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Application of Ujlayan-Dixit Fractional Exponential Probability Distribution

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Abstract This research aims to present the probability density functions of random variables for exponential distribution by applying a new technique of the Ujlayan-Dixit (UD) fractional derivative and to find some basic concepts related to probability distributions of random variables, which is density, cumulative distribution, survival and hazard functions. In addition, we provides the UD fractional isotopes with the expected values, r^{th}-moments, r^{th}-central moments, mean, variance, skewness and kurtosis. Finally, we give the UD fractional analogs to some entropies measures such as Shannon, Renyi, and Tsallis entropy.

Keywords Probability Distribution, Exponential Distribution, UD Fractional Derivative, Entropy

AMS 2010 subject classifications 26A33, 62E10

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1. Introduction

Fractional calculus is a generalization of classical calculus that develops the concepts of integrals and derivatives to non-integer orders. It allows us to compute integrals and derivatives of arbitrary orders [1, 2, 3]. This idea has many applications in various scientific and engineering fields; for more details, see the books of Hilfer [4], Tarasov [5], Samko *et al.* [6] and Miller *et al.* [7].

Fractional derivatives provide a modern mathematical framework that extends classical calculus to better model complex and real-world phenomena, see [8, 9, 10, 11, 12, 13, 14] to get a full overview about a framework and its applications. Among the intriguing advancements in this area is the notion of a ”Ujlayan-Dixit (UD) fractional derivative” introduced and developed by Dixit and Ujlayan [15, 16] in 2020, which has different properties and applications for modeling physical and natural phenomena.

In general, the UD fractional derivative is designed to provide a balance between local and non-local behavior, and aims to obtain some specific computational or physical advantages, depending on its application in fractional differential equations or fractional order systems. Due to its novelty, this concept may not be applied or studied as extensively as more established fractional derivatives (e.g. Riemann-Liouville, Caputo, and Conformable fractional derivative). Applications of the UD fractional derivative can be found in areas such as viscoelasticity, anomalous diffusion, and control theory, which rely heavily on fractional calculus. Unlike classical operators such as the Caputo and Riemann–Liouville derivatives, which often involve singular kernels or complex integral forms, the UD derivative offers a simpler structure that enables closed-form solutions for probability distributions and related statistical measures.

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Probability distributions are essential to probability theory and statistics, explaining how probabilities are allocated to various outcomes of a random variable (see [17, 18, 19]). Comprehending these distributions and their properties allows for accurate modeling and analysis of real-world data across diverse fields. In Probability distributions, fractional derivatives are especially helpful for modeling random processes with long-range dependencies, memory effects, or anomalous diffusion, by generalizing classical stochastic models. This makes fractional calculus an important tool for advancing the understanding of probabilistic systems with memory; see [20, 21]. Recently, researchers have used fractional derivatives in probability distributions and achieved impressive results, especially the conformable fractional derivative (presented by Khalil *et al.* [22] in 2014) where they studied and created most of the applications to conformable fractional probability distributions; for more information, can be found in the references [20, 23].

The exponential distribution is one of the most frequently employed continuous probability distributions in probability theory and statistics. It models the time between events in a Poisson process, where events occur independently and continuously at a constant average rate. The exponential distribution is a key idea in probability theory and statistics, since it is used to describe waiting times and life spans in many fields. Moreover, it has become indispensable in both theoretical research and practical applications in various fields and has received great attention from many scholars; see the references [20, 24, 25] for example.

In [26], Alhribat *et al.* used the UD fractional differential equations to generate new fractional distributions based on some probability distributions, where they established the UD fractional probability distribution functions for the exponential, Pareto, Levy, and Lomax distributions. Motivated by aforementioned work, in this paper, we will define the UD fractional probability distribution functions for the α-exponential. In addition, we will establish the basic properties and applications of the UD fractional probability for the α-exponential distribution and with a study of the entropy measures in the fractional case.

2. The UD Fractional Derivative

This section contains definition and some properties of the UD fractional derivative; for more details, see the references [15, 16].

