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Jackson's Theorem for the Kontorovich-Lebedev-Clifford Transform

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Abstract In this paper, by using the Kontorovich-Lebedev-Clifford translation operators studied recently by A. Prasad and U.K. Mandal (The Kontorovich-Lebedev-Clifford transform, Filomat 35:14 (2021), 4811–4824.), we prove Jackson's theorem associated with the Kontorovich-Lebedev-Clifford transform.

Keywords Macdonald functions; Kontorovich-Lebedev-Clifford transform; Translation operators; Convolution; Jackson's theorem.

AMS 2010 subject classifications 26A15, 33E20, 42A38, 42B35, 44A20.

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1. Introduction

In mathematical analysis, the theory of approximation of functions plays an important role to approximate functions by others to be useful and easier to study. The translation operator is one of the key tools used to construct the modulus of continuity and smoothness. Both are the fundamental elements of direct and inverse theorems in approximation theory. The modulus of smoothness may appear for the study of the connections between the smoothness properties and the best approximations of a function in weight functional spaces, we refer the reader to see ($[1, 2, 4-7, 17-19, 27]$) and references therein for some results about approximation of a function by using the modulus of continuity and smoothness.

Let $\mathcal{W}_2^m$ be the Sobolev space constructed by the operator Δ,

$$\mathcal{W}_2^m = \left\{ f \in L^2(\mathbb{R}^n) : \Delta^j f \in L^2(\mathbb{R}^n);\ j = 1, 2, \ldots, m \right\},$$

where $\Delta^j f = \Delta\left(\Delta^{j-1} f\right), \quad j = 1 \ldots m, D$ is the differential operator given by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

with $\Delta^0 f = f$.

A type of direct and inverse theorem is the well-known Jackson inequality. It was firstly studied by D. Jackson in [14] and generalized by A. Zygmund [27], and next it was extended to the modulus of continuity.

If f is a continuous function on $[-1, 1]$ and ω_f its modulus of continuity, then the classical theorem of Jackson [15] states that there exists a sequence of polynomials (Q_n) such that the degree of Q_n is $\leq n$ and

$$\max_{|x|\leq 1} |Q_n(x) - f(x)| \leq C\omega_f(1/n),\ n = 1, 2, 3, \ldots$$

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Various direct, but more or less involved proofs of this result are now available in the literature (see [3,5,21]).
Let

$$\omega_2(f,\delta)=\sup_{|t|<\delta}\{\|f(x+t)-f(x)\|_2\}$$

be the modulus of continuity of a function $f\in L^2(\mathbb{R}^n)$. The Jackson inequality is defined in the space $L^2(\mathbb{R}^n)$ by

$$E^2_\beta(f)\le c\beta^{-2m}\omega_2\left(\Delta^m f,1/\beta\right),$$

where $E^2_\beta(f)=\inf_{g\in\mathcal{I}_\beta}\|f-g\|_2$, is the best approximation of a function f in the set of all functions $f\in L^2(\mathbb{R}^n)$ with bounded spectrum of order $\beta>0$ with $\widehat{f}(\lambda)=0$, for $|\lambda|>\beta$.

In this paper, we introduce the modulus of smoothness then we study Jackson's theorem for the Kontorovich-Lebedev-Clifford transform defined as in [20]. Similar results dealing with this type of inequality have been established by many researchers, see [1,11-13,26] for Dunkl transform, [6] for the spherical mean operator, [7] for the Jacobi transform, [9] for the Heisenberg group.

We now briefly summarize the contents of the remaining sections of the paper. In §2 we recall some background material related to the Kontorovich-Lebedev-Clifford transform. We introduce the modulus of smoothness for the Kontorovich-Lebedev-Clifford transform. Moreover, we define Sobolev spaces for this transform. In the last section, we study Jackson's theorem which gives the error of the best approximation based on the modulus of continuity for the Kontorovich-Lebedev-Clifford transform.

