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Bayes estimators for the parameters of truncated Campbell distribution using Lindley's approximation

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Abstract In this research, the Campbell distribution (maximum value) was truncated by deleting a part of the distribution domain so that the distribution function maintains its probability properties, to obtain the truncated Campbell distribution (maximum value) (TC). Also, the maximum likelihood function (MLE) and Bayes estimators for the scale and location parameters were derived using the Lindley approximation by taking different loss functions, which are the squared loss function (SEL) and the general entropy function (GEL). We also used the simulation method to generate many sample sizes (n= 10, 60, 120, 150) with many different values of the scale parameter and the location parameter, and the estimators were compared using the mean square error (MSE) measure.

Keywords Campbell distribution, Truncated Campbell distribution, Maximum likelihood estimates, Bayesian estimation, Lindley's approximation.

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1. Introduction

The importance of truncated distributions appears in many scientific and applied fields, especially in Bayesian statistics and probabilistic programming to impose non-linear or physical constraints on parameters, as well as in reliability, for example, when analyzing devices within a specific time period. The statistical model of the output of the ensemble model based on the distribution of generalized truncated extreme values to calibrate the ensemble wind speed forecasts. Peng and Wan (2022) [13] presented the estimation of the truncation point in truncated normal samples and its applications. Peng and Wan (2022) [14] presented the estimation of truncation points for the truncated normal distribution. Cao et al. (2014)[15] presented the truncated distribution of the link travel time. Horrace,(2015) [16] introduced the moments of the truncated normal distribution. Nadarajah, (2009) [17], presented some truncated distribution. Aryuyuen (2018)[18] , introduced an application for the truncated Lindley distribution. Kouadria and Halim,(2025) [19], presented the truncated new XLindley distribution. Beirlant et al. (2016)[20], introduced the truncated and non-truncated Pareto distribution. Hassan et al. (2020)[21] , presented the properties and estimation for the new family of upper truncated distributions. Zhang, et al. (2011)[22] , introduced an upper truncated Weibull distribution. Chopin (2011)[23] derived the truncated Gaussian distributions. Krenek et al. (2017)[24] presented truncated normal and truncated skew normal distributions. Krishnakumari, and Dais (2024)[25] , introduced a new truncated distribution: properties and application.

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2. Campbell Distribution

The Campbell distribution (CD) has been introduced by Emil Julius Campbell [6]. The (GD) can be used to forecast the likelihood of an extreme earthquake, flood, or other natural disaster. Then the probability density function (pdf) and the cumulative distribution function (cdf) are defined by [1],[3],[7].

$$G(x) = 1 - exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right);\ -\infty < x < \infty \tag{1}$$

$$g(x) = \frac{1}{\alpha}\, exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)} + \left(\frac{x-\beta}{\alpha}\right)\right]\beta,\ location\ (real),\ \alpha > 0,\ scale\ (real) \tag{2}$$

3. Truncated Campbell TC Distribution

The concept of truncation is an important concept in probability distributions because of its importance in aligning the probability distribution with the study population. A left-truncated Campbell distribution is formed if the Campbell distribution is truncated only to the left. In this case (cdf) and (pdf) are given by.

$$F(x) = \frac{G(x) - G(0)}{G(\infty) - G(0)};\quad 0 < x < \infty \tag{3}$$

$$f(x) = \frac{f(x)}{G(\infty) - G(0)}; 0 \leq x < \infty \tag{4}$$

By substituting equation1 into equations 3, we get:

$$F_{TG}(x) = \frac{1 - exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right) - \left(1 - exp\left(-e^{\frac{(0-\beta)}{\alpha}}\right)\right)}{1 - exp\left(-e^{\frac{(\infty-\beta)}{\alpha}}\right) - \left(1 - exp\left(-e^{\frac{(0-\beta)}{\alpha}}\right)\right)}$$

$$= \frac{1 - exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right) - 1 + exp\left(-e^{\frac{-\beta}{\alpha}}\right)}{1 - 0 - \left(1 - exp\left(-e^{\frac{-\beta}{\alpha}}\right)\right)}$$

$$= \frac{-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right) + exp\left(-e^{\frac{-\beta}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}$$

$$= \frac{-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right) + exp\left(-e^{\frac{-\beta}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}$$

$$= \frac{-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} + \frac{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}$$

$$= \frac{-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} + 1$$

Then the (cdf) of the TC distribution is described as follows:

$$F_{TG}\left(x\right)=1-\frac{exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)};\qquad 0<x<\infty \tag{5}$$

By substituting equation 2 into equations4 , we get:

$$f\left(x\right)=\frac{\frac{1}{\alpha}\,exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)}+\left(\frac{x-\beta}{\alpha}\right)\right]}{1-exp\left(-e^{\frac{(\infty-\beta)}{\alpha}}\right)-\left(1-exp\left(-e^{\frac{(0-\beta)}{\alpha}}\right)\right)}$$

$$=\frac{\frac{1}{\alpha}\,exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)}+\left(\frac{x-\beta}{\alpha}\right)\right]}{1-0-1+exp\left(-e^{\frac{-\beta}{\alpha}}\right)}$$

Then the (pdf) of the TC distribution is described of follows:

$$f\left(x\right)=\frac{\frac{1}{\alpha}\,exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)}+\left(\frac{x-\beta}{\alpha}\right)\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)};\qquad 0\leq x<\infty \tag{6}$$

The reliability function of the TC distribution is described of follows:

$$R_{TG}\left(x\right)=1-F_{TEE}\left(x\right)=\frac{exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \tag{7}$$