Definition 2.1
[16] Let $f : [0, +\infty) \to \mathbb{R}$ be a function and for $\alpha \in [0, 1]$. The UD fractional derivative of order α of f is defined by:

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} f\left(xe^{\frac{\varepsilon\alpha}{x}}\right) - f(x)}{\varepsilon}. \tag{1}$$

If limit exists. Also, if f is UD differentiable in the interval $(0, x)$ and for $x > 0$ and $\alpha \in [0, 1]$ such that $\lim_{x\to 0^+} f^\alpha(x)$ exist, then

$$f^\alpha(0) = \lim_{x\to 0^+} f^\alpha(x).$$

Note that,

$$D^\alpha f(x) = \frac{d^\alpha f}{dx^\alpha}.$$

Theorem 2.2
[15] Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function and for $\alpha \in [0, 1]$. Then, f is UD differentiable, and

$$D^\alpha f(x) = (1 - \alpha)f(x) + \alpha f'(x). \tag{2}$$

In particular, for $\alpha = 0$ we have $D^0 f(x) = f(x)$, and for $\alpha = 1$ we have $D^1 f(x) = f'(x)$.

Lemma 2.3
[16] The UD derivatives of order $\alpha \in [0, 1]$ of some elementary real-valued differentiable functions in $[0; \infty)$, can be expressed as follows:

1. $D^\alpha(c) = (1 - \alpha)c$, for each constant $c \in \mathbb{R}$.

2. $D^\alpha((ax+b)^n) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}$, for each $a, b \in \mathbb{R}$.
3. $D^\alpha(e^{ax+b}) = ((1-\alpha) + a\alpha)e^{ax+b}$, for each $a, b \in \mathbb{R}$.
4. $D^\alpha(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}$, for each $a, b \in \mathbb{R}$.
5. $D^\alpha(\sin(ax+b)) = (1-\alpha)\sin(ax+b) + a\alpha\cos(ax+b)$, for each $a, b \in \mathbb{R}$.
6. $D^\alpha(\cos(ax+b)) = (1-\alpha)\cos(ax+b) - a\alpha\sin(ax+b)$, for each $a, b \in \mathbb{R}$.

Proposition 2.4
[15] Let $f, g : [0, +\infty) \to \mathbb{R}$ be two functions and for $0 \leq \alpha, \beta \leq 1$. Then, the properties of the UD fractional derivative are as follows:

1. The UD fractional derivative D^α is a linear operator, such that:
$$D^\alpha(af(x) + bg(x)) = aD^\alpha f(x) + bD^\alpha g(x), \quad \text{for any } \ a, b \in \mathbb{R}.$$
2. The UD fractional derivative of the product rule of the functions f and g is given by:
$$D^\alpha(f(x).g(x)) = (D^\alpha f(x))g(x) + \alpha(D^\alpha g(x))f(x).$$
3. The UD fractional derivative of the quotient rule of the functions f and g is given by:
$$D^\alpha(f(x).g(x)) = \frac{(D^\alpha f(x))g(x) - \alpha(D^\alpha g(x))f(x)}{(g(x))^2}, \quad \text{with } \ g(x) \neq 0.$$
4. Commutativity:
$$D^\alpha(D^\beta f(x)) = D^\beta(D^\alpha f(x)).$$

Remark 2.5
[16] The UD fractional derivative of order $\alpha \in [0, 1]$ violates the Leibnitz's rule for fractional derivatives, i.e:
$$D^\alpha(f(x).g(x)) \neq g(x)D^\alpha f(x) + f(x)D^\alpha g(x), \quad \text{for all } \ x \in [0, +\infty).$$
It also violates the semi-group property, i.e:
$$D^\alpha(D^\beta f(x)) \neq D^{\alpha+\beta} f(x), \quad \text{for } \ 0 \leq \alpha, \beta \leq 1.$$

Definition 2.6
[15] Let $f : [a, b] \to \mathbb{R}$ be a function continuous and for $\alpha \in (0, 1]$. The UD fractional integral of order α is defined by:
$$I_a^\alpha f(x) = \frac{1}{\alpha}\int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f(t)dt.$$

3. The UD Fractional Exponential Distribution

In this section, we will review the UD fractional probability distribution functions for the α-exponential; for more information, see the paper of Alhribat *et al* [26].