2. Preliminaries

2.1. The Kontorovich-Lebedev-Clifford transform

Let $L^p_\mu(\mathbb{R}_+),1\le p\le\infty$, denote the space of measurable functions f on $\mathbb{R}_+:=(0,\infty)$ such that $\|f\|_{p,\mu}<\infty$, with

$$\|f\|_{p,\mu}=\begin{cases}\left(\int_0^\infty|f(x)|^p d\mu(x)\right)^{1/p}, & 1\le p<\infty,\\ \operatorname*{ess\,sup}_{x\in\mathbb{R}_+}|f(x)|, & p=\infty,\end{cases}$$

where $d\mu(x)$ is the measure defined on $(0,\infty)$ by

$$d\mu(x)=\frac{dx}{2x}. \tag{1}$$

Similary we define the spaces $L^p_\nu(\mathbb{R}_+),1\le p\le\infty$, with the measure $d\nu(\tau)$ defined on $(0,\infty)$ by

$$d\nu(\tau)=\frac{4}{\pi^2}\sinh(2\pi\sqrt{\tau})d\tau. \tag{2}$$

The Kontorovich-Lebedev-Clifford (KLC) transform is an integral transform that extends the classical Kontorovich-Lebedev (KL) transform by incorporating Clifford algebra structures. The KLC transfrom of a function φ is defined on $(0,\infty)$, provided the integral exists, by (see [20])

$$\mathbb{K}\varphi(\tau)=\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\varphi(x)d\mu(x),\quad \tau\in\mathbb{R}_+, \tag{3}$$

where $K_{iy}(x),\ y\in\mathbb{R}_+$, is the Macdonald function (modified Bessel functions of the second kind), which naturally appear in problems with cylindrical symmetry and defined as [8]

$$K_{iy}(x)=\int_0^\infty e^{-x\cosh t}\cos(yt)dt,\quad x\in\mathbb{R}_+. \tag{4}$$

From [8], we have

$$|K_{iy}(x)| \le \int_0^\infty e^{-x\cosh t}dt = K_0(x). \tag{5}$$

The asymptotic behavior of the Macdonald function $K_{2i\sqrt{\tau}}(2\sqrt{x})$ with respect to x [20]

$$\begin{aligned}
&K_{2i\sqrt{\tau}}(2\sqrt{x}) = \frac{\sqrt{\pi}}{2x^{\frac{1}{4}}}e^{-2\sqrt{x}}\left[1+O\left(\frac{1}{\sqrt{x}}\right)\right],\ x\to\infty\\
&K_{2i\sqrt{\tau}}(2\sqrt{x}) = O(1),\ x\to 0\\
&K_0(2\sqrt{x}) = O(\ln x),\ x\to 0.
\end{aligned}$$

The KLC transform is a powerful mathematical tool that extends classical Fourier analysis to settings where traditional methods are inadequate. Its importance lies in its applications across approximation theory, differential equations, inverse problems, and mathematical physics.

Moreover, the differential operator

$$L = x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - x,$$

which resemble the Bessel differential operator and modifies the eigenvalue structure, is closely linked to Bessel functions and appears in problems with cylindrical or hyperbolic symmetry. It is best studied in weighted Sobolev spaces and acts as a spectral generator in the KLC setting.

$K_{2i\sqrt{\tau}}(2\sqrt{x})$ is an eigenfunction of the operator L:

$$LK_{2i\sqrt{\tau}}(2\sqrt{x}) = -\tau K_{2i\sqrt{\tau}}(2\sqrt{x}). \tag{6}$$

It also satisfies
$\left|K_{2i\sqrt{\tau}}(2\sqrt{x})\right| \le C'(b)x^{\frac{-b}{4}}[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}},\ 0<b<1/2.$

In the following, we assemble some fundamental properties of the Kontorovich-Lebedev-Clifford transform. The proof can be found in [20].

$$\mathbb{K}\left(L\varphi\right)(\tau) = -\tau\mathbb{K}\varphi(\tau). \tag{7}$$

The KLC transform has a more involved inversion formula using integrals of Bessel functions.

$$\varphi(x) = \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\mathbb{K}\varphi(\tau)d\nu(\tau), \quad x\in\mathbb{R}_+. \tag{8}$$

The Parseval equality is adapted to the weighted structure imposed by the Macdonald functions.

$$\int_0^\infty |\varphi(x)|^2 d\mu(x) = \int_0^\infty |\mathbb{K}\varphi(\tau)|^2 d\nu(\tau). \tag{9}$$

2.2. Translation operators and convolution structure in the KLC setting

Next, we define the translation and convolution structure for the KLC-transform. From [8], we have

$$\frac{2}{\pi^2}\int_0^\infty K_{i\eta}(p)K_{i\eta}(q)K_{i\eta}(r)\eta\sinh(\pi\eta)d\eta = T(p,q,r), \tag{10}$$

where $T(p,q,r)$ is symmetric in p,q,r and defined as:

$$T(p,q,r) = \frac{1}{2}\exp\left[\frac{-\left(p^2q^2+q^2r^2+r^2p^2\right)}{2pqr}\right], \quad p,q,r\in\mathbb{R}_+. \tag{11}$$