The failure rate function of the TC distribution is formulated as:

$$h_{TG}\left(x\right)=\frac{f_{TG}(x)}{R_{TG}(x)}=\frac{\frac{1}{\alpha}\,exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)}+\left(\frac{x-\beta}{\alpha}\right)\right]}{exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)} \tag{8}$$

The cumulative hazard rate function of the TC distribution is given as

$$H_{TG}\left(x\right)=-\ln R_{TG}\left(x\right) \tag{9}$$

The reverse hazard rate function of the TC distribution is given as

$$\xi_{TG}\left(x\right)=\frac{f(x)}{F(x)}=\frac{\frac{1}{\alpha}\,exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)}+\left(\frac{x-\beta}{\alpha}\right)\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)} \tag{10}$$

The odds ratio is useful in assessing the risk of a particular outcome when a particular factor is present. There for the Odds function of the TC distribution is given as

$$O_{TG}\left(x\right)=\frac{F_{TG}\left(x\right)}{R_{TG}\left(x\right)}=\frac{exp\left(-e^{\frac{-\beta}{\alpha}}\right)-exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)} \tag{11}$$

The Quantile Function is given in equation of the following:

$$u=F\left(x\right)\Rightarrow u=1-\frac{exp\left(-e^{\frac{(x-\beta)}{\alpha}}\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}\Rightarrow x=Q\left(u\right)=\beta+\alpha ln\left(e^{\frac{-\beta}{\alpha}}-\ln\left(1-u\right)\right) \tag{12}$$

$\boldsymbol{r^{th}}$**Distribution**

4. The r$^{t}hMomentofTCDistribution$

The moment r^{th} of the TC distribution can be derived as:

$$E(x^r)_{TG} = \int_0^{\infty} x^r f(x)_{TGD} dx = \int_0^{\infty} x^r \frac{\frac{1}{\alpha}\, exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)} + \left(\frac{x-\beta}{\alpha}\right)\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} dx \tag{13}$$

Using the transformation $z = \frac{x-\beta}{\alpha} \Rightarrow x = \alpha z + \beta \Rightarrow dx = \alpha dz$, then

$$E(x^r)_{TG} = \int_0^{\infty} (\alpha z + \beta)^r \frac{exp\left[-e^z + z\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} dZ = \int_0^{\infty} (\alpha z + \beta)^r \frac{e^z\, exp\left[-e^z\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} dZ \tag{14}$$

and based on See [8]

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{15}$$

$$E(x^r)_{TG} = \int_0^{\infty} (\alpha z + \beta)^r \frac{e^z \sum_{k=0}^{\infty} (-e^z)^k}{k! exp\left(-e^{\frac{-\beta}{\alpha}}\right)} dZ$$

$$= \int_0^{\infty} (\alpha z + \beta)^r \frac{e^z \sum_{k=0}^{\infty} (-1)^k (e^z)^k}{k! exp\left(-e^{\frac{-\beta}{\alpha}}\right)} dZ$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} (\alpha z + \beta)^r e^{(1+k)z} dZ$$

Using the Newton Binomial series, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, $n \geq 0$, then

$$E(x^r)_{TG} = \sum_{k,i=0}^{\infty} \frac{1}{k!\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} \binom{r}{i} (\alpha z)^{r-i} \beta^i e^{(1+k)z} dZ$$

$$= \sum_{k,i=0}^{\infty} \frac{(-1)^k}{k!\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} \binom{r}{i} \alpha^{r-i} z^{r-i} \beta^i e^{(1+k)z} dZ$$

Let $w = -(1+k)\, z \rightarrow z = \frac{-w}{(1+k)} \rightarrow dz = \left|\frac{-dw}{(1+k)}\right|$

$$E(x^r)_{TG} = \sum_{k,i=0}^{\infty} \frac{(-1)^k}{k!\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} \binom{r}{i} \alpha^{r-i} \left(\frac{-w}{(1+k)}\right)^{r-i} \beta^i e^{-w} \frac{dw}{(1+k)}$$

$$= \sum_{k,i=0}^{\infty} \frac{(-1)^k}{k!\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} \binom{r}{i} \alpha^{r-i} \frac{(-1)^{r-i} w^{r-i}}{(1+k)^{r-i}} \beta^i e^{-w} \frac{dw}{(1+k)}$$

$$= \sum_{k,i=0}^{\infty} \frac{(-1)^{k+r-i} \beta^i \alpha^{r-i} \binom{r}{i}}{k!\; (1+k)^{r+i+1}\; exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} w^{r-i} e^{-w} dw$$

$$E(x^r)_{TG} = \sum_{k,i=0}^{\infty} \frac{(-1)^{k+r-i}\beta^i\alpha^{r-i}\begin{pmatrix} r \\ i \end{pmatrix}}{k!\,(1+k)^{r+i+1}\, exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \int_0^{\infty} w^{r-i}e^{-w}dw$$

Then the r^{th} Moment of the TC Distribution is:

$$E(x^r)_{TG} = \sum_{k,i=0}^{\infty} \frac{(-1)^{k+r-i}\beta^i\alpha^{r-i}}{k!\,(1+k)^{r+i+1}\, exp\left(-e^{\frac{-\beta}{\alpha}}\right)} (r-i+1) \tag{16}$$

5. Estimation Methods

5.1. Maximum likelihood estimator

If $x_1.\ x_2.\ \ldots.\ x_n$be a random sample of size n from the TC distribution with parameters α, β, then the maximum likelihood function ($\mathcal{L}$) for observations is given in the following formula.

