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Modeling Medical and Reliability Data Sets Using a Novel Reciprocal Weibull Distribution: Estimation Methods and Sequential Sampling Plan Based on Truncated Life Testing

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Abstract An extended version of the reciprocal Weibull model is proposed and thoroughly analyzed. Key statistical properties of the model are derived, and various estimation techniques are employed to estimate the unknown parameters. A simulation study is conducted to evaluate the performance of these methods. Additionally, two real-world datasets are utilized to compare the effectiveness of the competing estimation methods. The significance of the proposed model is highlighted through these applications, demonstrating its superior performance over other competing models in fitting the datasets. A sequential sampling plan based on truncated life testing is introduced, leveraging a newly developed probabilistic model to enhance quality control decisions. The plan determines the acceptance or rejection of a lot based on life test outcomes within a specified truncation time, optimizing inspection efforts. A series of numerical experiments are conducted to validate the proposed approach, demonstrating its effectiveness in minimizing sample sizes while maintaining desired risk levels. The results highlight the impact of key parameters on the sampling process, ensuring a balance between producer and consumer risks.

Keywords Zero Truncated Poisson Distribution; Inverse Weibull Distribution; Maximum Likelihood; Bootstrapping Estimation; Kolmogorov Estimation; Modeling; Sequential Sampling; Truncated Life Testing; Producer's Risk; Operating Characteristic Function.

AMS 2010 subject classifications 60E05; 62N01; 62G05; 62N02; 62N05; 62E10; 62P30

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1. Introduction and physical motivation

The reciprocal Weibull (RW) distribution is effective in modeling failure hazard rates commonly observed in reliability analysis and biological research. De Gusmão et al. [43] introduced and examined a novel three-parameter RW distribution characterized by a monotonically decreasing and inverted failure rate. They explored its mathematical properties and proposed a location-scale regression model for analyzing real lifetime data. Recently, Refaie et al. [98] introduced an exponentiated Weibull model incorporating copulas for dependency modeling, expanding its real applications. Algamal and Basheer [8] employed particle swarm optimization for more accurate parameter estimation in a three-parameter Weibull distribution. Hybrid approaches like the Pareto-Weibull distribution (Rana et al. [90]) and the alpha power Weibull-Pareto model (Aljuhani et al. [10]) further extended the model's adaptability. Additionally, Malik and Ahmad [83] proposed a transmuted Weibull distribution,

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emphasizing its practical utility. These developments demonstrate the Weibull model's evolving versatility in accommodating diverse data complexities. Due to de Gusmo et al. [43], a random variable (r.v.) X is said to have the RW distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by (for $x \geq 0$)

$$w_c(x;\alpha,\beta) = cba^{\beta}x^{-\beta-1}\exp\left[-\left(\alpha x^{-1}\right)^{\beta}c\right],$$

and

$$\Upsilon_{\underline{\mathbf{P}}}(x) = \exp\left[-\left(\alpha x^{-1}\right)^{\beta}c\right]|_{\underline{\mathbf{P}}=\alpha,\beta,c},$$

respectively, where $\alpha > 0$ is a scale parameter, c and $\beta > 0$ are a shape parameters, respectively. We can easily prove that $h_c(x;\alpha,\beta)$ is a density function by substituting $u = -\left(\alpha x^{-1}\right)^{\beta}c$. The standard RW distribution is a special case of $w_c(x;\alpha,\beta)$ when $c = 1$. For $c = 1$, we get the standard RW distribution. The RW distribution can be simulated by using the nonlinear equation

$$x_u = \alpha\left\{\frac{1}{c}\left[-\log(u)\right]\right\}^{-\frac{1}{\beta}}$$

where u has the uniform $U(0,1)$ distribution. For $\beta = 2$, we get the generalized reciprocal Rayleigh distribution (GIR). For $\alpha = 1$, we have the generalized reciprocal exponential (GIEx) distribution. For $c = 1$ and $\beta = 2$, we get the standard IR distribution. For $c = \alpha = 1$ we get the standard REx model. For more details about the RW model see Gusmao et al. (2011), Harlow [64], Zaharim et al. [135], Krishna et.al. [81], Barreto-Souza et al. [35], Afify et al. [15], Korkmaz et al. [80], Yousof et al. ([123], [128], [133] and [130]), Chakraborty et al. [39], Elbiely and Yousof [46], Jahanshahi et al. [75] and Elsayed and Yousof [53]. Recentely, Yousof et al. [134] expanded the RW model by defining a new G family of distributions called extended odd reciprocal Weibull distribution (EOIW) and studied its properties, applications and then presented a regression mode based on the new family. Salah et al. [101] defined and studied a new RW model called the odd-Burr reciprocal Weibull (OBRW) model. In their study Salah et al. [101] presented some new bivariate type extensions using Farlie-Gumbel-Morgenstern copula, modified Farlie-Gumbel-Morgenstern copula, Clayton copula, and Renyi's entropy copula. Yousof et al. [132] defined and studied the two-parameter Xgamma reciprocal Weibull (XgRW) distribution with some Characterization results, different copulas and different classical estimation methods. Al-Babtain et al. [12] presented a new three parameter reciprocal Weibull model called the generalized odd generalized exponential reciprocal Weibull (GOGEIW) model with simple type copula, mathematical properties and some applications to breaking stress of carbon fibres and strengths data sets. Bhatti et al. [34] defined and studied the modified Burr XII reciprocal Weibull (BXIIRW) distribution. Goual et al. [58] studied the Lomax reciprocal Weibull (LxRW) model and its properties, applications and presented a modified chi-squared goodness-of-fit test for censored validation.

This paper introduces and examines a novel extension of the reciprocal Weibull (RW) distribution by incorporating the zero-truncated Poisson (ZTP) distribution. Consider a system composed of N independently functioning subsystems at a given time, where N follows a ZTP distribution with parameter $\lambda = 1$. The ZTP distribution represents the conditional distribution of a Poisson-distributed random variable given that its value is nonzero. The probability mass function (PMF) of N is defined as

$$\text{PMF}(N = n)|_{(n=1,2,\ldots)} = \frac{1}{\varrho[1]\,n!}\exp(-1), \tag{1}$$

where

$$\varrho[1] = 1 - \exp(-1).$$

The expected value ($\mathbf{E}(\mathbf{N}|\lambda = \mathbf{1})$) and variance ($\text{Var}(N|\lambda = 1)$) are, respectively, given by

$$\mathbf{E}(N|\lambda = 1) = \varrho^{-1}[1],$$

and

$$\text{Var}(N|\lambda = 1) = \left\{2 - \varrho^{-1}[1]\right\}\varrho^{-1}[1].$$

Suppose that the failure time of each subsystem has the Burr X reciprocal Weibull ("BXRW(c, α, β)" for short) defined by the cumulative distribution function (CDF) given by

$$G(x)=\left\{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}(x)\right]\right\}^2, \tag{2}$$

where

$$\mathbf{\Lambda}_{\underline{\mathbf{P}}}(x)=\frac{\Upsilon_{\underline{\mathbf{P}}}(x)}{1-\Upsilon_{\underline{\mathbf{P}}}(x)}.$$

Let X_v denote the failure time of the ith subsystem and let

$$x=\min\{X_1, X_2, \cdots, X_N\}. \tag{3}$$

Then the conditional CDF of x given N is

$$F(x|N)=1-\Pr(x>x\mid N)=1-[1-G(x)]^N. \tag{4}$$

Therefore using (4), the unconditional CDF of the PBXRW model can be expressed as

$$F_{\underline{\mathbf{\Phi}}}(x)=\frac{1}{\varrho[1]}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}(x)\right]\right\}^2\right)\right], \tag{5}$$

with the corresponding probability density function (PDF) as

$$f_{\underline{\mathbf{\Phi}}}(x)=\frac{4bc}{\varrho[1]}\frac{x^{-(\beta+1)}\left[1-\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{-3}}{\exp\left[2(\alpha/x)^{\beta}+\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}(x)\right]}\frac{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}(x)\right]}{\exp\left(\left\{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}(x)\right]\right\}^2\right)}. \tag{6}$$

The hazard rate function (HRF) of the new model can be calculated via $f(x)/[1-F(x)]$. For $c=1$, the PBXRW model will reduce to the four parameter PBXRW. For $\beta=2$, the PBXRW model will reduce to PBXRR model. For $\alpha=1$, the PBXRW model will reduce to PBXREx model. For $c=1, \beta=2$ the PBXRW model will reduce to the three parameter PBXRR model. For $c=\alpha=1$, the PBXRW model will reduce to the three parameter PBXEx model. The PDF of the new model can be right skewed and unimodal with symmetric and asymmetric shapes (see Figure 1) also it can be left skewed (see Table 1). The HRF of the new model can be decreasing-constant-increasing (U-shape or bathtub shape), increasing-constant-increasing, monomaniacal decreasing, monomaniacal increasing (see Figure 2).

Recently, many researchers have been keen to derive new probability distributions, but they have taken care of some applied aspects in practical fields such as insurance and actuarial science, and we mention them, for example, see Mohamed et al. ([91], [92] and [93]). While others were concerned with discretizing the continuous probability distributions (continuous G families of probability distributions) and applying the new discrete distributions (discrete G families of probability distributions) to different count and zero-inflated data, for more details see Aboraya et al. [2], Yousof et al. [131], Ibrahim et al. [70], Eliwa et al. [51] and Chesneau t al. [41]. By examining the statistical literature in the field of statistical hypothesis tests, we find that there are many practical applications for commonly used tests and new ones using many of the new probability distributions. For example, many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test and modified Nikulin-Rao-Robson goodness-of-fit test are considered by Ibrahim et al. [74], Goual et al. ([57], [58]), Mansour et al. ([84], [85], [86], [87], [88], [89]), Yadav et al. [122], Goual and Yousof [56], and Ibrahim et al. [67] among others. However, the Bagdonavičius-Nikulin goodness-of-fit test and the modified Bagdonavičius-Nikulin goodness-of-fit test are considered by Aidi et al [7], Ibrahim et al. [68], Yousof et al. ([124], [129] and [127]).

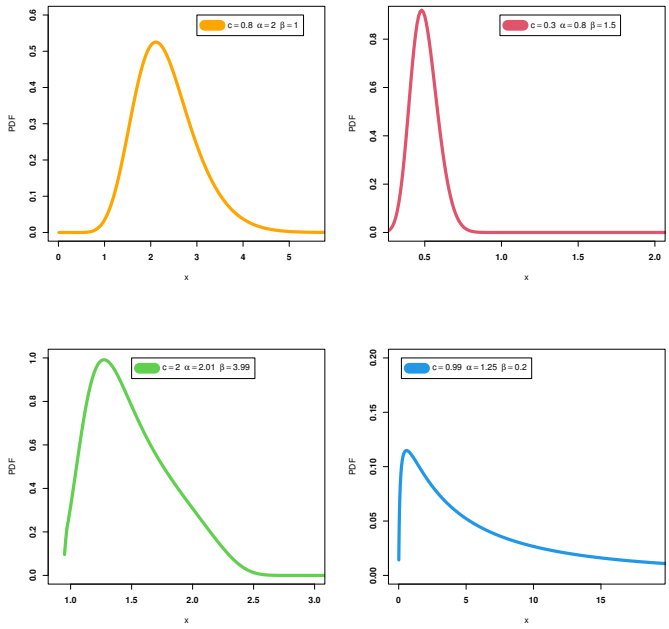

Figure 1. Plots of the new PDF for selected values of the parameter.

