)$ is

$$J(E_0) = \begin{bmatrix} r_1 - z_1(1-\delta) & 0 \\ 0 & -r_2 - z_2 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix $J(E_0)$ are $\lambda_1 = -(r_2+z_2) < 0$ and $\lambda_2 = r_1 - z_1(1-\delta)$. For $E_0 = (0,0)$ to be stable, both eigen values must be negative.When $r_1 < z_1(1-\delta)$, then the eigenvalue λ_2 is negative. Consequently, we get $|arg(\lambda_1)| = |arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. Hence E_0 is asymptotically stable. Therefore, $E_0 = (0,0)$ is stable if $r_1 < z_1(1-\delta)$ meaning that the natural growth rate of the prey is less than the rate at which it is reduced by harvesting.

If $r_1 > z_1(1-\delta)$, the one eigenvalue is positive and the other negative then $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$.Therefore E_0 is a saddle point and hence unstable. □

Theorem 4.5
The predator free equilibrium point $E_1 = (\frac{r_1-z_1(1-\delta)}{b}, 0)$ is stable if $\frac{c\nu(1-\delta)(r_1-z_1(1-\delta)}{b+\eta(1-\delta)(r_1-z_1(1-\delta)} < r_2 + z_2$ while saddle point if $\frac{c\nu(1-\delta)(r_1-z_1(1-\delta)}{b+\eta(1-\delta)(r_1-z_1(1-\delta)} > r_2 + z_2$.

Proof
The Jacobian matrix for $E_1 = (\frac{r_1-z_1(1-\delta)}{b}, 0)$ is

$$J(E_1) = \begin{bmatrix} -r_1 + z_1(1-\delta) & -\frac{r_1f(r_1-z_1(1-\delta))}{b} - \frac{\nu(1-\delta)(r_1-z_1(1-\delta))}{b+\eta(1-\delta)(r_1-z_1(1-\delta))} \\ 0 & \frac{c\nu(1-\delta)(r_1-z_1(1-\delta))}{b+\eta(1-\delta)(r_1-z_1(1-\delta))} - r_2 - z_2 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix $J(E_1)$ are $\lambda_1 = -r_1 + z_1(1-\delta)$ and $\lambda_2 = \frac{c\nu(1-\delta)(r_1-z_1(1-\delta))}{b+\eta(1-\delta)(r_1-z_1(1-\delta))} - r_2 - z_2$. Since $r_1 > z_1(1-\delta)$ is the existence of equilibrium E_1, so λ_1 is always a negative real number and hence $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. If $\frac{c\nu(1-\delta)(r_1-z_1(1-\delta)}{b+\eta(1-\delta)(r_1-z_1(1-\delta)} < r_2 + z_2$, $\lambda_2 < 0$. Thus $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. We can conclude that under the above condition, the predator free equilibrium E_1 is stable. This condition means that the conversion efficiency and predation rate of the predator, adjusted for the effects of refuge, must be less than the natural death rate of the predator combined with its harvesting rate. Otherwise, if $\frac{c\nu(1-\delta)(r_1-z_1(1-\delta)}{b+\eta(1-\delta)(r_1-z_1(1-\delta)} > r_2 + z_2, \lambda_2 > 0$. Thus $|arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$. Hence the predator free equilibrium E_1 is saddle point. □

Prior to an examination of the stability criterion for the interior equilibrium point $E_2 = (P^*, Q^*)$, it is necessary to formulate the Jacobian matrix J at this point as follows:

$$J(E_2) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}.$$

where,

$$\begin{aligned} J_{11} &= -bP^* + \frac{\eta\nu(1-\delta)^2 P^* Q^*}{(1+\eta(1-\delta)P^*)^2}, \\ J_{12} &= -\frac{r_1 f P^*}{(1+fQ^*)^2} - \frac{r_2+z_2}{c}, \\ J_{21} &= \frac{c\nu(1-\delta)Q^*}{(1+\eta(1-\delta)P^*)^2}. \end{aligned}$$

Theorem 4.6
The interior equilibrium point $E_2 = (P^*, Q^*)$ is locally asymptotically stable if at least one of the following conditions is satisfied:

1. $J_{11} \leq 0$,
2. $J_{11} > 0, J_{11}^2 - 4K < 0$ and $\sqrt{4K - J_{11}^2} > J_{11}\tan\left(\frac{\alpha\pi}{2}\right)$, with $K = -J_{12}J_{21} > 0$.

Proof
The characteristic equation of the $J(E_2)$ can be written as

$$\lambda^2 - J_{11}\lambda - J_{12}J_{21} = 0. \tag{5}$$

The roots of equation (5) are

$$\begin{aligned} \lambda_{1,2} &= \frac{J_{11} \pm \sqrt{J_{11}^2 - 4(-J_{12}J_{21})}}{2}, \\ &= \frac{J_{11} \pm \sqrt{J_{11}^2 - 4K}}{2}. \end{aligned}$$

If $J_{11} \leq 0$, then we have three sub-cases. The first sub-case is $J_{11} = 0$. In this sub-case, it is evident that both eigenvalues are purely complex conjugate,thus $|arg(\lambda_{1,2})| = \frac{\pi}{2} > \frac{\alpha\pi}{2}$. Therefore equilibrium point E_2 is locally asymptotically stable. The second sub-case is $J_{11} < 0$ and $J_{11}^2 - 4K \geq 0$. In this second sub-case, we can see that both eigenvalues are negative real numbers and we have $|arg(\lambda_{1,2})| = \pi > \frac{\alpha\pi}{2}$. Hence equilibrium point E_2 is locally asymptotically stable. The last sub-case is $J_{11} < 0$ and $J_{11}^2 - 4K < 0$, then the eigenvalues are conjugate complex with negative real part. Hence $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$, and equilibrium point E_2 is locally asymptotically stable.

If $J_{11} > 0, J_{11}^2 - 4K < 0$ and $\sqrt{4K - J_{11}^2} > J_{11}\tan\left(\frac{\alpha\pi}{2}\right)$, with $K = -J_{12}J_{21} > 0$, we obtain that $Im(\lambda_1) = -Im(\lambda_2) = \frac{1}{2}\sqrt{4K - J11^2} > 0$ and $Re(\lambda_1) = -Re(\lambda_2) = \frac{J11^2}{2}$. Hence, it is clear $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$ and equilibrium point E_2 is locally asymptotically stable. □

Theorem 4.7
The interior equilibrium point $E_2 = (P^*, Q^*)$ is unstable if any one of the following conditions hold:

1. $J_{11} > 0$ and $J_{11}^2 - 4K \geq 0$,
2. $J_{11} > 0, J_{11}^2 - 4K < 0$ and $\sqrt{4K - J_{11}^2} < J_{11} \tan\left(\frac{\alpha\pi}{2}\right)$.

Proof
The first case, $J_{11} > 0$ and $J_{11}^2 - 4K \geq 0$. Both eigenvalues in this case are positive real numbers, such that $|arg(\lambda_{1,2})| = 0 < \frac{\alpha\pi}{2}$. The second case is $J_{11} > 0, J_{11}^2 - 4K < 0$ and $\sqrt{4K - J_{11}^2} < J_{11} \tan\left(\frac{\alpha\pi}{2}\right)$. In this second case, we get a couple of complex conjugate eigenvalues and hence $Im(\lambda_1) = -Im(\lambda_2) = \frac{1}{2}\sqrt{4K - J11^2} > 0$ and $|arg(\lambda_{1,2})| = \tan^{-1} \frac{\sqrt{4K - J_{11}^2}}{J_{11}} < \frac{\alpha\pi}{2}$. Hence, it is clear that equilibrium point E_2 is unstable in both case. □

4.5. Global Stability

This section focuses on examining the global stability of two equilibrium points: E_1, which represents the predator-free state, and E_2, which represents an interior equilibrium. In our work, we employ the Generalized Lasalle Invariance Principle as described in Huo et al's publication (reference: [51]), and we also utilize Lemma 3.1 from Vargas de Leon's work [52].

Theorem 4.8
The predator free equilibrium point E_1 is globally asymptotically stable if $r_1 \leq \frac{c\nu z_1(1-\delta)^2 + (r_2+z_2)b}{c\nu(1-\delta)}$.

Proof
Define function of Lyapunov,

$$V_1(P, Q) = P - P_1 - P_1 log\frac{P}{P_1} + \frac{1}{c}Q,$$

where

$$P_1 = \frac{r_1 - z_1(1-\delta)}{b} > 0.$$

According Lemma 3.1 in Vargas De Leon [52] we can write,

$$\begin{aligned}
{}^C D^\alpha V_1 &\leq \left(\frac{P - P_1}{P}\right) {}^C D^\alpha P + \frac{1}{c} {}^C D^\alpha Q, \\
&= (P - P_1)\left(\frac{r_1}{1 + fQ} - bP - \frac{\nu(1-\delta)Q}{1 + \eta(1-\delta)P} - z_1(1-\delta)\right) + \frac{1}{c}\left(\frac{c\nu(1-\delta)PQ}{1 + \eta(1-\delta)P} - (r_2 + z_2)Q\right), \\
&= -\frac{b}{1 + fQ}(P - P_1)^2 + Q\left(\frac{\nu(1-\delta)P_1}{1 + \eta(1-\delta)P} - \frac{(r_2 + z_2)}{c}\right), \\
&\leq -\frac{b}{1 + fQ}(P - P_1)^2 + Q\left(\nu(1-\delta)P_1 - \frac{(r_2 + z_2)}{c}\right).
\end{aligned}$$

If we choose $r_1 < \frac{c\nu z_1(1-\delta)^2 + (r_2+z_2)b}{c\nu(1-\delta)}$, then ${}^C D^\alpha V_1 \leq 0$ for any $(P, Q) \in R_+^2$. If $P = P_1$ and $Q = 0$ then ${}^C D^\alpha V_1 = 0$. Also if we choose $r_1 = \frac{c\nu z_1(1-\delta)^2 + (r_2+z_2)b}{c\nu(1-\delta)}$,the Lyapunov function $V_1(P, Q)$ satisfies global stability condition. Hence, we can conclude that the predator free equilibrium point E_1 is globally asymptotically stable under condition $r_1 \leq \frac{c\nu z_1(1-\delta)^2 + (r_2+z_2)b}{c\nu(1-\delta)}$. □

Theorem 4.9
The interior equilibrium point $E_2 = (P^*, Q^*)$ is globally asymptotically stable if The interior equilibrium point E_2 is locally asymptotically stable.