$$\mathcal{L}\left(\alpha\ ,\ \ \beta|x\right) = \prod_{i=1}^{n} \frac{\frac{1}{\alpha}\, exp\left[-e^{\left(\frac{x-\beta}{\alpha}\right)} + \left(\frac{x-\beta}{\alpha}\right)\right]}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}$$

$$\mathcal{L}\left(\alpha\ ,\ \ \beta|x\right) = \frac{\left(\frac{1}{\alpha}\right)^n \left(exp\sum_{i=1}^{n}\left[-e^{\left(\frac{x_i-\beta}{\alpha}\right)} + \left(\frac{x_i-\beta}{\alpha}\right)\right]\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)} \tag{17}$$

$$ln\mathcal{L}\left(\alpha\ ,\ \ \beta|x\right) = -n\ln(\alpha)\ + \sum_{i=1}^{n}\left[-e^{\left(\frac{x_i-\beta}{\alpha}\right)} + \left(\frac{x_i-\beta}{\alpha}\right)\right] + e^{\frac{-\beta}{\alpha}} \tag{18}$$

$$\frac{\partial ln\mathcal{L}\left(\alpha\ ,\ \ \beta|x\right)_{TGD}}{\partial\alpha} = -\frac{n}{\alpha} + \frac{\sum_{i=1}^{n} e^{-\frac{\beta}{\alpha}}\left(\left((\beta-x_i)\, e^{\frac{x_i-\beta}{\alpha}} - \beta + x_i\right) e^{\frac{\beta}{\alpha}} - \beta\right)}{\alpha^2} \tag{19}$$

$$\frac{\partial ln\mathcal{L}\left(\alpha,\ \beta|x\right)_{TGD}}{\partial\beta} = \frac{\sum_{i=1}^{n} e^{\frac{x_i-\beta}{\alpha}}}{\alpha} - \frac{e^{\frac{-\beta}{\alpha}}}{\alpha} - \frac{1}{\alpha} \tag{20}$$

The maximum likelihood estimators of the parameters: α_{ML} and β_{ML}, are obtained by numerically solving the nonlinear equations $\frac{\partial ln\mathcal{L}(\alpha,\ \beta|x)_{TEV}}{\partial\alpha} = \frac{\partial ln\mathcal{L}(\alpha,\ \beta|x)_{TEV}}{\partial\beta} = 0$
In this case the best method in numerical analysis is Newton-Raphson method, since it is a method for finding successively better approximations to the roots of a real-valued function, then an iterative step is a technique of sequential approximations, and each approximation is called iteration.

5.2. Bayes estimation

The Bayesian method of estimation relies on the intuition of dealing with parameters as random variables that requires obtaining prior information about them, and this prior information can be formulated in the form of a prior probability distribution. The Gamma distribution was chosen as an a priori distribution because it is flexible and mathematically compatible with many distributions, especially exponential ones.
Consider that the prior distributions of α and β of TC are taken to be independent distributions with pdf.

$$v_1\left(\alpha\right) = \frac{b^k}{(k)}\alpha^{k-1}\ e^{-b\alpha} \tag{21}$$

$$v_2(\beta) = \frac{1}{\beta} \tag{22}$$

Thus, the joint prior distribution for β and α is:

$$v(\alpha\,,\ \beta) = \frac{b^k}{\beta\ (k)}\alpha^{k-1}\ e^{-b\alpha} \tag{23}$$

$$(\alpha\,,\ \beta\,|x\,)\ \propto\ \mathcal{L}\,(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta}\,|x\,)\,.v\,(\alpha\,,\ \beta) = C\,.\mathcal{L}\,(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta}\,|x\,)\,.v\,(\alpha\,,\ \beta) \tag{24}$$

where $C^{-1} = \int_0^\infty \int_0^\infty \mathcal{L}\,(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta}\ |x\,)\,.v\,(\alpha\,,\ \beta)\ .d\,(\alpha\,,\beta)$
Substituting (23) and (30) in (31) we get the posterior density function $(\boldsymbol{\alpha},\boldsymbol{\beta})$ as

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta}\,|x\,) = \frac{\frac{\left(\frac{1}{\alpha}\right)^n \left(exp\sum_{i=1}^n\left[-e^{\left(\frac{x_i-\beta}{\alpha}\right)}+\left(\frac{x_i-\beta}{\alpha}\right)\right]\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}\frac{b^k}{\beta\ (k)}\alpha^{k-1}\ e^{-b\alpha}}{\int_0^\infty\int_0^\infty \frac{\left(\frac{1}{\alpha}\right)^n \left(exp\sum_{i=1}^n\left[-e^{\left(\frac{x_i-\beta}{\alpha}\right)}+\left(\frac{x_i-\beta}{\alpha}\right)\right]\right)}{exp\left(-e^{\frac{-\beta}{\alpha}}\right)}\frac{b^k}{\beta\ (k)}\alpha^{k-1}\ e^{-b\alpha}d\,(\alpha,\beta)} \tag{25}$$

5.2.1. Bayes Estimator by using Lindley Approximation Lindley's approximation is an important analytical tool in Bayesian statistics, and many researchers have used it to estimate integral ratios in cases where accurate dimensional integrals are difficult to calculate. Some studies that have relied on this approximation include: For instance, Predo et al. (2010) [9] introduced the "Bayes estimators of modified Weibull distribution parameters using Lindley's approximatin". Ilhan and Merve (2020)[13] introduced two approximation techniques, specifically Lindley's approximation and the Tierney-Kadane (TK) method. Sana and Mohammad (2021)[10] introduced the Bayesian Estimation using Lindley's approximation. Hasanain et al. (2022)[5] introduced the "Bayes estimation of Lomax parameters under different loss functions using Lindley's approximation." The Lindley's approximation is expresed as:

$$\frac{\int w\,(\vartheta)\,e^{\mathcal{L}(\vartheta)}d\vartheta}{\int v\,(\vartheta)e^{\mathcal{L}(\vartheta)}d\vartheta} \tag{26}$$