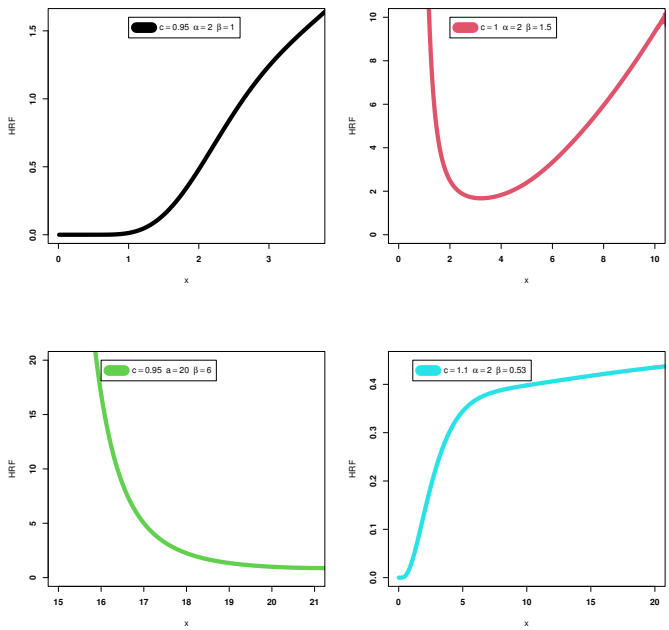

Figure 2. Plots of the new HRF for selected values of the parameter.

2. Properties

2.1. Expansions

Starting from (6) and using the power series, the new PDF in (6) can be expressed as

$$f_{\underline{\mathbf{\Phi}}}(x)=\sum_{h=0}^{\infty}\frac{4ba^{\beta}c\,(-1)^{h}}{h!\varrho\,[1]}x^{-(\beta+1)}\frac{\exp\left[-2\,(\alpha/x)^{\beta}\right]}{\exp\left[\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}(x)\right]\left[1-\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{3}}\left\{1-\exp\left[-\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}(x)\right]\right\}^{\varpi.-1}. \tag{7}$$

Then, if $\left|\frac{\tau_1}{\tau_2}\right|<1$ and $\tau_3>0$ is a real non-integer, the following power series holds

$$\left(1-\frac{\tau_1}{\tau_2}\right)^{\tau_3-1}=\sum_{\tau_4=0}^{\infty}\frac{(-1)^{\tau_4}\,\Gamma(\tau_3)}{\tau_4!\,\Gamma(\tau_3-\tau_4)}\left(\frac{\tau_1}{\tau_2}\right)^{\tau_4}. \tag{8}$$

Applying (8) to (7) we have

$$f_{\underline{\mathbf{\Phi}}}(x)=4\,\frac{\beta\alpha^{\beta}cy^{-(\beta+1)}}{\varrho\,[1]}\sum_{h,\upsilon=0}^{\infty}\frac{(-1)^{h+\upsilon}\,\Gamma(\varpi.)}{\upsilon!\,\Gamma(\varpi.-\upsilon)}\exp\left[-2\,(\alpha/x)^{\beta}\right]\frac{\exp\left[-(\upsilon+1)\,\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}(x)\right]}{\left[1-\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{3}}. \tag{9}$$

Applying the power series to the term $\exp\left[-(\upsilon+1)\,\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}(x)\right]$, equation (9) becomes

$$f_{\underline{\mathbf{\Phi}}}(x)=4\beta\alpha^{\beta}cy^{-(\beta+1)}\sum_{h,\upsilon,\upsilon=0}^{\infty}\frac{(-1)^{h+\upsilon+\upsilon}\,\Gamma(\varpi.)\,(\upsilon+1)^{\upsilon}\left[\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{2j+2}}{\upsilon!\,\upsilon!\varrho\,[1]\,\Gamma(\varpi.-\upsilon)\left[1-\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{2j+3}}\,|_{\varpi.=(1+h)2}. \tag{10}$$

Consider the series expansion

$$\left(1-\frac{\tau_1}{\tau_2}\right)^{-\tau_3}\,|_{\left(\left|\frac{\tau_1}{\tau_2}\right|<1,\;c>0\right)}=\sum_{d=0}^{\infty}\frac{\Gamma(\tau_3+d)}{d!\Gamma(\tau_3)}\left(\frac{\tau_1}{\tau_2}\right)^{d}. \tag{11}$$

Applying the expansion in (11) to (10) for the term $\left[1-\Upsilon_{\underline{\mathbf{P}}}(x)\right]^{2j+3}$, Equation (10) becomes

$$\begin{aligned}f_{\underline{\mathbf{\Phi}}}(x) &= \sum_{h,\upsilon,\upsilon,d=0}^{\infty}\frac{4\theta c\,(-1)^{h+\upsilon+\upsilon}\,(\upsilon+1)^{\upsilon}}{\upsilon!\,\upsilon!d!\,[2\,(\upsilon+1)+d]\,\varrho\,[1]}\,\frac{\Gamma(\varpi.)\,\Gamma(3+2j+d)}{\Gamma(\varpi.-\upsilon)\,\Gamma(2j+3)}\\ &\quad\times\beta\alpha^{\beta}x^{-(\beta+1)}\,[2\,(\upsilon+1)+d]\exp\left\{-[2\,(\upsilon+1)+d]\,(\alpha/x)^{\beta}\right\}.\end{aligned}$$

This can be written as

$$f_{\underline{\mathbf{\Phi}}}(x)=\sum_{\upsilon,d=0}^{\infty}\varsigma_{\upsilon,d}\,w_{c.}(x;\alpha,\beta)|_{c.=c[2(\upsilon+1)+d]}, \tag{12}$$

where

$$\varsigma_{\upsilon,d}=\frac{4\theta c\,(-1)^{\upsilon}\,\Gamma(3+2j+d)}{\upsilon!d!\varrho\,[1]\,\Gamma(2j+3)\,[2\,(\upsilon+1)+d]}\sum_{h,\upsilon=0}^{\infty}\frac{(-1)^{h+\upsilon}\,\Gamma(\varpi.)\,(\upsilon+1)^{\upsilon}}{\upsilon!\,\Gamma(\varpi.-\upsilon)},$$

and $w_{c.}(x;\alpha,\beta)$ is the IW PDF with scale parameter $\alpha c^{1/\beta}$ and and shape parameter β. Similarly, the CDF of the PBXRW model can also be expressed as

$$F_{\underline{\mathbf{\Phi}}}(x)=\sum_{\upsilon,d=0}^{\infty}\varsigma_{\upsilon,d}\,W_{c.}(x;\alpha,\beta), \tag{13}$$

where $W_{c.}(x;\alpha,\beta)$ is the the IW CDF with scale parameter $\alpha c^{1/\beta}$ and shape parameter β.

2.2. Some properties

The quantile function (QF) of x, where $X \sim$PBXRW (c, α, β), is obtained by inverting (5) as

$$Q(u) = \alpha \sqrt[\beta]{(-\ln C)^{-1}}. \tag{14}$$

where

$$C = \left[\left(1 + \sqrt{-\frac{1}{c}\ln B}\right)\right],$$
$$B = \left[1 - \sqrt[2]{(-\ln A)}\right],$$

and

$$A = 1 - u\varrho[1].$$

Simulating the PBXRW r.v. is straightforward. If U is a uniform variate on the unit interval $(0, 1)$, then the r.v. $x = Q(U)$ follows (6). The s^{th} ordinary moment of x, say $\mu'_{s,x}$, follows from (12) as

$$\mu'_{s,x}|_{(s<\beta)} = \mathbf{E}(X^s) = \sum_{\upsilon,d=0}^{\infty} \varsigma_{\upsilon,d}\, \alpha^s c.^{s/\beta}\, \Gamma(1 - s/\beta). \tag{15}$$

Setting $s = 1$ in (15) gives the mean of x as

$$\mathbf{E}(X)|_{(1<\beta)} = \sum_{\upsilon,d=0}^{\infty} \varsigma_{\upsilon,d}\, ac.^{1/\beta}\, \Gamma(1 - 1/\beta),$$

where $\Gamma(1 + \tau)|_{(\tau\in\mathbb{R}^+)} = \tau!$, and

$$\Gamma(\tau) = \int_0^{\infty} x^{\tau-1} \exp(-t)\, dt.$$

The flexibility of the new distribution is influenced by the degrees of the skew coefficient, kurtosis coefficient. The s^{th} incomplete moment of X is defined by $m_{s,x}(X) = \int_{-\infty}^{x} x^s f(x)dy$. We can write from (12)

$$m_{s,x}(X)|_{(s<\beta)} = \gamma\left(1 - s/\beta, (\alpha/t)^{\beta}\right) \sum_{\upsilon,d=0}^{\infty} \varsigma_{\upsilon,d}\, \alpha^s c.^{s/\beta}. \tag{16}$$

Setting $s = 1$ in (16) gives the 1^{st} incomplete moment of x as

$$m_{1,x}(X)|_{(1<\beta)} = \gamma\left(1 - 1/\beta, (\alpha/t)^{\beta}\right) \sum_{\upsilon,d=0}^{\infty} \varsigma_{\upsilon,d}\, ac.^{1/\beta},$$

where $\gamma(\varkappa_1, \varkappa_2)$ is the incomplete gamma function, where

$$\begin{aligned}
\gamma(\varkappa_1, \varkappa_2)|_{(\varkappa_1\neq 0,-1,-2,\ldots)} &= \int_0^{\varkappa_2} t^{\varkappa_1-1} \exp(-t)\, dt \\
&= \frac{1}{\varkappa_1}\varkappa_2^{\varkappa_1}\left\{{}_1\mathbf{F}_1[\varkappa_1; \varkappa_1\alpha + 1; -\varkappa_2]\right\} \\
&= \sum_{d=0}^{\infty} \varkappa_2^{\varkappa_1+d} \frac{(-1)^d}{d!\,(\varkappa_1 + d)} \\
&= \Gamma(\varkappa_1) - \Gamma(\varkappa_1, \varkappa_2),
\end{aligned}$$

the function ${}_1\mathbf{F}_1\left[\cdot,\cdot,\cdot\right]$ is a called the confluent hypergeometric function and

$$\Gamma\left(\varkappa_1,\varkappa_2\right)=\int_{\varkappa_2}^{\infty}t^{\varkappa_1-1}\exp\left(-t\right)dt,$$

The moment generating function (MGF) of x, say $M(t)=\mathbf{E}\left(\exp\left(t\,x\right)\right)$, is obtained from (12) as

$$M_X(t)|_{(s<\beta)}=\sum_{\upsilon,d,s=0}^{\infty}\varsigma_{\upsilon,d}\frac{t^s}{s!}\alpha^s c.^{s/\beta}\,\Gamma\left(1-s/\beta\right).$$

3. Estimation

In fact, the statistical literature contains many estimation methods and all of them are of interest and are appreciated by many researchers. In this Section, different estimation methods such as the maximum likelihood estimation method, Cramér–von Mises estimation method, the bootstrapping estimation method, Kolmogorov estimation method and Anderson Darling method (the left-tail of the second order) are used for estimating the unknown parameters. In this work we have neglected many of the methods of appreciation for me for their insignificance but because we must focus our attention only on some of the most famous and efficient methods in order to be able to compare them. But of course there are many ways that can be taken into consideration in future work. For more methods, see Ali et al. [20], Ali et al. [19], Alizadeh et al. [22], Yousof et al. [126], El-Morshedy et al. [52] and Yousof et al. [17].