The probability density function of a random variable X for the exponential distribution with parameter $\lambda > 0$ is defined as:
$$f(x, \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0. \tag{3}$$
Now, let's $y = \lambda e^{-\lambda x}$. Therefore, the first derivative of y is given as follows:
$$y' = -\lambda\left(\lambda e^{-\lambda x}\right) = -\lambda y.$$
Then, it can be written in the form:
$$y' + \lambda y = 0. \tag{4}$$

Thus, the equation (4) is a first order ordinary differential equation (DE).

Next, we consider the α-order (DE) with respect to the UD derivative in the following manner:

$$\begin{aligned} y^{(\alpha)} + \lambda y &= 0, \\ (1-\alpha)y + \alpha y' + \lambda y &= 0, \\ \alpha y' + (1-\alpha+\lambda)y &= 0, \\ y' + \left(\frac{1-\alpha+\lambda}{\alpha}\right) y &= 0. \end{aligned} \tag{5}$$

So, the equation (5) is a linear first-order differential equation with an integrating factor

$$\begin{aligned} u(x) &= e^{\int\left(\frac{1-\alpha+\lambda}{\alpha}\right)dx}, \\ &= e^{\left(\frac{1-\alpha+\lambda}{\alpha}\right)x}. \end{aligned}$$

The general solution of equation (5) is given by:

$$\begin{aligned} y &= \frac{\mathcal{A}}{u(x)}, \\ &= \mathcal{A}e^{\frac{\alpha-(\lambda+1)}{\alpha}x}. \end{aligned}$$

Hence, the new probability distribution will be as follows:

$$f_\alpha(x) = \mathcal{A}e^{\frac{\alpha-(\lambda+1)}{\alpha}x}. \tag{6}$$

To find the normalizing constant $\mathcal{A}$, we can be solve the following equation:

$$\int_0^{+\infty} f_\alpha(x)dx = 1.$$

Thus, the normalizing constant $\mathcal{A}$ is given by:

$$\mathcal{A} = \frac{(\lambda+1)-\alpha}{\alpha}, \quad \alpha \leq \lambda+1. \tag{7}$$

It is important to note that the condition $\alpha \leq \lambda + 1$ ensures the convergence of the integral used to normalize the probability density function. However, under our assumption that $0 < \alpha < 1$ and $\lambda > 0$, this condition is always satisfied, since $\lambda + 1 > 1 > \alpha$. Therefore, the constraint is not restrictive in practice. If α were to exceed $\lambda + 1$, the exponential term in the density would grow without bound, causing the integral to diverge and the function to fail normalization. Hence, the condition safeguards the validity of the probability model.

Lastly, we can write the UD fractional probability density function (UDFPDF) of the α-exponential distribution as follows:

$$f_\alpha(x) = \frac{(\lambda+1)-\alpha}{\alpha}e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}, \quad x > 0,\ \lambda > 0,\ 0 < \alpha < 1. \tag{8}$$

It's evident that, $f_\alpha(x)$ is once more an α-exponential distribution with $\lambda^* = \frac{(\lambda+1)-\alpha}{\alpha}$. Note that, we have

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \lambda e^{-\lambda x} = f(x). \tag{9}$$

Actually, we can plot the UD fractional probability distribution function (UDPDF) of α-exponential as shown in Figure 1 and Figure 2, respectively. By taking different values of α according to $\lambda = 0.5$ and $\lambda = 1.5$, respectively.