Where $\vartheta = (\vartheta_1,\ \vartheta_2,\ldots,\vartheta_n)$ are parameters, $\mathcal{L}\,(\vartheta)$ represents the likelihood function's logarithm, $w(\vartheta)$and $v(\vartheta)$ are any arbitrary functions for parameters.
Let $v(\vartheta)$ be the prior distribution of ϑ and $w\,(\vartheta) = v\,(\vartheta)\,.u(\vartheta)$. From (32) we can get posterior expectation which is as follow:

$$\hat{u} = E\,(u\,(\boldsymbol{\alpha}\,,\boldsymbol{\beta})) = \frac{\int_{(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})} u\,(\boldsymbol{\alpha}\,,\ \ \boldsymbol{\beta})\,.e^{\mathcal{L}(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})+G(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})}.\,d\,(\boldsymbol{\alpha}\,,\ \ \boldsymbol{\beta})}{\int_{(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})} e^{\mathcal{L}(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})+G(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta})}.d\,(\boldsymbol{\alpha}\,,\boldsymbol{\beta})} \tag{27}$$

Where $u\,(\boldsymbol{\alpha}\,,\boldsymbol{\beta})$ is a function of α and β only; $\mathcal{L}\,(\boldsymbol{\alpha}\,,\boldsymbol{\beta})$ is a log of likelihood function; $G\,(\boldsymbol{\alpha}\,,\ \boldsymbol{\beta}) = \log v\,(\alpha\,,\beta)$.
The Lindley approximation method yields Bayes estimators, which for the TC distribution expresses in [2],[5],[10] as follows:

$$\hat{u} = u\left(\widehat{\alpha},\ \widehat{\beta}\right) + \frac{1}{2}\left[A + \mathcal{L}_{111}B_{12} + \mathcal{L}_{222}B_{21+}\mathcal{L}_{21}C_{12} + \mathcal{L}_{12}C_{21}\right] + \rho_1 A_{12} + \rho_2 A_{12} \tag{28}$$

Where $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\boldsymbol{\beta}}$ are the MLE of α and β respectively. And

$$\left.\begin{array}{l}
A = \sum_{i=1}^{2}\sum_{i=1}^{2}\sigma_{ij}T_{ij}:\ \ i+j=3\\
\mathcal{L}_{ij} = \frac{\partial^{i+j}\mathcal{L}(\alpha,\beta)}{\partial\alpha^i\beta^i}\\
\rho_1 = \frac{\partial\rho}{\partial\alpha},\ \ \rho_2 = \frac{\partial\rho}{\partial\beta},\ \ \rho = \ln v\,(\alpha,\beta)\\
\sigma_1 = \frac{\partial u}{\partial\alpha},\ \sigma_2 = \frac{\partial u}{\partial\beta},\ \sigma_{ij} = \frac{\partial^2 R}{\partial\alpha_i\partial\beta_j}\\
A_{ij} = \sigma_i.T_{ij} + \sigma_j.T_{ji}\\
B_{ij} = (\sigma_i.T_{ij} + \sigma_j.T_{ij})\,.T_{ii}\\
C_{ij} = 3\sigma_i.T_{ii}.T_{ij} + \sigma_i\left(T_{ii}.T_{jj} + 2T_{ij}^2\right)
\end{array}\right\} \tag{29}$$

The numbers 1 and 2 on the right side of each symbol indicate to $\alpha$and β respectively. Let $\vartheta_1 = \alpha \ and \ \vartheta_2 = \beta$
$\boldsymbol{\rho_i = \frac{\partial \rho}{\partial \vartheta_i}, \ u_i = \frac{\partial(\vartheta_1, \ \vartheta_2, \ )}{\partial \vartheta_i} :}$ i=1, 2

$$\boldsymbol{\sigma_{ij} = \frac{\partial^2 u\left(\vartheta_1, \ \vartheta_2\right)}{\partial \vartheta_i \partial \vartheta_j}}; \mathcal{L}_{ij} = \frac{\partial^2 \mathcal{L}\left(\vartheta_1 . \ \vartheta_2\right)}{\partial \vartheta_i \partial \vartheta_j} : \ i, \ j = 1, \ 2; \mathcal{L}_{ijk} = \boldsymbol{\frac{\partial^2 \mathcal{L}\left(\vartheta_1, \ \vartheta_2\right)}{\partial \vartheta_i \partial \vartheta_j \vartheta_k} \ : \ i, \ \ j, \ \ k = 1, \ 2}$$

where $_{\text{ij}}$ is the $(\boldsymbol{i} \ , \ \ \boldsymbol{j})$ th element of the inverse of the matrix $[\mathcal{L}_{\boldsymbol{i} \ \boldsymbol{j}}]$ all evaluated at the MLE of the parameters. From the joint prior distribution (30) we have

$$\rho = \ln v\left(\alpha . \ \delta . \ \beta\right) \ = k \ln b \ - \ln \beta \ - \ln \delta \ - \ln\left(k\right) \ + \left(k-1\right) \ln \alpha \ - b\alpha$$