3.1. The maximum likelihood estimation (MLE) method

A statistical method known as maximum likelihood estimation (MLE) is used to estimate the parameters of a probability distribution that has been assumed in light of certain observed data. To do this, a likelihood function is maximized to increase the probability of the observed data under the presumptive statistical model. The parameter space position where the likelihood function is maximized is known as the maximum likelihood estimate. Since its justification is understandable and flexible, maximum likelihood is a well-liked method for drawing statistical conclusions. If the likelihood function is differentiable, then maxima can be determined using the derivative test. For instance, the ordinary least squares estimator increases the likelihood of the linear regression model, enabling in some cases to explicitly solve the first-order conditions of the likelihood function. However, it is frequently necessary to employ numerical methods to ascertain the maximum of the probability function. From the perspective of Bayesian inference, MLE is often equivalent to maximum a posteriori (MAP) estimates under a uniform prior distribution of the parameters. When likelihood serves as the goal function in frequentist inference, MLE is a special illustration of an extremum estimator. Consider a random sample from the PBXRW, then the log likelihood function can be expressed as

$$\begin{aligned}\log\mathbf{L} \;=\;& 2n\log\left[2\right]+n\log\beta+nb\log\alpha+n\log c\\ &-n\log\left[\varrho\left[1\right]\right]-(\beta+1)\sum_{\upsilon=1}^{n}\log x_\upsilon-3\log\left[1-\Upsilon_{\underline{\mathbf{P}}}\left(x_\upsilon\right)\right]\\ &+2\sum_{\upsilon=1}^{n}\log\Upsilon_{\underline{\mathbf{P}}}\left(x_\upsilon\right)-\sum_{\upsilon=1}^{n}\left\{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}\left(x_\upsilon\right)\right]\right\}^2\\ &-\sum_{\upsilon=1}^{n}\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}\left(x_\upsilon\right)+\sum_{\upsilon=1}^{n}\log\left\{1-\exp\left[-\mathbf{\Lambda}^2_{\underline{\mathbf{P}}}\left(x_\upsilon\right)\right]\right\}.\end{aligned}$$

The maximum likelihood method and its procedures are available in the literature with details. The components of the score vector are $\mathbf{U}\left(\underline{\mathbf{\Phi}}\right)=\frac{\partial\ell}{\partial\underline{\mathbf{\Phi}}}=\left(\frac{\partial\log\mathbf{L}}{\partial\alpha},\frac{\partial\log\mathbf{L}}{\partial\beta},\frac{\partial\log\mathbf{L}}{\partial c}\right)^{\intercal}$.

3.2. The Cramér–von Mises estimation (CVME) method

The CVME of the parameter vector $\underline{\mathbf{\Phi}}$ are obtained via minimizing the following expression with respect to α, β and c , where

$$\mathbf{CVM}_{(\underline{\mathbf{\Phi}})} = \frac{1}{12}n^{-1} + \sum_{v=1}^{n}\left[F_{\underline{\mathbf{\Phi}}}\left(x_v\right) - c_{(v,n)}^{[1]}\right]^2,$$

and $c_{(v,n)}^{[1]} = \frac{2i-1}{2n}$, then

$$\mathbf{CVM}_{(\underline{\mathbf{\Phi}})} = \sum_{v=1}^{n}\left[F_{\underline{\mathbf{\Phi}}}\left(x_v\right) - c_{(v,n)}^{[1]}\right]^2.$$

Then, CVME of the parameters α, β and c are obtained by solving the following non-linear equations

$$\sum_{v=1}^{n}\left\{\frac{1}{\varrho\left[1\right]}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}\left(x_v\right)\right]\right\}^2\right)\right] - c_{(v,n)}^{[1]}\right\}\mathcal{D}_{(\alpha)}(x_v,\underline{\mathbf{\Phi}}) = 0,$$

$$\sum_{v=1}^{n}\left\{\frac{1}{\varrho\left[1\right]}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}\left(x_v\right)\right]\right\}^2\right)\right] - c_{(v,n)}^{[1]}\right\}\mathcal{D}_{(\beta)}(x_v,\underline{\mathbf{\Phi}}) = 0,$$

and

$$\sum_{v=1}^{n}\left\{\frac{1}{\varrho\left[1\right]}\left[1-\exp\left(-\left\{1-\exp\left[-\mathbf{\Lambda}_{\underline{\mathbf{P}}}^{2}\left(x_v\right)\right]\right\}^2\right)\right] - c_{(v,n)}^{[1]}\right\}\mathcal{D}_{(c)}(x_v,\underline{\mathbf{\Phi}}) = 0,$$

where $\mathcal{D}_{(\alpha)}(x_v,\underline{\mathbf{\Phi}}) = \partial F_{\underline{\mathbf{\Phi}}}\left(x_v\right)/\partial\alpha$, $\mathcal{D}_{(\beta)}(x_v,\underline{\mathbf{\Phi}}) = \partial F_{\underline{\mathbf{\Phi}}}\left(x_v\right)/\partial\beta$ and $\mathcal{D}_{(c)}(x_v,\underline{\mathbf{\Phi}}) = \partial F_{\underline{\mathbf{\Phi}}}\left(x_v\right)/\partial c$ are the first derivatives of the CDF of PBXRW distribution with respect to α, β and c respectively.

3.3. Bootstrapping estimation (Boot-E) method

The wider category of resampling techniques includes bootstrapping, a form of test or measure that uses random sampling with replacement to replicate the sampling procedure. Bootstrapping provides sample estimates with accuracy ratings for bias, variance, confidence intervals, prediction error, and other factors. This method provides estimate of the sample distribution for almost any statistic using random sampling techniques. One popular choice for an approximation distribution is the empirical distribution function of the observed data. When a set of observations can be assumed to come from a separate population with a same distribution, a few resamples with replacement of the observed dataset can be created (and of equal size to the observed dataset). So, the bootstrapping method is a powerful statistical technique which is useful especially when the sample size is small. Under the normal circumstances, sample sizes of less than 40 cannot be dealt with by assuming a "normal" or a "t " distribution. Bootstraping techniques work quite well with samples that have less than 40 observation. The reason for this is that bootstrapping involves resampling. These kinds of techniques assume nothing about the distribution of our data. Bootstrapping has become more popular as computing resources have become more readily available. This is because for bootstrapping to be practical a computer must be used (see Efron and Tibshirani [44] and Hesterberg [66]).

3.4. KE method

The Kolmogorov estimates (KEs) $\widehat{\alpha}, \widehat{\beta}$ and $\widehat{c}$ of α, β and c are obtained by minimizing the function

$$K = \max_{1\leq v\leq n}\left\{\frac{v}{n} - F_{\underline{\mathbf{\Phi}}}\left(x_{v:n}\right), F_{\underline{\mathbf{\Phi}}}\left(x_{v:n}\right) - c_{(v,n)}^{[2]}\right\},$$

where $c_{(v,n)}^{[2]} = \frac{v-1}{n}$.

3.5. The Left-Tail of the Second-Order Anderson Darling method

The Left-Tail of the Second-Order Anderson Darling estimates (LTSOAD) $\widehat{\alpha}_{(\text{LTSOAD})}$, $\widehat{\beta}_{(\text{LTSOAD})}$and $\widehat{c}_{(\text{LTSOAD})}$ of α, β and c are obtained by minimizing

$$\text{LTSOAD}\left(\underline{\mathbf{\Phi}}\right) = 2\sum_{v=1}^{n} \log\left[F_{\underline{\mathbf{\Phi}}}\left(x_{v:n}\right)\right] + \frac{1}{n}\sum_{v=1}^{n} \frac{2i-1}{F_{\underline{\mathbf{\Phi}}}\left(x_{v:n}\right)}.$$

Then, the parameter estimates of $\widehat{\alpha}_{(\text{LTSOAD})}$, $\widehat{\beta}_{(\text{LTSOAD})}$and $\widehat{c}_{(\text{LTSOAD})}$ can be obtained by solving the nonlinear equations

$$\partial\left[\text{LTSOAD}\left(\underline{\mathbf{\Phi}}\right)\right]/\partial\alpha = 0, \partial\left[\text{LTSOAD}\left(\underline{\mathbf{\Phi}}\right)\right]/\partial\beta = 0$$

and

$$\partial\left[\text{LTSOAD}\left(\underline{\mathbf{\Phi}}\right)\right]/\partial c = 0.$$

4. Simulation for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N$= 1000$ generated data sets from the PBXRW version where $n = 50, 100, 150$ and 300. The estimates are compared in terms of their Average values (AVs) and mean squared errors MSEs($\underline{\mathbf{\Phi}}$). From Tables 1, 2, 3 and 4 we note that the MSE($\underline{\mathbf{\Phi}}$) tend to zero when n increases which means incidence of consistency property. The analysis of the tables for different sample sizes ($n =$ 50, 100, 200, 300) reveals distinct patterns in estimation accuracy across various methods as the MLE, CVM, Boot-E, KE, and LTSOAD. The primary goal is to determine the method with the lowest Mean Squared Error (MSE) for each parameter (c, α, β). Generally, MLE emerges as the most reliable method across all sample sizes, consistently exhibiting the lowest MSE in most cases. However, exceptions arise, particularly as sample size increases, where Boot-E and KE occasionally outperform MLE. For small sample sizes ($n = 50$), Boot-E proves advantageous for parameters $\beta = 1.5$, $c = 1.2$, and $\alpha = 0.9$, while MLE remains dominant for the rest. LTSOAD, on the other hand, exhibits significantly higher MSE values across all cases, making it the least preferred method.

As sample size increases to $n = 100$, the trend observed at $n = 50$ largely holds, but some refinements emerge. MLE continues to be the strongest estimator in most cases, but KE begins to show promise, particularly for $\beta =$ 1.5, where it outperforms other methods. Boot-E remains a strong alternative, especially for $c = 0.5$, though it does not consistently surpass MLE in accuracy. The improvement in estimation methods with increasing sample sizes underscores a crucial statistical principle: larger sample sizes tend to reduce estimation errors. Nevertheless, the KE method starts gaining prominence, especially for β-related estimates, suggesting its suitability for certain parameter distributions. CVM remains consistently close to MLE in performance but rarely surpasses it, making it a reasonable alternative but not the best choice overall.

For $n = 200$, KE becomes a serious competitor, particularly for $\beta = 1.5$, where it achieves the lowest MSE. MLE still dominates for most parameters, but Boot-E begins to show its strength, especially for $c = 1.2$ and $\beta =$ 0.5. The increasing accuracy of Boot-E suggests that resampling techniques provide better approximations when sufficient data is available. The persistence of KE as a strong candidate for $\beta = 1.5$ indicates that kernel-based estimation methods may be more effective for certain types of parameter distributions. As expected, LTSOAD continues to exhibit the highest MSE across all parameters, reinforcing its inefficiency compared to other methods. The refinement of estimation methods as sample sizes grow demonstrates the advantages of hybrid approaches, where multiple methods should be considered rather than relying solely on one.

At $n = 300$, the final trend solidifies. MLE remains the dominant method overall, but KE outperforms all others for $\beta = 1.5$ and $c = 1.2$. This result highlights the increasing accuracy of kernel-based methods for larger sample sizes, particularly for specific parameter distributions. Boot-E maintains its usefulness in some cases but does not

Table 1: MSEs for $n = 50$.

