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An efficient technique for solving the two-dimensional heat equation

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Abstract In this paper, we apply the semi analytic iterative method to the solution of the two-dimensional heat equation. By means of some numerical examples we show that this method is efficient and accurate in producing exact to near-exact solutions. Even where the exact solution is unknown, we were able to obtain it through the fast convergence of the method.

Keywords Two-Dimensional Heat Equation, Semi-Analytic Iterative Method, Convergence, Iterative Laplace Transform Method

AMS 2010 subject classifications 35A20, 35K05, 35K10, 35K15

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1. Introduction

Differential equations of real or complex order are highly valued by researchers due to their significant applications in science and technology. The applicability of these equations to real-world problems is evident in diverse fields like fluid dynamics, chemical physics, and control systems. The heat equation is one of the fundamental partial differential equations (PDEs) in mathematical physics and engineering that describes the variation in temperature in a specified area over a period of time [6]. It describes how heat propagates through a defined area over time. Ullah et al. [24] have pointed out the numerous applications of the heat equation and underscored the fact that this equation has been studied in various dimensions using techniques such as the Laplace and Fourier transforms. With wide use in thermodynamics, fluid dynamics, materials science, and engineering simulations, the two-dimensional (2D) heat equation is vital for modelling surface heat conduction. Many practical scenarios highlight its indispensable and significant relevance.

In cases of complex domains or boundary conditions, numerical methods are essential for solving the two-dimensional heat equation because analytical solutions are often impossible. Traditional numerical techniques such as finite difference methods (FDM), finite element methods (FEM), and finite volume methods (FVM) have been widely used to solve this equation [18]. However, the computational cost and time associated with these techniques, especially for large-scale problems, have led to the search for more efficient algorithms.

Recent developments focus on combining traditional methods with advanced computational strategies such as adaptive grids, spectral techniques, and machine learning-based approaches to improve computational efficiency while maintaining accuracy [20]. In particular, large-scale simulations have seen a significant increase in computational speed, thanks to the integration of multigrid methods and parallel computing. Furthermore, radial basis function (RBF) approximations, a type of meshless method, have proven effective for efficient solutions to multidimensional heat equations.

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Moreover, modern techniques like physics-informed neural networks (PINNs) are gaining traction for solving PDEs, including the heat equation, as they leverage deep learning to approximate solutions without relying on traditional discretization methods [17]. These methods are particularly advantageous for high-dimensional problems or when experimental data is integrated into the model. In this work, we present an efficient numerical technique to solve the two-dimensional heat equation that balances computational efficiency and accuracy. The proposed method builds upon recent advancements in numerical and computational techniques to reduce computational time and address limitations associated with conventional methods.

A simple 2D heat conduction model is shown in Figure 1 [12]. The left and right bars behave as heat sources and sinks, respectively. Heat flows by the process of convection from a region of higher temperature (heat source) to a region of lower temperature (heat sink). Encouraged by the excellent performance of the semi-analytic iterative

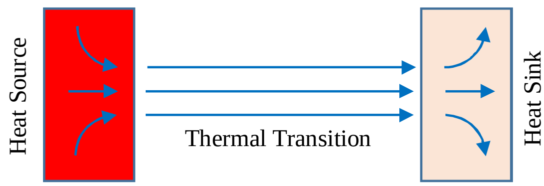


Figure 1. Engineering model of heat conduction (adapted from Mastoi et al. [12])

method, in this paper we apply this method to the solution of the two-dimensional heat equation which in $\{x, y\}$ space is given by

$$\rho c_p T_t = \frac{\partial}{\partial x}(k_x T_x) + \frac{\partial}{\partial x}(k_y T_y) + Q, \tag{1}$$

where ρ is the density, c_p is the specific heat or heat capacity, $k_{x,y}$ the thermal conductivities in the x and y directions, respectively, and Q the heat source term which describes the heat release process if $Q > 0$ and the heat absorption process if $Q < 0$. In particular, a positive Q represents the radiogenic heat production (RHP).

If the thermal conductivity is isotropic (i.e., $k_x = k_y = k$) and constant, then equation (1) can be rewritten as

$$T_t = \kappa (T_{xx} + T_{yy}) + \frac{Q}{\rho c_p}, \tag{2}$$

where $\kappa = \frac{k}{\rho c_p} > 0$, called the *thermal diffusivity*, is a material-specific quantity depending on the thermal conductivity, the density and the specific heat. Equation (2) is the two-dimensional heat equation being considered in this paper, subject to the initial condition $T(x, y, 0) = g(x, y)$. If there is no RHP (i.e., when $Q = 0$), then we have the homogeneous 2D heat equation

$$T_t = \kappa (T_{xx} + T_{yy}). \tag{3}$$

The 2D heat equation represents changes in the temperature $T(x, y, t)$ with respect to time t in a region of space having coordinates (x, y). Thus, in this equation T_{xx} and T_{yy} are the thermal conductions in the x and y directions, respectively. In more compact form, equation (2) can be expressed as

$$T_t = \kappa \Delta T + \frac{Q}{\rho c_p}, \tag{4}$$

where Δ is the Laplacian operator defined as

$$\Delta T = T_{xx} + T_{yy}.$$

1.1. Brief Review of Existing Methods

Equation (2), with initial condition $T(x, y, 0) = g(x, y)$, has been solved using a variety of methods and schemes in the literature. These include the method of fundamental solutions (MFS) [8], the three-level explicit time-split MacCormack method [13], the tri-diagonal matrix algorithm (TDMA) [9], finite difference methods [16], the iterative Laplace transform (ILT) method [24] and the shifting function method [7], to mention but a few.

1.2. Contributions

The main contribution of this paper is a very simple, straightforward and accurate algorithm that requires less computational work and rapidly converges to the exact solution of the 2D heat equation.

1.3. Organization

The rest of the paper is organized as follows: Section 2 describes the semi analytic iterative method. In Section 4 the results of numerical experiments based on a selection of test problems are presented and compared with the exact solution and solutions from previous methods used in the literature and in Section 5 some conclusions are drawn.