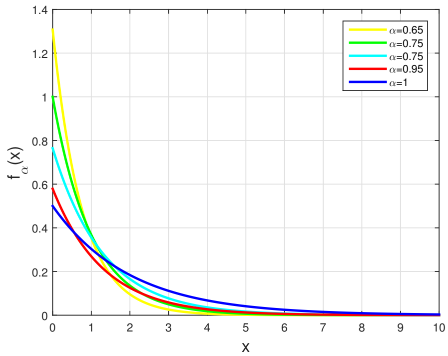

Figure 1. The UDPDF of α-exponential for different values of α and $\lambda = 0.5$.

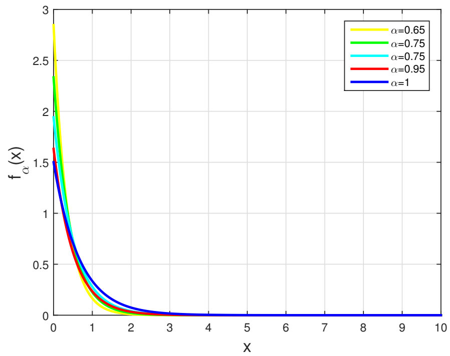

Figure 2. The UDPDF of α-exponential for different values of α and $\lambda = 1.5$.

It should be noted that Figure 1 shows how the UD fractional probability density function (PDF) behaves for different values of α when $\lambda = 0.5$. As α increases toward 1, the PDF approaches the classical exponential form. On the other hand, Figure 2 presents the same behavior for $\lambda = 1.5$, showing that higher λ values cause the PDF to decay faster, while the impact of α still controls the fractional effect.

4. Applications To The UD Fractional Exponential Distribution

In this section, we establish the basic properties and applications of the UD fractional probability for the α-exponential distribution; based on these references [20, **?**, 19].

4.1. The UD fractional cumulative distribution function (UDFCDF)

The UD fractional cumulative distribution function (UDFCDF) for the α-exponential distribution is given by:

$$F_\alpha(x) = \frac{(\lambda+1)-\alpha}{\lambda\alpha}\left(e^{\frac{(\alpha-1)}{\alpha}x} - e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}\right). \tag{10}$$

In effect,

$$\begin{aligned}
F_\alpha(x) &= P_\alpha(X \leq x), \\
&= I_0^\alpha f(x), \\
&= \frac{1}{\alpha}\int_0^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f_\alpha(t)dt, \\
&= \frac{(\lambda+1)-\alpha}{\alpha^2}\int_0^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} e^{-\frac{(\lambda+1)-\alpha}{\alpha}t}dt, \\
&= \frac{(\lambda+1)-\alpha}{\alpha^2} e^{\frac{(\alpha-1)}{\alpha}x}\int_0^x e^{-\frac{\lambda}{\alpha}t}dt, \\
&= \frac{(\lambda+1)-\alpha}{\lambda\alpha} e^{\frac{(\alpha-1)}{\alpha}x}\left(1 - e^{-\frac{\lambda}{\alpha}x}\right), \\
&= \frac{(\lambda+1)-\alpha}{\lambda\alpha}\left(e^{\frac{(\alpha-1)}{\alpha}x} - e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}\right).
\end{aligned}$$

Note that,

$$\lim_{\alpha\to 1^-} F_\alpha(x) = 1 - e^{-\lambda x} = F(x). \tag{11}$$

Then, we can plot The UD fractional cumulative distribution function (UDFCDF) of α-exponential as shown in Figure 3 and Figure 4, respectively. By taking different values of α according to $\lambda = 0.5$ and $\lambda = 1.5$, respectively.

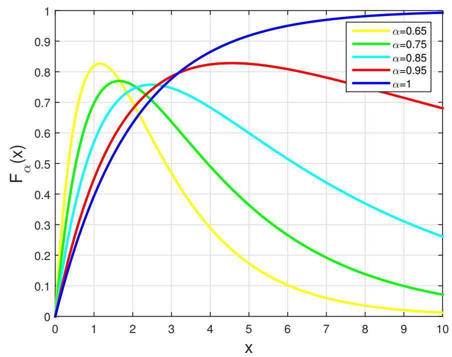

Figure 3. The UDFCDF of α-exponential for different values of α and $\lambda = 0.5$.