Now, putting $p = 2\sqrt{x}, q = 2\sqrt{y}, r = 2\sqrt{z}$ and $\eta = 2\sqrt{\tau}$ in (10) and (11), we get

$$\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\tau}}(2\sqrt{y})K_{2i\sqrt{\tau}}(2\sqrt{z})d\nu(\tau) = D(x,y,z), \tag{12}$$

where

$$D(x,y,z) = \frac{1}{2}\exp\left[\frac{-(xy+yz+zx)}{\sqrt{xyz}}\right], \quad x,y,z \in \mathbb{R}_+. \tag{13}$$

Clearly from (13), $D(x,y,z)$ is symmetric in x,y,z. Since

$$\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{z}} + \sqrt{\frac{zx}{y}} \geq \sqrt{x}\left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\right) \geq 2\sqrt{x}.$$

Thus

$$\exp\left(-\left(\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{z}} + \sqrt{\frac{zx}{y}}\right)\right) \leq \exp(-2\sqrt{x}). \tag{14}$$

From (13) and (14)

$$|D(x,y,z)| \leq 1/2\exp(-2\sqrt{x}). \tag{15}$$

By using (8), (9) and (12), the product of Macdonald functions can be written as:

$$K_{2i}\sqrt{\tau}(2\sqrt{x})K_{2i}\sqrt{\tau}(2\sqrt{y}) = \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{z})D(x,y,z)d\mu(z) = \mathbb{K}(D(x,y,.))(\tau). \tag{16}$$

The translation operator $\mathfrak{T}_x$ of a function φ is defined by

$$\mathfrak{T}_x(\varphi)(y) = \int_0^\infty D(x,y,z)\varphi(z)d\mu(z). \tag{17}$$

This operator does not just shift the function, it reconstructs it in an integral way using Macdonald functions. It is like watching a ripple expand outward, adapting its shape dynamically.

The corresponding convolution operator is defined as:

$$(\varphi\#\psi)(x) = \int_0^\infty \mathfrak{T}_x(\varphi)(y)\psi(y)d\mu(y) \tag{18}$$

$$= \int_0^\infty\int_0^\infty D(x,y,z)\varphi(z)\psi(y)d\mu(z)d\mu(y). \tag{19}$$

Note that

$$\int_0^\infty D(x,y,z)d\mu(z) = K_0\left(2\sqrt{x+y}\right)$$

The Kontorovich-Lebedev-Clifford of translation and convolution operators are as follows:

$$\mathbb{K}\left(\mathfrak{T}_x(\varphi)\right)(\tau) = K_{2i\sqrt{\tau}}(2\sqrt{x})\mathbb{K}\varphi(\tau) \tag{20}$$

and

$$\mathbb{K}(f\#g)(\tau) = \mathbb{K}f(\tau)\mathbb{K}g(\tau).$$

3. Jackson's theorem for the Kontorovich-Lebedev-Clifford transform

In this section, we prove Jackson's theorem, which gives the error of the best approximation based on the modulus of continuity for the Kontorovich-Lebedev-Clifford transform. It plays a fundamental role in approximation theory by providing precise bounds on the best possible approximation of a function in terms of its smoothness and regularity. We denote by

- $\Delta_h(f) = \mathfrak{T}_h(f) - f,\ f \in L^2_\mu(\mathbb{R}_+)$.
- $\mathcal{I}_a,\ a > 0$, the set of all functions $\varphi \in L^2_\mu(\mathbb{R}_+)$ with bounded spectrum of order a with $\mathbb{K}\varphi(\tau) = 0$, for $|\tau| > a$.
- $$P_a(f) = \mathbb{K}^{-1}\left(\mathbb{K}f.\, \mathbf{1}_{[0,a]}\right),\ f \in L^2_\mu(\mathbb{R}_+),\ a > 0, \tag{21}$$
  where $\mathbf{1}_{[0,a]}$ is the caracteristic function of the interval $[0, a]$.
- $\mathcal{W}^m_2 = \{f \in L^2_\mu(\mathbb{R}_+)\,; L^j f \in L^2_\mu(\mathbb{R}_+),\ 1 \le j \le m\},\ m = 1, 2, ...$

Definition 3.1: We define for $f \in L^2_\mu(\mathbb{R}_+)$

1. The modulus of continuity of f: $\omega_2(f, \delta) = \sup\limits_{0<h\le\delta} \|\Delta_h(f)\|_{2,\mu},\ \delta > 0$.
2. The best approximation of f: $E^2_a(f) = \inf\limits_{g\in\mathcal{I}_a} \|f - g\|_{2,\mu},\ a > 0$.