Parameters	MLE	CVM	Boot-E	KE	LTSOAD
$c = 0.5$	0.00039	0.00048	0.00048	0.00403	0.00800
$\alpha = 0.5$	0.00018	0.00021	0.00022	0.00020	0.00524
$\beta = 1.5$	0.01539	0.03166	0.00772	0.00805	0.04830
$c = 1.2$	0.00248	0.00286	0.00225	0.00267	0.00709
$\alpha = 0.9$	0.0558	0.00653	0.00504	0.00587	0.01425
$\beta = 0.5$	0.00088	0.00150	0.00090	0.00183	0.00317

Table 2: MSEs for $n = 100$.

Parameters	MLE	CVM	Boot-E	KE	LTSOAD
$c = 0.5$	0.00021	0.00024	0.00020	0.00204	0.01290
$\alpha = 0.5$	0.00009	0.00010	0.00009	0.00010	0.00851
$\beta = 1.5$	0.00773	0.01430	0.01058	0.00367	0.05619
$c = 1.2$	0.00125	0.00143	0.00150	0.00141	0.00374
$\alpha = 0.9$	0.00281	0.00326	0.00349	0.00313	0.00759
$\beta = 0.5$	0.00043	0.00072	0.00053	0.00091	0.00182

Table 3: MSEs for $n = 200$.

Parameters	MLE	CVM	Boot-E	KE	LTSOAD
$c = 0.5$	0.00011	0.00012	0.00039	0.00112	0.02968
$\alpha = 0.5$	0.00005	0.00005	0.00017	0.00006	0.01937
$\beta = 1.5$	0.00431	0.00677	0.00739	0.00202	0.10274
$c = 1.2$	0.00064	0.00066	0.00063	0.00067	0.00189
$\alpha = 0.9$	0.00143	0.00148	0.00143	0.00149	0.00396
$\beta = 0.5$	0.00021	0.00033	0.00020	0.00042	0.00097

consistently outperform MLE. The persistence of MLE's superiority in most cases reinforces its reliability as a default estimation method. However, the shift in the best-performing method for certain parameters as sample sizes grow emphasizes the importance of method selection based on data characteristics. Ultimately, while MLE is the safest and most effective method in general, KE and Boot-E should be considered in cases where their accuracy surpasses MLE, particularly for β-related estimates at higher sample sizes.

5. Real data modeling

5.1. Real data modeling for comparing competitive estimation methods

The first dataset, originally collected by Bjerkedal [37], consists of 72 recorded survival times of Guinea pigs that were injected with varying doses of tubercle bacilli. This dataset has been extensively studied and analyzed in statistical literature, primarily because of two significant characteristics: an increasing-constant failure rate and the presence of extreme observations. These features make the dataset particularly challenging and interesting for fitting probability distributions, which is why numerous researchers have employed various statistical models

Table 4: MSEs for $n = 300$

Parameters	MLE	CVM	Boot-E	KE	LTSOAD
$c = 0.5$	0.00007	0.00008	0.00008	0.00073	0.03904
$\alpha = 0.5$	0.00003	0.00004	0.00003	0.00004	0.02526
$\beta = 1.5$	0.00273	0.00460	0.00301	0.00128	0.12400
$c = 1.2$	0.00048	0.00047	0.00065	0.00046	0.00123
$\alpha = 0.9$	0.00091	0.00107	0.00151	0.00103	0.00257
$\beta = 0.5$	0.00014	0.00023	0.00017	0.00028	0.00068

to analyze it. The increasing-constant failure rate suggests that the risk of failure changes over time, making it an important case for testing the applicability of different survival models. Moreover, the presence of extreme observations can heavily influence parameter estimation and goodness-of-fit assessments, leading researchers to explore robust statistical methods that account for such anomalies (see, for example, Afify et al. [14], Ibrahim et al. [73] and Almazah et al. [23]). In this study, we focus on two widely used goodness-of-fit statistics, namely the Cramér-Von Mises statistic $W^{\cdot}$ and the Anderson-Darling statistic $A^{\cdot}$, to evaluate the adequacy of different probability models in describing the failure times data. These statistics provide insights into how well theoretical distributions align with empirical data, with Anderson-Darling placing more weight on the tails of the distribution, making it particularly relevant for datasets with extreme values. To visually compare the performance of various estimation methods, see also [96], [112] and [97] for more applications.

The second dataset, originally obtained from Smith and Naylor [114], consists of 63 recorded observations of the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory in England. However, the original paper does not specify the exact units of measurement, which has led to some ambiguity in interpreting the results. Despite this limitation, the dataset has been extensively studied and analyzed using various probability distributions due to its intriguing statistical properties. One of the key reasons for its widespread use is the presence of a monotonically increasing failure rate, which suggests that the probability of failure rises consistently as stress levels increase. Additionally, this dataset contains a significant number of extreme observations, making it a valuable test case for evaluating the robustness and applicability of different statistical models. The presence of such outliers presents challenges in parameter estimation and goodness-of-fit analysis, prompting researchers to explore alternative probability distributions that can accurately capture the behavior of the data. As a result, this dataset has become a benchmark in reliability analysis and survival modeling, with numerous studies investigating the best-fitting distributions to describe its failure characteristics. Its significance extends beyond theoretical interest, as understanding the strength distribution of glass fibers has practical implications in materials science and engineering, particularly in predicting material failure under stress.

Table 5 presents the estimated values of the parameters c, α, and β along with the goodness-of-fit statistics $W^{\cdot}$ (Cramér–von Mises) and $A^{\cdot}$ (Anderson-Darling) for the two real datasets under different estimation methods. For Dataset I, the Maximum Likelihood Estimation (MLE), Cramér–von Mises Estimation (CVME), and Bootstrap estimation (Boot-E) methods yield relatively close parameter estimates, with $W^{\cdot}$ and $A^{\cdot}$ values suggesting moderate model fit. The KE method, however, produces significantly lower parameter estimates, particularly for c and α, and achieves the smallest goodness-of-fit statistics, indicating a better overall fit. The Least Trimmed Squares with Adaptive Outlier Detection (LTSOAD) method shows results comparable to CVME and Boot-E, but its parameter estimates suggest a different model structure. For Dataset II, the MLE method results in substantially higher values for both $W^{\cdot}$ and $A^{\cdot}$, indicating a poor fit to the data. KE again produces significantly smaller parameter estimates, but its goodness-of-fit statistics, particularly $A^{\cdot}$, remain higher than in Dataset I, suggesting that extreme values in Dataset II impact model performance. CVME and Boot-E show similar behavior, with moderate values of $W^{\cdot}$ and $A^{\cdot}$, indicating a better fit compared to MLE. Interestingly, LTSOAD produces the lowest parameter estimates for c, α, and β, but its $W^{\cdot}$ and $A^{\cdot}$ values remain relatively high. Overall, KE appears to provide the best fit for Dataset I,

Table 5: The values of estimators A$^{\cdot}$ and W$^{\cdot}$ under the two real data sets.

		Estimates			Statistics	
Data	Methods	$\widehat{c}$	$\widehat{\alpha}$	$\widehat{\beta}$	W$^{\cdot}$	A$^{\cdot}$
I	MLE	21.93470	0.01719	0.40782	0.19663	1.06055
	CVME	24.96021	0.03898	0.47268	0.21103	1.13882
	KE	0.72308	0.00250	0.00372	0.15782	0.85519
	Boot-E	24.62419	0.02036	0.42666	0.19918	1.07448
	LTSOAD	17.91781	0.04404	0.43308	0.20148	1.08674
II	MLE	33.02478	0.06889	1.25409	7.58831	45.6557
	CVME	43.81864	0.18962	1.96449	0.53241	2.89427
	KE	0.71447	0.00250	0.00373	0.70052	3.82839
	Boot-E	43.70824	0.07406	1.36327	0.56032	3.06441
	LTSOAD	12.56901	0.04271	0.79994	0.61110	3.34998

as indicated by its smallest W$^{\cdot}$ and A$^{\cdot}$ values. However, for Dataset II, none of the methods produce a particularly strong fit, though CVME and Boot-E demonstrate improved performance compared to MLE. This suggests that while KE is effective in datasets with fewer extreme observations, datasets with a high number of outliers or an increasing failure rate may benefit from alternative robust estimation techniques such as CVME or Boot-E.

5.2. Real data modeling for comparing competitive distributions

In this section, we present two practical applications of the PBXRW distribution using real data sets to evaluate its performance and applicability. The first application involves comparing the PBXRW distribution with several closely related models, including the odd log-logistic RW (OLLRW), the Marshall-Olkin RW (MORW), the Kumaraswamy RW (KRW), the beta RW (BRW), the Kumaraswamy Marshall-Olkin reciprocal exponential (KMORE), the Kumaraswamy Marshall-Olkin reciprocal Rayleigh (KMORR), and the RW distributions. This comparative analysis aims to determine the best-fitting model for the given dataset by analyzing statistical measures and fit criteria. The inclusion of multiple competing models provides a comprehensive framework to assess whether the PBXRW distribution offers improved flexibility in capturing the behavior of the data, particularly in scenarios where the failure rate is complex or exhibits variations over time. In the second application, we extend the analysis to another real dataset, comparing the PBXRW distribution against a subset of related models, namely the MORW, BRW, KMOIR, and RW distributions. This second evaluation allows us to examine how the PBXRW model performs under different data conditions and whether it maintains its effectiveness across various scenarios. By employing goodness-of-fit statistics, probability plots, and model selection criteria, we aim to provide insights into the strengths and limitations of the PBXRW distribution relative to alternative models. Ultimately, these applications help establish whether the PBXRW distribution serves as a robust and reliable choice for modeling real-world failure time data.

In this section, we present a comprehensive graphical and numerical analysis of the two real data sets under consideration. The visual representations, including the total time on test (TTT) plot, quantile-quantile (Q-Q) plot, box plot, and nonparametric Kernel density estimation (KDE) plot for the first data set, are illustrated in Figure 3. The TTT plot provides insights into the empirical hazard rate function (HRF), which for the first data set exhibits an ”upside-down then increasing” pattern, indicating a non-monotonic behavior in failure rates. The KDE plot further supports this observation, revealing a bimodal distribution that is right-skewed with a heavy tail, suggesting the presence of significant variations and extreme values in the data. Similarly, the TTT plot, Q-Q plot, box plot, and KDE plot for the second real data set are depicted in Figure 4. In this case, the TTT plot suggests an ”increasing” empirical HRF, indicating a failure rate that grows over time. The KDE plot for this data set also exhibits a bimodal distribution but is left-skewed with a heavy tail, highlighting the presence of extreme values on the lower end of the distribution. To rigorously compare different probability distributions and assess

their fit to the data, we employ several well-established model selection criteria, including the maximized log-likelihood (-2ℓ), the Akaike Information Criterion ($Cr^{[1]}$), the consistent Akaike Information Criterion ($Cr^{[5]}$), the Bayesian Information Criterion ($Cr^{[2]}$), and the Hannan-Quinn Information Criterion ($Cr^{[3]}$). These criteria provide objective measures to determine the best-fitting model for each data set. Tables 6 and 8 summarize the maximum likelihood estimates (MLEs) along with their corresponding standard errors (SEs), ensuring statistical reliability in parameter estimation. Additionally, Tables 7 and 9 present the numerical values of the model selection criteria (-2ℓ, $Cr^{[1]}$, $Cr^{[2]}$, $Cr^{[3]}$, and $Cr^{[5]}$), allowing for a comparative evaluation of different models. Finally, Figures 5 and 6 illustrate the fitted probability density function (PDF), cumulative distribution function (CDF), and hazard rate function (HRF) for the two data sets, respectively. These visualizations offer a clear representation of how well each model captures the underlying data structure, further supporting our analysis.