Proof
Define function of Lyapunov

$$V_2(P,Q) = P - P^* - P^* log\frac{P}{P^*} + s_1(Q - Q^* - Q^* log\frac{Q}{Q^*}), s_1 > 0.$$

According Lemma 3.1 in Vargas De Leon [52],

$$\begin{aligned}
{}^C D^\alpha V_2(P,Q) &\leq \left(\frac{P-P^*}{P}\right) {}^C D^\alpha P(t) + s_1 \left(\frac{Q-Q^*}{Q}\right) {}^C D^\alpha Q(t), \\
&= (P-P^*)\left(\frac{r_1}{1+fQ} - bP - \frac{\nu(1-\delta)Q}{1+\eta(1-\delta)P}\right) + s_1(Q-Q^*)\left(\frac{c\nu(1-\delta)P}{1+\eta(1-\delta)P} - (r_2+z_2)\right), \\
&\leq -\left(b - \frac{\eta\nu(1-\delta)^2 Q^*}{1+\eta(1-\delta)P^*}\right)(P-P^*)^2 + \left(\frac{s_1 c\nu(1-\delta)}{1+\eta(1-\delta)P} - \nu(1-\delta)\right)\left(\frac{(P-P^*)(Q-Q^*)}{1+\eta(1-\delta)P}\right).
\end{aligned}$$

Choose s_1 for which $\frac{s_1 c\nu(1-\delta)}{1+\eta(1-\delta)x} = \nu(1-\delta)$. Thus, we have

$${}^C D^\alpha V_2 \leq -\left(b - \frac{\eta\nu(1-\delta)^2 Q^*}{1+\eta(1-\delta)P^*}\right)(P-P^*)^2. \tag{6}$$

Consider the function $V_3(P,Q) = \frac{1}{2}\left[(P-P^*) + \frac{1}{c}(Q-Q^*)\right]^2$ and $V_4(P,Q) = (P-P^*) + \frac{1}{c}(Q-Q^*)$.
Choosing α-order Caputo derivative of $V_4(P,Q)$, it gives

$$\begin{aligned}
{}^C D^\alpha V_4 &= P\left(\frac{r_1}{1+fQ} - bP - \frac{\nu(1-\delta)Q}{1+\eta(1-\delta)P} - z_1(1-\delta)\right) + \frac{1}{c}Q\left(\frac{c\nu(1-\delta)P}{1+\eta(1-\delta)P} - (r_2+z_2)\right), \\
&= -bP(P-P^*) + \frac{Q^*(P-P^*) - P^*(Q-Q^*)}{1+\eta(1-\delta)P^*}.
\end{aligned} \tag{7}$$

Now choosing α-order Caputo derivative of $V_3(P,Q)$, we have

$$\begin{aligned}
{}^C D^\alpha V_3 &\leq \left((P-P^*) + \frac{1}{c}(Q-Q^*)\right) {}^C D^\alpha V_4, \\
&= \left((P-P^*) + \frac{1}{c}(Q-Q^*)\right)\left(-bP(P-P^*) + \frac{Q^*(P-P^*) - P^*(Q-Q^*)}{1+\eta(1-\delta)P^*}\right), \\
&= -bP(P-P^*)^2 - \frac{P^*(Q-Q^*)^2}{c(1+\eta(1-\delta)P^*)} + \left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c}\right)(P-P^*)^2, \\
&- \frac{bQ}{c}(P-P^*)^2 + \left(\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right)(P-P^*)(Q-Q^*) \\
&\leq \left(\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right)(P-P^*)(Q-Q^*) \\
&+ \left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c}\right)(P-P^*)^2 - \frac{P^*(Q-Q^*)^2}{c(1+\eta(1-\delta)P^*)}.
\end{aligned} \tag{8}$$

Now it is obvious that,

$$\begin{aligned}
&\left(\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right)(P-P^*)(Q-Q^*), \\
&\leq \frac{\left[\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right]^2}{\frac{2P^*}{c(1+\eta(1-\delta)P^*)}}(P-P^*)^2 + \frac{P^*(Q-Q^*)^2}{2c(1+\eta(1-\delta)P^*)}.
\end{aligned}$$

Suppose that $z = \frac{\left(\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right)^2}{\frac{2P^*}{c(1+\eta(1-\delta)P^*}}$. Thus, we have

$$\left(\frac{Q^*}{c(1+\eta(1-\delta)P^*)} - \frac{P^*}{1+\eta(1-\delta)P^*} - \frac{bP^*}{c}\right)(P-P^*)(Q-Q^*) \leq z(P-P^*)^2 + \frac{P^*(Q-Q^*)^2}{2c(1+\eta(1-\delta)P^*)}.$$

Therefore, from equation (8) we get,

$$^{C}D^{\alpha}V_3 \leq \left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c} + z\right)(P-P^*)^2 - \frac{P^*(Q-Q^*)^2)}{c(1+\eta(1-\delta)P)}. \tag{9}$$

Now consider the Lyapunov function

$$V(P,Q) = V_2(P,Q) + s_2V_3(P,Q), s_2 > 0.$$

By selecting the α-order Caputo derivative of $V(P,Q)$ and applying the linearity property, as well as utilizing equations (8) and (9), we obtain:

$$\begin{aligned} ^{C}D^{\alpha}V \leq &-\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right)(P-P^*)^2 \\ &+ s_2\left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c} + z\right)(P-P^*)^2 - \frac{s_2P^*(Q-Q^*)^2}{c(1+\eta(1-\delta)P^*)}, \\ &= -\left(\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right) - s_2\left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c} + z\right)\right)(P-P^*)^2 \\ &- \frac{s_2P^*(Q-Q^*)^2}{c(1+\eta(1-\delta)P^*)}. \end{aligned} \tag{10}$$

Now choose the constant s_2 such that $\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right) - s_2\left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c} + z\right) = \frac{1}{2}\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right)$, which gives that $\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right) = 2s_2\left(\frac{Q^*(P-P^*)^2}{1+\eta(1-\delta)P^*} + \frac{bQ^*}{c} + z\right) > 0$. Then, from equation (10) we have,

$$^{C}D^{\alpha}V \leq -\frac{1}{2}\left(b - \frac{\eta\nu(1-\delta)^2Q^*}{1+\eta(1-\delta)P^*}\right)(P-P^*)^2 - \frac{s_2P^*(Q-Q^*)^2}{c(1+\eta(1-\delta)P^*)}.$$

Thus it gives $^{C}D^{\alpha}V < 0$ for all $(P,Q) \neq (P^*,Q^*)$, and $^{C}D^{\alpha}V = 0$ for $(P,Q) = (P^*,Q^*)$. Hence the interior equilibrium point is globally asymptotically stable. □

4.6. *Bifurcation Analysis*

In this section, we focused on the condition of Hopf-bifurcation in fractional order system. We consider the following fractional order system:

$$^{C}D^{\alpha}P = g(\omega, P), \tag{11}$$

where $\alpha \in (0,1), P \in R^2$. Let E^* is an equilibrium point of system (11), such that the system (11) undergoes Hopf-bifurcation around equilibrium point E^* w.r.t the parameter μ at $\mu = \mu^*$ if this condition hold as follows:

1. The Jacobian matrix of system (11) at the equilibrium point E^* has a couple of complex conjugate eigenvalues $\lambda_{1,2} = x_j \pm iy_j$ become purely imaginary at $\mu = \mu^*$,
2. $\psi_{1,2}(\alpha, \mu^*) = 0$,
3. $\frac{\partial\psi_{1,2}}{\partial\alpha}\Big|_{\mu=\mu^*\neq 0}$,

with $\psi_i(\alpha,\mu) = \frac{\alpha\pi}{2} - min_{i=1,2}|arg(\lambda_i(\mu))|$.

Based on the previous sections, we have determined that the stability of the system of dynamics is influenced by the order of derivative. Now, we will analyze the parameter α as a Hopf-bifurcation parameter in the following manner:

1. The Jacobian matrix of system (3) at the equilibrium point E^* has a couple of complex conjugate eigenvalues $\lambda_{1,2} = x_j \pm iy_j$ become purely imaginary at $\alpha = \alpha^*$,
2. $\theta_{1,2}(\alpha^*) = 0$,
3. $\frac{\partial\theta_{1,2}}{\partial\alpha}\Big|_{\alpha=\alpha^*\neq 0}$,

with $\theta_i(\alpha) = \frac{\alpha\pi}{2} - min_{i=1,2}|arg(\lambda_i(\alpha))|$.
Next, we analyze the Hopf-bifurcation conditions of system (3) in the following theorem.

Theorem 4.10
The system (3) undergoes Hopf-bifurcation around $E^*(P^*,Q^*)$ at $\alpha = \alpha^* = \tan^{-1}\left|\frac{\sqrt{4K-J_{11}^2}}{J_{11}}\right|$ where $4K > J_{11}^2, J_{11} \neq 0$.

