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Unit New Half Logistic Distribution: Theory, Estimation, and Applications with Novel Regression Analysis

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Abstract In this paper, a new unit distribution is introduced. Some statistical properties of the proposed distribution are analyzed, including moments, Bonferroni and Lorenz curves, etc. Six estimation methods are investigated to estimate the three parameters of the proposed distribution. The performance of these estimators is compared using bias, mean square error, average absolute bias, and mean relative error using the Monte Carlo simulation. In addition, some real data analysis is performed using data sets on the amount of water from the California Shasta reservoir, the average failure times of a fleet of air conditioning systems, and skewed to-right data. A novel regression analysis is proposed based on the new distribution. A practical example illustrates its effectiveness and applicability compared to existing methods, including the beta, Kumaraswamy, and log-extended exponential geometric regression analyses.

Keywords Beta regression analysis, OECD data, maximum likelihood estimation, Monte Carlo simulation, bounded distribution.

AMS 2010 subject classifications 60E05, 62F10, 62G30, 62H12

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1. Introduction

The statistical literature offers limited distributions for modeling data observed as percentages or proportions, such as mortality rates, recovery rates, and educational attainment measures. Commonly used bounded distributions, including the Kumaraswamy and beta distributions, may not always provide accurate models for proportional data. This limitation has motivated the development of new unit distributions. Consequently, existing lifetime distributions have been transformed to the interval (0, 1), constraining them to a bounded range. Some of the unit distributions studied in recent years include those proposed by [1], [5], and [16]. The primary objective of this study is to introduce a new unit distribution based on the new half logistic (NHL) distribution presented by [10].

The beta regression model is widely regarded as a popular tool for analyzing data where the response variable is constrained to the interval (0, 1). In recent years, several alternative regression models have been developed as substitutes for the beta regression model, including those proposed by [5], [12], and [11]. This study presents a novel regression model based on the proposed distribution, deriving its probability density function (pdf) and cumulative distribution function (cdf). The effectiveness of the new regression model is demonstrated through an application to educational attainment data from Organisation for Economic Co-operation and Development (OECD) countries, highlighting its superior performance compared to existing regression models.

The structure of the paper is organized as follows: Section 2 introduces the new distribution and examines its mathematical properties. Section 3 discusses various estimation methods for the three unknown parameters of the proposed distribution. Comprehensive Monte Carlo simulations evaluating the performance of these estimators are presented in Section 4. Section 5 introduces a novel regression model. In Section 6, the applicability of the

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proposed distribution and regression model is demonstrated through four practical examples. Finally, Section 7 concludes the paper with a discussion.

2. The bounded version of the new half logistic distribution

Let X be a random variable (rv) following NHL distribution, with its pdf and cdf defined, respectively, as

$$f_{NHL}\left(x;\alpha,\beta\right)=\frac{\alpha\exp\left(-x\right)+\beta\exp\left(-\beta x\right)+\left(\beta-\alpha\right)\exp\left(-\left(\beta+1\right)x\right)}{\left(1+\exp\left(-x\right)\right)^{\alpha+1}},\tag{1}$$

and

$$F_{NHL}\left(x;\alpha,\beta\right)=\frac{\left(1-\exp\left(-\beta x\right)\right)}{\left(1+\exp\left(-x\right)\right)^{\alpha}},x>0,\alpha>0,\beta>0,$$

respectively, where α,β,θ is the scale parameters. Let the rv X has pdf in Equation (1), then the pdf and cdf of the rv $Y=\exp\left(\frac{-X}{\theta}\right)$ are obtained

$$f_{UNHL}\left(y;\alpha,\beta,\theta\right)=\frac{\left(\alpha y^{\theta}+\beta y^{\beta\theta}+\left(\beta-\alpha\right)y^{\left(\left(\beta+1\right)\theta\right)}\right)\theta}{y\left(1+y^{\theta}\right)^{\alpha+1}},\tag{2}$$

and

$$F_{UNHL}(y;\alpha,\beta,\theta)=1-\frac{\left(1-y^{\beta\theta}\right)}{\left(1+y^{\theta}\right)^{\alpha}},\tag{3}$$

respectively, where $y\in(0,1)$ and $\alpha>0,\beta>0,\theta>0$ are the distribution parameters. The distribution with pdf in Equation (2), is referred to as the unit new half logistic (UNHL) distribution, denoted by $UNHL\left(\alpha,\beta,\theta\right)$. Moreover, the survival function (sf) of the UNHL distribution can also be easily obtained as $S_{UNHL}\left(y;\alpha,\beta,\theta\right)=1-F_{UNHL}\left(y;\alpha,\beta,\theta\right)$. The hazard rate function (hrf) of the UNHL distribution is expressed as

$$h_{UNHL}(y;\alpha,\beta,\theta)=\frac{\left(\alpha y^{\theta}+\beta y^{\beta\theta}+\left(\beta-\alpha\right)y^{\left(\left(\beta+1\right)\theta\right)}\right)\theta\left(1+y^{\theta}\right)^{\alpha}}{y\left(1+y^{\theta}\right)^{\alpha+1}\left(1-y^{\beta\theta}\right)}.$$

The plots of the pdf and hrf are presented in Figures 1 and 2, respectively. Figure 1 demonstrates that the pdf of the UNHL distribution exhibits various patterns, including increasing, decreasing, and unimodal shapes. Figure 2 reveals that the hrf of the UNHL distribution displays increasing, decreasing, and unimodal trends, which indicates its versatility in capturing different types of real-world phenomena, such as risk patterns that change over time.