Now, we state and prove Jackson's theorem for the Kontorovich-Lebedev-Clifford transform.

Theorem 3.2: Let $m \in \mathbb{N}$ and $a > 0$. For all $f \in \mathcal{W}^m_2$, there exits $c > 0$ such that

$$E^2_\alpha(f) \le c\, a^{-m}\omega_2\left(L^m f, 1/a\right). \tag{22}$$

Proof: According to the Parseval equality (9), and the relation (21), we obtain

$$\begin{aligned}\|f - P_a(f)\|^2_{2,\mu} &= \left\|(1 - \mathbf{1}_{[0,a]}(\tau))\mathbb{K}f(\tau)\right\|^2_{2,\nu}\\ &= \int_{\tau>a} |\mathbb{K}f(\tau)|^2\, d\nu(\tau)\end{aligned}$$

By using the fact that $\left|1 - K_{2i\sqrt{\tau}}(2\sqrt{h})\right| \ge c_1$ for $0 < h \le 1/a$, we get

$$\begin{aligned}&\|f - P_a(f)\|^2_{2,\mu}\\ &\le c_1^{-2}\int_{\tau>a}\left|1 - K_{2i\sqrt{\tau}}(2\sqrt{h})\right|^2 |\mathbb{K}f(\tau)|^2\, d\nu(\tau)\\ &= c_1^{-2}\int_{\tau>a}(-\tau)^{-2m}\left|1 - K_{2i\sqrt{\tau}}(2\sqrt{h})\right|^2 |\mathbb{K}(L^m f)(\tau)|^2\, d\nu(\tau).\end{aligned}$$

It follows that

$$\|f - P_a(f)\|^2_{2,\mu} \le c_1^{-2}a^{-2m}\int_0^{+\infty}\left|1 - K_{2i\sqrt{\tau}}(2\sqrt{h})\right|^2 |\mathbb{K}(L^m f)(\tau)|^2\, d\nu(\tau).$$

Then, according to the relation (20), we get

$$\begin{aligned}\|f - P_a(f)\|^2_{2,\mu} &\le c_1^{-2}a^{-2m}\int_0^{+\infty} |\mathbb{K}(\Delta_h)(L^m f)(\tau)|^2\, d\nu(\tau)\\ &= c_1^{-2}a^{-2m}\left\|\Delta_h\left(L^m f\right)\right\|_{2,\mu}.\\ \|f - P_a(f)\|_{2,\mu} &\le ca^{-2m}\omega_2\left(L^m f, 1/a\right),\end{aligned}$$

with $c = c_1^{-1}$.

By using the fact that $P_a(f) \in \mathcal{I}_a$, we obtain the relation (22).

4. Applications

4.1. Case where $f(x) = e^{-x}$

For $f(x) = e^{-x}$, the approximation error decays very rapidly with a for an exponentially regular function.

Proposition Let $f(x) = e^{-x}$. The best spectral approximation satisfies the bound

$$E_a^2(e^{-x}) \leq c'\, a^{-2m}.$$

Proof: We wish to evaluate the Jackson bound in the particular case where $f(x) = e^{-x}$. To do this, we follow the steps below.

Let L be the differential operator associated with the Kontorovich-Lebedev-Clifford (KLC) transform. In general, for $f(x) = e^{-x}$, we have:

$$L^m e^{-x} = P_m(x) e^{-x},$$

where $P_m(x)$ is a polynomial depending on m. For example, we observe that:

$$Le^{-x} = -e^{-x}, \quad L^2 e^{-x} = e^{-x}.$$

Thus, we formally deduce that

$$L^m e^{-x} = (-1)^m e^{-x}.$$

By definition, the modulus of continuity is given by

$$\omega_2(L^m e^{-x}, \delta) = \sup_{0<h\leq\delta} \left\| \Delta_h(L^m e^{-x}) \right\|_{2,\mu},$$

where the difference operator is defined as

$$\Delta_h(L^m e^{-x}) = \mathfrak{T}_h(L^m e^{-x}) - L^m e^{-x}.$$

In many contexts, the translation operator $\mathfrak{T}_h$ acts as a smoothing operator, and one can obtain the estimate

$$\omega_2(L^m e^{-x}, \delta) \approx \delta^m \left\| L^m e^{-x} \right\|_{2,\mu}.$$