Proof
Since, $\lambda_{1,2} = \upsilon_1 \pm i\upsilon_2$, where $\upsilon_1 = \frac{J_{11}}{2}, \upsilon_2 = \frac{\sqrt{4b-J_{11}^2}}{2}$. Hence,
$\theta_{1,2}(\alpha^*) = \frac{\alpha^*\pi}{2} - \tan^{-1}\left|\frac{\upsilon_2}{\upsilon_2}\right| = \tan^{-1}\left|\frac{\upsilon_2}{\upsilon_2}\right| - \tan^{-1}\left|\frac{\upsilon_2}{\upsilon_2}\right| = 0$,
and $\frac{\partial\theta_{1,2}}{\partial\alpha}\Big|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0$. So that, all the conditions of Hopf-bifurcation are satisfied. □

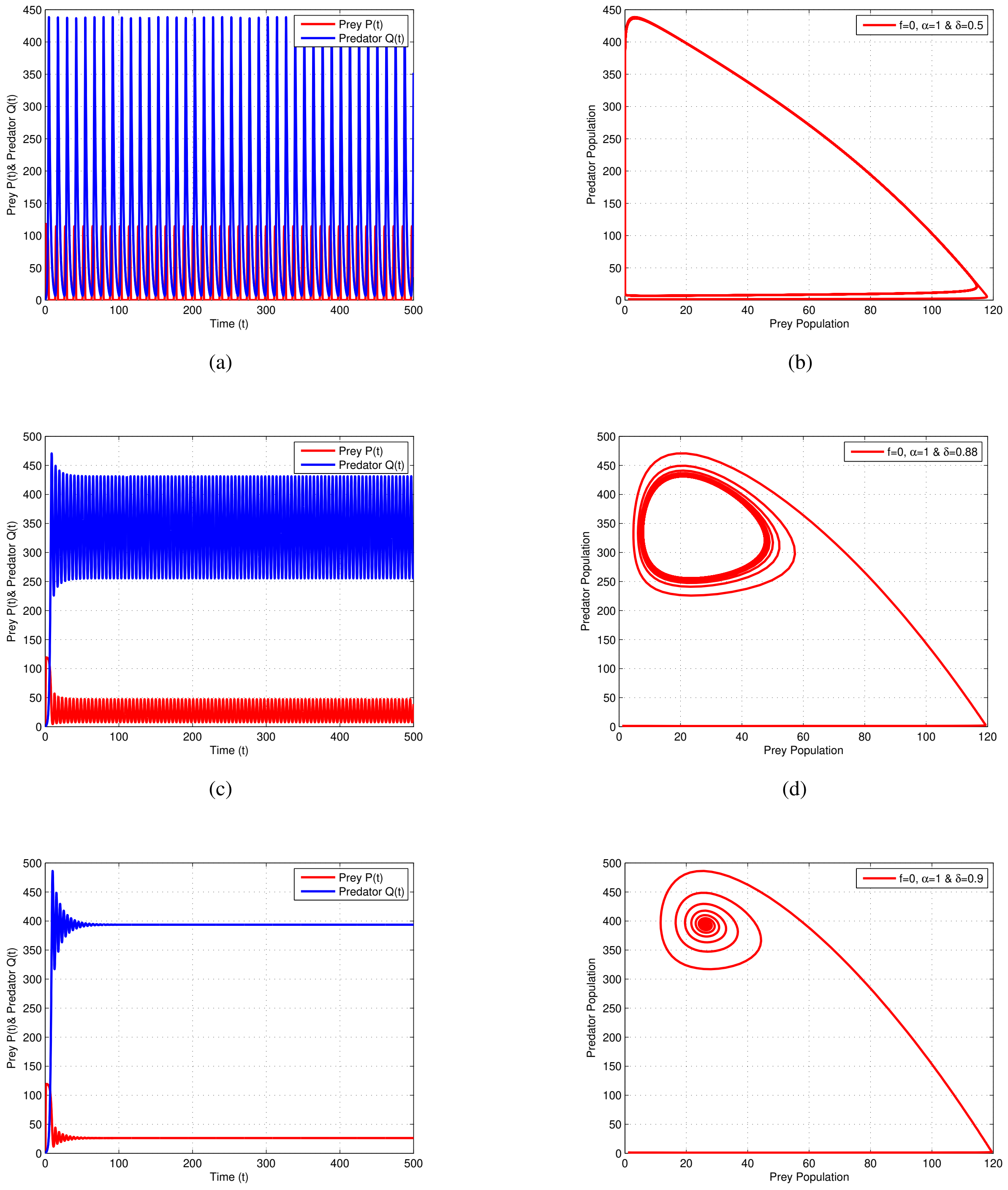
5. Numerical Simulation and Discussion

In this section, we conduct numerical simulations of the model (3) using the Grünwald–Letnikov scheme [53].The purpose of numerical simulations is to validate the outcome of a dynamic analysis. The parameter values are $r_1 = 2, f = 2, \nu = 0.6, c = 0.8, \delta = 0.5, \eta = 0.2, z_1 = 0.2, z_2 = 0.1, r_2 = 0.03$ and $b = 0.01$. The parameter values are chosen hypothetically as we currently lack any data pertaining to our model. The following case has been used to categorize various numerical outcomes of model simulation and their corresponding observations:

Case 1 No fear effect. In Case 1, we analyzed the dynamics of the model system (3) in the absence of fear, by varying the parameters δ and α. This study examines the impact of fear on both integral and fractional order systems, emphasizing the influence of fear's absence on system behavior. In the first sub-case (1a), we examined prey refuge within the integer order system, maintaining α at 1 while systematically increasing the prey refuge parameter δ from 0 to values nearing 1. As δ increases, the system transitions from unstable periodic oscillations to a stable state near the interior equilibrium, as illustrated in Figure 1. Figures 1(a) and 1(b) present the time series and phase portrait for $\delta = 0.5$, 1(c) and 1(d) present the time series and phase portrait for $\delta = 0.88$, whereas Figures 1(e) and 1(f) depict these for $\delta = 0.9$, with both sets indicating a stable steady state.

Biologically, increasing δ, which represents prey refuge (e.g., the availability of shelters), enhances prey survival by reducing exposure to predators. This leads to an increase in prey biomass. However, predators, relying on prey availability as their primary food source, reach maximum biomass when δ is between 0.89 and 0.97. Beyond this threshold ($\delta > 0.97$), the available prey becomes insufficient to sustain the predator population, ultimately leading to predator extinction. Ecologically, this can result in imbalances such as unchecked prey population growth and its cascading effects on community dynamics.

Numerically, the system's stabilization as δ increases indicates that refuge availability can suppress oscillatory dynamics, steering the system towards equilibrium. This outcome suggests a strong dependence of predator-prey interactions on spatial or behavioral refuge mechanisms in real ecosystems.



(a)

(b)

(c)

(d)

(e)

(f)

Figure 1. Dynamical behavior of the model system for different values of prey refuge δ, and fear effect $f = 0$ in the system. The graphs (a) and (b) represent the time series and phase portrait of the solutions of the model when $\delta = 0.5$; the graphs (c) and (d) represent the time series phase portrait of the solutions of the model when $\delta = 0.88$; the graphs (e) and (f) represent the time series phase portrait of the solutions of the model when $\delta = 0.9$.

In the second sub-case (1b), we maintained the fear effect and prey refuge values at $f = 0$ and $\delta = 0.5$, subsequently reducing α to zero incrementally. The decrease in α enhances system stability, as illustrated in Figure 2. Figures 2(a) and 2(b) illustrate the time series and phase portrait for $\alpha = 0.8$, while Figures 2(c) and 2(d) depict these for $\alpha = 0.6$. Reducing α results in a shift from instability to stability, attributed to the prey species' robust memory, enabling defensive adaptations in response to predators, particularly under conditions of limited food resources for the predators.

Biologically, lower α reflects reduced predator memory or their ability to ”remember” prey locations effectively. This gives an advantage to the prey species, as predators become less efficient in hunting. Prey can develop defensive adaptations, such as avoiding certain areas or increasing the use of refuges, allowing them to survive better, especially under conditions where predator food resources are limited.

Ecologically, the findings imply that reduced predator efficiency, possibly due to environmental stressors or reduced cognitive abilities, can stabilize predator-prey dynamics, mitigating over-predation risks. Numerical simulations corroborate this by illustrating how gradual memory decay (α reduction) shifts system dynamics from oscillatory to stable.

Overall, the numerical simulations validate the analytical results, highlighting the critical roles of refuge and memory in determining ecosystem stability. Such insights can inform strategies for habitat design and conservation, focusing on enhancing natural refuges and mitigating factors that impair predator hunting efficiency.

Case 2 No Prey Refuge. In Case 2, we examined the dynamics of the model system devoid of prey refuge by modifying the parameters f and α, concentrating on the impact of the lack of prey refuge on both integral and fractional order systems. In the initial sub-case (2a), we examined the influence of fear within the integer order system, maintaining α at 1 while systematically increasing the fear effect parameter f from 0 to 5. The behavior of the model system near the interior equilibrium point $E_2 = (2.6316, 5.6566)$ transitions from initially unstable periodic oscillations to a stable state, as illustrated in Figure 3. Figures 3(a) and 3(b) illustrate the time series and phase portrait for $f = 1$, figures 3(c) and 3(d) illustrate the time series and phase portrait for $f = 2.1$ whereas Figures 3(e) and 3(f) depict these for $f = 4$, indicating a stable state at $E_2 = (2.6316, 5.6566)$. In the absence of prey refuge, the system exhibits instability under conditions of low fear effect. In the biology perspective, f increases, it reflects heightened fear in prey due to predator presence, resulting in reduced prey movement, activity, and reproduction rates. This behavioral adaptation minimizes predation risk but also decreases the prey population's reproductive output. For predators, reduced prey availability due to fear-induced inactivity stabilizes population dynamics by preventing over-predation.