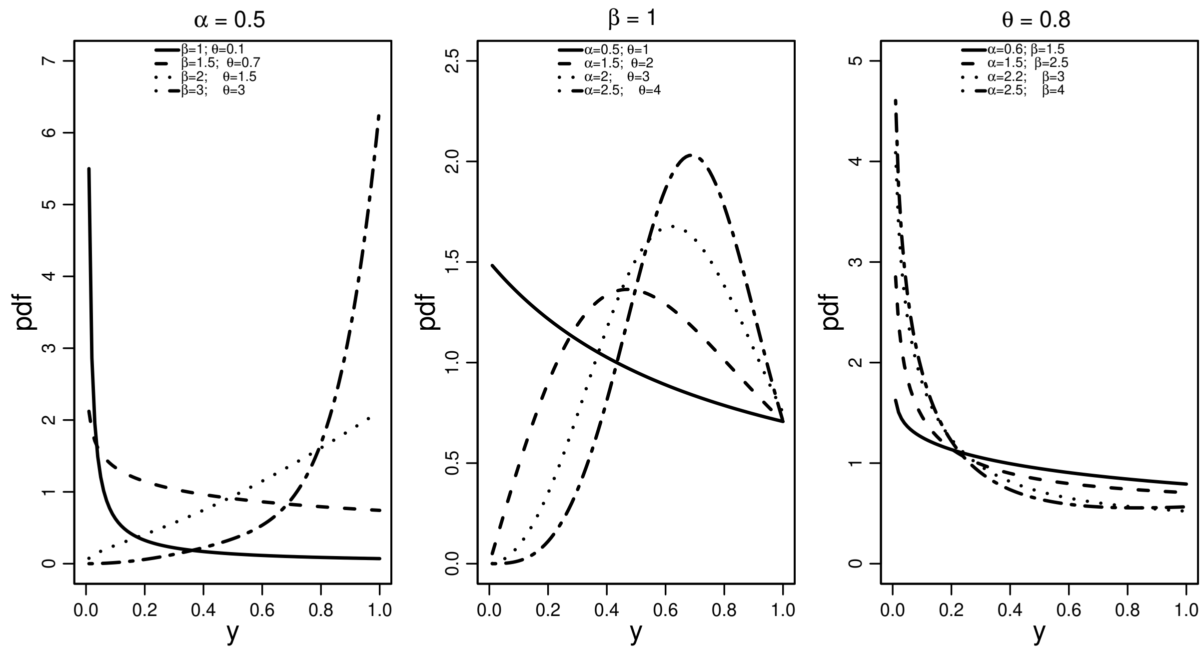

Figure 1. The pdf of UNHL distribution for various parameter choices

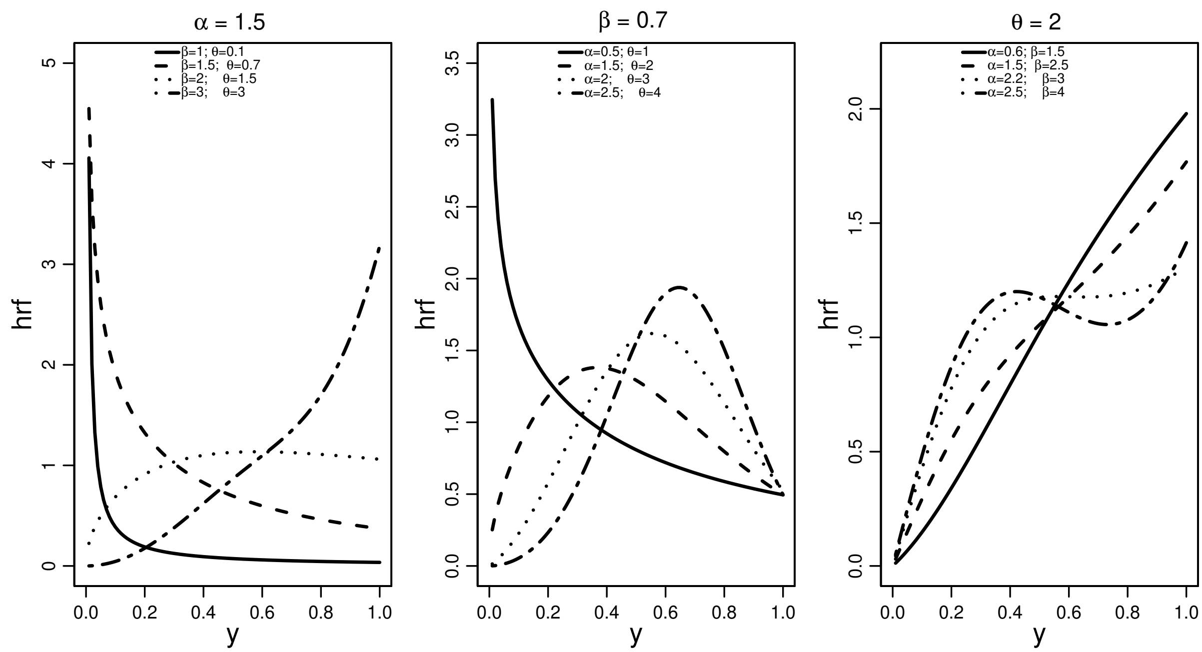

Figure 2. The hrf of UNHL distribution for various parameter choices

2.1. Moments

Let Y be a $UNHL\,(\alpha, \beta, \theta)$ rv with pdf in Equation (2). The r-th moment of Y is obtained as

$$\begin{aligned}
\mu_r &= \int_0^1 y^r f(y)\,dy \\
&= \frac{1}{2}\int_{-1}^{1}\left(\frac{z+1}{2}\right)^r f\left(\frac{z+1}{2}\right)dz \\
&= \frac{1}{2}\int_{-1}^{1}\left(\frac{z+1}{2}\right)^r \frac{\left(\alpha\left(\frac{z+1}{2}\right)^{\theta}+\beta\left(\frac{z+1}{2}\right)^{\beta\theta}+(\beta-\alpha)\left(\frac{z+1}{2}\right)^{((\beta+1)\theta)}\right)\theta}{\left(\frac{z+1}{2}\right)\left(1+\left(\frac{z+1}{2}\right)^{\theta}\right)^{\alpha+1}}dz \\
&\simeq \frac{1}{2}\sum_{l=1}^{N}\varpi_\ell\left(\frac{z_\ell+1}{2}\right)^r \frac{\left(\alpha\left(\frac{z_\ell+1}{2}\right)^{\theta}+\beta\left(\frac{z_\ell+1}{2}\right)^{\beta\theta}+(\beta-\alpha)\left(\frac{z_\ell+1}{2}\right)^{((\beta+1)\theta)}\right)\theta}{\left(\frac{z_\ell+1}{2}\right)\left(1+\left(\frac{z_\ell+1}{2}\right)^{\theta}\right)^{\alpha+1}},
\end{aligned}$$

where z_l and ϖ_l are the zeros and the corresponding Christoffel numbers of the Legendre-Gauss quadrature formula on the interval $(-1,1)$, respectively, as detailed in [4]. The ϖ_l is given by

$$\varpi_l = \frac{2}{(1-z_\ell)^2\left[L'_{N+1}(z_\ell)\right]^2}, \tag{4}$$

where

$$L'_{N+1}(z_\ell) = \frac{dL_{N+1}(z)}{dz}$$

at $z = z_\ell$ and $L_{N+1}(\cdot)$ is the Legendre polynomial of degree N.

2.2. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves, introduced by [3], have significant applications in the fields of economics and insurance. Now, we present the Bonferroni and Lorenz curves of the $UNHL(\alpha,\beta,\theta)$ distribution. Let the rv Y follow the $UNHL(\alpha,\beta,\theta)$ distribution with the pdf given by Equation (2). The Bonferroni and Lorenz curves are presented

$$\begin{aligned}
BC(\xi) &= \frac{q^2}{4\xi\mu_1}\sum_{l=0}^{N}\varpi_l(z_\ell+1)f\left(\frac{q}{2}(z_\ell+1)\right) \\
&= \frac{q^2}{4\xi\mu_1}\sum_{l=0}^{N}\varpi_l(z_\ell+1)\left(\frac{\left(\alpha\left(\frac{q}{2}(z_\ell+1)\right)^{\theta}+\beta\left(\frac{q}{2}(z_\ell+1)\right)^{\beta\theta}+(\beta-\alpha)\left(\frac{q}{2}(z_\ell+1)\right)^{((\beta+1)\theta)}\right)\theta}{\left(\frac{q}{2}(z_\ell+1)\right)\left(1+\left(\frac{q}{2}(z_\ell+1)\right)^{\theta}\right)^{\alpha+1}}\right),
\end{aligned}$$

and

$$\begin{aligned}
LC(\xi) &= \frac{q^2}{4\mu_1}\sum_{l=0}^{N}\varpi_l(z_\ell+1)f\left(\frac{q}{2}(z_\ell+1)\right) \\
&= \frac{q^2}{4\mu_1}\sum_{l=0}^{N}\varpi_l(z_\ell+1)\left(\frac{\left(\alpha\left(\frac{q}{2}(z_\ell+1)\right)^{\theta}+\beta\left(\frac{q}{2}(z_\ell+1)\right)^{\beta\theta}+(\beta-\alpha)\left(\frac{q}{2}(z_\ell+1)\right)^{((\beta+1)\theta)}\right)\theta}{\left(\frac{q}{2}(z_\ell+1)\right)\left(1+\left(\frac{q}{2}(z_\ell+1)\right)^{\theta}\right)^{\alpha+1}}\right),
\end{aligned}$$

respectively, where ϖ_l is given in Equation (4) and $q = F^{-1}(\xi)$.

2.3. Order statistics

In this subsection, we examine some results related to the order statistics of the UNHL distribution. Let Y_1, $Y_2, \ldots, Y_n$ be a random sample from the UNHL distribution and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ denote the corresponding order statistics. The cdf and pdf of $Y_{(r)}$ are provided in their general forms, respectively, as follows:

$$\begin{aligned} F_{Y_{(r)}}(y;\alpha,\beta,\theta) &= \sum_{i=r}^{n} \binom{n}{i} F(y;\alpha,\beta,\theta)^{i} \{1 - F(y;\alpha,\beta,\theta)\}^{n-i} \\ &= \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^{j} \binom{n}{i} \binom{n-i}{j} F(y;\alpha,\beta,\theta)^{i+j}, \end{aligned}$$

and

$$\begin{aligned} f_{Y_{(r)}}(y;\alpha,\beta,\theta) &= \frac{1}{B(r,n-r+1)} F(y;\alpha,\beta,\theta)^{r-1} \{1 - F(y;\alpha,\beta,\theta)\}^{n-r} f(y;\alpha,\beta,\theta) \\ &= \frac{1}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^{i} \binom{n-r}{i} F(y;\alpha,\beta,\theta)^{r+i-1} f(y;\alpha,\beta,\theta), \end{aligned}$$

where $r = 1, 2, \ldots, n$ and $B(.;.)$ is the classical beta function. The cdf and pdf of the $Y_{(r)}$ order statistic of the UNHL distribution are also obtained by

$$\begin{aligned} F_{Y_{(r)}}(y;\alpha,\beta,\theta) &= \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^{j} \binom{n}{i} \binom{n-i}{j} \\ &\quad \times \left\{ 1 - \frac{(1 - y^{\beta\theta})}{(1 + y^{\theta})^{\alpha}} \right\}^{i+j}, \end{aligned}$$

and

$$\begin{aligned} f_{Y_{(r)}}(y;\alpha,\beta,\theta) &= \frac{\left(\alpha y^{\theta} + \beta y^{\beta\theta} + (\beta - \alpha) y^{((\beta+1)\theta)}\right)\theta}{y(1 + y^{\theta})^{\alpha+1} B(r,n-r+1)} \\ &\quad \times \sum_{i=0}^{n-r} (-1)^{i} \binom{n-r}{i} \left\{ 1 - \frac{(1 - y^{\beta\theta})}{(1 + y^{\theta})^{\alpha}} \right\}^{r+i-1}. \end{aligned}$$

When $r = 1$ and $r = n$, the distributions of $\min\{Y_1, Y_2, \ldots, Y_n\}$ and $\max\{Y_1, Y_2, \ldots, Y_n\}$ statistics can be easily achieved, respectively.