Table 6 presents the maximum likelihood estimates (MLEs) and their corresponding standard errors (SEs) for the first data set across different models. The PBXRW model estimates c =0.01719 and α =0.40782 with relatively small SEs. The OLLGRW model shows a high estimate for β at 13.9901, though its SE is quite large. The KRW model has an extremely high value for β =45.7326, while BRW estimates α =19.9786 with significant standard errors. The KMORE model exhibits a notably high β value of 68.1393, with small SEs for other parameters. KMORR and MORW models estimate c around $1.6 - 1.8$ and 3b13b1 in the range of $13 - 58$. The RW model has the highest estimate for c =54.1888 but lacks values for α and β. Each model displays considerable variation in parameter estimates, indicating differences in their adaptability to the data. The standard errors also vary widely, suggesting different levels of confidence in the estimated parameters.

Table 7 provides the values of $-2\widehat{\ell}$, $Cr^{[1]}$, $Cr^{[2]}$, $Cr^{[3]}$ and $Cr^{[5]}$ for the first data set using different models. The PBXRW model achieves the lowest -2ℓ value of 674.77, making it the best-fitting model based on all criteria. In terms of AIC, BIC, HQIC, and CAIC, it continues to perform the best, with scores of 680.77, 687.60, 683.49, and 681.12, respectively. The OLLGRW model has higher values, with the -2ℓ ranging from 779.27 to 780.51 and its respective scores showing a similarly increased complexity. The KRW, BRW, and KMORE models show marginally worse fits than OLLGRW, with -2ℓ values between 780.51 and 782.75. The RW model's results are even higher, indicating a poorer fit, with its -2ℓ value of 791.35 and higher penalty values. The MORW model, although having a similar range of values, still scores higher than PBXRW. Finally, the KMORR model demonstrates the worst performance, with the highest -2ℓ value of 800.26, along with the highest AIC, BIC, HQIC, and CAIC scores, indicating that this model is the least suitable for the data. Overall, PBXRW stands out as the best model in terms of fit and parsimony.

Table 8 presents the MLEs and their corresponding SEs for the second data set across various models. For the PBXRW model, the estimated parameters with their respective standard errors being relatively large for θ (34.099), but smaller for c (0.0161) and α (0.1170). The OLLGRW model shows a much higher value for θ (28.31) with significant standard errors, particularly for θ (17.17), which suggests a less stable estimation. The BRW model displays some unusual results with very high estimates for 3b13b1 and β (19.591 and 30.411, respectively), along with larger standard errors, indicating a poor fit. The KMORR model provides relatively reasonable estimates, with θ =1 and smaller standard errors for its parameters, especially for c (0.079) and α (0.034). The MORW model presents lower values for its parameters and standard errors, suggesting a more stable fit, though not optimal. Lastly, the RW model gives estimates for θ (2.888) and c (1.264), with moderate standard errors, indicating a reasonable model fit but with some uncertainty in the parameters. Overall, the PBXRW model provides the most stable and well-fitting estimates, while models like BRW exhibit high uncertainty in their parameter estimates.

Table 9 presents the model comparison metrics for the second data set, including values for $-2\widehat{\ell}$, $Cr^{[1]}$, $Cr^{[2]}$, $Cr^{[3]}$ and $Cr^{[5]}$ for various models. Among the models, PBXRW exhibits the lowest -2ℓ value (48.036) and comparatively favorable criteria values across all information criteria, particularly in $Cr^{[1]}$ (54.036), $Cr^{[2]}$ (60.466), $Cr^{[3]}$ (56.565 and $Cr^{[5]}$ (54.443), which suggest it provides the best fit for the data. The BRW and KMORR models show even higher values for -2ℓ, with BRW reaching 61.74 and KMORR 67.32, and both exhibit poor performance across the information criteria, with $Cr^{[2]}$ and $Cr^{[3]}$ values rising significantly. The RW and MORW models provide

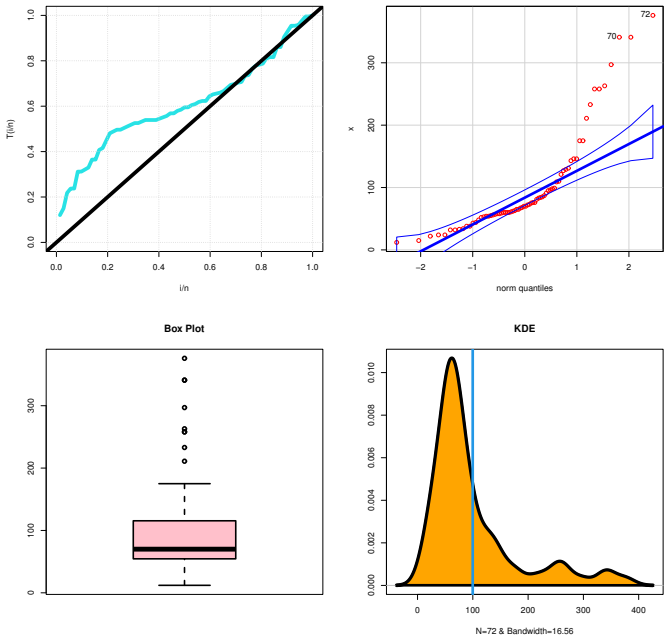

Figure 3. TTT, QQ, box, KDE plots for the 1^{st} real data.

the worst fits, as indicated by their very high -2ℓ values (93.74 and 95.76, respectively) and elevated information criteria values, making them the least favorable options. Thus, the PBXRW model consistently stands out as the most appropriate choice for fitting this data set, based on the log-likelihood and information criteria metrics.

The PBXRW model proves to be the best fit for the data, as demonstrated by its significantly lower -2ℓ value compared to other models, indicating a superior fit. It also performs best in the information criteria, particularly in $Cr^{[2]}$, with the lowest values across all criteria. In contrast, models like OLLGRW, BRW, and KMORR show higher -2ℓ and information criterion values, signaling poorer performance. The PBXRW model strikes an optimal balance between fit and model complexity, ensuring that it captures the data effectively without overfitting. This makes PBXRW the most suitable model for the given datasets, outperforming other alternatives in both theoretical and practical aspects. Therefore, based on both statistical measures and model comparison, PBXRW is the best model for the data.

For the first application, the result of the likelihood ratio test (LRT) comparing the PBXRW and OLLGRW models shows an LRT statistic of -104.5 with a p-value of 1. Typically, a negative LRT statistic indicates an issue, as the unrestricted model (with more parameters) should always fit the data at least as well as the restricted one. In other words, the $-2\widehat{\ell}$ of the simpler model should be greater than or equal to that of the more complex model. The p-value of 1 suggests that there is no significant evidence to favor the more complex OLLGRW model over the simpler PBXRW model. This could indicate a mistake in model specification, parameter counting, or a misinterpretation of nested model relationships. It is recommended to re-evaluate the model definitions, ensure that PBXRW is truly nested within OLLGRW, and double-check the $-2\widehat{\ell}$ values. Proper nesting is crucial for a valid LRT, so revisiting these aspects will help ensure accurate results.

For the second application, the LRT between the PBXRW and BRW models shows an LRT statistic of -13.704 with a p-value of 1. In the context of LRT, a negative test statistic generally indicates a miscalculation or an incorrect nesting of the models. By definition, the restricted (simpler) model should have a higher $-2\widehat{\ell}$ value, leading to a

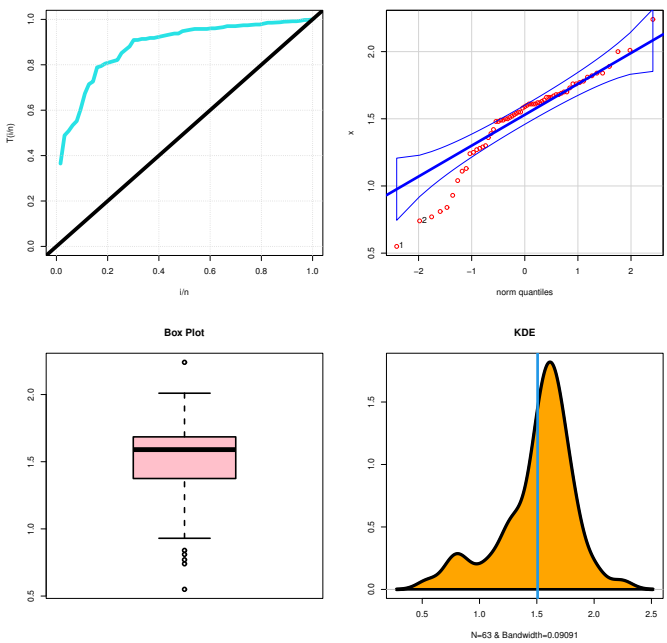

Figure 4. TTT, QQ, box, KDE plots for the 2^{nd} real data.

Table 6: MLEs and their SEs for the 1^{st} data.

Model	$\widehat{\theta}$	$\widehat{c}$	$\widehat{\alpha}$	$\widehat{\beta}$
PBXRW		21.9347	0.01719	0.40782
		(9.9682)	(0.01859)	(0.0334)
OLLGRW	4.7989	1.3108	13.9901	0.38
	(5.1585)	(1.889)	(55.23)	(0.404)
KRW	0.6207	0.7111	45.7326	8.2723
	(0.003)	(0.013)	(0.092)	(0.979)
BRW	0.322	24.5032	19.9786	20.1331
	(0.0012)	(0.087)	(7.246)	(7.26)
KMORE	8.8727	0.1758	68.1393	2.6258
	(1.174)	(0.000)	(0.020)	(0.512)
KMORR	9.993	1.6788	58.4697	0.6389
	(1.972)	(0.001)	(0.105)	(0.098)
MORW	14.9816	1.7855	13.991	
	(4.631)	(0.193)	(2.96)	
RW	1.4148	54.1888		
	(0.003)	(0.111)		

non-negative LRT statistic. The fact that the p-value is 1 further suggests no significant evidence against the simpler PBXRW model. This outcome may imply a mistake in specifying the model parameters or mislabeling the models as nested. It's essential to double-check the parameter counts, ensure proper nesting, and re-evaluate the $-2\widehat{\ell}$ values to validate the comparison. If the models are not truly nested, an LRT is not appropriate.

Table 7: $-2\widehat{\ell}$, $Cr^{[1]}$, $Cr^{[2]}$, $Cr^{[3]}$ and $Cr^{[5]}$ for 1^{st} data.