2. Description of the Proposed Method of Solution

The SAIM was used by Al-Jawary and Al-Razaq [2] to solve Duffing equations, by Yassein [25] to solve higher order integro-differential equations and by Yassein and Aswhad [26] to solve KdV equations. This method uses an iterative approach together with analytical computations to provide a solution of a modified reformulated linear problem. The SAIM was inspired by the homotopy analysis method (HAM) which is a general approximate analytical approach for obtaining convergent series solutions of strongly nonlinear problems without the need for linearization [23]. The SAIM offers several advantages over existing methods such as Picard's successive approximations method (SAM) and the Adomian decomposition method (ADM) in that it is very easy to implement since it avoids the calculation of Adomian polynomials for the nonlinear term in the ADM or Lagrange multipliers in He's variational iteration method (VIM), thus demanding less computational work [10, 14]. In this paper we propose to use the SAIM to solve the linear 2D heat equation of the form (2), with initial condition $T(x, y, 0) = g(x, y)$, which can be expressed as

$$LT + NT = \tilde{Q} \tag{5}$$

with the condition $C\left(T, \frac{\partial T}{\partial t}\right) = 0$, where $LT = T_t$, $NT = -\kappa(T_{xx} + T_{yy})$ and $\tilde{Q} = \frac{Q}{\rho c_p}$ is the heat source term. The algorithm of the SAIM is represented by the flowchart in Figure 2 below. The first step in the implementation of the SAIM is to find the initial approximation by solving

$$L[T_0(x, y, t)] - \tilde{Q} = 0 \text{ with } C\left(T_0, \frac{\partial T_0}{\partial t}\right) = 0 \tag{6}$$

It should be noted that

$$T_0(x, y, t) = T(x, y, 0) + tT_t(x, y, 0) = g(x, y).$$

The next iteration to the solution can be obtained by solving

$$L[T_1(x, y, t)] + N[T_0(x, y, t)] - \tilde{Q} = 0 \text{ with } C\left(T_1, \frac{\partial T_1}{\partial t}\right) = 0 \tag{7}$$

After several iterations we obtain the general form of the SAIM solution which is

$$L[T_{n+1}(x, y, t)] + N[T_n(x, y, t)] - \tilde{Q} = 0 \text{ with } C\left(T_{n+1}, \frac{\partial T_{n+1}}{\partial t}\right) = 0, \tag{8}$$

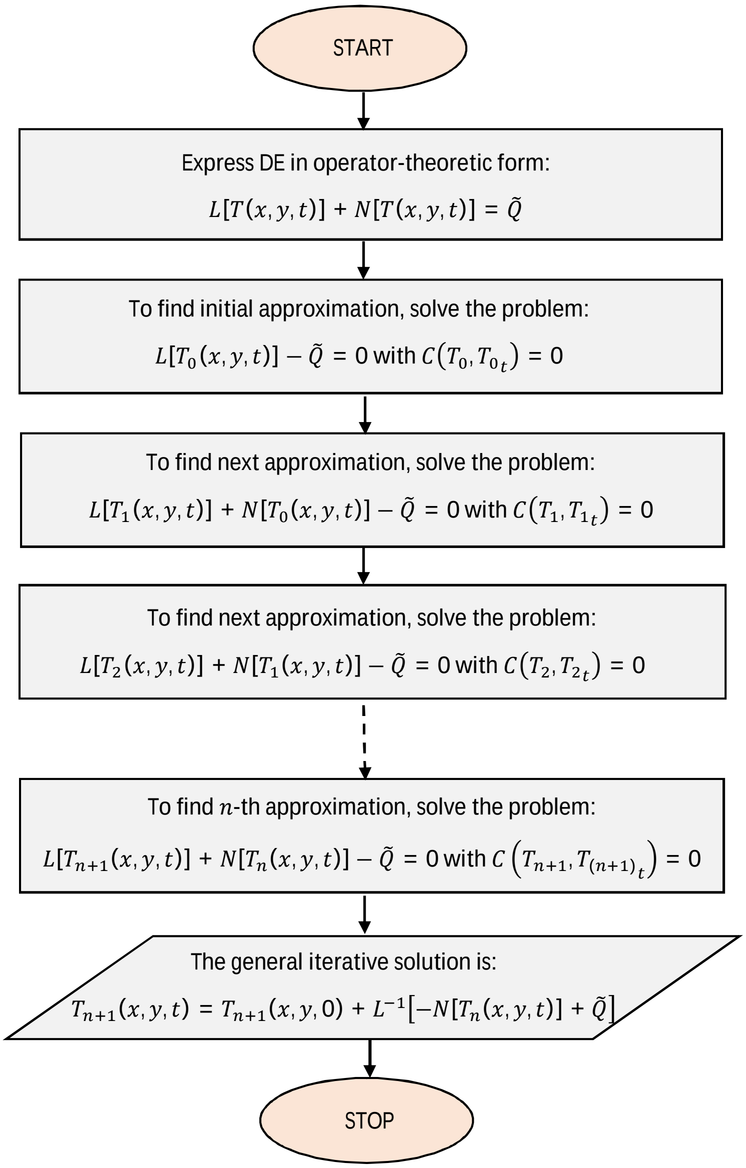


Figure 2. Flowchart of the SAIM algorithm

from which the general iterative formula for solving the 2D heat equation (2) is

$$T_{n+1}(x,y,t) = T_{n+1}(x,y,0) + L^{-1}\left[-N[T_n(x,y,t)] + \tilde{Q}\right], \tag{9}$$

where $L^{-1} = \int_0^t (\cdot) ds$. Each iteration of the function $T_n(x,y,t)$ effectively represents a complete solution for equation (5). This iterative procedure is very easy and has the advantage that any iterative solution is an improvement of the previous iterate, and as more and more iterations are obtained, the iterative solution converges to the exact solution of equation (2).

3. Convergence Analysis for the Proposed Method

To give the convergence analysis for the semi analytic iterative method, we begin with the following:

$$\begin{cases} \varsigma_0 = T_0(x,y,t), \\ \varsigma_1 = \Theta[\varsigma_0], \\ \varsigma_2 = \Theta[\varsigma_0 + \varsigma_1], \\ \vdots \\ \varsigma_{n+1} = \Theta[\varsigma_0 + \varsigma_1 + \cdots + \varsigma_n]. \end{cases}$$

Now we can define the operator $\Theta[T(x,y,t)]$ as

$$\Theta[\varsigma_n] = T_n(x,y,t) - \sum_{i=0}^{n-1} T_i(x,y,t),\ n = 1,2,3,\ldots, \tag{10}$$

where $T(x,y,t)$ represents the analytical solution of the 2D heat equation from the SAIM. The following theorems, all of whose proofs are presented in Odibat and Momani [15], provide sufficient conditions for the convergence of the SAIM.

Theorem 3.1

The series solution $T(x,y,t) = \sum_{n=0}^{\infty} T_n(x,y,t)$ will represent the exact solution to the nonlinear problem being solved if the series solution is convergent.