3. Different point estimation methods

In this section, several estimators for the parameters of the UNHL distribution are proposed. We discuss the estimators, such as maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramer-von Mises (CvM), Anderson-Darling (AD), and maximum product spacing (MPS). Let Y_1, Y_2, ..., Y_n denote random samples from the $UNHL(\alpha,\beta,\theta)$ distribution, and let y_1, y_2, ..., y_n represent the observed values of the sample. Further, let $Y_{(1)}$, $Y_{(2)}$, ..., $Y_{(n)}$ denote the order statistics based on the sample y_1, y_2, ..., y_n, with the realizations $y_{(1)}$, $y_{(2)}$, ..., $y_{(n)}$. The likelihood and log-likelihood functions can be expressed, respectively, as

$$L(\Xi) = \prod_{i=1}^{n} \frac{\left(\alpha y_i^{\theta} + \beta y_i^{\beta\theta} + (\beta - \alpha) y_i^{((\beta+1)\theta)}\right)\theta}{y_i \left(1 + y_i^{\theta}\right)^{\alpha+1}}$$

and

$$\begin{aligned} \ell(\Xi) &= \sum_{i=1}^{n} \log\left(\alpha y_i^{\theta} + \beta y_i^{\beta\theta} + (\beta - \alpha) y_i^{((\beta+1)\theta)}\right) \\ &\quad + n\log(\theta) - \sum_{i=1}^{n} \log(y_i) - (\alpha+1)\sum_{i=1}^{n} \log\left(1 + y_i^{\theta}\right), \end{aligned}$$

where $\Xi = (\alpha, \beta, \theta)$. Then, the ML of $\widehat{\Xi} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right)$ for Ξ given as

$$\begin{aligned} \widehat{\Xi}_1 &= \arg\max\ \ell(\Xi). \\ (\alpha, \beta, \theta) &\in (0, \infty) \times (0, \infty) \times (0, \infty) \end{aligned}$$

Let us address the following functions to derive other estimators

$$\theta_{LS}(\Xi) = \sum_{i=1}^{n} \left(\left(1 - \frac{\left(1 - y_{(i)}^{\beta\theta}\right)}{\left(1 + y_{(i)}^{\theta}\right)^{\alpha}}\right) - \frac{i}{n+1} \right)^2, \tag{5}$$

$$\theta_{WLS}(\Xi) = \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left(\left(1 - \frac{\left(1 - y_{(i)}^{\beta\theta}\right)}{\left(1 + y_{(i)}^{\theta}\right)^{\alpha}}\right) - \frac{i}{n+1} \right)^2, \tag{6}$$

$$\begin{aligned} \theta_{AD}(\Xi) &= -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left[\log\left\{ \left(1 - \frac{\left(1 - y_{(i)}^{\beta\theta}\right)}{\left(1 + y_{(i)}^{\theta}\right)^{\alpha}}\right) \right\} \right. \\ &\quad \left. + \log\left\{ 1 - \left(1 - \frac{\left(1 - y_{(n+i-1)}^{\beta\theta}\right)}{\left(1 + y_{(n+i-1)}^{\theta}\right)^{\alpha}}\right) \right\} \right], \end{aligned} \tag{7}$$

$$\theta_{CvM}(\Xi) = \frac{1}{12n} + \sum_{i=1}^{n} \left[\left(1 - \frac{\left(1 - y_{(i)}^{\beta\theta}\right)}{\left(1 + y_{(i)}^{\theta}\right)^{\alpha}}\right) - \frac{2i-1}{2n} \right]^2, \tag{8}$$

and

$$\theta_{MPS}(\Xi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log\left[F(x_{(i)}|\Xi) - F(x_{(i-1)}|\Xi)\right]. \tag{9}$$

Then the LS, WLS, AD, CvM, and MPS are obtained by minimizing or maximizing Equations (5)-(9), respectively, as

$$\begin{aligned} \widehat{\Xi}_2 &= \arg\min \theta_{LS}(\Xi), \\ (\alpha, \beta, \theta) &\in (0, \infty) \times (0, \infty) \times (0, \infty) \end{aligned}$$

$$\begin{aligned}\widehat{\Xi}_3 &= \arg\min \theta_{WLS}(\Xi), \\ (\alpha,\beta,\theta) &\in (0,\infty)\times(0,\infty)\times(0,\infty)\end{aligned}$$

$$\begin{aligned}\widehat{\Xi}_4 &= \arg\min \theta_{AD}(\Xi), \\ (\alpha,\beta,\theta) &\in (0,\infty)\times(0,\infty)\times(0,\infty)\end{aligned}$$

$$\begin{aligned}\widehat{\Xi}_5 &= \arg\min \theta_{CvM}(\Xi), \\ (\alpha,\beta,\theta) &\in (0,\infty)\times(0,\infty)\times(0,\infty)\end{aligned}$$

and

$$\begin{aligned}\widehat{\Xi}_6 &= \arg\max \theta_{MPS}(\Xi). \\ (\alpha,\beta,\theta) &\in (0,\infty)\times(0,\infty)\times(0,\infty)\end{aligned}$$

These equations cannot be solved for minimization or maximization analytically. Therefore the Nelder-Mead or BFGS methods are employed in R, and the optimization is performed using the optim function in R [17].

4. Simulation study for point estimates

In this section, a Monte Carlo simulation experiment with 5000 runs is conducted to evaluate the bias, mean square error (MSE), average absolute bias (ABB), and mean relative error (MRE) of the ML, LS, WLS, AD, CvM, and MPS estimators for the unknown parameters of the UNHL distribution. Five distinct parameter settings are considered, and the sample sizes are selected as $n =$ 25, 50, 100, 200, 500, and 1000. The parameters of the proposed distribution under various scenario conditions are presented in Table 1.

Table 1. The different scenario conditions for α, β, and θ parameters.

Scenario	α	β	θ
1	0.5	0.9	0.6
2	1	1.5	0.9
3	2	1	1.5
4	1.5	2	3
5	2.5	3	3.5

The simulation results are reported in Tables 2 and 5. It is observed from these tables that the bias, MSE, ABB, and MRE of all estimators tend to decrease towards zero as the sample size increases. Based on the simulation results obtained for the α parameter, it is observed that the best estimation methods are ML, LS, and MPS for the bias criterion; ML, WLS, and MPS for the MSE criterion; and WLS for the ABB and MRE criteria. Based on the simulation results for the β parameter, it is observed that the best estimation methods are ML, LS, and WLS for the bias criterion; ML, WLS, and MPS for the MSE, ABB, and MRE criteria. Based on the simulation results for the θ parameter, it is observed that the best estimation methods are AD, CvM, and MPS for the bias criterion; AD, and MPS for the MSE, ABB, and MRE criteria.

Table 2. The bias values for α, β, and θ parameters.