Model	$-2\widehat{\ell}$	$Cr^{[1]}$	$Cr^{[2]}$	$Cr^{[3]}$	$Cr^{[5]}$
PBXRW	674.77	680.77	687.60	683.49	681.12
OLLGRW	779.27	787.49	796.56	791.00	788.79
KRW	780.51	788.57	797.67	792.16	789.19
BRW	780.62	788.67	797.74	792.38	789.28
KMORE	782.75	790.74	799.85	794.34	791.36
RW	791.35	795.37	799.95	797.14	795.54
MORW	790.17	796.14	802.96	798.83	796.59
KMORR	800.26	808.27	817.37	811.82	808.81

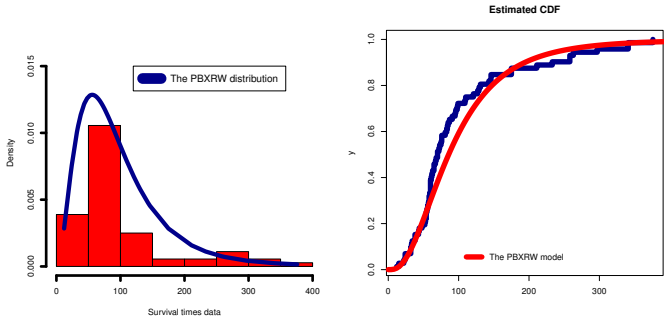

Figure 5. The fitted PDF, CDF and estimated HRF and for the first data set.

Table 8: MLEs and their SEs for the 2^{nd} data.

Model	$\widehat{\theta}$	$\widehat{c}$	$\widehat{\alpha}$	$\widehat{\beta}$
PBXRW		33.02478	0.06889	1.25409
		(34.099)	(0.0161)	(0.1170)
BRW	0.685	1.331	19.591	30.411
	(0.181)	(1.085)	(18.115)	(18.238)
KMORR	1	2.7498	0.5971	5.7974
	(0.192)	(0.079)	(0.034)	(0.008)
MORW	0.4816	2.3876	1.5441	
	(0.252)	(0.253)	(0.226)	
RW	2.888	1.264		
	(0.234)	(0.059)		

Table 9: $-2\widehat{\ell}$, $Cr^{[1]}$, $Cr^{[2]}$, $Cr^{[3]}$ and $Cr^{[5]}$ for 2^{nd} data.

Model	$-2\widehat{\ell}$	$Cr^{[1]}$	$Cr^{[2]}$	$Cr^{[3]}$	$Cr^{[5]}$
PBXRW	48.036	54.036	60.466	56.565	54.443
BRW	61.74	69.695	78.295	73.095	70.436
KMORR	67.32	75.337	83.883	78.735	76.457
RW	93.74	97.737	102.19	99.425	97.918
MORW	95.76	101.73	108.29	104.46	102.18

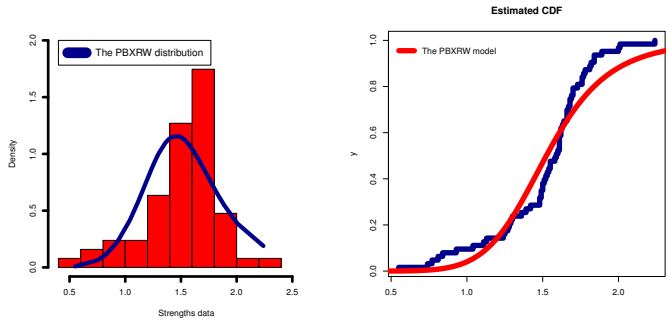

Figure 6. The fitted PDF, CDF and estimated HRF and for the second data set.

6. Sequential sampling plan based on truncated life test

A sequential sampling plan is a statistical technique widely used in various industries, including manufacturing, pharmaceuticals, and agriculture, quality control and reliability testing. Unlike standard fixed-sample size plans, sequential sampling allows you to make decisions about a batch or population after analyzing each item or a small group of them. This dynamic technique can result in significant reductions in inspection labor and time. The fundamental principle is that decisions are made sequentially, using the accumulated information obtained during inspection. Following each examination, a determination is made to either: Accept the batch, Reject the batch or Continue sampling. In many cases, sequential sampling might result in a smaller average sample size compared to fixed-sample plans (single- double- multiple) This is because decisions can be made early when strong evidence of acceptance or rejection accumulates. The sequential sampling plan is distinguished by the potential for lower sample size, faster decision-making in many circumstances, and efficient use of resources. Implementation may be more challenging than with fixed-sample plans, and the number of inspections is not fixed, making planning tricky. The process follows these steps:

Start: The process begins.

Take a unit of sample: A sample is taken for evaluation.

Decision Based on $d(n)$: If

$$\sum_{n=1}^{z} d(n) \geq x_R,$$

the sample is rejected. If

$$\sum_{n=1}^{z} d(n) \leq x_A,$$

the sample is accepted. If

$$x_A < \sum_{n=1}^{z} d(n) < x_R,$$

the sample is retested, and the process repeats until a decision is reached.

In numerous industry sectors, waiting for all units to fail is unworkable. Therefore, the goal of research is to design time truncated tests in which choices are made within a specific timeframe (for example, testing a product for 500 hours rather than waiting for its whole lifecycle). Allow industries to detect early failures and keep defective products from reaching customers. Research on sequential sampling for the PBXRW distribution contributes to better quality assurance, reduced costs, and improved reliability predictions across various industries. It provides scientifically backed methodologies for making faster and more accurate acceptance/rejection decisions

while balancing risks and efficiency. The purpose of this section is to obtain design parameters for sequential sampling plans. Suppose that the lifetimes of the test units follow PBXRW (c, α, β) distribution which has the CDF given in (5). It is presumed that the shape parameter is known.

When choosing an acceptance sampling plan, practitioners are likely to be familiar with the model parameters. Most manufacturers use an approximate shape parameter for each product. So, when the shape parameter is unknown, they can utilize the estimated value Aslam and Jun [32]. Designing a sampling plan ensures that the product's lifetime median is at least υ_0. The lot is accepted if there is enough evidence that $\upsilon \geq \upsilon_0$ at a particular degree of the producer's risk $\mathcal{P}^*$ and consumer's risk C^*. It is easier to determine the termination time x_0 as a multiple of the stated life υ_0. To simplify, we will consider $x_0 = \varepsilon\upsilon_0$ for a constant ε (termination time ratio), for example, $\varepsilon = 0.25$, indicates that the experiment time is just quarter the stipulated life. The CDF of the PBXRW (c, α, β) distribution at x_0 is used to calculate the probability of failure for each item as follows

$$\Pi = \frac{1}{\varrho\,[1]}\left[1-\exp\left(-\left\{1-\exp\left[-\left[\frac{\varsigma\left(\frac{\upsilon}{\upsilon_0}|\frac{E}{C};\varepsilon,c,\beta\right)}{1-\varsigma\left(\frac{\upsilon}{\upsilon_0}|\frac{E}{C};\varepsilon,c,\beta\right)}\right]^2\right]\right\}^2\right)\right], \tag{17}$$

where

$$\varsigma\left(\frac{\upsilon}{\upsilon_0}|\frac{E}{C};\varepsilon,c,\beta\right) = \exp\left(-\left\{\frac{\frac{\upsilon}{\upsilon_0}}{\varepsilon\beta\left[-\ln(\frac{E}{C})\right]^{\frac{1}{2}}}\right\}^{\beta}c\right).$$

The quality of an item is often indicated by its real median/mean or, quantiles are occasionally employed. We can express the quality level of a product in terms of ratio of its median lifetime to the specified life

$$\mathcal{M} = \frac{1}{\upsilon_0}\upsilon.$$

Many studies depend on the SSPs such as Rasay et al. [99], Liu, et al. [82], Zoramawa et al. [136], Zoramawa et al.[137], singh and Buttar [113] and Opperman and Ning [95].

6.1. Design of Sequential sampling plan

Designing the SSPs is the process of developing an organized procedure for examining or testing units from a batch in a step-by-step manner. Instead of deciding based on a predetermined sample size, sequential sampling allows decisions (accept, reject, or continue sampling) to be made incrementally as data is gathered about number of defects. The design method comprises determining the values of x_A and x_R that meet consumer risk $\mathcal{C}^*$ at a lot of tolerance reliability level (LTRL $= \pi_1$) and producer risk $\mathcal{P}^*$ at an acceptable reliability level (ARL $= \pi_2$). Wald [121] and the Statistical Research Group [115] determined that four values must be specified before constructing a sequential-sampling plan. They are π_1, π_2, $\mathcal{P}^*$, and $\mathcal{C}^*$. Using these values, the following equation can be used to generate the two lines that divide the sequential-sampling chart into three regions:

$$\text{Lower Line} \qquad x_A = nS - h_1,$$

$$\text{Upper Line} \qquad x_A = nS + h_2$$

The continuation region for SPRT of strength $(\pi_1, \mathcal{P}^*,\ \pi_2, \mathcal{C}^*\ )$ is given by the inequality

$$x_A < d_{(n)} < x_R,$$

where
$d_{(n)}$:The number of defects observed during inspection of the first n unit.
x_A : Acceptance limit line

x_R : Rejection limit line
n : Sequential unit taken from the lot of sample items taken
S : Equilibrium quality or slope of the acceptance and rejection boundaries, where

$$S = S_{\pi_1,\pi_2} = \frac{1}{\varrho_{\pi_1,\pi_2}} \ln\left[\frac{1}{1-\pi_2}(1-\pi_1)\right].$$

h_1: Intercept of the acceptance line, where

$$h_1 = h_{1,\mathcal{C}^*,\mathcal{P}^*} = \frac{1}{\varrho_{\pi_1,\pi_2}} \ln\left[\frac{1}{\mathcal{C}^*}(1-\mathcal{P}^*)\right].$$

h_2 : Intercept of the rejection lines, where

$$h_2 = h_{2,\mathcal{C}^*,\mathcal{P}^*} = \frac{1}{\varrho_{\pi_1,\pi_2}} \ln\left[\frac{1}{\mathcal{P}^*}(1-\mathcal{C}^*)\right].$$

and

$$\varrho = \varrho_{\pi_1,\pi_2} = \ln\left[\frac{1}{\pi_1(1-\pi_2)}\pi_2(1-\pi_1)\right]$$

With these metrics, SSPs for attribute examinations may be readily constructed and specified.

6.2. Performance measures of the SSps

Performance metrics help to monitor how each sequential sampling plan is being implemented. The following formulae can be used to compute the acceptance lot (L_a), non-conforming fraction (Π), and average sample number (ASN), where

$$L_a = \frac{\left[\frac{1}{\mathcal{P}^*}(1-\mathcal{C}^*)\right]^\delta - 1}{\left[\frac{1}{\mathcal{P}^*}(1-\mathcal{C}^*)\right]^\delta - \left[\frac{1}{(1-\mathcal{P}^*)}\mathcal{C}^*\right]^\delta}, \tag{18}$$

$$\Pi = \frac{1-\left[\frac{1}{(1-\pi_1)}(1-\pi_2)\right]^\delta}{\left(\frac{1}{\pi_1}\pi_2\right)^\delta - \left[\frac{1}{(1-\pi_1)}(1-\pi_2)\right]^\delta} | -\infty < \delta < \infty, \delta \neq 0. \tag{19}$$

These two equations determine different locations on the operating characteristic (OC) function for varying δ values. The three values of δ are of special relevance. For $\delta = 1$, $\Pi = \pi_1$ and $L_a = (1-\mathcal{P}^*)$. For $\delta = 0$, $\Pi = S$ and $L_a = \frac{1}{h_1+h_2}h_2$. For $\delta = -1$, $\Pi = \pi_2$ and $L_a = \mathcal{C}^*$. By combining L_a and Π, we may obtain the generic formula for ASN.

$$ASN = \frac{L_a \log\left[\frac{1}{(1-\mathcal{P}^*)}\mathcal{C}^*\right] + (1-L_a)\log\left[\frac{1}{\mathcal{P}^*}(1-\mathcal{C}^*)\right]}{\Pi \log\left(\frac{1}{\pi_1}\pi_2\right) + (1-\Pi)\log\left[\frac{1}{1-\pi_1}(1-\pi_2)\right]}. \tag{20}$$

Where L_a and Π are the probability of acceptance lot and the failure probability. The ASN in a sequential sampling plan serves as a standard for maximizing inspection efficiency, managing expenses, and guaranteeing consistent quality control. The following optimization problem is used to derive SSP parameters.