And then we receive

$$\rho_1 = \frac{k-1}{\alpha} - b, \rho_2 = \frac{-1}{\beta},$$

The values of $\mathcal{L}_{\boldsymbol{ij}}$ can be obtained as follows for i, j =1, 2

$$ln\mathcal{L}\left(x\right) == -n \ln \alpha \ + \sum_{i=1}^{n}\left[-e^{\left(\frac{x_i-\beta}{\alpha}\right)} + \left(\frac{x_i-\beta}{\alpha}\right)\right] + e^{\frac{-\beta}{\alpha}}$$

$$\mathcal{L}_1 = -\frac{n}{\alpha} + \frac{\sum_{i=1}^{n} e^{-\frac{\beta}{\alpha}}\left(\left(\left(\beta - x_i\right) e^{\frac{x_i-\beta}{\alpha}} - \beta + x_i\right) e^{\frac{\beta}{\alpha}} - \beta\right)}{\alpha^2}$$

$$\mathcal{L}_2 = \frac{\sum_{i=1}^{n} e^{\frac{x_i-\beta}{\alpha}}}{\alpha} - \frac{e^{\frac{-\beta}{\alpha}}}{\alpha} - \frac{1}{\alpha}$$

$$\mathcal{L}_{11} = \frac{n}{\alpha^2} - \frac{\sum_{i=1}^{n}\left(\left(\left(\left(2\beta - 2x_i\right)\alpha - \beta^2 + 2x_i\beta - x_i^2\right) e^{\frac{x_i-\beta}{\alpha}} + \left(2x_i - 2\beta\right)\alpha\right) e^{\frac{\beta}{\alpha}} - 2\beta\alpha + \beta^2\right)}{\alpha^4}$$

$$\mathcal{L}_{12} = -2\left(\left(\left(\beta - \alpha - x_i\right) e^{\frac{x_i-\beta}{\alpha}} + \beta + \alpha - x_i\right) e^{\frac{\beta}{\alpha}} - \beta + \alpha\right)$$

$$\mathcal{L}_{21} = \frac{-\mathrm{e}^{\frac{-\beta}{\alpha}}(((-\mathrm{xi})e^{\frac{x_i-\beta}{\alpha}})e^{\frac{\beta}{\alpha}})}{\alpha^3}$$

$$\mathcal{L}_{22} = -\frac{e^{\frac{x_i-\beta}{\alpha}}}{\alpha^2} + \frac{e^{\frac{-\beta}{\alpha}}}{\alpha^2}$$

$$\mathcal{L}_{222} = \frac{e^{\frac{x_i-\beta}{\alpha}}}{\alpha^3} - \frac{e^{\frac{-\beta}{\alpha}}}{\alpha^3}$$

$$\mathcal{L}_{221} = \frac{\mathrm{e}^{\frac{-\beta}{\alpha}}((2-\mathrm{xi})e^{\frac{x_i}{\alpha}}-2)}{\alpha^4} = \mathcal{L}_{212} = \mathcal{L}_{122}$$

$$\boldsymbol{\mathcal{L}_{111} = \frac{2n}{\alpha^3} - \frac{e^{\frac{-\beta}{\alpha}} \sum_{i=1}^{n}\left(\left(\begin{array}{c}\left(\left(6\beta - 6x_i\right)\alpha^2 + \left(-6\beta^2 + 12x_i\beta - 6x_i^2\right)\alpha + \beta^3 - 3x_i\beta^2 + 3x_i^2\beta - x_i^3\right) \\ e^{\frac{x_i-\beta}{\alpha}} + \left(6x_i - 6\beta\right)\alpha^2 \\ e^{\frac{\beta}{\alpha}} - 6\beta\alpha^2 + 6\beta^2\alpha + \beta^3\end{array}\right)\right)}{\alpha^6}}$$

$$\mathcal{L}_{112} = \frac{2((-\mathrm{xi})e^{\frac{\mathrm{xi}-}{\alpha}}-\mathrm{xi})\mathrm{e}^{\frac{\beta}{\alpha}})}{\alpha^4}$$

$$\mathcal{L}_{121} = \frac{2\left(\left(\left(\alpha^2 - x_i\alpha + x_i\beta - x_i^2\right) e^{\frac{x_i-\beta}{\alpha}} - \alpha^2 + \beta\alpha + \beta^2 - x_i\beta\right) e^{\frac{\beta}{\alpha}} - \alpha^2\right)}{\alpha^2}$$

$$\mathcal{L}_{211} = \frac{e^{\frac{-\beta}{\alpha}}\left(\left(\left(2\alpha^2 + \left(4x_i - 4\beta\right)\alpha + \beta^2 - 2x_i\beta + x_i^2\right) e^{\frac{x_i-\beta}{\alpha}} - 2\alpha^2\right) e^{\frac{\beta}{\alpha}} - 2\alpha^2 + 4\beta\alpha - \beta^2\right)}{\alpha^5}$$

We can now determine the value of the Bayesian estimates of the various parameters.

Legendre (1805) proposed the quadratic loss function, and Gauss (1810) created the idea of least squares. The formula for the loss function of θ is[9] :

$$L\left(\widehat{\theta}-\theta\right)=\left(\widehat{\theta}-\theta\right)^{2} \tag{30}$$

Where $\widehat{\theta}$ is the estimate of the parameter θ. Under the above loss function the Bayes estimator of θ given by

1. If $u\left(\widehat{\alpha},\ \widehat{\beta}\right)=\alpha$, then $\sigma_1=1$ and $\sigma_2=\sigma_{12}=\sigma_{21}=\sigma_{11}=\sigma_{22}=0$

$$\widehat{\alpha}_{BSEL}=E\left(u\left(\alpha,\beta\right)\right)=\widehat{\alpha}_{MLE}+\frac{1}{2}\left[A+\mathcal{L}_{111}B_{12}+\mathcal{L}_{222}B_{21+}\mathcal{L}_{21}C_{12}+\mathcal{L}_{12}C_{21}\right]+\rho_1A_{12}+\rho_2.A_{12}$$

$$A=\sigma_{12}.T_{12}+\ \sigma_{21}.T_{21}=0$$

$$A_{12}=\sigma_1.T_{12}+\sigma_2.T_{21}=T_{12}$$

$$A_{21}=\sigma_2.T_{21}+\sigma_1.T_{12}=T_{21}$$

$$B_{12}=\left(\sigma_1.T_{12}+\sigma_2.T_{12}\right)T_{11}=T_{12}.T_{11}$$

$$B_{21}=\left(\sigma_2.T_{21}+\sigma_1.T_{21}\right).T_{22}=T_{21}.T_{22}$$

$$C_{12}=3\sigma_1.T_{11}.T_{12}+\sigma_1.\left(T_{11}.T_{22}+2T_{12}^2\right)=3T_{11}T_{12}+\left(T_{11}.T_{22}+2T_{12}^2\right)$$

$$C_{21}=3\sigma_2T_{22}.T_{21}+\sigma_2.\left(T_{22}.T_{11}+2T_{21}^2\right)=0$$