		α						β						θ					
n	**Scenario**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**
25	1	1.6756	0.0083	0.5433	0.5171	-0.1811	0.5271	1.1347	-0.1661	0.7823	0.9046	-0.1545	0.6649	1.0070	-0.1395	0.5604	0.1817	-0.1086	0.2793
50		0.8020	-0.0348	0.4296	0.3312	-0.1567	0.4572	0.4955	-0.1210	0.6066	0.5040	-0.1302	0.5700	0.5468	-0.1379	0.4779	0.1597	-0.0916	0.2511
100		0.3928	-0.0432	0.3204	0.1952	-0.1287	0.3790	0.2782	-0.0896	0.4263	0.2837	-0.0930	0.4159	0.2989	-0.1162	0.3853	0.1251	-0.1057	0.2496
200		0.2353	-0.0523	0.2567	0.1461	-0.0923	0.2997	0.1830	-0.0675	0.3118	0.1831	-0.0721	0.3024	0.1904	-0.0894	0.3064	0.0967	-0.1168	0.2298
500		0.1206	-0.0336	0.1495	0.0838	-0.0559	0.1969	0.0957	-0.0427	0.1694	0.0965	-0.0468	0.1749	0.0994	-0.0562	0.1998	0.0435	-0.0942	0.1455
1000		0.0847	-0.0042	0.0779	0.0652	-0.0248	0.1240	0.0706	-0.0095	0.0944	0.0730	-0.0131	0.0984	0.0727	-0.0252	0.1259	0.0256	-0.0651	0.0894
25	2	0.6315	0.3746	0.3561	0.4719	0.8747	0.2277	0.5188	1.3128	0.3217	0.4903	0.9591	0.2464	0.6878	0.4086	0.4382	0.0290	1.2465	-0.0393
50		0.2273	0.1434	0.1928	0.2746	0.4769	0.1674	0.2581	0.6269	0.1218	0.2445	0.4138	0.1354	0.3398	0.2649	0.2583	-0.0363	0.7531	-0.0511
100		0.1519	0.2109	0.0926	0.1770	0.2752	0.0816	0.1783	0.3571	0.0553	0.1795	0.2763	0.0601	0.2089	0.1909	0.1247	-0.0811	0.4203	-0.0649
200		0.0128	0.0105	0.0645	0.1153	0.1568	0.0463	-0.0465	-0.0145	0.0281	0.0212	0.0443	0.0553	0.1343	0.1225	0.0643	-0.0694	0.1586	-0.0379
500		0.0168	0.0160	0.0402	-0.0025	0.0015	0.0263	-0.0475	-0.0392	0.0187	-0.0167	-0.0183	0.0249	0.0116	-0.0031	0.0336	-0.0458	0.0115	-0.0114
1000		-0.0326	-0.0342	0.0286	-0.0403	-0.0379	0.0201	-0.0600	-0.0621	0.0202	-0.0372	-0.0391	0.0251	-0.0327	-0.0399	0.0241	-0.0392	-0.0086	-0.0070
25	3	1.5492	0.0540	0.5558	0.6222	0.2213	0.2703	1.0248	0.3187	0.4672	0.7471	0.1073	0.3508	1.0580	0.0761	0.4491	0.2624	0.2752	0.1813
50		0.5206	0.1761	0.3000	0.2961	0.1071	0.1858	0.2826	0.0483	0.3150	0.3436	0.0957	0.2103	0.4761	0.0465	0.2756	0.0715	0.1196	0.1188
100		0.1914	-0.0243	0.1893	0.1422	-0.0039	0.1831	0.1305	-0.0357	0.2156	0.1330	-0.0545	0.2134	0.2178	-0.0348	0.2279	0.0437	0.0297	0.1070
200		0.1341	0.0049	0.1126	0.0976	-0.0036	0.1384	0.0659	-0.0488	0.1535	0.1454	0.0386	0.1074	0.1313	-0.0190	0.1571	0.0336	-0.0009	0.0924
500		0.0405	-0.0310	0.0810	0.0042	-0.0552	0.1120	0.0156	-0.0455	0.0953	0.0215	-0.0431	0.0965	0.0202	-0.0575	0.1171	0.0051	-0.0214	0.0659
1000		0.0512	-0.0043	0.0542	0.0133	-0.0384	0.0863	0.0072	-0.0434	0.0733	0.0653	0.0144	0.0595	0.0206	-0.0402	0.0890	-0.0128	-0.0438	0.0567
25	4	0.4299	1.5940	0.5028	0.2550	1.2376	0.2730	0.3221	2.8970	0.2021	0.3134	1.5632	0.3137	0.4162	0.7642	0.6718	-0.0925	1.8002	-0.3318
50		0.2342	0.9875	0.3004	0.1758	0.8864	0.2028	0.2235	1.4075	0.1455	0.1539	0.8979	0.1717	0.2376	0.6344	0.4007	-0.1229	1.0922	-0.2981
100		0.2111	0.7013	0.1684	0.1795	0.6340	0.1410	0.1840	0.7220	0.1285	0.1905	0.6644	0.1509	0.2128	0.5284	0.2264	-0.1134	0.5727	-0.2304
200		0.1196	0.4061	0.1119	0.1713	0.4820	0.1439	0.0971	0.3694	0.0998	0.1405	0.4178	0.1221	0.1904	0.4417	0.1839	-0.0995	0.2775	-0.1389
500		0.0701	0.2301	0.0547	0.1250	0.2979	0.1047	0.0897	0.2431	0.0758	0.0906	0.2384	0.0779	0.1364	0.2887	0.1223	-0.0734	0.0863	-0.0616
1000		0.0191	0.1125	0.0304	0.1000	0.2045	0.0884	0.0563	0.1468	0.0575	0.0253	0.1111	0.0470	0.1072	0.2023	0.0979	-0.0437	0.0413	-0.0279
25	5	0.2304	3.1515	0.1945	-0.0443	1.3814	-0.1152	-0.0035	5.2226	-0.0699	0.0604	2.0321	0.0317	0.1863	1.0039	0.1669	-0.2832	2.5269	-0.4022
50		0.0603	1.8590	0.0614	-0.0882	1.0492	-0.1052	-0.0911	2.0386	-0.0846	-0.0301	1.4648	-0.0134	0.0236	0.8769	0.0273	-0.2811	1.3457	-0.3065
100		-0.0300	0.8539	0.0169	-0.1117	0.5612	-0.0712	-0.1126	0.7516	-0.0537	-0.0837	0.6813	-0.0238	-0.0560	0.4845	-0.0052	-0.2454	0.6324	-0.1909
200		-0.0756	0.3154	-0.0138	-0.1089	0.3647	-0.0581	-0.1113	0.2953	-0.0464	-0.1069	0.2589	-0.0382	-0.0836	0.3218	-0.0268	-0.2084	0.1986	-0.1292
500		-0.0822	0.0537	-0.0299	-0.1126	0.0697	-0.0615	-0.0930	0.0564	-0.0443	-0.0942	0.0426	-0.0425	-0.1023	0.0568	-0.0490	-0.1386	-0.0076	-0.0806
1000		-0.0608	0.0048	-0.0213	-0.0703	0.0226	-0.0356	-0.0593	0.0163	-0.0262	-0.0600	0.0091	-0.0256	-0.0654	0.0163	-0.0296	-0.0780	-0.0228	-0.0439

Table 3. The MSE values for α, β, and θ parameters.

		α						β						θ					
n	**Scenario**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**
25	1	27.4739	0.8394	2.7124	2.7483	0.1916	1.0185	27.4497	0.3680	2.8421	4.4897	0.2007	1.7363	3.8474	0.1807	1.1063	3.2482	0.1447	0.8191
50		5.8977	0.1290	1.9450	1.1408	0.1783	0.8058	2.4906	0.2088	1.9046	1.2670	0.1928	1.4056	1.3743	0.1741	0.8480	1.1300	0.1230	0.6308
100		0.7424	0.1149	1.1660	0.4281	0.1622	0.6114	0.5577	0.1787	1.0259	0.4596	0.1744	0.8712	0.4845	0.1588	0.6299	0.3708	0.1189	0.4374
200		0.1812	0.1087	0.5912	0.1752	0.1494	0.4136	0.1822	0.1491	0.5931	0.1756	0.1439	0.5246	0.1878	0.1474	0.4232	0.1321	0.1083	0.2857
500		0.0643	0.0848	0.2160	0.0712	0.1216	0.2213	0.0659	0.1082	0.1917	0.0669	0.1092	0.1956	0.0733	0.1205	0.2220	0.0459	0.0770	0.1145
1000		0.0339	0.0627	0.0636	0.0404	0.0979	0.1043	0.0400	0.0831	0.0744	0.0363	0.0831	0.0770	0.0415	0.0975	0.1050	0.0220	0.0521	0.0471
25	2	8.8148	4.7172	1.5008	1.2922	4.7554	0.6634	5.8744	19.6454	1.5889	1.3494	10.1435	0.5707	2.2390	2.8812	1.1457	0.7070	7.0432	0.3625
50		1.5510	1.5719	0.6856	0.4776	2.2976	0.3886	0.7373	4.3451	0.4107	0.5088	2.6031	0.2361	0.5646	1.6974	0.5086	0.3773	3.4718	0.2276
100		0.3139	0.9961	0.1403	0.2815	0.9690	0.1035	0.5240	1.4318	0.1200	0.3510	0.9913	0.0542	0.2928	0.8410	0.1367	0.2385	1.3724	0.1012
200		0.1886	0.3153	0.0659	0.2247	0.4469	0.0323	0.2954	0.2184	0.0515	0.2338	0.3748	0.0630	0.2258	0.3987	0.0368	0.1473	0.4357	0.0436
500		0.1649	0.1888	0.0353	0.1777	0.1489	0.0107	0.1936	0.1095	0.0124	0.1747	0.1257	0.0090	0.1785	0.1459	0.0109	0.0588	0.0687	0.0090
1000		0.1204	0.1152	0.0087	0.1488	0.0982	0.0055	0.1413	0.0839	0.0049	0.1421	0.1066	0.0072	0.1484	0.0977	0.0056	0.0301	0.0239	0.0034
25	3	43.8217	1.0033	2.8184	3.5513	0.9481	0.9306	24.9673	8.2725	2.6598	4.2894	0.9532	1.0264	5.5599	0.5442	1.2100	3.1234	1.4581	1.0613
50		3.1351	1.9528	1.0537	1.0766	0.4025	0.4897	1.4156	2.2970	1.0739	0.9393	0.4536	0.4332	1.4348	0.3174	0.5686	0.6199	0.4519	0.4381
100		0.4125	0.1150	0.2735	0.4121	0.1506	0.3012	0.3929	0.1434	0.3117	0.3664	0.1297	0.2802	0.4823	0.1410	0.3268	0.2306	0.1387	0.1987
200		0.1769	0.1002	0.0970	0.1903	0.1247	0.1613	0.1842	0.1061	0.1433	0.2029	0.1201	0.1044	0.2071	0.1177	0.1669	0.1033	0.1002	0.1044
500		0.0811	0.0533	0.0402	0.1005	0.0784	0.0730	0.0908	0.0656	0.0529	0.1101	0.0665	0.0526	0.1022	0.0761	0.0735	0.0459	0.0593	0.0444
1000		0.0549	0.0486	0.0200	0.0739	0.0681	0.0416	0.0592	0.0491	0.0291	0.0701	0.0724	0.0265	0.0737	0.0667	0.0418	0.0266	0.0359	0.0246
25	4	1.4997	36.0222	2.7344	1.1458	7.1109	2.5944	5.4024	85.7554	3.8175	0.8949	16.2179	2.0449	1.3399	4.8511	3.6594	0.8214	13.8448	2.0352
50		0.5940	13.6454	1.4051	0.5055	4.2240	1.2414	0.7795	16.8203	1.4142	0.5183	6.8455	0.8925	0.5037	3.2855	1.6513	0.6047	6.2447	1.2207
100		0.4414	3.3294	0.4440	0.3913	2.1605	0.3805	0.4451	3.1029	0.4480	0.4287	2.7446	0.3921	0.3734	1.8615	0.4224	0.4753	2.1441	0.6876
200		0.3204	1.2694	0.2941	0.2999	1.4041	0.2182	0.2966	1.2138	0.2241	0.3200	1.2661	0.2085	0.2935	1.3098	0.2186	0.3406	0.9877	0.3487
500		0.2146	0.4785	0.1101	0.2283	0.5992	0.1282	0.2082	0.5158	0.1082	0.2081	0.5025	0.1095	0.2253	0.5813	0.1282	0.1901	0.3474	0.1142
1000		0.1568	0.2746	0.0602	0.1656	0.3543	0.0807	0.1449	0.3004	0.0616	0.1689	0.3006	0.0668	0.1645	0.3491	0.0812	0.1165	0.1925	0.0514
25	5	1.5969	118.3512	1.0845	1.2503	9.5317	1.0514	2.8723	264.3543	1.3876	1.1623	26.2675	0.8993	1.4210	7.3918	1.2360	1.3691	31.4299	1.3877
50		0.7192	54.7035	0.4611	0.7440	7.1404	0.5098	0.7888	50.5449	0.5473	0.7607	19.9892	0.4646	0.7444	6.2890	0.4879	0.9752	14.4072	0.7755
100		0.4797	12.2785	0.2513	0.4823	3.4410	0.2732	0.5075	8.1428	0.2717	0.4990	6.4244	0.2482	0.4651	3.1298	0.2616	0.6626	5.7121	0.3785
200		0.3161	2.5964	0.1349	0.3261	2.4677	0.1606	0.3291	2.3842	0.1439	0.3246	2.2496	0.1379	0.3161	2.3079	0.1548	0.4089	2.3184	0.1759
500		0.1917	0.8679	0.0652	0.2073	0.9609	0.0815	0.1875	0.8712	0.0680	0.1887	0.8603	0.0676	0.2023	0.9384	0.0790	0.2093	0.8363	0.0734
1000		0.1157	0.4598	0.0368	0.1206	0.4951	0.0458	0.1072	0.4484	0.0375	0.1072	0.4464	0.0374	0.1188	0.4889	0.0450	0.0982	0.3925	0.0351