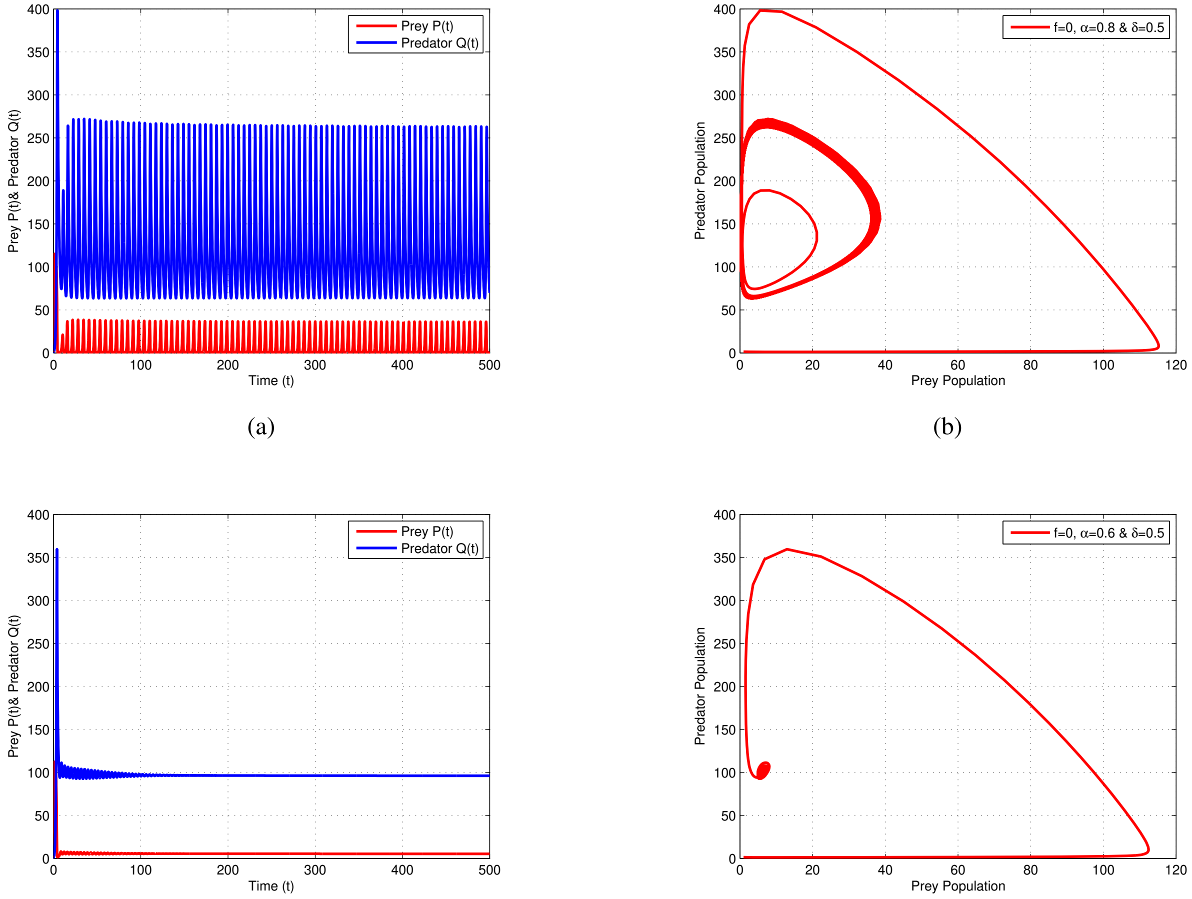
In the ecological perspective,the absence of prey refuge exacerbates the effects of low fear levels, leading to instability. However, with higher fear, the predator-prey interaction stabilizes, balancing populations. This suggests that fear acts as a regulatory mechanism in ecosystems where physical refuges are unavailable.

In the second sub-case (2b), with prey refuge and fear factor established at $\delta = 0$ and $f = 1$, we systematically decreased α. This reduction enhances stability within the model system, as demonstrated in Figure 4. Figures 4(a) and 4(b) present the time series and phase portrait for $\alpha = 0.98$, whereas Figures 4(c) and 4(d) illustrate these for $\alpha = 0.92$, highlighting the stabilizing effect of a reduction in α on the system dynamics.

Case 3 No Harvesting, In this case, we removed harvesting from both populations and examined the effect of refuge on the system's dynamics. As the level of refuge increased, the system moved from unstable periodic behavior to a stable steady state. For example, at low refuge values ($\delta = 0.1$), the population showed oscillatory behavior (see Figure 5 in (a) and (b)), but as refuge increased to $\delta = 0.4$, the system stabilized (see Figure 5 in (c) and (d)). Decreasing α further increased system stability, as shown in Figure 7, where lower values of α lead to more stable population levels.

Refuge plays a critical role in moderating population dynamics by providing a safe space for individuals to escape predation or adverse conditions. At low levels of refuge ($\delta = 0.1$), the oscillatory behavior observed in the population indicates a high level of predator-prey interaction or inter-species competition, which can lead to boom-and-bust cycles. These cycles are characteristic of systems where individuals are highly exposed to environmental pressures or predation risks.

As the level of refuge increases to $\delta = 0.4$, the stabilization of the system suggests that the availability of safe areas reduces the intensity of these interactions, allowing populations to recover more effectively and maintain



(a) (b)

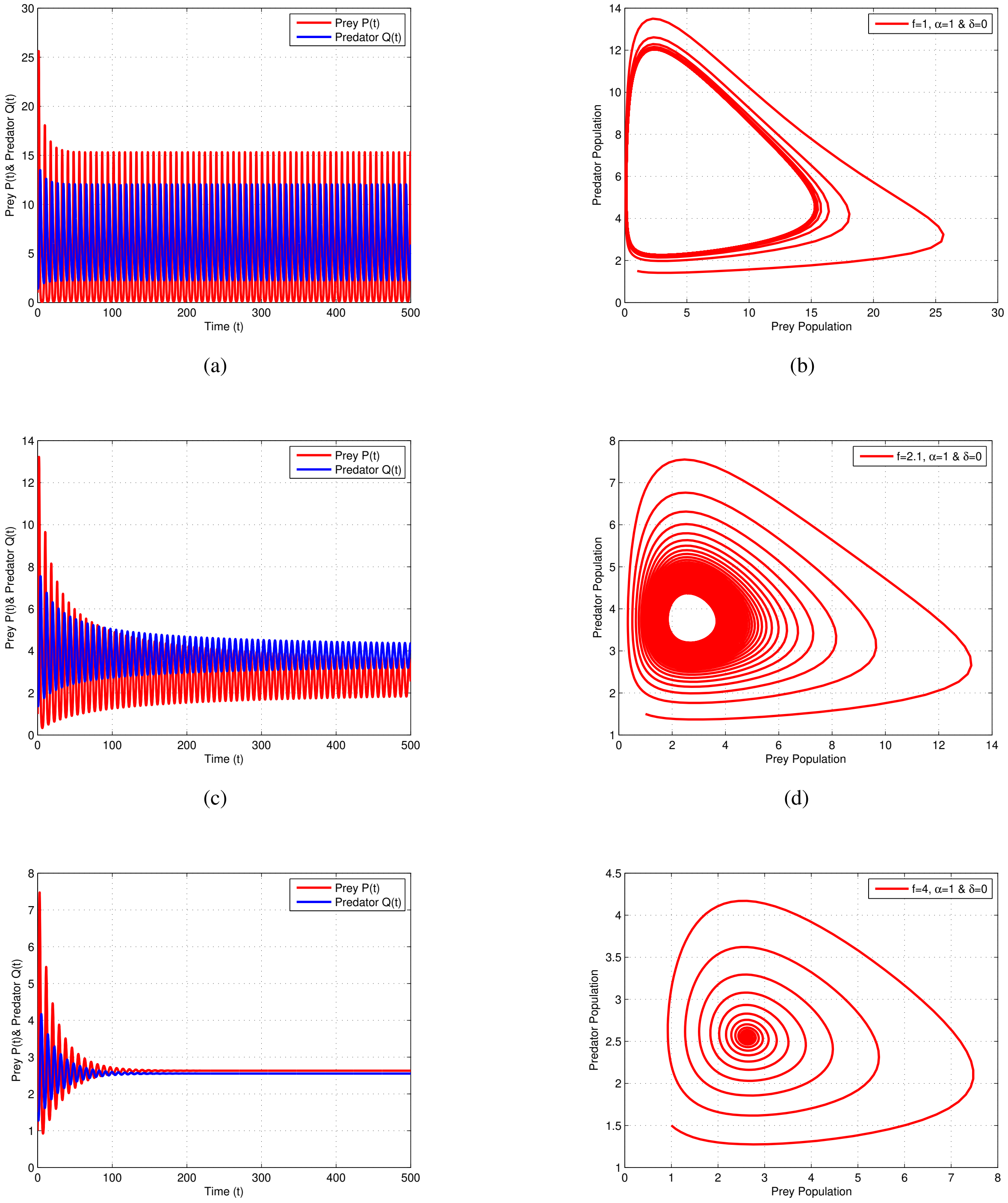
(c) (d)

Figure 2. The dynamic behavior of the model system is examined in the context of varying values of the prey refuge δ and the fear effect $f = 0, \delta = 0.5$. The time series and phase portrait of the model's solutions are depicted in the graphs (a) and (b), respectively, when $\alpha = 0.8$. The time series and phase portrait of the model's solutions are depicted in the graphs (c) and (d) when $\alpha = 0.6$.

more consistent levels over time. This demonstrates the importance of refuge in buffering populations against environmental fluctuations and preventing extreme population crashes or explosions.