Theorem 3.2

Suppose that Θ, defined in equation (10), is an operator from $\mathcal{H}$ to $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Then the series solution $T(x,y,t) = \sum_{i=0}^{n} T_i(x,y,t)$ converges if $\exists\ 0 < \delta < 1$ such that $\|\Theta[\varsigma_0 + \varsigma_1 + \cdots + \varsigma_{n+1}]\| \leq \delta\Theta[\varsigma_0 + \varsigma_1 + \cdots + \varsigma_n]\ \forall\, \delta \in \mathbb{N} \cup \{0\}$.

This idea is a special case of the fixed point concept and is considered a sufficient condition for establishing the convergence of the SAIM.

Theorem 3.3

If the series solution $\sum_{n=0}^{\infty} T_n(x,y,t)$ converges to $T(x,y,t)$, then the maximum error $E_n(x,y,t)$ is given by

$$E_n(x,y,t) \leq \frac{1}{1-r} r^{n+1} \left\|T_0\right\|,$$

where the truncated series $\sum_{i=0}^{n} T_i(x,y,t)$ is used for tackling a broad range of nonlinear problems and r ensures contraction.

In summary, it can be stated that the solution obtained by the SAIM converges to the exact solution provided that $\exists\ 0 < \delta < 1$ such that

$$C_n = \begin{cases} \frac{\|\varsigma_{n+1}\|}{\|\varsigma_n\|} & \text{if } \|\varsigma_n\| \neq 0 \\ 0 & \text{if } \|\varsigma_n\| = 0 \end{cases}$$

When $0 \leq C_n < 1\ \forall\ n = 0,1,2,\ldots$, the power series solution $\sum_{n=0}^{\infty} T_n(x,y,t)$ converges to the exact solution $T(x,y,t)$ [3, 5].

4. Numerical Experiments

In this section we present some numerical examples illustrating the applicability of the SAIM for solving the 2D heat equation. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz with 6.0GB and 64-bit operating system (Windows 8). Unless otherwise stated, a fixed $t = 0.5$ was used throughout and the figures were constructed using MATLAB R2016a. The results are presented in tables and figures accompanying the discussion.

Example 4.1
Consider the homogeneous 2D heat equation [8]:

$$T_t = \Delta T, \; T(x, y, 0) = x^2 + y^2, \tag{11}$$

with exact solution $T(x, y, t) = 4t + x^2 + y^2$. We solve (11) using the SAIM by first rewriting it as

$$LT + NT = 0$$

where $LT = T_t$ and $NT = -\Delta T$. Here, the source term is zero. The primary problem is to find the initial approximation by solving the equation

$$L[T_0(x, y, t)] = 0, \text{ with } T(x, y, 0) = x^2 + y^2 \tag{12}$$

Using the initial condition, the solution of the primary problem is

$$T_0(x, y, t) = T(x, y, 0) = x^2 + y^2.$$

The general recursive relation for solving (11) is

$$L[T_{n+1}(x, y, t)] = -N[T_n(x, y, t)], \text{ with } T_{n+1}(x, y, 0) = x^2 + y^2, \tag{13}$$

i.e.,

$$T_{n+1}(x, y, t) = T_{n+1}(x, y, 0) + \int_0^t \left[T_{n_{xx}} + T_{n_{yy}}\right] ds. \tag{14}$$

From this recursive relation, we obtain the approximations

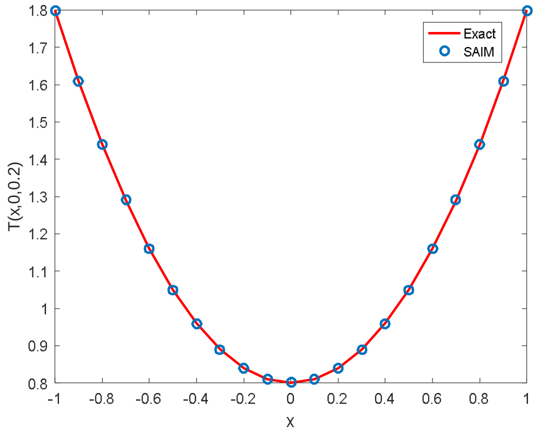
$$\begin{aligned}
T_0(x, y, t) &= x^2 + y^2 \\
T_1(x, y, t) &= x^2 + y^2 + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}}\right] ds = x^2 + y^2 + 4t \\
T_2(x, y, t) &= x^2 + y^2 + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}}\right] ds = x^2 + y^2 + 4t \\
&\vdots \\
T_{n+1}(x, y, t) &= x^2 + y^2 + 4t, \; n \geq 1.
\end{aligned}$$

This is the exact solution of the given 2D heat equation. It can be seen here that the SAIM converged very fast to the exact solution. The results are shown in Table 1. Figure 3 compares the exact and SAIM solutions $T(x, 0, 0.2)$, $T(x, 0, 0.5)$ and $T(x, 0, 0.8)$ while Figure 4 shows the surface plot of the SAIM solution for a fixed $t = 0.2$ and $-1 \leq x, y \leq 1$. The results for $t = 0.2$ and $t = 0.8$ were also obtained using the method of fundamental solutions (MFS) [8].

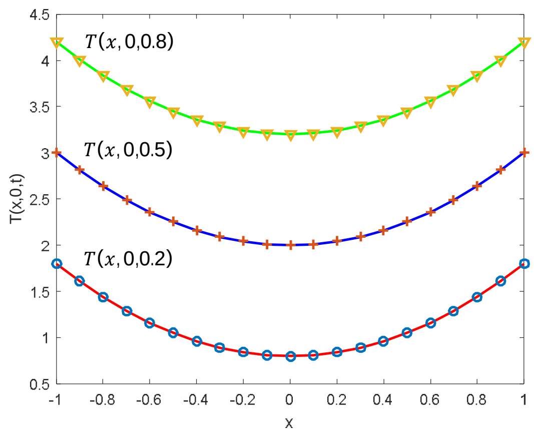
Example 4.2
Consider the following nonhomogeneous 2D heat equation with the given initial condition adapted from Ullah et

Table 1. Comparison of exact and approximate solutions from SAIM for Example 4.1 ($y = 0,\ t = 0.2$)

x	$T(x, y, t)$	$T_{\text{SAIM}}(x, y, t)$
0	0.8	0.8
0.2	0.84	0.84
0.4	0.96	0.96
0.6	1.16	1.16
0.8	1.44	1.44
1.0	1.8	1.8



(a)



(b)