Table 4. The ABB values for α, β, and θ parameters.

		α						β						θ					
n	**Scenario**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**
25	1	1.7350	0.3359	0.6052	1.1425	0.3812	0.6275	1.7358	0.4257	0.8802	1.2580	0.3955	0.7415	1.3331	0.3705	0.6376	1.0877	0.3251	0.4663
50		0.8738	0.2987	0.4920	0.7463	0.3701	0.5452	0.8216	0.4040	0.6929	0.7574	0.3871	0.6412	0.8175	0.3654	0.5498	0.6715	0.2974	0.3980
100		0.4786	0.2871	0.3854	0.4874	0.3534	0.4684	0.4943	0.3725	0.5190	0.4935	0.3699	0.5028	0.5145	0.3498	0.4662	0.4257	0.2910	0.3559
200		0.3008	0.2797	0.3154	0.3188	0.3471	0.3860	0.3095	0.3415	0.3974	0.3075	0.3352	0.3865	0.3310	0.3439	0.3859	0.2606	0.2776	0.2987
500		0.1948	0.2497	0.2084	0.2050	0.3139	0.2816	0.1954	0.2921	0.2458	0.1960	0.2930	0.2505	0.2084	0.3122	0.2806	0.1668	0.2296	0.1982
1000		0.1429	0.2158	0.1362	0.1533	0.2836	0.2028	0.1479	0.2569	0.1675	0.1457	0.2577	0.1699	0.1550	0.2827	0.2024	0.1167	0.1836	0.1332
25	2	0.8814	0.9669	0.4838	0.7396	1.3123	0.4361	0.9531	1.8282	0.5838	0.7622	1.4318	0.4256	0.8852	1.0253	0.5656	0.5855	1.5861	0.3825
50		0.5357	0.6656	0.3030	0.5135	0.8805	0.3111	0.6222	0.9466	0.3010	0.5493	0.8085	0.2699	0.5384	0.7709	0.3582	0.4765	1.0395	0.2894
100		0.4410	0.5734	0.1851	0.4296	0.5754	0.1875	0.5549	0.6016	0.1805	0.4760	0.5417	0.1554	0.4373	0.5454	0.2057	0.3858	0.6341	0.2059
200		0.3541	0.3986	0.1293	0.3910	0.4132	0.1195	0.4291	0.3260	0.1360	0.3938	0.3889	0.1346	0.3932	0.4003	0.1247	0.2899	0.3374	0.1292
500		0.3342	0.3237	0.0780	0.3589	0.3022	0.0784	0.3636	0.2734	0.0780	0.3520	0.2814	0.0726	0.3609	0.3029	0.0796	0.1795	0.1553	0.0667
1000		0.2938	0.2775	0.0547	0.3370	0.2744	0.0579	0.3257	0.2603	0.0540	0.3244	0.2727	0.0547	0.3380	0.2754	0.0587	0.1222	0.1076	0.0455
25	3	1.9225	0.3559	0.7344	1.1175	0.5390	0.6112	1.5982	0.6738	0.7668	1.1836	0.4383	0.6087	1.4171	0.4367	0.6814	0.9626	0.5850	0.5925
50		0.7938	0.4391	0.4411	0.6891	0.3916	0.4410	0.7638	0.3918	0.5163	0.6855	0.3785	0.3930	0.7891	0.3599	0.4680	0.5509	0.4014	0.4032
100		0.4684	0.2411	0.2968	0.4600	0.3129	0.3609	0.4615	0.3066	0.3581	0.4529	0.2928	0.3477	0.4960	0.2999	0.3725	0.3570	0.3072	0.2975
200		0.3296	0.2282	0.1933	0.3392	0.2878	0.2633	0.3366	0.2632	0.2534	0.3551	0.2802	0.2111	0.3514	0.2781	0.2658	0.2485	0.2599	0.2251
500		0.2252	0.1831	0.1383	0.2556	0.2375	0.1865	0.2406	0.2162	0.1613	0.2539	0.2177	0.1588	0.2571	0.2336	0.1859	0.1704	0.1945	0.1518
1000		0.1852	0.1740	0.0980	0.2182	0.2220	0.1422	0.1958	0.1912	0.1221	0.2081	0.2185	0.1134	0.2174	0.2194	0.1419	0.1305	0.1472	0.1125
25	4	0.7447	2.1049	0.9579	0.6680	1.7585	0.9709	0.9219	3.3655	1.1277	0.7263	2.0720	0.9302	0.7186	1.4568	1.1233	0.6974	2.1771	1.0182
50		0.5858	1.4301	0.7092	0.5659	1.3347	0.7097	0.6863	1.7836	0.7438	0.5763	1.3837	0.6599	0.5543	1.1840	0.7622	0.6238	1.4320	0.7834
100		0.5331	1.0198	0.4921	0.5080	0.9953	0.4834	0.5356	1.0837	0.5013	0.5241	1.0358	0.4778	0.4927	0.9280	0.4924	0.5437	0.9073	0.5789
200		0.4581	0.7411	0.3748	0.4351	0.7962	0.3709	0.4416	0.7605	0.3568	0.4543	0.7669	0.3559	0.4276	0.7706	0.3738	0.4589	0.6378	0.4068
500		0.3812	0.5201	0.2595	0.3890	0.5797	0.2852	0.3727	0.5437	0.2609	0.3725	0.5369	0.2596	0.3841	0.5708	0.2857	0.3437	0.4276	0.2520
1000		0.3322	0.4224	0.1962	0.3371	0.4713	0.2262	0.3179	0.4400	0.1970	0.3350	0.4436	0.2002	0.3342	0.4673	0.2264	0.2671	0.3273	0.1752
25	5	0.8596	3.9910	0.7262	0.8280	2.2175	0.7742	0.8932	6.1484	0.8191	0.8479	3.0073	0.7218	0.8684	1.9625	0.7874	0.9077	3.2575	0.8633
50		0.6730	2.7239	0.5231	0.6875	1.9286	0.5585	0.7093	3.0377	0.5613	0.7015	2.4685	0.5303	0.6860	1.7999	0.5470	0.7761	2.2689	0.6452
100		0.5562	1.7200	0.3931	0.5606	1.4365	0.4118	0.5775	1.7170	0.4061	0.5760	1.6376	0.3962	0.5512	1.3752	0.4029	0.6447	1.6278	0.4589
200		0.4471	1.1238	0.2910	0.4657	1.1715	0.3195	0.4611	1.1487	0.3010	0.4600	1.1210	0.2959	0.4599	1.1398	0.3133	0.5014	1.1285	0.3254
500		0.3329	0.7292	0.2024	0.3588	0.7825	0.2286	0.3363	0.7436	0.2073	0.3375	0.7399	0.2070	0.3559	0.7740	0.2254	0.3448	0.7114	0.2129
1000		0.2453	0.5266	0.1489	0.2629	0.5572	0.1684	0.2437	0.5253	0.1521	0.2441	0.5244	0.1518	0.2619	0.5539	0.1672	0.2287	0.4746	0.1449

Table 5. The MRE values for α, β, and θ parameters.