Minimum ASN
Subject to:

$$L_a\left(\pi_1 \left| \frac{1}{v_0}v = \mathcal{M}_1; \mathcal{C}^*\right.\right) \leq \mathcal{C}^*,$$

$$L_a\left(\pi_2\left|\frac{1}{v_0}v=\ \mathcal{M}_2;\mathcal{P}^*\right.\right)\geq 1-\mathcal{P}^*,$$

$$\mathcal{M}_1<\mathcal{M}_2 \text{ and } v\geq v_0.$$

The parameters $\mathcal{M}_1$ and $\mathcal{M}_2$ represent the life ratios at the consumer and producer risks, respectively. In this investigation, the ratio $\mathcal{M}_1$ was fixed to one. The products lifetime is considered a quality attribute and is expressed using the ratio $\mathcal{M}$. To compute the probability of batch acceptance in the OC function, the ratio $\mathcal{M}$ is employed. Increasing $\mathcal{M}$ increases the probability of batch acceptance. The technique suggested to calculate ASN and L_a is as follows:

1. Use (17) to determine the value of Π for a given $\mathcal{M}$.
2. Apply the value of Π to (19) to obtain the equivalent value of δ.
3. The value of L_a is calculated using the value of δ from (18).
4. To calculate ASN, enter the values of L_a and Π into (20).

6.3. An illustrative example

The numerical outcomes for X ˜ PBXRW $(x;c,\beta)$ when $\beta=0.5$ and $c=2$ are given in Tables 10, 11 and 12. Table 10 displays the results of acceptance and rejection limit lines for SSPs with parameter $\beta=0.5$ and $c=2$ under median of PBXRW. Tables 11 and 12 display the results of OC and ASN for SSPs with parameter $\beta=0.5$ and $c=2$ under median of PBXRW when $\mathcal{M}_2=2$ and 4 respectively.

Assume that the lifetime of a product follows a PBXRW distribution with the shape parameters $\beta=0.5$ and $c=2$. A manufacturer of bulbs wants to ensure that their products meet a certain lifespan standard. They need to verify that the median lifespan of their product exceeds a specified Target lifespan of 10 hours. Traditional life testing can be time-consuming, so they want to use a sequential sampling plan to make decisions efficiently. The customer prefers the risk of accepting the lot with the median value of 1000 to become less than 0.25, and the manufacturer wants the risk of refusing the lot with the median value of 2,000 to become less than 0.05. The test will be stopped at 500 hours (Truncation time). Based on this information, it is found that $v_0=1000h$, $\varepsilon=0.5$, ARL $=2000h$, LTRL $=1000$, $\mathcal{M}_1=1$, $\mathcal{M}_2=2$, $\mathcal{C}^*=0.25$ and $\mathcal{P}^*=0.05$. To set up the test: A sample of bulbs is selected randomly from the production lot. Each bulb is placed under continuous operation in a controlled environment. The time until each bulb fails is recorded. As bulbs fail, the cumulative failure data is analyzed. After each failure (or after a predetermined time interval), a decision is made:

1. If the data is equal to or greater than $2000h$ falls within the acceptance region, the lot is accepted, and testing is stopped.
2. If the data is less than 1000 hours falls within the rejection region, the lot is rejected, and testing is stopped.
3. If the data falls within the continuation region, another bulb is tested, and the process repeats.

If a decision hasnt been reached before X, the test is stopped, and a final decision is made based on the data collected up to that point. This prevents excessively long testing times. The initial operation of designing a SSPs in this test is to compute calculate π_1, π_2. From the equations of S, h_1, h_2, and ϱ are $0.091, 0.631, 1.281$ and 2.115, respectively. The acceptance limit line (x_A) and rejection limit line (x_R) are as follows:

$$x_A=0.091\ n-0.631, \tag{21}$$

and

$$x_R=0.091\ n+1.281. \tag{22}$$

The results of x_A and x_R for the SSPs are illustrated in Table 10. For example, take the situation of obtaining the acceptance and rejection limit line for $n = 10$. Substituting $n = 10$ into (21) and (22) gets the following

$$x_A = 0.091\ (10) - 0.631 = 0.279$$

and

$$x_R = 0.091\ (10) + 1.281 = 2.191.$$

Table 10 presents the acceptance limit (x_A) and rejection limit (x_R) for SSPs under a PBXRW distribution with parameters $\beta = 0.5$ and $c = 2$. The table outlines decision thresholds based on sample size (n), indicating whether a batch should be accepted, rejected, or require additional sampling. For smaller sample sizes ($n = 1$ to 6), the x_A remains undefined ("$\varnothing$"), suggesting that a decision cannot be made until further samples are tested. As n increases, the acceptance and rejection limits rise progressively, reflecting more stringent criteria for larger sample sizes. For instance, at $n = 10$, the acceptance and rejection limits are $x_A = 0$ and $x_R = 2$, whereas at $n = 60$, they reach $x_A = 5$ and $x_R = 7$, respectively. This structured increase ensures that decisions are dynamically adjusted based on accumulated data, optimizing inspection efforts while maintaining reliability standards. The table is instrumental in practical applications of SSPs, particularly in manufacturing, reliability testing, and quality control, where efficient decision-making is critical in identifying defective products while minimizing unnecessary sampling efforts. Table 10 presents the acceptance (x_A) and rejection (x_R) limit lines for sequential sampling plans under the median of the PBXRW distribution with parameters β =0.5 and c =2. The values indicate the decision thresholds at different sample sizes (n), where a batch is either accepted, rejected, or requires further sampling. Notably, the acceptance limit is undefined for small sample sizes ($n = 1$ to 6), suggesting that acceptance is not possible at these early stages. Additionally, the rejection limit at $n = 20$(x_R =37) appears significantly higher than its neighboring values, which may require verification or further explanation. The table is structured in a compact format, but a clearer presentation, such as better column alignment, could improve readability. Moreover, an explanation of why the range extends to $n = 60$ and how these thresholds influence real-world decision-making in quality control or reliability testing would enhance its practical relevance.

Table 10: The results of acceptance and rejection limit lines for SSPs under median PBXRW distribution with parameters $\beta = 0.5$ and $c = 2$.

n	x_A	x_R	n	x_A	x_R	n	x_A	x_R	n	x_A	x_R	n	x_A	x_R	n	x_A	x_R
1	$\varnothing$	1	11	0	2	21	1	3	31	2	4	41	3	5	51	4	6
2	$\varnothing$	1	12	0	2	22	1	3	32	2	4	42	3	5	52	4	6
3	$\varnothing$	2	13	1	2	23	1	3	33	2	4	43	3	5	53	4	6
4	$\varnothing$	2	14	1	3	24	2	3	34	2	4	44	3	5	54	4	6
5	$\varnothing$	2	15	1	3	25	2	4	35	3	4	45	3	5	55	4	6
6	$\varnothing$	2	16	1	3	26	2	4	36	3	5	46	4	5	56	4	6
7	0	2	17	1	3	27	2	4	37	3	5	47	4	6	57	5	6
8	0	2	18	1	3	28	2	4	38	3	5	48	4	6	58	5	7
9	0	2	19	1	3	29	2	4	39	3	5	49	4	6	59	5	7
10	0	2	20	1	37	30	2	4	40	3	5	50	4	6	60	5	7

Cells with ($\varnothing$) in Table 10 indicate that acceptance is not allowed at stage i. Also, the acceptance and rejection limits must be integers. Thus, the values of x_A and x_R are rounded to the closest whole. Thus, the acceptance and rejection limits for $n = 10$ are 0 and 2, respectively. Given this outcome, for $n = 10$, If the total number of faulty items until this phase is zero, accept the batch; if the total number of recorded faulty items up to this point is one, the sampling procedure should be continued; otherwise, if the total number of faulty items until this phase is two or more, reject the batch.

Table 11 presents the OC values and ASN for the SSPs under the median PBXRW distribution with parameters $\beta = 0.5, c = 2$, and $M_{\mathbf{2}} = 2.0$. The table is structured based on different levels of consumer risk ($\mathcal{C}^*$), varying from

Table 11: OC value and ASN for SSPs under median PBXRW at $\beta = 0.5$, $c = 2$ and $M_{\mathbf{2}} = 2.0$.

		$\varepsilon = 0.5$				$\varepsilon = 1.0$			
		$\mathcal{P}^* = 0.05$		$\mathcal{P}^* = 0.01$		$\mathcal{P}^* = 0.05$		$\mathcal{P}^* = 0.01$	
$\mathcal{C}^*$	$\mathcal{M}$	OC	ASN	OC	ASN	OC	ASN	OC	ASN
0.25	2	0.95	8.696	0.99	10.127	0.95	2.992	0.99	3.484
	4	0.999	7.127	1	7.360	1	1.904	1	1.965
	6	1	6.995	1	7.212	1	1.798	1	1.854
	8	1	6.976	1	7.191	1	1.777	1	1.832
	10	1	6.972	1	7.187	1	1.771	1	1.825
0.1	2	0.95	15.308	0.99	17.076	0.95	5.267	0.99	5.876
	4	1	12.03	1	12.261	1	3.213	1	3.273
	6	1	11.797	1	12.014	1	3.032	1	3.088
	8	1	11.764	1	11.979	1	2.996	1	3.051
	10	1	11.757	1	11.972	1	2.986	1	3.041
0.05	2	0.95	20.342	0.99	22.34	0.95	6.999	0.99	7.686
	4	1	15.737	1	15.968	1	4.203	1	4.263
	6	1	15.43	1	15.646	1	3.966	1	4.022
	8	1	15.386	1	15.601	1	3.919	1	3.974
	10	1	15.377	1	15.592	1	3.905	1	3.96
0.01	2	0.95	32.062	0.99	34.567	0.95	11.032	0.99	11.894
	4	1	24.343	1	24.575	1	6.50	1	6.561
	6	1	23.864	1	24.080	1	6.134	1	6.189
	8	1	23.796	1	24.011	1	6.061	1	6.115
	10	1	23.782	1	23.997	1	6.040	1	6.095

0.25 to 0.01, and includes results for two error levels (ε =0.5 and ε =1.0). For each combination of parameters, the OC value represents the probability of accepting a lot, while ASN indicates the expected sample size required to reach a decision. As $\mathcal{C}^*$ decreases (indicating a stricter sampling requirement), ASN values generally increase, demonstrating the need for more extensive sampling to ensure compliance. Similarly, lower producer risk ($\mathcal{P}^*$) results in a slightly larger ASN, ensuring better discrimination between conforming and non-conforming lots. For instance, at $\mathcal{C}^*$ =0.01 and $\mathcal{P}^*$ =0.01, the ASN reaches its highest values, such as 34.567 for ε =0.5 and 11.894 for ε =1.0, reflecting the rigorous nature of the sampling process under these conditions. Meanwhile, OC values approach 1 as $\mathcal{M}$ increases, indicating that nearly all lots are accepted under larger sample sizes, ensuring high reliability. The table effectively captures the trade-off between OC and ASN, emphasizing the balance between minimizing sample size and maintaining stringent acceptance criteria. However, while the data presentation is comprehensive, it could benefit from additional explanation regarding the practical implications of the findings, especially for real applications in reliability testing or quality control. Furthermore, a graphical representation of the relationship between ASN and different risk levels might improve clarity and aid in visual interpretation.