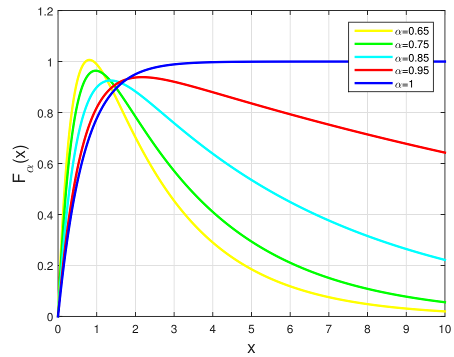

Figure 4. The UDFCDF of α-exponential for different values of α and $\lambda = 1.5$.

Obviously, Figure 3 illustrates the UD fractional cumulative distribution function (CDF) for various α values with $\lambda = 0.5$. As α increases, the CDF rises more sharply, approaching the classical exponential CDF. On the other hand, Figure 4 shows the CDF behavior for $\lambda = 1.5$, where the distribution accumulates probability more rapidly, and the role of α remains consistent in controlling the fractional dynamics.

4.2. The UD fractional survival function (UDFSF)

The UD fractional survival distribution function (UDFSDF) of X is defined by:

$$\begin{aligned} S_\alpha(x) &= 1 - F_\alpha(x), \\ &= 1 - \frac{(\lambda+1)-\alpha}{\lambda\alpha}\left(e^{\frac{(\alpha-1)}{\alpha}x} - e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}\right). \end{aligned} \tag{12}$$

Note that,

$$\lim_{\alpha\to 1^-} S_\alpha(x) = e^{-\lambda x} = S(x). \tag{13}$$

4.3. The UD fractional hazard function (UDFHF)

The UD fractional hazard distribution function (UDFHDF) of X is defined by:

$$\begin{aligned} h_\alpha(x) &= \frac{f_\alpha(x)}{S_\alpha(x)}, \\ &= \frac{\frac{(\lambda+1)-\alpha}{\alpha}\, e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}}{1 - \frac{(\lambda+1)-\alpha}{\lambda\alpha}\left(e^{\frac{\alpha-1}{\alpha}x} - e^{-\frac{(\lambda+1)-\alpha}{\alpha}x}\right)}, \\ &= \frac{\lambda}{\frac{\lambda\alpha}{(\lambda+1)-\alpha}\, e^{-\frac{(\lambda+1)-\alpha}{\alpha}x} - e^{\frac{\lambda}{\alpha}x} + 1}. \end{aligned} \tag{14}$$

Note that, we have

$$\lim_{\alpha\to 1^-} h_\alpha(x) = \lambda = h(x). \tag{15}$$

Thus, we can plot The UD fractional hazard distribution function (UDFHDF) of α-exponential as shown in Figure 5 according to $\lambda = 0.5$ and to different values of α.

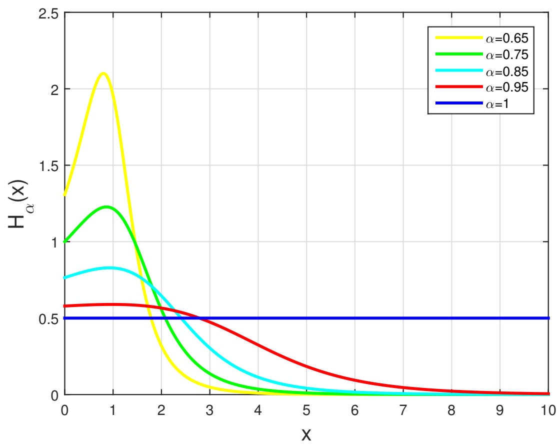

Figure 5. The UDFHDF of α-exponential for different values of α and $\lambda = 0.5$.

Clearly, Figure 5 displays the UD fractional hazard function for various α values with $\lambda = 0.5$. As α increases, the hazard rate transitions smoothly toward the constant rate of the classical exponential distribution, highlighting the influence of α on the aging behavior of the process.