To compute the value of $T_{11},\ T_{22},\ T_{12}$ by the equation such that:

$$\left.\begin{array}{c} T_{11}=\frac{M}{DM-V^2} \\ T_{22}=\frac{D}{DM-V^2} \\ T_{1\,2}=T_{2\,1}=\frac{-V}{DM-V^2} \end{array}\right\}$$

Where $M=\mathcal{L}_{11}\ ,\ D=\mathcal{L}_{22}\ ,\ V=\mathcal{L}_{12} Thentheestimator \alpha$ can be written as:

$$\widehat{\alpha}_{BSEL}=\widehat{\alpha}_{MLE}+\frac{1}{2}\left[\mathcal{L}_{111}.T_{12}.T_{11}+\mathcal{L}_{222}.T_{21}.T_{22}+\mathcal{L}_{21}.\left(3T_{11}.T_{12}+\left(T_{11}.T_{22}+2T_{12}^2\right)\right)+\right. \tag{31}$$

$\text{L}_{12}.\left(3T_{11}.T_{12}+\left(T_{11}.T_{22}+2T_{12}^2\right)\right)+$
$\left(\frac{k-1}{\alpha}-b\right).T_{12}-\frac{1}{\beta}.T_{21}$

Assuming that b any positive number

1. If $\sigma=u\left(\widehat{\alpha},\ \widehat{\beta}\right)=\beta$, then $\sigma_2=1$ and $\sigma_1=\sigma_{12}=\sigma_{21}=\sigma_{11}=\sigma_{22}=0$

$$\widehat{\beta}_{BSEL}=E\left(u\left(\alpha,\beta\right)\right)=\widehat{\beta}_{MLE}+\frac{1}{2}\left[A+\mathcal{L}_{111}B_{12}+\mathcal{L}_{222}B_{21+}\mathcal{L}_{21}C_{12}+\mathcal{L}_{12}C_{21}\right]+\rho_1A_{12}+\rho_2A_{21}$$

$$A=\sigma_{12}.T_{12}+\ \sigma_{21}.T_{21}=0$$

$$A_{12}=\sigma_1.T_{12}+\sigma_2.T_{21}=T_{12}$$

$$A_{21}=\sigma_2.T_{21}+\sigma_1.T_{12}=T_{21}$$

$$B_{12}=\left(\sigma_1.T_{12}+\sigma_2.T_{12}\right).T_{11}=T_{12}.T_{11}$$

$$B_{21}=\left(\sigma_2.T_{21}+\sigma_1.T_{21}\right).T_{22}=T_{21}.T_{22}$$

$$C_{12}=3\sigma_1.T_{11}.T_{12}+\sigma_1.\left(T_{11}.T_{22}+2T_{12}^2\right)=0$$

$$C_{21}=3\sigma_2.T_{22}.T_{21}+\sigma_2.\left(T_{22}.T_{11}+2T_{21}^2\right)=3T_{22}.T_{21}+\left(T_{22}.T_{11}+2T_{21}^2\right)$$

Then the estimator β written as follows:

$\widehat{\beta}_{BSEL}=\widehat{\beta}_{MLE}+\frac{1}{2}\left[\mathcal{L}_{111}.T_{12}.T_{11}+\mathcal{L}_{222}.T_{21}.T_{22}+\mathcal{L}_{12}.\left(3T_{22}.T_{21}+\left(T_{22}.T_{11}+2T_{21}^2\right)\right)\right]+$
$\left(\frac{k-1}{\alpha}-b\right).T_{12}-\frac{1}{\beta}.T_{21}$

5.2.3. Bayesian estimator with General Entropy Loss (GEL) function The General Entropy Loss function of θ is[4] :

$$L\left(\widehat{\theta},\ \theta\right)=\left(\frac{\widehat{\theta}}{\theta}\right)^{k}-k\ln\left(\frac{\widehat{\theta}}{\theta}\right)-1\quad ;k>0 \tag{32}$$

1. If $u\left(\widehat{\alpha},\ \widehat{\beta}\right)=\alpha^{-k}$then $\sigma_1=-k\alpha^{-k-1},\ \sigma_{11}=k\left(k+1\right)\alpha^{-k-2},\ \sigma_2=\sigma_{12}=\sigma_{21}=0$

$$\widehat{\alpha}_{BGEL}=E\left(u\left(\alpha,\beta\right)\right)=\widehat{\alpha}_{MLE}+\frac{1}{2}\left[A+\mathcal{L}_{111}B_{12}+\mathcal{L}_{222}B_{21+}\mathcal{L}_{21}C_{12}+\mathcal{L}_{12}C_{21}\right]+\rho_1A_{12}+\rho_2A_{12}$$