Additionally, decreasing α, which represents factors such as interaction strength or environmental carrying capacity, further enhances stability. Lower α values could imply reduced competition for resources or less frequent encounters between predators and prey, leading to a more balanced and sustainable population dynamic. These findings highlight the critical role of habitat features, such as refuge and resource availability, in shaping the long-term stability and resilience of ecological systems.

Case 4 Combined Effects of Fear, Refuge, and Harvesting, Dynamics of the model system in presence of fear effect, prey refuge and harvesting.In this case, we demonstrate three simulation i.e. stability of equilibrium point E_0, E_1, and E_2 in Figure 6, 8, and 9. Furthermore, in the last simulation, Lastly, adjusting the fractional-order parameter $\alpha = 0.98$, and $\alpha = 0.93$ further highlighted changes in stability, as seen in Figure 10.

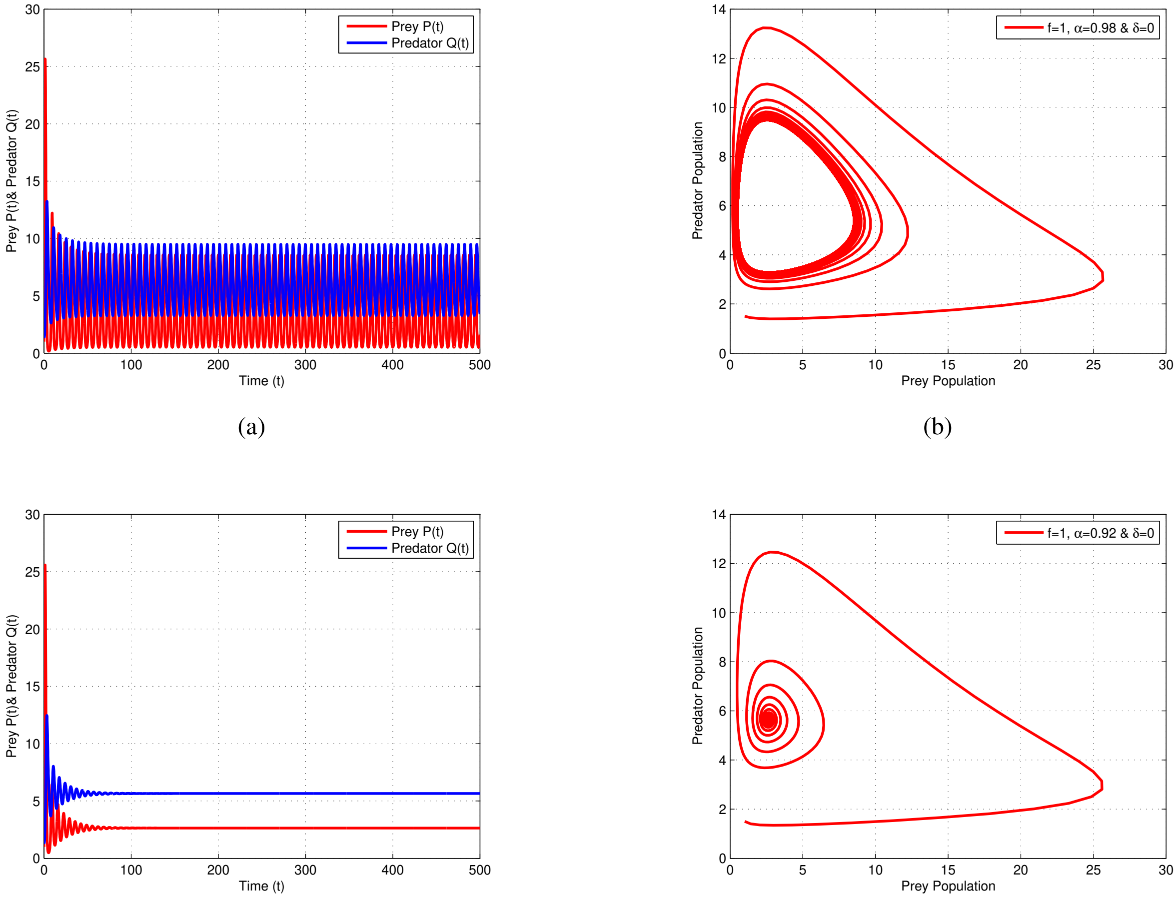


(a) (b)

(c) (d)

(e) (f)

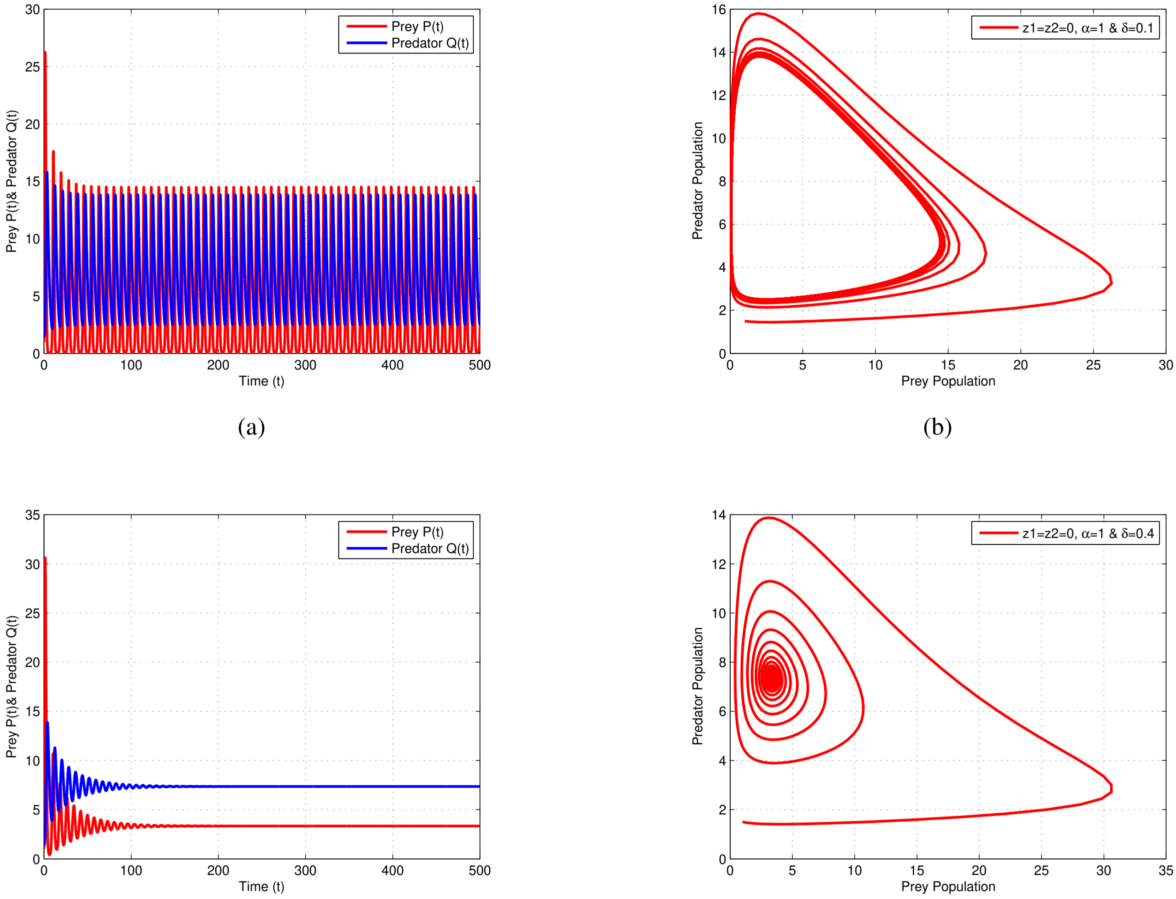
Figure 3. The dynamical behavior of the model system is analyzed for varying values of fear f and with prey refuge set to $\delta = 0$. The graphs (a) and (b) illustrate the time series and phase portrait of the model's solutions for $f = 1$, the graphs (c) and (d) present the time series and phase portrait for $f = 2.1$, whereas the graphs (e) and (f) present the time series and phase portrait for $f = 4$.



(a) (b)

(c) (d)

Figure 4. Dynamical behavior of the model system for varying values of fear f and prey refuge $\delta = 0, f = 1$ in the system. The graphs (a) and (b) illustrate the time series and phase portrait of the model's solutions when $\alpha = 0.98$, while the graphs (c) and (d) illustrate the time series and phase portrait of the model's solutions when $\alpha = 0.92$.



(a) (b)

(c) (d)

Figure 5. The dynamic behavior of the model system is examined in the context of varying prey refuge values δ and harvesting $z_1 = z_2 = 0$. The time series and phase portrait of the model's solutions at $\delta = 0.1$ are depicted in the graphs (a) and (b),respectively. The time series and phase portrait of the model's solutions at $\delta = 0.4$ are depicted in the graphs (c) and (d).