4.4. The UD fractional expectation (UDFE)

4.4.1. n^{th} α-Moment $(E_\alpha(X^n))$ The UD fractional expectation E_α of a function (X^n) of continuous random variable X whose UDFPDF $f_\alpha(x)$ is given by:

$$\begin{aligned} E_\alpha(X^n) &= \int_0^\infty x^n f_\alpha(x)dx, \\ &= \int_0^\infty x^n \frac{(\lambda+1)-\alpha}{\alpha} e^{-\frac{(\lambda+1)-\alpha}{\alpha}x} dx, \\ &= \frac{n!\alpha^n}{((\lambda+1)-\alpha)^n}. \end{aligned} \tag{16}$$

Constantly, we can find the first and second moments as follows:

$$E_\alpha(X) = \frac{\alpha}{(\lambda+1)-\alpha} = \mu. \tag{17}$$

$$E_\alpha(X^2) = \frac{2\alpha^2}{((\lambda+1)-\alpha)^2}. \tag{18}$$

Note that,

$$\lim_{\alpha\to 1^-} E_\alpha(X^n) = \frac{n!}{\lambda^n} = E(X^n). \tag{19}$$

4.4.2. The n^{th} UD fractional central moment $E_\alpha(X-\mu)^n$ The n^{th} UD fractional central moment $E_\alpha(X-\mu)^n$ of X is defined by:

$$E_\alpha(X-\mu)^n = \int_0^\infty (x-\mu)^n f_\alpha(x)dx. \tag{20}$$

Based on the formula (20), we can find the list of central moments as follows:

1. First central moment:

$$E_\alpha(X-\mu) = 0. \tag{21}$$

2. Second central moment:

$$E_\alpha(X-\mu)^2 = \frac{\alpha^2}{((\lambda+1)-\alpha)^2}. \tag{22}$$

3. Third central moment:

$$E_\alpha(X-\mu)^3 = \frac{2\alpha^3}{((\lambda+1)-\alpha)^3}. \tag{23}$$

4. Fourth central moment:

$$E_\alpha(X-\mu)^4 = \frac{9\alpha^4}{((\lambda+1)-\alpha)^4}. \tag{24}$$

4.4.3. The UD fractional variance σ_α^2 The UD fractional variance σ_α^2 of X is given by:

$$\begin{aligned}\sigma_\alpha^2 &= E_\alpha(X^2)-(E_\alpha(X))^2,\\ &= \frac{\alpha^2}{((\lambda+1)-\alpha)^2}.\end{aligned} \tag{25}$$

4.4.4. The UD fractional standard deviation σ_α The UD fractional standard deviation σ_α of X is square root of the variance and given by:

$$\begin{aligned}\sigma_\alpha &= \sqrt{\sigma_\alpha^2},\\ &= \sqrt{\frac{\alpha^2}{((\lambda+1)-\alpha)^2}},\\ &= \frac{\alpha}{((\lambda+1)-\alpha)}.\end{aligned} \tag{26}$$

4.4.5. The UD fractional skewness SK_α The UD fractional skewness SK_α of X defined by:

$$SK_\alpha = \frac{E_\alpha(X-\mu)^3}{(\sigma_\alpha(X))^3}. \tag{27}$$

By using the equations (23) and (26), we get:

$$SK_\alpha = \frac{\frac{2\alpha^3}{((\lambda+1)-\alpha)^3}}{\left(\frac{\alpha}{((\lambda+1)-\alpha)}\right)^3} = 2. \tag{28}$$

4.4.6. The UD fractional kurtosis KU_α The UD fractional kurtosis KU_α of X defined by:

$$KU_\alpha = \frac{E_\alpha(X-\mu)^4}{(\sigma_\alpha(X))^4}. \tag{29}$$

According to equations (24) and (26), we find:

$$KU_\alpha = \frac{\frac{9\alpha^4}{((\lambda+1)-\alpha)^4}}{\left(\frac{\alpha}{((\lambda+1)-\alpha)}\right)^4} = 9. \tag{30}$$