Figure 3. Comparison of approximate and exact solutions for the 2D heat equation in Example 4.1 for $-1 \leq x \leq 1$, $y = 0$ with (a) $t = 0.2$, (b) $t = 0.2,\ 0.5,\ 0.8$

al. [24] with fractional order $\theta = 1$:

$$T_t = \Delta T + x + y + 1,\ T(x, y, 0) = \mathrm{e}^{-(x+y)}. \tag{15}$$

The exact solution of (15) is not known. Rewriting (15) as

$$LT + NT = \tilde{Q},$$

where $LT = u_t$, $NT = -\Delta T$ and $\tilde{Q} = x + y + 1$, the general recursive relation is given by

$$L[T_{n+1}(x, y, t)] = -N[T_n(x, y, t)] + \tilde{Q},\ \text{with}\ T_{n+1}(x, y, 0) = \mathrm{e}^{-(x+y)}. \tag{16}$$

We then use the iteration

$$T_{n+1}(x, y, t) = T_{n+1}(x, y, 0) + \int_0^t \left[T_{n_{xx}} + T_{n_{yy}} + x + y + 1\right] ds \tag{17}$$

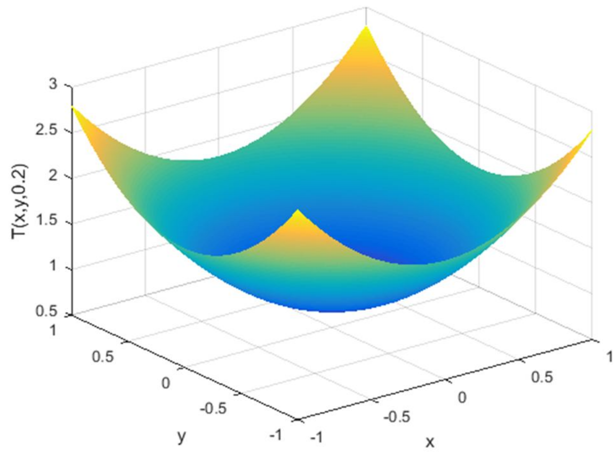


Figure 4. Surface plot for SAIM solution for $-1 \leq x, y \leq 1$ and $t = 0.2$

to obtain the following successive approximations:

$$\begin{aligned}
T_0(x,y,t) &= T(x,y,0) = \mathrm{e}^{-(x+y)} \\
T_1(x,y,t) &= \mathrm{e}^{-(x+y)} + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}} + x + y + 1\right] ds = (1+2t)\mathrm{e}^{-(x+y)} + (x+y+1)t \\
T_2(x,y,t) &= \mathrm{e}^{-(x+y)} + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}} + x + y + 1\right] ds = (1+2t+2t^2)\mathrm{e}^{-(x+y)} + (x+y+1)t \\
T_3(x,y,t) &= \mathrm{e}^{-(x+y)} + \int_0^t \left[T_{2_{xx}} + T_{2_{yy}} + x + y + 1\right] ds = \left(1+2t+2t^2+\frac{4}{3}t^3\right)\mathrm{e}^{-(x+y)} + (x+y+1)t
\end{aligned}$$

and so on. The rest of the approximations may be computed in the same manner. As $n \to \infty$ this will tend to the exact solution

$$T(x,y,t) = \mathrm{e}^{2t-x-y} + (x+y+1)t$$

These results compare favourably with those obtained by Ullah et al. [24] using the iterative Laplace transform (ILT) method. Table 2 shows the results for fixed $t = 0.5$ and $0 \leq x, y \leq 10$ and Figure 5 shows the results for $0 \leq x, y \leq 10$. Absolute errors are also shown in Figure 8(a) and show the relative superiority of the ILT over the SAIM for small values of x.

Example 4.3
Consider the nonhomogeneous 2D heat equation adapted from Ullah et al. [24] with fractional order $\theta = 1$:

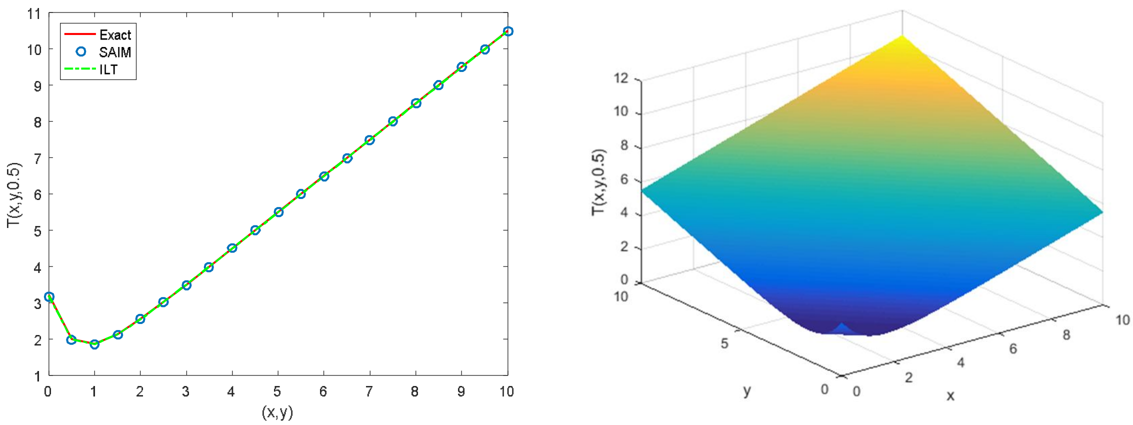
$$T_t = \Delta T + x + y + t^2,\ T(x,y,0) = \sin(x+y). \tag{18}$$

with unknown exact solution. Here, $LT = T_t$, $NT = -\Delta T$ and $\tilde{Q} = x + y + t^2$. Since the primary problem $LT_0 = 0$, with $T_0(x,y,0) = \sin(x+y)$, has a solution $T_0(x,y,t) = \sin(x+y)$, equation (18) can be solved using the general iterative scheme

$$T_{n+1}(x,y,t) = T_{n+1}(x,y,0) + \int_0^t \left[T_{n_{xx}} + T_{n_{yy}} + x + y + s^2\right] ds. \tag{19}$$

Table 2. Comparison of approximate solutions from SAIM and ILT for Example 4.2 ($t = 0.5$)