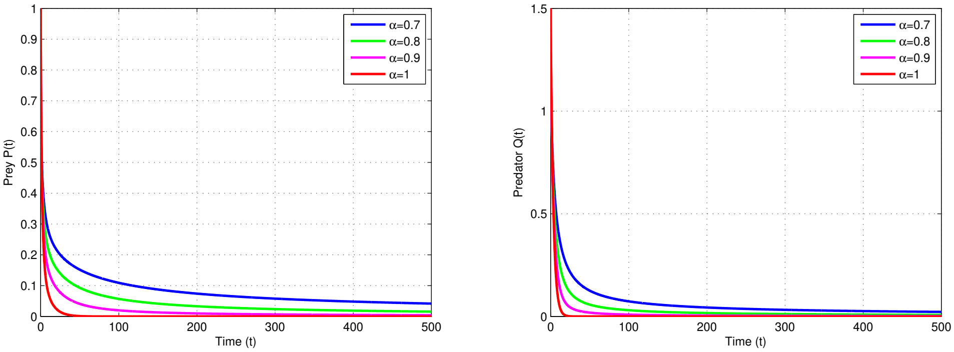
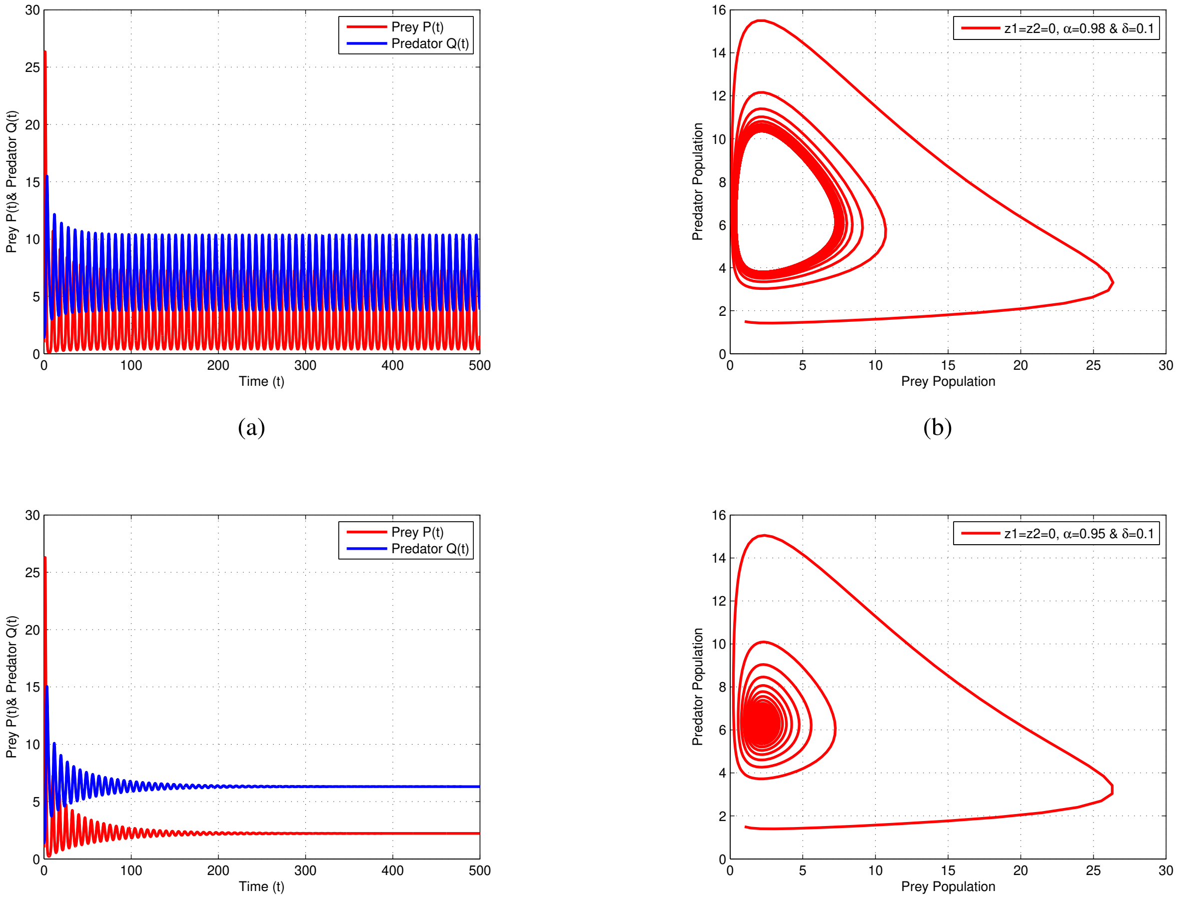


Figure 6. Graphic Solution system (3) with respect to time t for case $r_1 = 0.001 < z_1(1-\delta) = 0.0990$ (Stability of E_0)



(a) (b)

(c) (d)

Figure 7. The dynamic behavior of the model system is investigated for varying values of prey refuge δ and harvesting $z_1 = z_2 = 0, \delta = 0.1$ in the system. The time series and phase portrait of the model's solutions are depicted in the graphs (a) and (b), respectively, when $\alpha = 0.98$. The time series and phase portrait of the model's solutions are depicted in the graphs (c) and (d) when $\alpha = 0.95$.

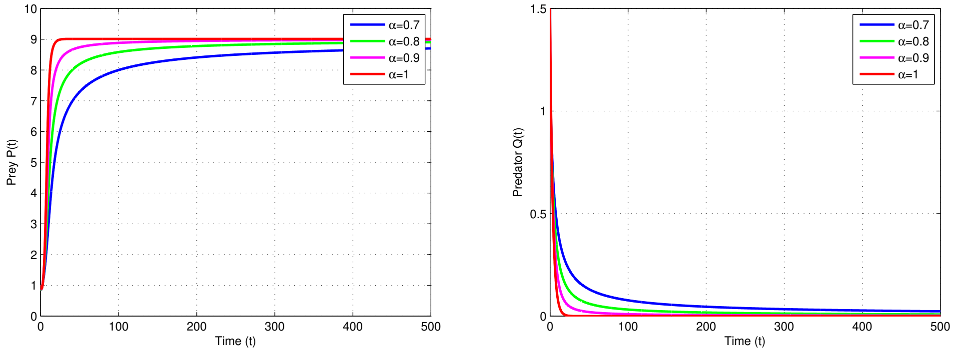


Figure 8. Graphic Solution system (3) with respect to time t when case $r_1 = 1 > z_1(1-\delta) = 0.0990$ and $\frac{c\nu(1-\delta)(r_1-z_1(1-\delta)}{b+\eta(1-\delta)(r_1-z_1(1-\delta)} = 0.0141 < r_2 + z_2 = 0.3$ (Existence and the stability of equilibrium point $E_1(9.01, 0)$

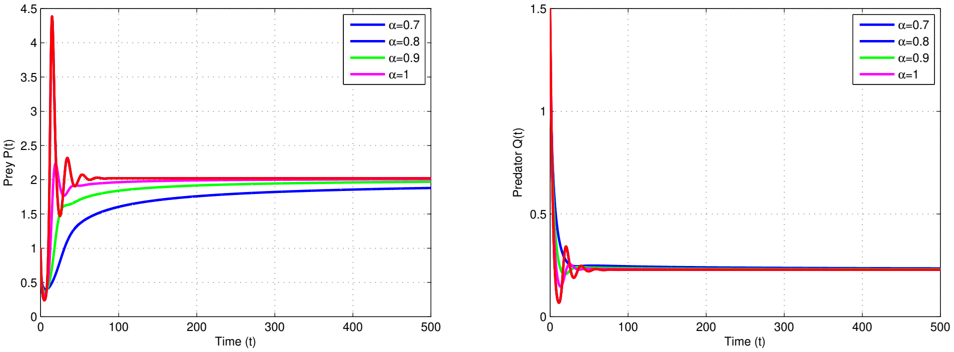


Figure 9. Graphic Solution system (3) with respect to time t are shown when the case $a_2 < 0$ and $J_{11} = -0.1925 < 0$. This demonstrates the existence and stability of the interior equilibrium point $E_2(2.0208, 0.2316)$.