4.5. The UD fractional Entropy Measures

4.5.1. The UD fractional Shannon entropy αH The UD fractional Shannon entropy αH of a random variable X is given by:

$$\begin{aligned} \alpha H(X) &= -\int_0^{\infty} f_\alpha(x)\log(f_\alpha(x))dx, \\ &= -E_\alpha\Big(log(f_\alpha(X))\Big), \\ &= 1-\log\left(\frac{(\lambda+1)-\alpha}{\alpha}\right). \end{aligned} \tag{31}$$

Note that,

$$\lim_{\alpha\to 1^-}\alpha H(X) = 1-\ln(\lambda) = H(X). \tag{32}$$

4.5.2. The UD fractional Tsallis entropy αT_q The UD fractional Tsallis entropy αT_q of a random variable X is given by:

$$\begin{aligned} \alpha T_q(X) &= \frac{1}{q-1}\left(1-\int_0^{+\infty}[f_\alpha(x)]^q dx\right), \\ &= \frac{1}{q-1}\Big(1-E_\alpha\left([f_\alpha(X)]^{q-1}\right)\Big), \\ &= \frac{1}{q-1}\left(1-\frac{1}{q}\left(\frac{(\lambda+1)-\alpha}{\alpha}\right)^{q-1}\right). \end{aligned} \tag{33}$$

Note that,

$$\lim_{\alpha\to 1^-}\alpha T_q(X) = \frac{1}{q-1}\left(1-\frac{\lambda^{q-1}}{q}\right) = T_q(X). \tag{34}$$

As $q \to 1$, the UD fractional Tsallis entropy reduces to the UD fractional Shannon entropy, i.e:

$$\lim_{q\to 1}\alpha T(X) = \alpha H(X). \tag{35}$$

4.5.3. The UD fractional Rényi entropy αR_q The UD fractional Rényi entropy αR_q of a random variable X is given by:

$$\begin{aligned} \alpha R_q(X) &= \frac{1}{1-q}\log\left(\int_0^{+\infty}[f_\alpha(x)]^q dx\right), \\ &= \frac{1}{1-q}\log\Big(E_\alpha\left([f_\alpha(X)]^{q-1}\right)\Big), \\ &= \frac{\log(q)}{q-1}-\log\left(\frac{(\lambda+1)-\alpha}{\alpha}\right). \end{aligned} \tag{36}$$

Note that,

$$\lim_{\alpha\to 1^-}\alpha R_q(X) = \frac{\log(q)}{q-1}+\log\left(\frac{1}{\lambda}\right) = R_q(X). \tag{37}$$

As $q \to 1$, the UD fractional Rényi entropy reduces to the UD fractional Shannon entropy, i.e:

$$\lim_{q\to 1} \alpha R_q(X) = \alpha H(X). \tag{38}$$

It is worth noting that all three entropy measures—Shannon, Tsallis, and Rényi—decrease as α approaches 1, and converge to their classical counterparts. This reflects a reduction in uncertainty or randomness as the fractional behavior fades. For smaller α, the entropies are larger, indicating that the UD fractional exponential distribution captures more dispersion or irregularity in the system compared to the classical exponential distribution.

5. Conclusion

In this research, we have introduce the probability density functions of random variables of exponential distribution using a new technique of the UD fractional derivative, and also establish the basic properties and applications of the UD fractional probability for the α-exponential distribution such as density, cumulative distribution, survival and hazard functions. Additionally, we have give the UD fractional isotopes with the expected values, r^{th}-moments, r^{th}-central moments, mean, variance, skewness and kurtosis, and study of the entropy measures in the fractional case. This model may find applications in areas such as reliability theory, survival analysis, and queuing systems, where memory and non-local effects are significant. Future work could explore the use of UD fractional distributions in real-world data fitting, model comparison with other fractional approaches, and extensions to multivariate or time-dependent distributions.

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