		α						β						θ					
n	**Scenario**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**	**ML**	**LS**	**WLS**	**AD**	**CvM**	**MPS**
25	1	3.4701	0.3732	1.0086	2.2849	0.4235	1.0458	3.4715	0.4731	1.4670	2.5161	0.4395	1.2358	2.6661	0.4117	1.0627	2.1754	0.3612	0.7772
50		1.7475	0.3318	0.8200	1.4925	0.4112	0.9087	1.6432	0.4488	1.1548	1.5149	0.4301	1.0687	1.6349	0.4060	0.9164	1.3431	0.3305	0.6634
100		0.9572	0.3190	0.6424	0.9748	0.3927	0.7806	0.9886	0.4138	0.8649	0.9871	0.4111	0.8380	1.0290	0.3887	0.7770	0.8514	0.3234	0.5931
200		0.6016	0.3108	0.5257	0.6375	0.3856	0.6433	0.6191	0.3794	0.6623	0.6150	0.3725	0.6442	0.6619	0.3821	0.6432	0.5211	0.3084	0.4978
500		0.3896	0.2775	0.3474	0.4099	0.3488	0.4693	0.3908	0.3246	0.4096	0.3920	0.3256	0.4175	0.4167	0.3469	0.4677	0.3337	0.2552	0.3304
1000		0.2859	0.2398	0.2269	0.3066	0.3151	0.3381	0.2958	0.2855	0.2792	0.2913	0.2863	0.2832	0.3101	0.3141	0.3373	0.2335	0.2040	0.2220
25	2	0.8814	0.6446	0.5375	0.7396	0.8748	0.4846	0.9531	1.2188	0.6487	0.7622	0.9546	0.4729	0.8852	0.6835	0.6285	0.5855	1.0574	0.4250
50		0.5357	0.4438	0.3367	0.5135	0.5870	0.3456	0.6222	0.6311	0.3345	0.5493	0.5390	0.2998	0.5384	0.5139	0.3980	0.4765	0.6930	0.3216
100		0.4410	0.3822	0.2056	0.4296	0.3836	0.2083	0.5549	0.4010	0.2005	0.4760	0.3611	0.1726	0.4373	0.3636	0.2286	0.3858	0.4228	0.2288
200		0.3541	0.2657	0.1437	0.3910	0.2754	0.1328	0.4291	0.2174	0.1512	0.3938	0.2593	0.1496	0.3932	0.2669	0.1385	0.2899	0.2250	0.1436
500		0.3342	0.2158	0.0867	0.3589	0.2015	0.0871	0.3636	0.1822	0.0867	0.3520	0.1876	0.0807	0.3609	0.2019	0.0885	0.1795	0.1035	0.0741
1000		0.2938	0.1850	0.0607	0.3370	0.1830	0.0643	0.3257	0.1736	0.0601	0.3244	0.1818	0.0608	0.3380	0.1836	0.0652	0.1222	0.0717	0.0505
25	3	0.9612	0.3559	0.4896	0.5588	0.5390	0.4075	0.7991	0.6738	0.5112	0.5918	0.4383	0.4058	0.7086	0.4367	0.4543	0.4813	0.5850	0.3950
50		0.3969	0.4391	0.2941	0.3446	0.3916	0.2940	0.3819	0.3918	0.3442	0.3427	0.3785	0.2620	0.3946	0.3599	0.3120	0.2755	0.4014	0.2688
100		0.2342	0.2411	0.1979	0.2300	0.3129	0.2406	0.2307	0.3066	0.2387	0.2264	0.2928	0.2318	0.2480	0.2999	0.2483	0.1785	0.3072	0.1984
200		0.1648	0.2282	0.1289	0.1696	0.2878	0.1756	0.1683	0.2632	0.1689	0.1776	0.2802	0.1408	0.1757	0.2781	0.1772	0.1243	0.2599	0.1501
500		0.1126	0.1831	0.0922	0.1278	0.2375	0.1243	0.1203	0.2162	0.1075	0.1269	0.2177	0.1059	0.1286	0.2336	0.1240	0.0852	0.1945	0.1012
1000		0.0926	0.1740	0.0654	0.1091	0.2220	0.0948	0.0979	0.1912	0.0814	0.1041	0.2185	0.0756	0.1087	0.2194	0.0946	0.0652	0.1472	0.0750
25	4	0.4964	1.0525	0.3193	0.4454	0.8792	0.3236	0.6146	1.6827	0.3759	0.4842	1.0360	0.3101	0.4790	0.7284	0.3744	0.4650	1.0886	0.3394
50		0.3906	0.7150	0.2364	0.3773	0.6674	0.2366	0.4575	0.8918	0.2479	0.3842	0.6918	0.2200	0.3695	0.5920	0.2541	0.4159	0.7160	0.2611
100		0.3554	0.5099	0.1640	0.3387	0.4977	0.1611	0.3571	0.5419	0.1671	0.3494	0.5179	0.1593	0.3285	0.4640	0.1641	0.3624	0.4536	0.1930
200		0.3054	0.3705	0.1249	0.2901	0.3981	0.1236	0.2944	0.3803	0.1189	0.3028	0.3834	0.1186	0.2851	0.3853	0.1246	0.3059	0.3189	0.1356
500		0.2541	0.2601	0.0865	0.2593	0.2899	0.0951	0.2484	0.2719	0.0870	0.2483	0.2684	0.0865	0.2561	0.2854	0.0952	0.2291	0.2138	0.0840
1000		0.2215	0.2112	0.0654	0.2247	0.2357	0.0754	0.2119	0.2200	0.0657	0.2233	0.2218	0.0667	0.2228	0.2337	0.0755	0.1780	0.1637	0.0584
25	5	0.3438	1.3303	0.2075	0.3312	0.7392	0.2212	0.3573	2.0495	0.2340	0.3391	1.0024	0.2062	0.3473	0.6542	0.2250	0.3631	1.0858	0.2466
50		0.2692	0.9080	0.1494	0.2750	0.6429	0.1596	0.2837	1.0126	0.1604	0.2806	0.8228	0.1515	0.2744	0.6000	0.1563	0.3104	0.7563	0.1843
100		0.2225	0.5733	0.1123	0.2243	0.4788	0.1177	0.2310	0.5723	0.1160	0.2304	0.5459	0.1132	0.2205	0.4584	0.1151	0.2579	0.5426	0.1311
200		0.1788	0.3746	0.0831	0.1863	0.3905	0.0913	0.1845	0.3829	0.0860	0.1840	0.3737	0.0845	0.1840	0.3799	0.0895	0.2006	0.3762	0.0930
500		0.1332	0.2431	0.0578	0.1435	0.2608	0.0653	0.1345	0.2479	0.0592	0.1350	0.2466	0.0591	0.1424	0.2580	0.0644	0.1379	0.2371	0.0608
1000		0.0981	0.1755	0.0425	0.1052	0.1857	0.0481	0.0975	0.1751	0.0434	0.0977	0.1748	0.0434	0.1047	0.1846	0.0478	0.0915	0.1582	0.0414

5. A new regression analysis

In this section, a novel regression analysis is introduced based on the UNHL distribution. Let $\log(\theta_i) = z_i^T \boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_p)^T \in \mathbb{R}^p$ and $z_i = (1, z_{i1}, z_{i2}, \ldots, z_{ip})$ and $i = 1, 2, \ldots, n$. The reparameterization forms of the cdf and pdf are given, by,

$$F_{RUNHL}(y_i;\boldsymbol{\Psi}) = 1 - \frac{\left(1 - y_i^{\beta \exp\left(z_i^T \boldsymbol{\gamma}\right)}\right)}{\left(1 + y_i^{\exp\left(z_i^T \boldsymbol{\gamma}\right)}\right)^{\alpha}},$$

and

$$f_{RUNHL}\left(y_i;\boldsymbol{\Psi}\right) = \frac{\left(\alpha y_i^{\exp\left(z_i^T \boldsymbol{\gamma}\right)} + \beta y_i^{\beta \exp\left(z_i^T \boldsymbol{\gamma}\right)} + \left(\beta - \alpha\right) y_i^{\left(\left(\beta+1\right) \exp\left(z_i^T \boldsymbol{\gamma}\right)\right)}\right) \exp\left(z_i^T \boldsymbol{\gamma}\right)}{y_i \left(1 + y_i^{\exp\left(z_i^T \boldsymbol{\gamma}\right)}\right)^{\alpha+1}},$$

respectively, where $\boldsymbol{\Psi} = (\alpha, \beta, \boldsymbol{\gamma})$. In the remainder of the manuscript, the rv Y_i will be denoted by $Y_i \sim RUNHL\left(\boldsymbol{\Psi}\right)$.