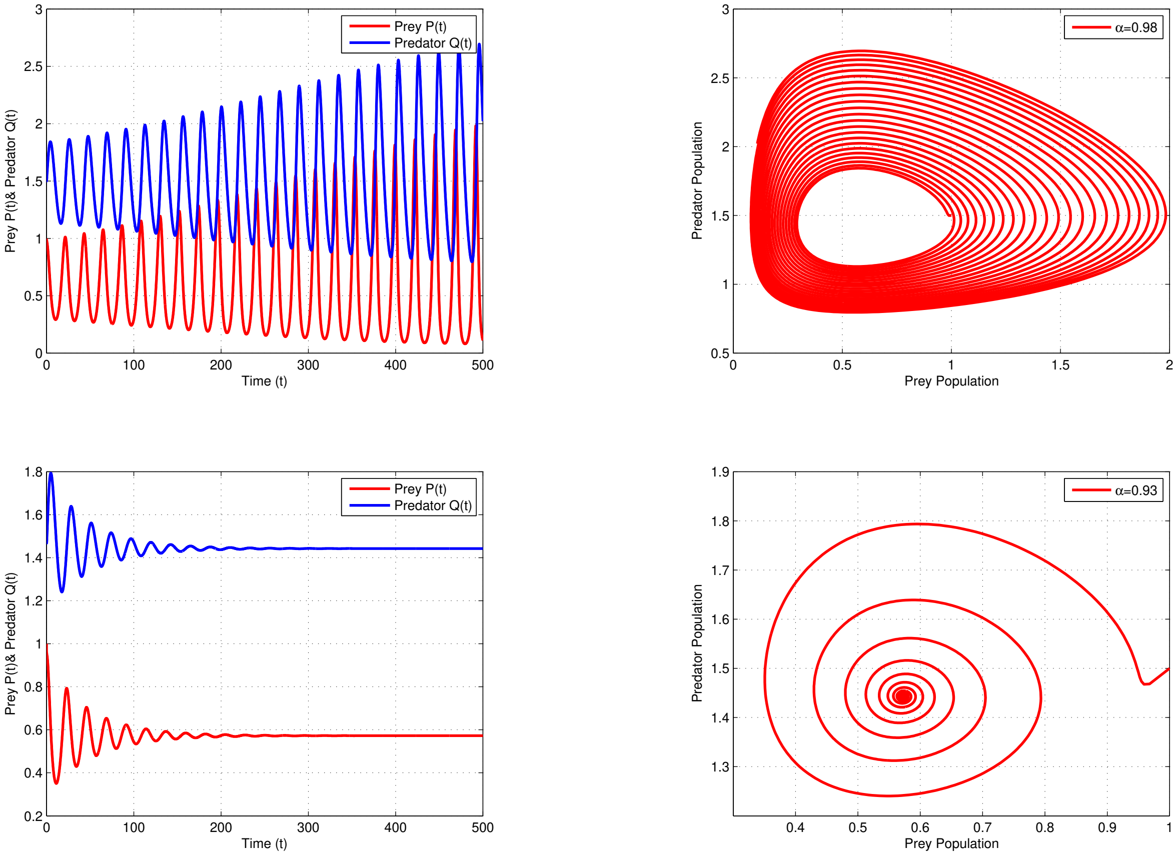


Figure 10. The time series and phase portrait of the model's solutions are depicted in graphs (a) and (b), respectively, when $\alpha = 0.98$. Similarly, the time series and phase portrait of the model's solutions are depicted in graphs (c) and (d) at the interior equilibrium point $E_2 = (0.5727, 1.4402)$ when $\alpha = 0.92$.

6. Conclusions

This study introduced a fractional-order predator-prey model incorporating the effects of fear, refuge, and harvesting using the fractional Caputo derivative. By capturing memory effects in predator-prey dynamics, the model provides a more accurate depiction of population interactions over time. Rigorous analysis confirmed the existence, uniqueness, and biological viability of the model's solutions, ensuring non-negativity and boundedness. Stability analysis also revealed conditions for equilibrium or periodic oscillations through Hopf bifurcation, highlighting the value of fractional calculus in improving traditional predator-prey models.

The findings of this study offer significant insights into the management and conservation of ecological systems. By incorporating factors such as fear effects, prey refuge, and harvesting into the predator-prey model, this research underscores the importance of considering both direct and indirect interactions in ecosystems. For instance, the fear-induced stress on prey populations, while reducing predation risk, also impacts reproductive rates, which could guide the design of protected areas to balance prey safety and ecosystem stability. Our findings on the effects of fear, refuge, and optimal harvesting provide actionable insights for designing conservation policies. For example, controlling predator populations to balance prey species survival can mitigate over-predation and maintain biodiversity. Additionally, the identification of optimal harvesting rates serves as a guideline for policymakers to implement sustainable harvesting strategies that prevent ecosystem destabilization while supporting economic activities. The model highlights the importance of incorporating behavioral dynamics, such as fear-induced changes in prey behavior, into conservation planning. These factors can inform the design of protected areas and the regulation of human activities in sensitive ecosystems.

Moreover, the inclusion of harvesting parameters highlights the delicate balance required in resource management. Excessive harvesting, as demonstrated in the model, risks destabilizing ecosystems, potentially leading to the collapse of both predator and prey populations. These insights could inform regulatory strategies for sustainable harvesting practices, emphasizing the need for adaptive management that considers ecological thresholds. Future applications of this model could involve exploring its predictions in the context of real-world conservation strategies. For example, it could assist in optimizing the size and placement of wildlife corridors or assessing the impact of selective harvesting practices. Such interdisciplinary approaches would enhance the utility of this model in addressing complex ecological challenges.

The physical interpretation of the model shows that fear induces stress in prey, reducing reproductive rates, while refuge offers protection but can destabilize the system if overutilized, leading to predator extinction. Harvesting, representing human intervention, introduces additional mortality rates, and the model demonstrates that excessive harvesting could destabilize ecosystems by collapsing both predator and prey populations. These findings have significant implications for ecosystem management, emphasizing the need for sustainable harvesting practices and adaptive strategies that account for both direct and indirect effects of predation and human activity on population dynamics.

Despite its valuable insights, the model has limitations. It assumes constant environmental conditions, focuses on a single predator-prey pair, and does not consider spatial heterogeneity or multi-species interactions, which are essential for understanding complex ecological systems.Future research can explore incorporating dynamic elements, such as spatial variability and variable harvesting strategies. There is also potential to integrate interdisciplinary approaches, such as noise modeling from electrical systems, as seen in the study of SiGe bipolar phototransistors [54] , or using algorithms for managing complex systems like battery datasets [55], which could offer insights into sustainable ecological practices. Additionally, deep learning techniques, such as those used in sentiment analysis on social networking data [56], could be applied to simulate complex ecological networks and predict outcomes based on various environmental factors. These interdisciplinary methods may further enhance the model's applicability to real-world scenarios, including wildlife conservation, sustainable resource management, and ecological restoration.

Appendix

Explanation note Proof of Theorem 4.2 (second equation). From the second equation, we have

$$\begin{aligned}
\frac{dQ}{dt} &= -r_2Q + \frac{c\nu(1-\delta)PQ}{1+\eta(1-\delta)P} - z_2Q,\\
\frac{dQ}{Q} &= \left(-r_2 + \frac{c\nu(1-\delta)P}{1+\eta(1-\delta)P} - z_2\right)dt,\\
\ln(Q) &= \int_0^t \left(-r_2 + \frac{c\nu(1-\delta)P(\tau)}{1+\eta(1-\delta)P(\tau)} - z_2\right)d\tau + \ln P(0),\\
Q(t) &= Q(0)\exp\left\{\int_0^t \left(-r_2 + \frac{c\nu(1-\delta)P(\tau)}{1+\eta(1-\delta)P(\tau)} - z_2\right)d\tau\right\} \geq 0.
\end{aligned}$$

Acknowledgement

This work was funded by the Balai Pembiayaan Pendidikan Tinggi (BPPT) [202209090769]; Lembaga Pengelola Dan Pendidikan (LPDP)[202209090769].

REFERENCES

1. B. Mondal, S. Roy, U. Ghost and P.K. Tiwari, *A systematic study of autonomous and nonautonomous predator–prey models for the combined effects of fear, refuge, cooperation and harvesting*, The European Physical Journal Plus, vol. 137, no. 1, pp. 724–746, 2022.
2. X. Liu, Q. Huang, *The dynamics of a harvested predator–prey system with Holling type IV functional response*, Biosystems, vol. 169, no. 1, pp. 26–39, 2018.
3. K. Sarkar, B. Mondal, *Dynamic analysis of a fractional-order predator–prey model with harvesting*, Int. Journal of Dynamics and Control, vol. 11, pp. 1518–1531, 2023.
4. C. Zhang, X. Li, *Dynamics of a Discrete Leslie–Gower Model with Harvesting and Holling-II Functional Response*, Mathematics, vol. 11, pp. 3303–3321, 2023.
5. A.R.M. Jamil, R.K. Naji, *Modeling and Analysis of the Influence of Fear on the Harvested Modified Leslie–Gower Model Involving Nonlinear Prey Refuge*, Mathematics, vol. 10, pp. 2857, 2023.
6. R. Han, G. Mandal, L.N. Guin, S. Chakravarty, *Dynamical response of a reaction–diffusion predator–prey system with cooperative hunting and prey refuge*, Journal of Statistical Mechanics: Theory and Experiment, vol. 10, pp. 103502, 2022.
7. J. Ghosh, B. Sahoo, S. Poria, *Prey-predator dynamics with prey refuge providing additional food to predator*, Chaos, Solitons & Fractals, vol. 96, pp. 110–119, 2017.
8. S. Wang, Z. Ma, W. Wang, *Dynamical behavior of a generalized eco-epidemiological system with prey refuge*, Advances in Difference Equations, vol. 2018, pp. 1–20, 2018.
9. S. N. Raw, B. P. Sarangi, *Qualitative and dynamical analysis of a bionomic fishery model with prey refuge*, Acta Biotheoretica, vol. 70, pp. 11–48, 2022.
10. M. Rayungsari, A. Suryanto, W. M. Kusumawinahyu, I. Darti, *Dynamical analysis of a predator-prey model incorporating predator cannibalism and refuge*, Axioms, vol. 11, pp. 116–133, 2022.
11. H. Deng, F. Chen, Z. Zhu, Z. Li, *Dynamic behaviors of Lotka–Volterra predator–prey model incorporating predator cannibalism*, Advances in Difference Equations, vol 2019, pp. 1–17, 2019.
12. F. Zhang, Y. Chen, J. Li, *Dynamical analysis of a stage-structured predator-prey model with cannibalism*, Mathematical Biosciences, vol. 307, pp. 33–41, 2019.
13. M. Rayungsari, A. Suryanto, W. M. Kusumawinahyu, I. Darti, *Dynamics analysis of a predator–prey fractional-order model incorporating predator cannibalism and refuge*, Frontiers in Applied Mathematics and Statistics,vol. 9, pp. 1122330–1122341, 2023.
14. A. S. Abdulghafour, R. K. Naji, *Modeling and analysis of a prey-predator system incorporating fear, predator-dependent refuge, and cannibalism*, Commun. Math. Biol. Neurosci.,vol. 2022, pp. 2052–2541, 2022.
15. S. Roy, P.K. Tiwari, H. Nayak, M. Martcheva, *Effects of fear, refuge and hunting cooperation in a seasonally forced eco-epidemic model with selective predation*, The European Physical Journal Plus, vol 137, pp. 528–558, 2022.
16. F.B. Yousef, A. Yousef, C. Maji, *Effects of fear in a fractional-order predator-prey system with predator density-dependent prey mortality*, Chaos, Solitons & Fractals, vol. 145, pp. 110711–110720, 2021.
17. X. Wang, L. Zanette, X. Zou, *Modelling the fear effect in predator–prey interactions*, Journal of mathematical biology, vol. 73, pp. 1179–1204, 2016.
18. X. Dong, B. Niu, *On a diffusive predator–prey model with nonlocal fear effect*, Applied Mathematics Letters,vol. 132 ,pp. 108156–108163, 2022.
19. Y. Li, M. He, Z. Li, *Dynamics of a ratio-dependent Leslie–Gower predator–prey model with Allee effect and fear effect*, Mathematics and Computers in Simulation, vol. 201, pp. 417–439, 2022.