$$A=\sigma_{12}T_{12}+\ \sigma_{21}T_{21}=0$$

$$A_{12}=\sigma_1T_{12}+\sigma_2T_{21}=-k\alpha^{-k-1}T_{12}$$

$$A_{21}=\sigma_2T_{21}+\sigma_1T_{12}==-k\alpha^{-k-1}T_{12}$$

$$B_{12}=\left(\sigma_1T_{12}+\sigma_2T_{12}\right)T_{11}==-k\alpha^{-k-1}T_{12}T_{11}$$

$$B_{21}=\left(\sigma_2T_{21}+\sigma_1T_{21}\right)T_{22}==-k\alpha^{-k-1}T_{21}T_{22}$$

$$C_{12}=3\sigma_1T_{11}T_{12}+\sigma_1\left(T_{11}T_{22}+2T_{12}^2\right)=-3\left(k\alpha^{-k-1}\right)T_{11}T_{12}-k\alpha^{-k-1}\left(T_{11}T_{22}+2T_{12}^2\right)$$

$$C_{21}=3\sigma_2T_{22}T_{21}+\sigma_2\left(T_{22}T_{11}+2T_{21}^2\right)=0$$

Then the estimator α can be written as:

$$\widehat{}_{\text{BGEL}}=\widehat{\alpha}_{MLE}+\frac{1}{2}\left[-k\alpha^{-k-1}\mathcal{L}_{111}T_{12}T_{11}+k\alpha^{-k-1}\mathcal{L}_{222}T_{21}T_{22}+\mathcal{L}_{21}\left(3T_{11}T_{12}+\left(T_{11}T_{22}+2T_{12}^2\right)\right)\right. \tag{33}$$

-3k$\alpha^{-k-1}\mathcal{L}_{12}T_{11}T_{12}-k\alpha^{-k-1}\left(T_{11}T_{22}+2T_{12}^2\right)+\left(\frac{k-1}{\alpha}-b\right)T_{12}-\frac{1}{\beta}T_{21}$

1. If $\sigma=u\left(\widehat{\alpha},\ \widehat{\beta}\right)=\beta^{-k}$, then $\sigma_2=-k\beta^{-(k+1)}$ and $\sigma_{22}=k\left(k+1\right)\beta^{-(k+2)},\ \sigma_{12}=\sigma_{21}=\sigma_{11}=0$

$$\widehat{\beta}_{BSEL}=E\left(u\left(\alpha,\beta\right)\right)=\widehat{\beta}_{MLE}+\frac{1}{2}\left[A+\mathcal{L}_{111}B_{12}+\mathcal{L}_{222}B_{21+}\mathcal{L}_{21}C_{12}+\mathcal{L}_{12}C_{21}\right]+\rho_1A_{12}+\rho_2A_{21}$$

$$A=\sigma_{12}.T_{12}+\ \sigma_{21}.T_{21}=0$$

$$A_{12}=\sigma_1.T_{12}+\sigma_2.T_{21}=-k.\beta^{-(k+1)}.T_{21}$$

$$A_{21}=\sigma_2.T_{21}+\sigma_1.T_{12}=-k.\beta^{-(k+1)}.T_{21}$$

$$B_{12}=\left(\sigma_1.T_{12}+\sigma_2.T_{12}\right)T_{11}=-k.\beta^{-(k+1)}.T_{12}.T_{11}$$

$$B_{21}=\left(\sigma_2.T_{21}+\sigma_1.T_{21}\right).T_{22}=-k.\beta^{-(k+1)}.T_{21}.T_{22}$$

$$C_{12}=3\sigma_1.T_{11}.T_{12}+\sigma_1.\left(T_{11}.T_{22}+2T_{12}^2\right)=0$$

$$C_{21}=3\sigma_2.T_{22}.T_{21}+\sigma_2.\left(T_{22}.T_{11}+2T_{21}^2\right)=-3k.\beta^{-(k+1)}.T_{22}.T_{21}-k.\beta^{-(k+1)}.\left(T_{22}.T_{11}+2T_{21}^2\right)$$

Then the estimator β written as follows:

$$\widehat{\beta}_{BGEL}=E\left(u\left(\alpha\ \ ,\beta\right)\right)=\widehat{\beta}_{MLE}-\frac{1}{2}\left[K.\mathcal{L}_{111}.\beta^{-(k+1)}.T_{12}.T_{11}+\mathcal{L}_{222}.\beta^{-(k+1)}.T_{21}.T_{22}+\mathcal{L}_{12}.k.\beta^{-(k+1)}\right.$$

$.\left(3T_{22}.T_{21}+T_{22}.T_{11}+2T_{21}^2\right)-\left(\frac{k-1}{\alpha}-b\right).k.\beta^{-(k+1)}.T_{21}+k.\beta^{-(k+2)}.T_{21}$(34)

Table 1. MLE and Bayes Estimator for and by using Lindley Approximation when α=0.3; β= 0.3, 0.6, 1.2

	n	MLE		Bayes under SEL		Bayes under GEL		Best
		$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	
0.3	5	0.48292	0.49611	0.29654	0.38384	0.22781	0.38567	GEL
	10	0.48253	0.49584	0.29647	0.38356	0.22765	0.38562	GEL
	60	0.48166	0.48667	0.29258	0.38224	0.22572	0.38552	GEL
	120	0.47643	0.48387	0.26826	0.35583	0.22279	0.35321	GEL
	150	0.47223	0.48265	0.26659	0.35279	0.22243	0.35232	GEL
0.6	200	0.46865	0.47883	0.22785	0.33729	0.22183	0.33349	GEL
	5	0.45382	0.48762	0.24748	0.38661	0.23742	0.37274	GEL
	10	0.45341	0.48722	0.24732	0.38641	0.23738	0.37258	GEL
	60	0.45216	0.48533	0.24273	0.38355	0.23662	0.37694	GEL
	120	0.39943	0.43733	0.24154	0.38254	0.23541	0.37482	GEL
	150	0.39642	0.43286	0.24122	0.38215	0.23532	0.37421	GEL
1.2	200	0.41853	0.44265	0.24715	0.38624	0.23623	0.37823	GEL
	5	0.39523	0.44382	0.29468	0.38957	0.28573	0.37106	GEL
	10	0.39512	0.44379	0.29456	0.38946	0.28561	0.37993	GEL
	60	0.39238	0.44167	0.29023	0.38733	0.28337	0.37972	GEL
	120	0.39219	0.43793	0.29016	0.38728	0.28388	0.37633	GEL
	150	0.38492	0.43266	0.29005	0.38699	0.28173	0.37724	GEL
	200	0.87934	0.43022	0.28934	0.38376	0.27934	0.37428	GEL