Taking $\delta = \frac{1}{a}$, we have

$$\omega_2(L^m e^{-x}, 1/a) \approx a^{-m} \left\| L^m e^{-x} \right\|_{2,\mu}.$$

The weighted L^2-norm is defined by

$$\left\| L^m e^{-x} \right\|_{2,\mu} = \left(\int_0^\infty \left| L^m e^{-x} \right|^2 d\mu(x) \right)^{1/2},$$

with the measure

$$d\mu(x) = \frac{dx}{2x}.$$

Since $L^m e^{-x} = (-1)^m e^{-x}$, we obtain:

$$\left\| L^m e^{-x} \right\|_{2,\mu} = \left(\int_0^\infty e^{-2x} \frac{dx}{2x} \right)^{1/2}.$$

This integral converges and is denoted by

$$\left\| L^m e^{-x} \right\|_{2,\mu} = C_m,$$

where C_m is a constant depending on m.

The Jackson theorem provides the inequality

$$E_a^2(e^{-x}) \leq c\,a^{-m}\,\omega_2(L^m e^{-x}, 1/a).$$

Substituting the previous estimate, we obtain:

$$E_a^2(e^{-x}) \leq c\,a^{-m}\left(a^{-m}C_m\right) = c'\,a^{-2m},$$

where $c' = c\,C_m$ is a positive constant depending on m.

4.2. Application of Jackson's Theorem to the Heat Equation

The classical heat equation is a partial differential equation that describes the distribution of heat (or temperature) in a given region over time. For a function $u(x,t)$ that represents the temperature at position $x \in \mathbb{R}_+$ and time $t > 0$, the equation is:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \quad \text{with} \quad x \in \mathbb{R}_+,\ t > 0.$$

Here, Δ is the spatial Laplacian, and in one spatial dimension, it simplifies to the second derivative:

$$\Delta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

We are typically interested in solving this equation with an initial condition $u(x,0) = f(x)$, where $f(x)$ is a given function that represents the initial temperature distribution.

We solve this heat equation using spectral methods.

We will apply the KLC transform to express the solution in terms of its spectral components. The solution to the heat equation can be written as:

$$u(x,t) = \mathbb{K}^{-1}\left(\mathbb{K}f(\tau)e^{-\tau t}\right).$$

Thus, the solution is essentially a superposition of exponentially decaying modes (due to the $e^{-\tau t}$ factor), each of which corresponds to a particular frequency (or spectral component) determined by the KLC transform.

Now, we consider the approximation of $u(x,t)$ by truncating the spectral expansion. The idea is to approximate the solution by taking only the first a spectral components. This approximation is given by:

$$P_a u(x,t) = \mathbb{K}^{-1}\left(\mathbb{K}f(\tau)\mathbf{1}_{[0,a]}(\tau)e^{-\tau t}\right),$$

where $\mathbf{1}_{[0,a]}(\tau)$ is the characteristic function of the interval $[0,a]$, which truncates the spectral representation at $\tau = a$.

In the context of the heat equation, we can apply Jackson's theorem to the approximation of $u(x,t)$. We know that:

$$u(x,t) = \mathbb{K}^{-1}\left(\mathbb{K}f(\tau)e^{-\tau t}\right)$$

and the approximation using the truncated spectral series is:

$$P_a u(x,t) = \mathbb{K}^{-1}\left(\mathbb{K}f(\tau)\mathbf{1}_{[0,a]}(\tau)e^{-\tau t}\right).$$

The approximation error $E_a^2(u)$ for the solution to the heat equation is then given by:

$$E_a^2(u) = \|u(x,t) - P_a u(x,t)\|_{2,\mu}.$$

From Jackson's theorem, we know that:

$$E_a^2(u) \leq c\, a^{-m}\, \omega_2(L^m f, 1/a) e^{-at}.$$

The term e^{-at} reflects the heat diffusion in the solution, as the exponential factor causes the error to decay over time. This shows that the error in the approximation decreases both with the truncation parameter a and with time.

By applying Jackson's theorem to the heat equation, we obtain an error bound for the approximation of the solution in terms of the spectral expansion.

This method is especially useful in numerical simulations of the heat equation, where spectral methods can be used to solve PDEs efficiently and accurately, with a guaranteed error bound that improves over time.

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