5.1. Estimation of the parameters in the regression model

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of the $RUNHL\left(\boldsymbol{\Psi}\right)$ distribution. The corresponding log-likelihood function is given, by

$$\begin{aligned} \ell\left(\boldsymbol{\Psi}\right) &= \sum_{i=1}^{n} \log\left(\alpha y_i^{\exp\left(z_i^T \boldsymbol{\gamma}\right)} + \beta y_i^{\beta \exp\left(z_i^T \boldsymbol{\gamma}\right)} + \left(\beta - \alpha\right) y_i^{\left(\left(\beta+1\right) \exp\left(z_i^T \boldsymbol{\gamma}\right)\right)}\right) \\ &\quad + \sum_{i=1}^{n} \left(z_i^T \boldsymbol{\gamma}\right) - \sum_{i=1}^{n} \log\left(y_i\right) \\ &\quad - \left(\alpha+1\right) \sum_{i=1}^{n} \log\left(1 + y_i^{\exp\left(z_i^T \boldsymbol{\gamma}\right)}\right). \end{aligned} \tag{10}$$

The ML of $\boldsymbol{\Psi}$, say $\widehat{\boldsymbol{\Psi}} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}_0, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_p\right)$, is obtained by maximizing $\ell\left(\boldsymbol{\Psi}\right)$ in Equation (10). Under the some regularity conditions, the asymptotic distribution of $\left(\widehat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}\right)$ is a multivariate normal $N_{p+1}\left(0, J^{-1}\right)$ where J represents the expected information matrix. The observed information matrix is often used instead of J.

6. Real data examples

In this section, two real data analyses are carried out to demonstrate the applicability of the proposed distribution and regression model. The UNHL distribution is compared to the NHL [10], unit Weibull (UW) [6], unit Lindley (UL) [7], unit Teissier (UT) [2], unit Muth (UM) [16], unit Improved second-degree Lindley (UISDL) [5], Kumaraswamy (KM) [15], and unit Burr-XII (UB-XII) distributions [13]. The $\widehat{\ell}$, Akaike's information criteria (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS) statistics, and KS p-values are calculated both data sets.

6.1. First real data for the proposed distribution

The first dataset, the Better Life Index (BLI), can be accessed at https://stats.oecd.org/, and it is also analyzed by [9]. The BLI dataset encompasses 11 metrics, including housing, income, employment, culture, education, environment, civic engagement, health, life satisfaction, safety, and work-life balance, along with 24 variables. In this report, we utilize the long-term unemployment rate metric, which represents the proportion of individuals who are unemployed for one year or longer relative to the total labor force (comprising both employed and unemployed individuals). The dataset comprises 38 samples. The data are: 0.0131, 0.0184, 0.0354, 0.0077, 0.0079, 0.0104, 0.0131, 0.0192, 0.0213, 0.0400, 0.0157,0.1565, 0.0172, 0.0026, 0.0323, 0.0049, 0.0659, 0.0103, 0.0005, 0.0335, 0.0269, 0.0235, 0.0007, 0.0197, 0.0074, 0.0066, 0.0152, 0.0443, 0.0478, 0.0317, 0.0766, 0.0112, 0.0182, 0.0239, 0.0113,0.0066, 0.0159, 0.1646. The results are presented in Table 6. It can be observed from Table 6 that the new proposed UNHL distribution of the real data is better than the distributions available in the literature. Furthermore, the fit to UNHL distribution for better life index data can be observed in Figure 3 with fitted pdf, cdf, sf, and probability–probability (P–P) plots.

Table 6. Data analysis results for first data

	NHL	UNHL	UW	UL	UT	UM	UISD	KM	UB-XII
$\widehat{\ell}$	95.0006	97.6462	96.2029	96.5835	91.1525	91.1525	94.4305	97.1480	54.3641
AIC	-186.0003	-189.2923	-188.4059	-191.1669	-180.3050	-178.3050	-186.8611	-190.2959	-104,7282
BIC	-182.7252	-184.3796	-185.1307	-189.5293	-178.6674	-175.0298	-185.2235	-187.0207	-101,4531
CAIC	-185.6575	-188.5865	-188.0630	-191.0558	-180.1938	-177.9621	-186.7500	-189.9531	-104,3854
HQIC	-184.8351	-187.5444	-187.2406	-190.5843	-179.7223	-177.1397	-186.2784	-189.1306	-103,5630
KS	0.1743	0.1164	0.1374	0.1350	0.2421	0.2421	0,1465	0.1169	0.5290
KS p-values	0.1986	0.6816	0.4697	0.4930	0.0232	0.0232	0.3886	0.6764	0.0000
α	0.0549	34.1455	0.0042	33.5019	0.2634	1.0000	34.4456	0.9407	0.0108
β	51.9673	5.3809	3.6136			3.7961		28.1414	67.0075
θ		0.9781							

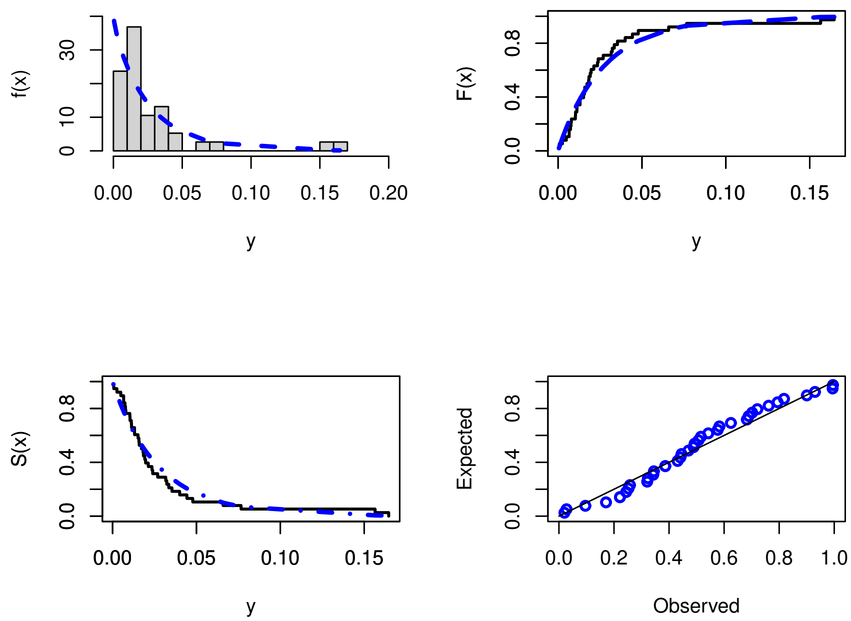

Figure 3. The fitted pdf, cdf, sf, and P–P plots for UNHL distribution of the better life index data

6.2. Second real data for the proposed distribution

The second dataset [18] includes the amount of water the California Shasta Reservoir could hold each month in February between 1991 and 2010. The data consists of 20 samples. The data are: 0.33894, 0.43192, 0.75993, 0.72463, 0.75758, 0.81156, 0.78534, 0.78366, 0.81563, 0.84741, 0.76801, 0.84349, 0.78741, 0.84987, 0.69597, 0.84232, 0.82869, 0.58019, 0.43068, 0.74256. The results are presented in Table 7. It can be observed from Table 7 that the new proposed UNHL distribution of the real data is better than the distributions available in the literature. Furthermore, the fit to UNHL distribution for the amount of water in the California Shasta Reservoir data can be observed in Figure 4 with fitted pdf, cdf, sf, and P–P plots.

Table 7. Data analysis results for second data

	NHL	UNHL	UW	UL	UT	UM	UISD	KM	UB-XII
$\widehat{\ell}$	-13.4595	19.7561	10.9570	13.8272	4.4541	8.9375	6.4021	13.4747	11.7443
AIC	30.9190	-33.5122	-17.9140	-25.6544	-6.9082	-13.8750	-10.8041	-22.9494	-19.4885
BIC	32.9105	-30.5250	-15.9225	-24.6586	-5.9125	-11.8835	-9.8084	-20.9579	-17.4971
CAIC	31.6249	-32.0122	-17.2081	-25.4321	-6.6860	-13.1691	-10.5819	-22.2435	-18.7827
HQIC	31.3077	-32.9290	-17.5252	-25.4600	-6.7139	-13.4862	-10.6097	-22.5607	-19.0998
KS	0.4070	0.1153	0.2416	0.2421	0.4843	0.2841	0.4848	0.2209	0.2255
KS p-values	0.0017	0.9258	0.1638	0.1621	0.0001	0.0638	0.0001	0.2447	0.2244
α	0.0757	243.9510	4.2071	0.4957	2.1581	0.3541	0.7079	6.3476	5.8563
β	1.4126	0.1191	1.5704			0.3559		4.4893	1.7653
θ		29.0460							