20. R. Yang, D. Jin, *Dynamics in a predator-prey model with memory effect in predator and fear effect in prey*, Electronic Research Archive,vol. 30, pp. 1322–1339, 2022.
21. M. Chen, Y. Takeuchi, J.F. Zhang, *Dynamic complexity of a modified Leslie–Gower predator–prey system with fear effect*, Communications in Nonlinear Science and Numerical Simulation, vol.119, pp. 107109–107126, 2023.
22. I. Benamara, A. E. Abdllaoui, *Bifurcation in a delayed predator–prey model with Holling type IV functional response incorporating hunting cooperation and fear effect*, International Journal of Dynamics and Control, vol. 11, pp. 2733–2750, 2023.
23. P.K. Tiwari, K.A.N.A. Amri, S. Samanta, Q.J.A. Khan, J Chattopadhyay, *A systematic study of autonomous and nonautonomous predator–prey models with combined effects of fear, migration and switching*, Nonlinear Dynamics, vol. 103, pp. 2125–2162, 2021.
24. P. Panday, N. Pal, S. Samanta, J. Chattopadhyay, *Stability and bifurcation analysis of a three-species food chain model with fear*, International Journal of Bifurcation and Chaos, vol. 28, pp. 1850009, 2018.
25. K.H. Elliott, G.S. Betini, D.R. Norris, *Fear creates an Allee effect: experimental evidence from seasonal populations*, Proceedings of the Royal Society B: Biological Sciences, vol. 284, pp. 20170878, 2017.
26. S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, *Fear effect in prey and hunting cooperation among predators in a Leslie-Gower model*, Mathematical Biosciences and Engineering, vol. 16, pp. 5146, 2019.
27. P. Panday, S. Samanta, N. Pal, J. Chattopadhyay, *Delay induced multiple stability switch and chaos in a predator–prey model with fear effect*, Mathematics and Computers in Simulation, vol. 172, pp. 134–158, 2020.
28. H. Zhang, Y. Cai, S. Fu, W. Wang, *Impact of the fear effect in a prey-predator model incorporating a prey refuge*, Applied Mathematics and Computation, vol. 356, pp. 328–337, 2019.
29. P. Paul, T.K. Kar, E. Das, *Reactivity in prey–predator models at equilibrium under selective harvesting efforts*, The European Physical Journal Plus, vol. 136, pp. 510, 2021.
30. B. Daşbaşı, *Stability analysis of mathematical model including pathogen-specific immune system response with fractional-order differential equations*, Computational and mathematical methods in medicine, vol. 2018, pp. 1–10, 2018.
31. H.A.A. El-Saka, A.A.M. Arafa, M.I. Gouda, *Dynamical analysis of a fractional SIRS model on homogenous networks*, Advances in Difference Equations, vol. 2019, pp. 1–15, 2019.
32. D. Mukherjee, *Fear induced dynamics on Leslie-Gower predator-prey system with Holling-type IV functional response*, Jambura Journal of Biomathematics (JJBM), vol 3, pp. 49–57, 2022.
33. E. Rahmi, I. Darti, A. Suryanto, Trisilowati, *A modified Leslie–Gower model incorporating Beddington–DeAngelis functional response, double Allee effect and memory effect*, Fractal and Fractional, vol. 5, pp. 84, 2021.
34. A. Suryanto, I. Darti, S. Anam, *Stability analysis of a fractional order modified Leslie-Gower model with additive Allee effect*, International Journal of Mathematics and Mathematical Sciences, vol.2017, pp. 1–10, 2017.
35. S. Karthikeyan, P. Ramesh, M. Sambath, *Stability analysis of fractional-order predator-prey model with anti-predator behaviour and prey refuge*, Journal of Mathematical Modeling, vol.11, pp.527–546, 2023.
36. M. Farman, K. Jamil, C. Xu, K.S. Nisar, A. Amjad, *Fractional order forestry resource conservation model featuring chaos control and simulations for toxin activity and human-caused fire through modified ABC operator*, Mathematics and Computers in Simulation, vol 227, pp. 282–302, 2025.
37. K.S. Nisar, M. Farman, E. Hincal, A. Hasan, P. Abbas, *Chlamydia infection with vaccination asymptotic for qualitative and chaotic analysis using the generalized fractal fractional operator*, Scientific Reports, vol 14, pp. 25938, 2024.
38. K.S. Nisar, M. Farman, K. Jamil, S. Jamil, E. Hincal, *Fractional-order PID feedback synthesis controller including some external influences on insulin and glucose monitoring*, Alexandria Engineering Journal, vol 113, pp. 60–73, 2025.
39. R.C. Mittal, S. Pandit, *A numerical algorithm to capture spin patterns of fractional Bloch nuclear magnetic resonance flow models*, Journal of Computational and Nonlinear Dynamics, vol. 14, pp. 81001, 2019.
40. S. Pandit, R.C. Mittal, *A numerical algorithm based on scale-3 Haar wavelets for fractional advection dispersion equation*, Engineering Computations, vol. 38, pp. 1706–1724, 2021.
41. R.C. Mittal, S. Pandit, *Quasilinearized Scale-3 Haar wavelets-based algorithm for numerical simulation of fractional dynamical systems*, Engineering Computations, vol. 35, pp. 1907–1931, 2019.
42. A. A. Themairi, M.A. Alqudah, *Existence and Uniqueness of Caputo Fractional Predator-Prey Model of Holling-Type II with Numerical Simulations*, Mathematical Problems in Engineering, vol 2021, pp. 1–6, 2021.
43. M. Das, G.P. Samanta, *A delayed fractional order food chain model with fear effect and prey refuge*, Mathematics and Computers in Simulation, vol. 178, pp. 218–245, 2020.
44. M. Higazy, S.A.M. Alsallami, S. Abdel-Khalek, A. El-Mesady, *Dynamical and structural study of a generalized Caputo fractional order Lotka-Volterra model*, Results in Physics, vol. 37, pp. 105478, 2022.
45. H. Jafari, R.M. Ganji, N.S. Nkomo, Y.P. Lv, *A numerical study of fractional order population dynamics model*, Results in Physics, vol.27, pp. 104456, 2021.
46. H.L. Li, L. Zhang, C. Hu, Y.L. Jiang, Z. Teng, *Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge*, Journal of Applied Mathematics and Computing,vol. 57, pp. 435–449, 2017.
47. Q. Wang, R. Han, *Dynamics of Fractional Order Predator Prey Models with Refuge and Allee Effect in Prey: With and Without Time Delay*, Qualitative Theory of Dynamical Systems, vol 24, pp. 49, 2025.
48. G.R. Kumar, K. Ramesh, A. Khan, K. Lakshminarayan, T. Abdeljawad, *Dynamical study of fractional order Leslie-Gower model of predator-prey with fear, Allee effect, and inter-species rivalry*, Results in Control and Optimization, vol 14, pp. 100403, 2024.
49. J. Cresson, A. Szafrańska, *Discrete and continuous fractional persistence problems–the positivity property and applications*, Communications in Nonlinear Science and Numerical Simulation, vol. 44, pp. 424–448, 2017.
50. M. Al-Refai, Y. Luchko, *Comparison principles for solutions to the fractional differential inequalities with the general fractional derivatives and their applications*, Journal of Differential Equations, vol. 319, pp. 312–324, 2022.
51. J. Huo, H. Zhao, L. Zhu, *The effect of vaccines on backward bifurcation in a fractional order HIV model*, Nonlinear Analysis: Real World Applications, vol. 26, pp. 289–305, 2015.
52. C. Vargas-De-León, *Volterra-type Lyapunov functions for fractional-order epidemic systems*, Communications in Nonlinear Science and Numerical Simulation, vol. 24, pp. 75–85, 2015.

53. A. El-Mesady, O. Bazighifan, S. Araci, *Insight into Caputo Fractional-Order Extension of Lotka–Volterra Model with Emphasis on Immigration Effect*, Journal of Mathematics, vol. 2023, pp. 8795660, 2023.
54. A.M. Diop, J.L. Polleux, C. Algani, S. Mazer, M. Fattah, M.E.L. Bekkali, *Design electrical model noise and perform nonlinearities of SiGe bipolar phototransistor*, International Journal of Innovative Research and Scientific Studies, vol.6, pp. 731–740, 2023.
55. G. Krishna, R. Singh, A. Gehlot, P. Singh, S. Rana, S.V. Akram, K. Joshi, *An imperative role of studying existing battery datasets and algorithms for battery management system*, Review of Computer Engineering Research, vol 10, pp. 28–39, 2023.
56. Z. Alsalami, *Modeling of Optimal Fully Connected Deep Neural Network based Sentiment Analysis on Social Networking Data*, Journal of Smart Internet of Things, vol 2022, pp. 114–132, 2022.