Table 12 presents the OC values and ASN for the SSPs under the median PBXRW distribution with parameters $\beta = 0.5, c = 2$, and $M_{\mathbf{2}} = 4.0$. Similar to the previous table, this one examines different consumer risk levels ($\mathcal{C}^*$) ranging from 0.25 to 0.01 under two error conditions (ε =0.5 and ε =1.0), along with varying producer risks ($\mathcal{P}^*$ =0.05 and $\mathcal{P}^*$ =0.01). The results show that as $\mathcal{C}^*$ decreases, the ASN generally increases, reflecting the need for larger sample sizes in more stringent quality control conditions. At higher sample sizes, the OC values approach 1, indicating a greater likelihood of lot acceptance, especially for more lenient conditions. Notably, for ε =1.0 and $\mathcal{P}^*$ =0.01, the ASN reaches its highest value of 23.273 at $\mathcal{C}^*$ =0.01, $\mathcal{M}$ =2, highlighting the rigorous nature of the sampling procedure in these conditions. Conversely, lower values of $\mathcal{M}$ and higher consumer risks yield significantly lower ASN values, reducing the inspection burden. The trend observed is consistent across different levels of $\mathcal{C}^*$, with ε =1.0 resulting in generally lower ASN values compared to ε =0.5, indicating that higher error tolerance reduces the required sample size. Overall, the data underscores the balance between risk management

Table 12: OC value and ASN for SSPs under median PBXRW at $\beta = 0.5$, $c = 2$ and $M_{\mathbf{2}} = 4.0$.

		$\varepsilon = 0.5$				$\varepsilon = 1.0$			
		$\mathcal{P}^* = 0.05$		$\mathcal{P}^* = 0.01$		$\mathcal{P}^* = 0.05$		$\mathcal{P}^* = 0.01$	
$\mathcal{C}^*$	$\mathcal{M}$	OC	ASN	OC	ASN	OC	ASN	OC	ASN
0.25	2	0.751	4.009	0.847	6.209	0.717	1.26	0.812	2.007
	4	0.95	5.431	0.99	6.325	0.95	1.364	0.99	1.588
	6	0.985	5.859	0.999	6.299	0.986	1.392	0.999	1.49
	8	0.994	6.002	1	6.293	0.995	1.402	1	1.465
	10	0.997	6.057	1	6.292	0.998	1.406	1	1.458
0.1	2	0.678	7.328	0.793	10.915	0.629	2.293	0.741	3.519
	4	0.95	9.561	0.99	10.666	0.95	2.401	0.99	2.678
	6	0.987	10.055	0.999	10.518	0.989	2.385	0.999	2.486
	8	0.995	10.2	1	10.489	0.996	2.38	1	2.442
	10	0.998	10.252	1	10.482	0.999	2.378	1	2.429
0.05	2	0.637	9.901	0.761	14.559	0.579	3.084	0.698	4.679
	4	0.95	12.705	0.99	13.953	0.95	3.19	0.99	3.503
	6	0.988	13.221	0.999	13.708	0.989	3.134	0.999	3.24
	8	0.996	13.369	1	13.662	0.997	3.117	1	3.181
	10	0.998	13.42	1	13.652	0.999	3.112	1	3.163
0.01	2	0.569	16.059	0.706	23.273	0.495	4.95	0.622	7.423
	4	0.95	20.026	0.99	21.591	0.95	5.028	0.99	5.421
	6	0.988	20.558	0.999	21.112	0.99	4.87	0.999	4.989
	8	0.996	20.716	1	21.03	0.997	4.829	1	4.896
	10	0.998	20.773	1	21.012	0.999	4.816	1	4.868

and sampling efficiency in the sequential acceptance process. However, while the table provides clear numerical insights, additional context on its practical applications and graphical representations could enhance understanding. Comparing these results with Table 11 reveals that increasing $\mathcal{M}_2$ from 2.0 to 4.0 leads to slightly lower OC values in certain cases, suggesting that stricter acceptance criteria may be in place, though the ASN remains within a comparable range.

The OC values describes the probability of accepting a lot as a function of the lot's actual quality level. In sequential sampling plans, OC values do not remain constant but change continually when items are examined one by one. A well-designed sequential sampling plan aims for an OC value that provides high acceptance probability for good lots and low acceptance probability for defective lots, ensuring effective quality control. These tables show that when the quality ratio grows, the OC values arrive at one. Reducing the value of $\mathcal{P}^*$ raises the value of OC while maintaining the customers risk. The value of OC falls as the value of ARL increases from 2 to 4. Decreasing $\mathcal{P}^*$ values increases the ASNs value. Decreasing Tthe $\mathcal{C}^*$ values improve the ASNs value. Enhancing the quality attribute level raises L_a while reducing ASN. Increasing time ratio reduces the ASN and OC values.

7. Concluding remarks and future points

A new extension of the reciprocal Weibull model is introduced and studied. Some of its statistical properties are derived. The density of the new model can be right skewed and unimodal with symmetric and asymmetric shapes also it can be left skewed. The failure rate function of the new model can be decreasing-constant-increasing (U-shape or bathtub shape), increasing-constant-increasing, increasing, upside down-bathtub, monomaniacal decreasing, upside down-increasing, monomaniacal increasing and upside down. Two real data applications are used for comparing competitive estimation methods. For modeling the survival times Guinea pigs, the the CVME method is the best method with W$^{\cdot}$ =0.11519 and A$^{\cdot}$ =0.65323 then MLE method with W$^{\cdot}$ =0.12540 and

$A^{\cdot}$ =0.69132. For modeling the strengths of glass fibres, the Boot-E method is the best method with $W^{\cdot}$ =0.56078 and $A^{\cdot}$ =3.06719 then the KE and LTSOAD methods with $W^{\cdot}$ =0.70103, 0.70039 and $A^{\cdot}$ =3.83111, 3.83573. The importance of the new model is demonstrated via two real data applications. The new model is much better than other competitive models in modeling two real data sets. Finally, a new sequential sampling plan for truncated life testing is proposed utilizing an advanced probabilistic model to improve quality assessment. The plan systematically decides whether to accept or reject a batch based on life test data within a predefined time limit, ensuring efficient inspection. Extensive numerical experiments are performed to assess the model's effectiveness, confirming its ability to reduce sample sizes while maintaining acceptable risk levels. The findings emphasize the influence of key factors on the sampling process, helping to balance the risks for both producers and consumers. This research offers practical insights for industries seeking to implement cost-effective and reliable quality control measures.

We may employ a variety of novel beneficial goodness-of-fit tests, such the Nikulin-Rao-Robson goodness-of-fit test and the Bagdonaviius-Nikulin goodness-of-fit test, for right censored validation as a potential future project.Bayesian analysis can also be considered using various loss functions and making a comparison between the loss function and each other. The new distribution can also be applied in the field of insurance and reinsurance, especially with regard to insurance claims and data on reinsurance returns. It is also possible to create a set of acceptance sampling plans (single, double and multiple), which are of great importance in solving problems of results, examination and statistical quality control (see Ahmed and Yousof [5] and Ahmed et al. [6]). Bayesian and classical inference of the reliability in the multicomponent stress-strength under the new model can also be considered (see Rasekhi et al. [100], Saber and Yousof [102], Saber et al. [104] and Saber et al. [105]). The new distribution qualifies to be used in censored regression modeling processes and their consequences and future predictions (see Korkmaz et al. ([77], [78] and [79]), Taghipour et al. [117], Ibrahim et al. [71], Hamedani et al. [61] and Altun et al. ([27], [28], [29], [30], [31] and [26])). Finally, the new distribution is flexible enough and it is expected that many researchers will be motivated to derive more bivariate and multivariate distributions accordingly. Recently, many bivariate and multivariate distributions have been presented using the copula methods, and the copulas in general are mathematical functions from which more than one bivariate (or multivariate) versions of the same distribution can be derived. Given the limited space in this work, we will mention some of these works, and for more details see Salah et al. ([106]), Al-babtain et al. [11], Ali et al. ([18] and [21]), Aboraya et al. [3], Elgohari and Yousof ([47], [48] and [49]), Saber et al. [103], Elgohari et al. ([50]), Shehata and Yousof ([108], [109] and [110]), Shehata et al. ([111]), Hamed et al. ([59]), Ibrahim et al. ([72]) and Chesneau et al. [40]. On the other hand, the new model can be employed on the reliability and risk analysis, see [62], [9], [107], [125], [69].

Future research could delve deeper into various advanced applications and extensions of these statistical distributions. Following Khalil and Ali [76] we could explore applying the new distribution to high-dimensional datasets or extreme value modeling, while following Alsultan [25] we may investigate our distribution's application in real-time reliability systems. Extending the PBXRW distribution to accommodate mixture models, as suggested by Daghistani et al. [42], could broaden its applicability. Afify et al. [4] applied a new heavy-tailed exponential distribution to financial risk modeling, revealing its potential for extreme event prediction. Aldahlan et al. [16] assessed the suitability for survival data with competing risks, while Gabanakgosi and Oluyede [55] expanded the Topp-Leone-Gompertz-G power series to analyze stochastic processes. Exploring the new probability class introduced by Hassan et al. [65] in the context of big data analytics or machine learning could yield novel insights. Verster and Mbongo [120] could apply the generalized Pareto distribution in environmental extreme value analysis to expand its practical utility. Abdelaziz et al. [1] might investigate their unified family of probabilistic models within a Bayesian hierarchical framework to enhance its inferential power. Future work by Almetwally et al. [13] could apply the Marshall-Olkin Alpha Power Rayleigh distribution to model reliability data in engineering. Chettri et al. [38] could extend their generalized power transformation to multivariate distributions for complex data scenarios. Benkhelifa [36] might explore the Beta Power Muth distribution for time series modeling, while ElSherpieny and Almetwally [54] could analyze their Exponentiated Generalized Alpha Power family in actuarial science. Almongy et al. [24] could apply a novel Lomax distribution in biostatistics, while Elbatal and Aldukeel [45] may investigate the Erlang-Truncated Exponential distribution for network traffic modeling. Bagci et al. [33]

might examine the Alpha Power Inverted Kumaraswamy distribution for failure time data, and Oluyede et al. [94] could apply the a new Burr III-G model in risk assessment for insurance. ul Haq et al. [119] explored the a new Burr III distribution for dynamic reliability systems. Finally, Teghri et al. [118] extended the Lindley-Frailty model to include dependent censoring schemes, enhancing its modeling capabilities. By following these and other applied papers, we can employ the new distribution in a variety of domains and test the robustness and flexibility of the new distribution against a variety of real datasets. Moreover, the new model can be employed in characterization studies (Hamedani et al. [63]), be generalized using the new unified family of Abdelaziz et al. [1], be employed in double group sampling plan for manufacturing industry (Hafeez et al. [60]), be discretized and copared with the poisson regression (Sultan et al. [116]).

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