(x, y)	$T(x, y, t)$	$T_{\text{SAIM}}(x, y, t)$	$T_{\text{ILT}}(x, y, t)$	e_{SAIM}	e_{ILT}
(0,0)	3.218281828	3.166666667	3.208333333	$5.16152E-02$	$9.94850E-03$
(1,1)	1.867879441	1.860894089	1.866533059	$6.98535E-03$	$1.34638E-03$
(2,2)	2.549787068	2.548841704	2.549604855	$9.45365E-04$	$1.82213E-04$
(3,3)	3.506737947	3.506610006	3.506713287	$1.27941E-04$	$2.46599E-05$
(4,4)	4.500911882	4.500894567	4.500908545	$1.73150E-05$	$3.33735E-06$
(5,5)	5.500123410	5.500121066	5.500122958	$2.34332E-06$	$4.51661E-07$
(6,6)	6.500016702	6.500016385	6.500016641	$3.17135E-07$	$6.11257E-08$
(7,7)	7.500002260	7.500002217	7.500002252	$4.29195E-08$	$8.27246E-09$
(8,8)	8.500000306	8.500000300	8.500000305	$5.80852E-09$	$1.11956E-09$
(9,9)	9.500000041	9.500000041	9.500000041	$7.86097E-10$	$1.51516E-10$
(10,10)	10.50000001	10.50000001	10.50000001	$1.06386E-10$	$2.05063E-11$



(a) (b)

Figure 5. (a) Comparison of approximate solutions from SAIM and ILT for the 2D heat equation in Example 4.2 for $0 \le x, y \le 10$ for fixed $t = 0.5$; (b) Surface plot for SAIM solution for $0 \le x, y \le 10$

Thus, the first four approximations are

$$
\begin{aligned}
T_0(x,y,t) &= \sin(x+y) \\
T_1(x,y,t) &= \sin(x+y) + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}} + x + y + s^2\right] ds \\
&= (1-2t)\sin(x+y) + (x+y)t + \frac{1}{3}t^3 \\
T_2(x,y,t) &= \sin(x+y) + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}} + x + y + s^2\right] ds \\
&= (1-2t+2t^2)\sin(x+y) + (x+y)t + \frac{1}{3}t^3 \\
T_3(x,y,t) &= \sin(x+y) + \int_0^t \left[T_{2_{xx}} + T_{2_{yy}} + x + y + s^2\right] ds \\
&= \left(1-2t+2t^2-\frac{4}{3}t^3\right)\sin(x+y) + (x+y)t + \frac{1}{3}t^3
\end{aligned}
$$

and so on. As $n \to \infty$, this converges to the exact solution

$$T(x, y, t) = \mathrm{e}^{-2t} \sin(x + y) + (x + y)t + \frac{1}{3}t^3.$$

Table 3 and Figures 6 and 7 compare the results from the SAIM with the solution obtained using the iterative Laplace transform method [24]. Absolute errors are also shown in Figure 8(b) and indicate that the SAIM is far more accurate than the ILT.

Table 3. Comparison of approximate solutions from SAIM and ILT for Example 4.3 ($t = 0.5$)

(x, y)	$T(x, y, t)$	$T_{\text{SAIM}}(x, y, t)$	$T_{\text{ILT}}(x, y, t)$	e_{SAIM}	e_{ILT}
(0,0)	0.041666667	0.041666667	0.750000000	0	0.708333333
(1,1)	1.376178496	1.344765809	2.053099142	0.031412687	0.676920646
(2,2)	1.763254588	1.789399168	2.497732502	0.026144581	0.734477914
(3,3)	2.938875449	2.948528167	3.656861501	0.009652718	0.717986051
(4,4)	4.405631226	4.371452749	5.079786082	0.034178477	0.674154857
(5,5)	4.841532484	4.860326296	5.568659630	0.018793812	0.727127145
(6,6)	5.844272521	5.862809027	6.571142361	0.018536506	0.726869839
(7,7)	7.406090747	7.371869119	8.080202452	0.034221629	0.674111705
(8,8)	7.935752955	7.945698894	8.654032228	0.009945939	0.718279272
(9,9)	8.765393898	8.791337584	9.499670918	0.025943686	0.73427702
(10,10)	10.37752046	10.34598175	11.05431508	0.031538705	0.676794628

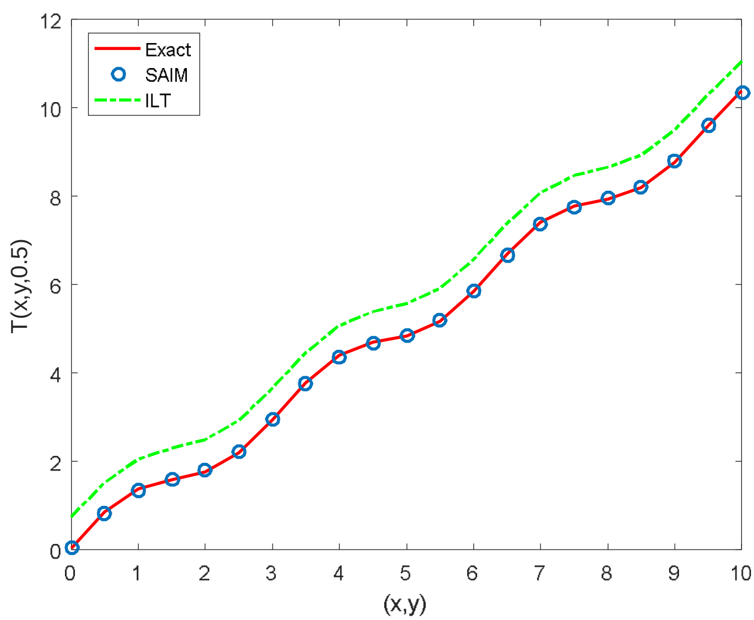
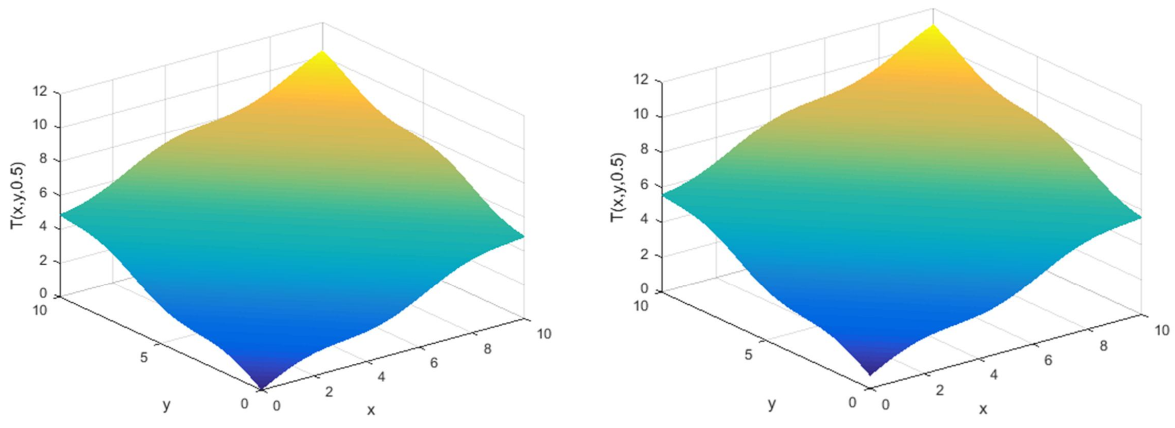
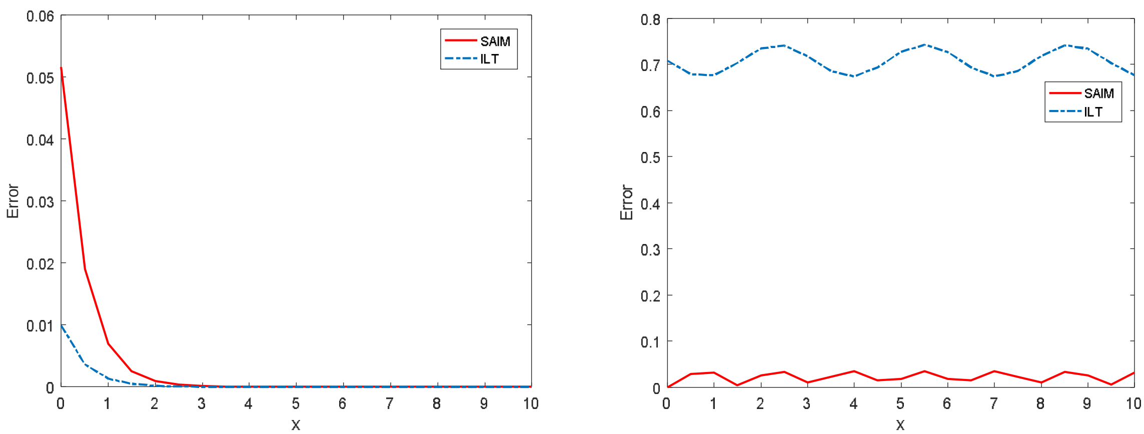