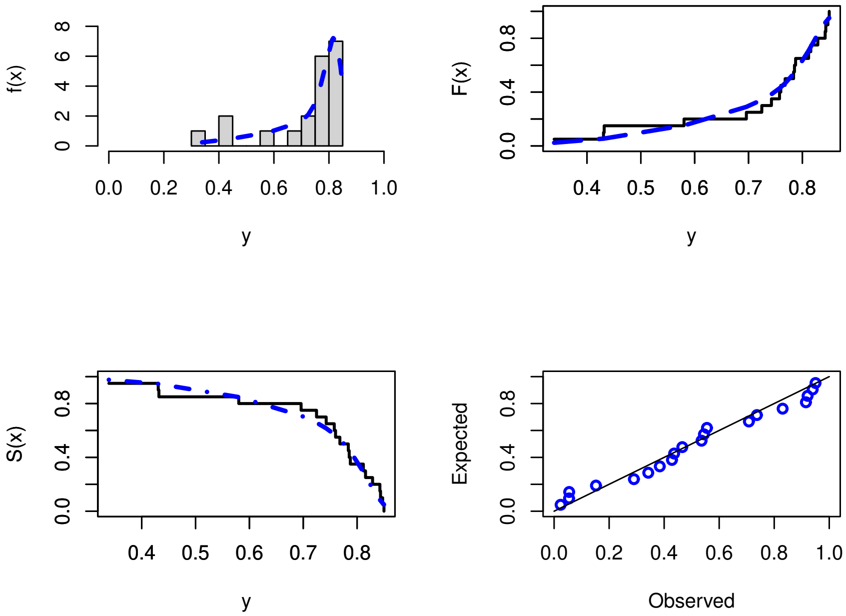

Figure 4. The fitted pdf, cdf, sf, and P–P plots for UNHL distribution of the amount of water the California Shasta Reservoir data

6.3. *Third real data for the proposed distribution*

The third dataset set contains information about the following set of skewed to-right data, discussed by [8]. The data consists of 50 samples. The data are: 0.445, 0.493, 0.285, 0.564, 0.76, 0.381, 0.69, 0.579, 0.636, 0.238, 0.149, 0.244, 0.126, 0.796, 0.405, 0.553, 0.78, 0.431, 0.184, 0.375, 0.198, 0.89, 0.192, 0.463, 0.486, 0.521, 0.366, 0.486, 0.116, 0.511, 0.612, 0.117, 0.384, 0.326, 0.057, 0.412, 0.586, 0.517, 0.57, 0.588, 0.497, 0.246, 0.234, 0.228, 0.552, 0.893, 0.403, 0.458, 0.134, 0.338. The results are presented in Table 8. It can be observed from Table 8 that the new proposed UNHL distribution of the real data is better than the distributions available in the literature. Furthermore, the fit to UNHL distribution for the skewed to-right data can be observed in Figure 5 with fitted pdf, cdf, sf, and P–P plots.

Table 8. Data analysis results for third data

	NHL	UNHL	UW	UL	UT	UM	UISD	KM	UB-XII
$\widehat{\ell}$	-7.7899	9.9058	9.9141	3.2816	3.1079	7.8382	-26.0713	9.8845	9.2374
AIC	19.5797	-13.8117	-15.8281	-4.5632	-4.2157	-11.6765	54.1426	-15.7690	-14.4747
BIC	23.4038	-8.0756	-12.0041	-2.6512	-2.3037	-7.8524	56.0546	-11.9449	-10.6507
CAIC	19.8351	-13.2899	-15.5728	-4.4799	-4.1324	-11.4212	54.2259	-15.5137	-14.2194
HQIC	21.0360	-11.6273	-14.3719	-3.8351	-3.4876	-10.2203	54.8707	-14.3128	-13.0185
KS	0.2332	0.0778	0.0950	0.1646	0.2490	0.1245	0.4454	0.0878	0.0868
KS p-values	0.0087	0.9228	0.7578	0.1330	0.0041	0.4203	0.0000	0.8354	0.8457
α	0.0000	4.0002	0.8251	1.1906	0.8744	0.6253	1.5815	1.8098	1.3879
β	2.3261	0.6655	1.7404			1.0030		2.8940	2.4224
θ		2.4790							

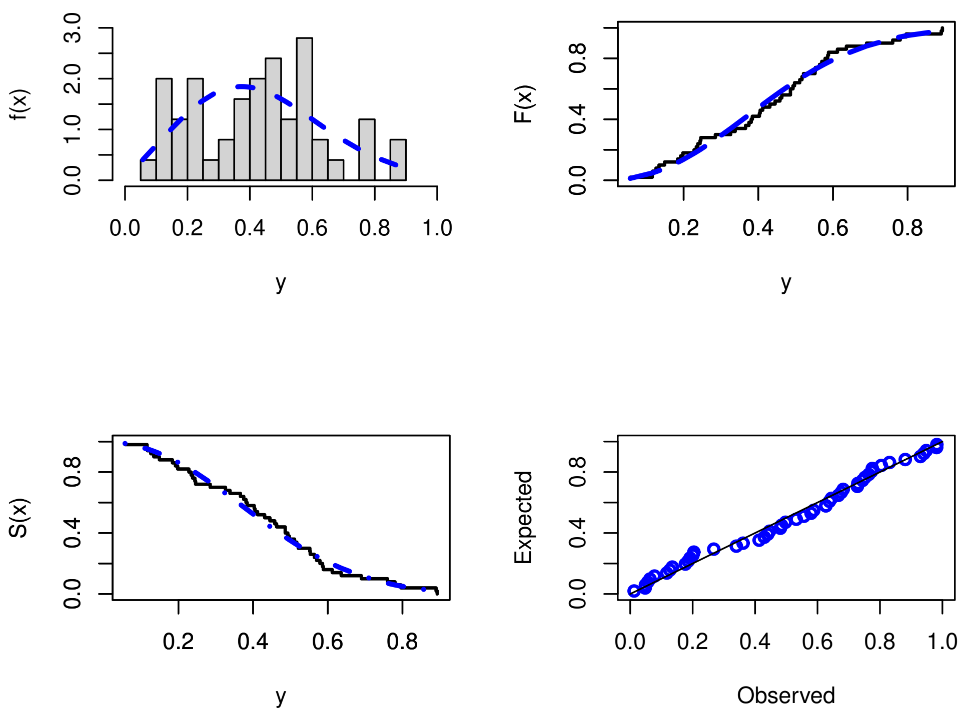

Figure 5. The fitted pdf, cdf, sf, and P–P plots for UNHL distribution of the skewed to-right data

6.4. Real data example for the proposed regression model

In this subsection, a real data application is conducted to evaluate the usability and superiority of the new regression model. The data used for the application consists of a percentage of educational attainment for OECD countries, along with other components such as percentage of voter turnout, homicide rate, and life satisfaction. The data can be accessed at https://stats.oecd.org/; further details about this dataset are provided in [12]. The beta regression (BR), Kumaraswamy regression (KR), and log-extended exponential geometric regression (LEEG) models ([14]) are considered for comparison purposes. The goal of this application is to model the percentage of education level values of OECD countries (variable y) with voter participation percentage (variable x_1), homicide rate (variable x_2), and life satisfaction (variable x_3). For each of the models, ML, standard error (SE), ℓ, and AIC are computed. The results are presented in Table 9, demonstrating that the RUNHL model exhibits the best modeling capacity and is a viable alternative to the beta and Kumaraswamy regression models commonly used in the literature. γ_1, γ_2, and γ_3 are statistically significant at the %5 level in the RUNET regression model. The positive coefficient of γ_3 indicates that it significantly increases the median response, whereas the negative coefficients of γ_1 and γ_2 suggest a decreasing effect on the median response.

Table 9. Data analysis results for OECD data.

	RUNHL			**BR**			**KR**			**LEEG**		
Parameters	**Estimate**	**SE**	**p-value**	**Estimate**	**SE**	**p-value**	**Estimate**	**SE**	**p-value**	**Estimate**	**SE**	**p-value**
γ_0	1.0516	0.9198	0.2529	0.9615	0.9685	0.3208	1.6318	1.1629	0.1606	0.3269	1.0711	0.7602
γ_1	-1.8288	0.8182	0.0254	-2.9211	1.0176	0.0041	-4.1024	1.3947	0.0033	-4.0914	1.4627	0.0052
γ_2	-0.0544	0.0186	0.0034	-0.0470	0.0178	0.0083	-0.0405	0.0167	0.0153	-0.0476	0.0145	0.0010
γ_3	0.3917	0.1212	0.0012	0.3794	0.1492	0.0110	0.4206	0.2532	0.0967	0.6215	0.1746	<0.0001
α	5.2014	2.2991	<0.0001	11.5900	2.6100	<0.0001	6.2166	1.0844	<0.0001	7.8374	1.7424	<0.0001
β	1.6725	1.9487	<0.0001			<0.0001			<0.0001			<0.0001
$\widehat{\ell}$	32.6706			30.9024			29.4339			28.6481		
AIC	-53.3411			-51.8048			-48.8678			-47.2962		

7. Concluding remarks

In this paper, we propose a new unit distribution derived from the NHL distribution and examine its mathematical properties, including moments, and stochastic ordering. Several estimators are introduced to estimate unknown parameters of the distribution, and their performances are evaluated through Monte Carlo simulation based on various criteria, such as bias, MSE, ABB, and MRE. The results from the Monte Carlo simulation show that all estimators perform well, particularly with large sample sizes. Additionally, the applicability of the proposed distribution and regression model is demonstrated through real data analysis. The proposed distribution and regression model offers an alternative approach that can be applied to a wide range of practical problems, delivering enhanced modeling capacity and greater flexibility.

Conflict of interest

The authors declare that they have no conflict of interest

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