Table 2. MLE and Bayes Estimator for and m by using Lindley Approximation when α=0.6; β=0.3, 0.6, 1.2

	n	MLE		Bayes under SEL		Bayes under GEL		Best
		$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	
0.3	5	0.34672	0.32856	0.44778	0.43697	0.03899	0.04986	GEL
	10	0.34657	0.32834	0.44756	0.43688	0.03893	0.04973	GEL
	60	0.34272	0.32826	0.44375	0.43634	0.03835	0.04733	GEL
	120	0.34211	0.32622	0.44278	0.43387	0.03622	0.04377	GEL
	150	0.34201	0.32552	0.44074	0.43321	0.03601	0.04332	GEL
0.6	200	0.3416	0.32534	0.44027	0.43278	0.03583	0.04301	GEL
	5	0.33852	0.32458	0.44948	0.43852	0.03836	0.04312	GEL
	10	0.33833	0.32456	0.44932	0.43845	0.03823	0.04639	GEL
	60	0.33723	0.32423	0.44836	0.43823	0.03804	0.04618	GEL
	120	0.33713	0.32287	0.44622	0.43378	0.03755	0.04522	GEL
	150	0.33710	0.32253	0.44489	0.43316	0.03539	0.04498	GEL
1.2	200	0.33678	0.32247	0.44474	0.43307	0.03524	0.04465	GEL
	5	0.33934	0.32951	0.44263	0.43945	0.02981	0.04568	GEL
	10	0.33925	0.32462	0.44256	0.43939	0.02983	0.04572	GEL
	60	0.32723	0.32412	0.43966	0.43823	0.02897	0.04539	GEL
	120	0.32549	0.32327	0.43345	0.42996	0.02366	0.04339	GEL
	150	0.32526	0.32288	0.43299	0.42723	0.02328	0.04266	GEL
	200	0.32487	0.32224	0.43257	0.42686	0.02312	0.04238	GEL

Table 3. MLE and Bayes Estimator for and by using Lindley Approximation when α=1.2; β=0.3, 0.6, 1.2

	n	MLE		Bayes under SEL		Bayes under GEL		Best
		$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	
0.3	5	0.23469	0.25938	0.37836	0.39659	0.09879	0.08071	GEL
	10	0.23464	0.25935	0.37832	0.39654	0.09876	0.08068	GEL
	60	0.23452	0.25911	0.37821	0.39638	0.09836	0.08051	GEL
	120	0.23355	0.25864	0.37735	0.39618	0.09321	0.08032	GEL
	150	0.23327	0.25831	0.37719	0.39603	0.09118	0.08014	GEL
0.6	200	0.23276	0.25713	0.37694	0.39539	0.09102	0.08010	GEL
	5	0.23793	0.25762	0.37812	0.39648	0.08983	0.08015	GEL
	10	0.23789	0.25755	0.37797	0.39597	0.08975	0.08008	GEL
	60	0.23743	0.25723	0.37756	0.39549	0.08925	0.07782	GEL
	120	0.23728	0.25714	0.37748	0.39515	0.08357	0.07729	GEL
	150	0.23715	0.25711	0.37721	0.39503	0.08198	0.07715	GEL
1.2	200	0.23681	0.25693	0.37674	0.39419	0.08117	0.07686	GEL
	5	0.23823	0.25672	0.37999	0.39976	0.07885	0.06063	GEL
	10	0.23812	0.25632	0.37997	0.39963	0.07879	0.06059	GEL
	60	0.24796	0.25617	0.37978	0.39923	0.07837	0.05839	GEL
	120	0.24587	0.25585	0.38889	0.39853	0.06389	0.05698	GEL
	150	0.24531	0.25513	0.38876	0.39828	0.06103	0.05286	GEL
	200	0.24514	0.25489	0.38856	0.39769	0.06087	0.05234	GEL

6. Simulation Study

In this section, the simulation study, was performed as empirical method to determine the behavior of the MLE and Bayes Estimator for α and β by using Lindley Approximation distribution parameters for sample sizes (n =5, 10, 60, 120, 150, 200), with α = 0.3,0.6 and 1.2; β = 0.3, 0.6 and 1.2. The mean square error (MSE) was used as a criterion for comparisons and evaluations. Using The simulation of (Monte Carlo), and repeat each experiment (1000) times, we obtained the results.
In Tables (1, 2 and 3): From the empirical results, we can see the following:

1. Table 1 shows that when (β=0.3),the output of the Bayesian estimator is Many (loss functions) of and are better than MLE. According to Table (2,3), the MLE's and Bayes estimators under (SEL) of α and βincreases irregularly, Bayes estimators of α and β under (GEL) decreases.
2. Tables (1,2,3) show that the Bayesian estimator with GEL is a better estimator compared to other estimators for all sample sizes.
3. Note that the increase in β is due to the (MSE) values for different β values and all sample sizes.
4. shows that the values associated with the parameters decrease with increasing sample size.

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