Figure 6. Comparison of approximate solutions from SAIM and ILT for the 2D heat equation in Example 4.3 for $0 \leq x, y \leq 10$ and $t = 0.5$



(a) (b)

Figure 7. Surface plots for $0 \leq x, y \leq 10$ and $t = 0.5$ for (a) SAIM and (b) ILT



(a) (b)

Figure 8. Comparison of absolute errors for SAIM and ILT for (a) Example 4.2 and (b) Example 4.3

Example 4.4
Consider the following homogeneous 2D heat equation [6]:

$$T_t = \Delta T,\ T(x, y, 0) = x(\pi - x)y(\pi - y) \tag{20}$$

The exact solution for this equation is unknown. However, a near-exact solution can be obtained using the SAIM. Equation (20) can be rewritten as:

$$LT + NT = 0$$

with $LT = T_t$ and $NT = -\Delta T$. The initial problem yields the solution $T_0(x, y, t) = x(\pi - x)y(\pi - y)$, so that the first few iterations give the approximations

$$\begin{aligned}
T_0(x,y,t) &= x(\pi-x)y(\pi-y) \\
T_1(x,y,t) &= x(\pi-x)y(\pi-y) + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}}\right] ds \\
&= x(\pi-x)y(\pi-y) - 2t\left[y(\pi-y) + x(\pi-x)\right] \\
T_2(x,y,t) &= x(\pi-x)y(\pi-y) + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}}\right] ds \\
&= x(\pi-x)y(\pi-y) - 2t\left[y(\pi-y) + x(\pi-x)\right] - 4t^2 \\
T_3(x,y,t) &= x(\pi-x)y(\pi-y) + \int_0^t \left[T_{2_{xx}} + T_{2_{yy}}\right] ds \\
&= x(\pi-x)y(\pi-y) - 2t\left[y(\pi-y) + x(\pi-x)\right] - 4t^2 \\
&\vdots \\
T_{n+1}(x,y,t) &= x(\pi-x)y(\pi-y) + \int_0^t \left[T_{n_{xx}} + T_{n_{yy}}\right] ds \\
&= x(\pi-x)y(\pi-y) - 2t\left[y(\pi-y) + x(\pi-x)\right] - 4t^2,\ n \geq 2
\end{aligned}$$

Since the SAIM has converged fast to this solution, it must be the exact solution. The results are shown in Table 4 and Figure 9.

Table 4. Comparison of exact and approximate solutions from SAIM for Example 4.4 ($t = 0.5$)

(x, y)	$T(x, y, t)$	$T_{\text{SAIM}}(x, y, t)$
(0,0)	-1.000000000	-1.000000000
(1,1)	-0.696766213	-0.696766213
(2,2)	-0.353435467	-0.353435467
(3,3)	-1.669119606	-1.669119606
(4,4)	17.65706953	17.65706953
(5,5)	103.9260201	103.9260201
(6,6)	327.4386202	327.4386202
(7,7)	782.4957581	782.4957581
(8,8)	1587.398322	1587.398322
(9,9)	2884.447200	2884.447200
(10,10)	4839.943280	4839.943280

Example 4.5
Consider the following homogeneous 2D heat equation [8]:

$$T_t = \Delta T,\ T(x, y, 0) = \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] \tag{21}$$

with exact solution $T(x, y, t) = \sqrt{2}e^{-\frac{\pi^2}{4}t}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right]$. Equation (21) can be rewritten as:

$$LT + NT = 0$$

with $LT = T_t$ and $NT = -\Delta T$. The initial problem yields the solution

$$T_0(x, y, t) = \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right].$$

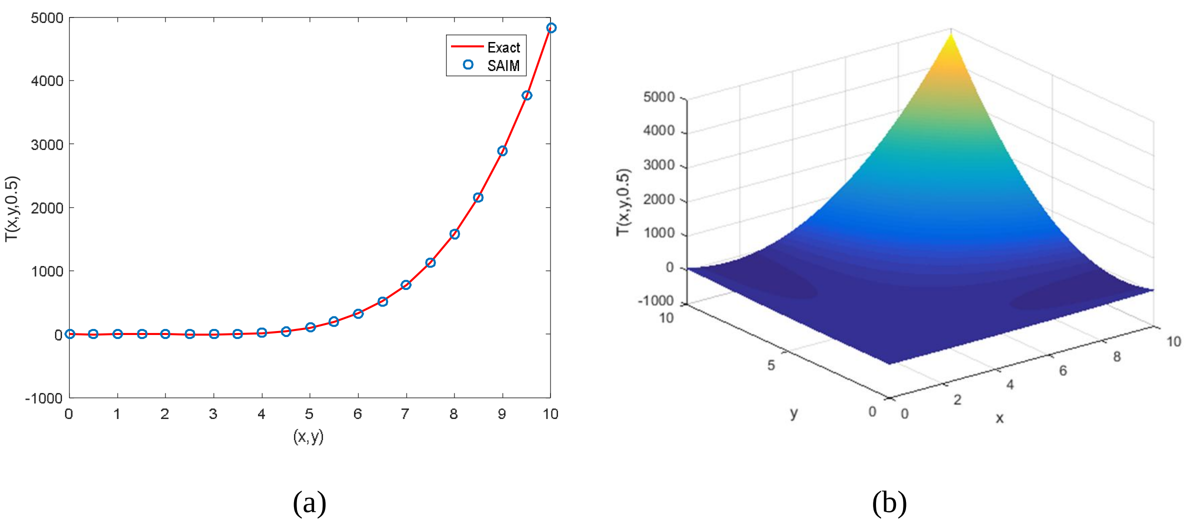


Figure 9. (a) Comparison of approximate and exact solutions for the 2D heat equation in Example 4.4 for $t = 0.5$; (b) Space-time surface plot for $0 \le x, y \le 10$ and $t = 0.5$ for approximate solution from SAIM

Thus, from the first few iterations we have the approximations

$$\begin{aligned}
T_0(x,y,t) &= T(x,y,0) = \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] \\
T_1(x,y,t) &= \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}}\right] ds \\
&= \sqrt{2}\left(1 - \frac{\pi^2}{4}t\right)\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] \\
T_2(x,y,t) &= \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}}\right] ds \\
&= \sqrt{2}\left(1 - \frac{\pi^2}{4}t + \frac{\pi^4}{32}t^2\right)\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] \\
T_3(x,y,t) &= \sqrt{2}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right] + \int_0^t \left[T_{2_{xx}} + T_{2_{yy}}\right] ds \\
&= \sqrt{2}\left(1 - \frac{\pi^2}{4}t + \frac{\pi^4}{32}t^2 - \frac{\pi^6}{384}t^3\right)\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right]
\end{aligned}$$

and so on. As $n \to \infty$, this converges to the exact solution

$$T(x,y,t) = \sqrt{2}\mathrm{e}^{-\frac{\pi^2}{4}t}\left[\cos\left(\frac{\pi x}{2} - \frac{\pi}{4}\right) + \cos\left(\frac{\pi y}{2} - \frac{\pi}{4}\right)\right]$$

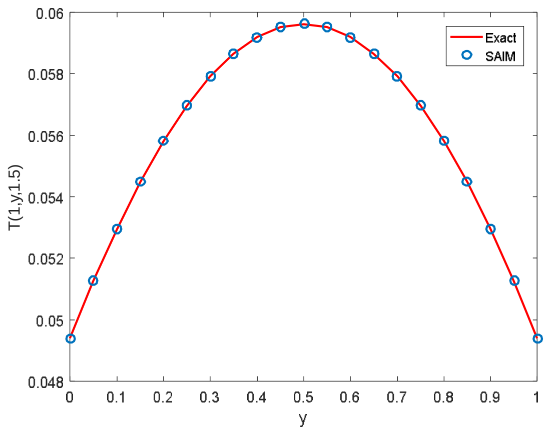
This same solution was also obtained by Johansson et al. [8] using the method of fundamental solutions (MFS). The results are obtained in Table 5 and Figure 10

Example 4.6
Consider the following homogeneous 2D heat equation [8]:

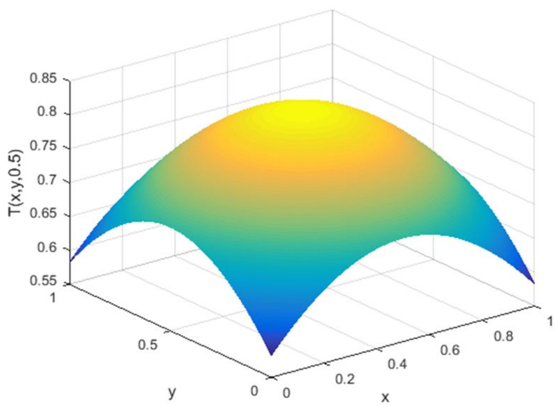
$$T_t = \Delta T, \; T(x,y,0) = \sin(\pi x)\sin(\pi y) \tag{22}$$

Table 5. Comparison of exact and approximate solutions from SAIM for Example 4.5 ($x = 1,\ t = 1.5$)

y	$T(x, y, t)$	$T_{\mathrm{SAIM}}(x, y, t)$
0	0.049392610	0.049392610
0.2	0.062238319	0.062238319
0.4	0.068991708	0.068991708
0.6	0.068991708	0.068991708
0.8	0.062238319	0.062238319
1.0	0.049392610	0.049392610



(a)



(b)

Figure 10. (a) Comparison of approximate and exact solutions for the 2D heat equation in Example 4.5 for $x = 1$ and $t = 1.5$; (b) Space-time surface plots for $0 \leq x, y \leq 1$ and $t = 0.5$ for SAIM solution

with exact solution $T(x, y, t) = \sin(\pi x)\sin(\pi y)\mathrm{e}^{-2\pi^2 t}$. This problem is solved in the domain $[0, 1] \times [0, 1]$ for $t = 0.2$. Equation (22) can be rewritten in operator-theoretic form as:

$$LT + NT = 0$$

with $LT = T_t$ and $NT = -\Delta T$. The solution to the initial problem is

$$T_0(x, y, t) = \sin(\pi x)\sin(\pi y)$$

Thus, from the first few iterations we have the approximations

$$\begin{aligned}
T_0(x,y,t) &= T(x,y,0) = \sin(\pi x)\sin(\pi y) \\
T_1(x,y,t) &= \sin(\pi x)\sin(\pi y) + \int_0^t \left[T_{0_{xx}} + T_{0_{yy}}\right] ds \\
&= (1-2\pi^2 t)\sin(\pi x)\sin(\pi y) \\
T_2(x,y,t) &= \sin(\pi x)\sin(\pi y) + \int_0^t \left[T_{1_{xx}} + T_{1_{yy}}\right] ds \\
&= (1-2\pi^2 t + 2\pi^4 t^2)\sin(\pi x)\sin(\pi y) \\
T_3(x,y,t) &= \sin(\pi x)\sin(\pi y) + \int_0^t \left[T_{2_{xx}} + T_{2_{yy}}\right] ds \\
&= (1-2\pi^2 t + 2\pi^4 t^2 - \frac{4}{3}\pi^6 t^3)\sin(\pi x)\sin(\pi y) \\
T_4(x,y,t) &= \sin(\pi x)\sin(\pi y) + \int_0^t \left[T_{3_{xx}} + T_{3_{yy}}\right] ds \\
&= (1-2\pi^2 t + 2\pi^4 t^2 - \frac{4}{3}\pi^6 t^3 + \frac{2}{3}\pi^8 t^4)\sin(\pi x)\sin(\pi y)
\end{aligned}$$

and so on. As $n \to \infty$, this converges to the exact solution

$$T(x,y,t) = \sin(\pi x)\sin(\pi y)\mathrm{e}^{-2\pi^2 t}$$

The results are shown in Figure 11. The approximate solution using radial basis functions (RBFs) is compared with the SAIM. Because the SAIM approximates to the exact solution even in this example, it is more accurate and efficient.

5. Conclusion

In this work we have used the semi analytic iterative method to solve the 2D heat equation and found that it produces exact to near-exact solutions to these kinds of PDEs. Interestingly, even where the exact solution was unknown, we were able to obtain the exact solution through the quick convergence of the SAIM. The paper has therefore confirmed the accuracy and efficiency of this method and its suitability for solving such linear PDEs. However, when it comes to solving differential equations with random function excitation, only few iterations of the SAIM can be calculated due to the difficulty of integrating random functions. In addition, the SAIM faces challenges when dealing with higher-order integro-differential equations involving complex nonlinearities as it often leads to nested integrals that are difficult to evaluate analytically [14]. Future work could therefore focus on the application of some modifications of the SAIM to address these limitations, such as the Discrete Temimi-Ansari Method (DTAM) which combines the classical SAIM with the finite difference numerical scheme [19]. The SAIM can also be modified to provide an alternative approach for handling nonlinearity. One such modification replaces the nonlinear portion of a differential equation with equivalent Taylor's series or Chebyshev polynomial approximations [1] which is advantageous in that it eliminates the need for restrictive assumptions when dealing with the nonlinear portion of a differential equation. Another modification of the SAIM in the literature involves the use of Boole's Rule to improve the performance of the proposed method [11]. The SAIM has also been enhanced by blending the classical SAIM with the Aboodh Transform (AT) method, resulting in the so-called Aboodh Temimi Ansari Transform Method, which effectively decreases the computing workload when implementing numerical solutions of systems of fractional PDEs [21]. Overall, for linear equations like the 2D heat equation, and even for some nonlinear PDEs, the SAIM is easy to implement, straightforward, direct and gives a better approximate solution which converges to the exact solution with only few iterations [4].

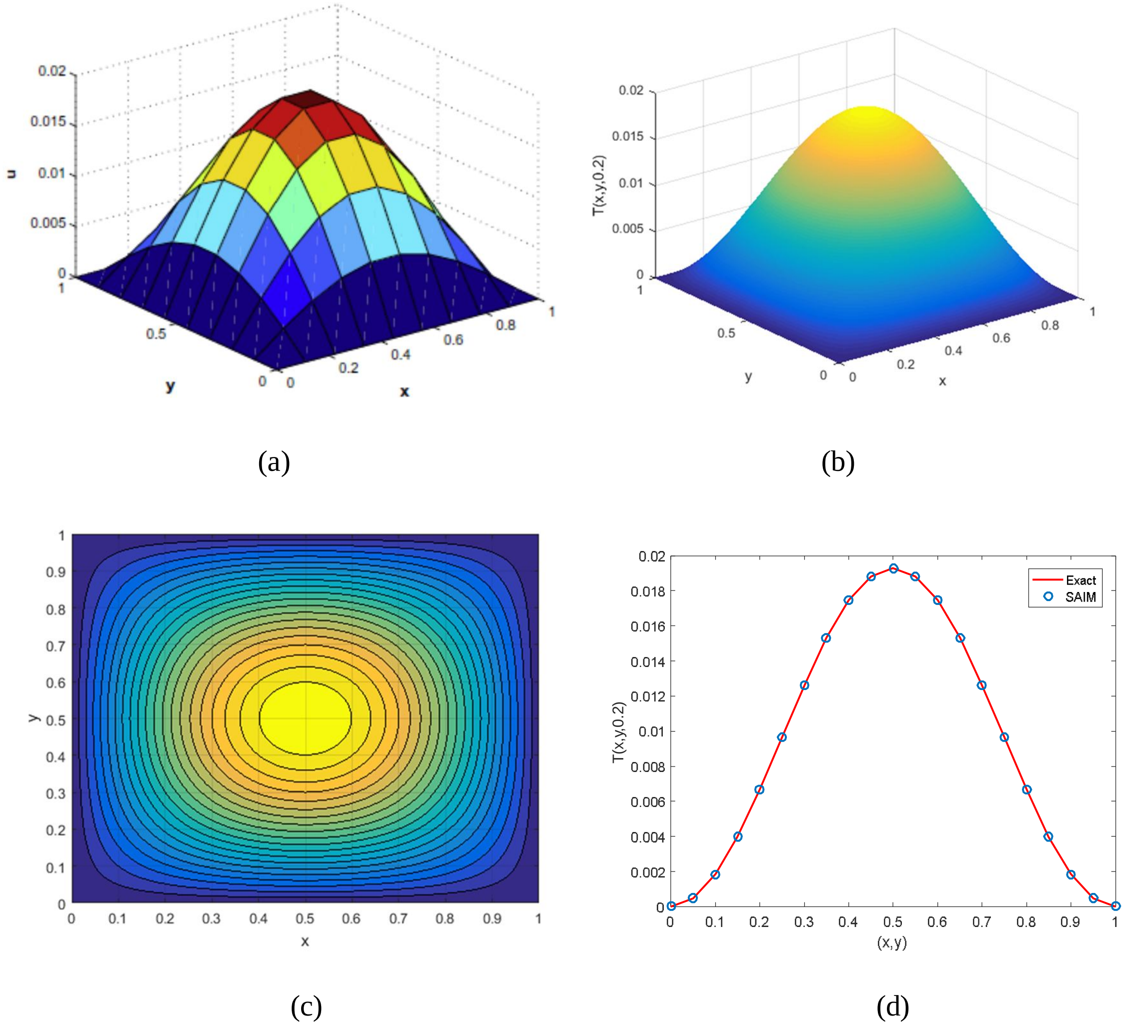


Figure 11. For the 2D heat equation in Example 4.6, (a) Surface plot for RBF approximate solution $((x, y) \in [0, 1] \times [0, 1],\ t = 0.2)$; (b) Space-time surface plot for exact solution; (c) Contour diagram for exact solution (d) Comparison of exact and SAIM solutions $((x, y) \in [0, 1] \times [0, 1],\ t = 0.2)$

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