STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–5.
Published online in International Academic Press (www.IAPress.org)

On the Local Multiset Dimension of Comb Product Graphs

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Abstract One of the topics of distance in graphs is resolving set problems. This topic has many application in science and technology namely the application of resolving set problems in networks is one of the describe navigation robots, chemistry structure, and computer sciences. Suppose the set $W = \{s_1, s_2, \ldots, s_k\} \subset V(G)$, the vertex representations of $\in V(G)$ is $r_m(x|W) = \{d(x, s_1), d(x, s_2), \ldots, d(x, s_k)\}$, where $d(x, s_i)$ is the length of the shortest path of the vertex x and the vertex in W together with their multiplicity. The set W is called a local m-resolving set of graphs G if $r_m(vW) \neq r_m(u|W)$ for $uv \in E(G)$. The local m-resolving set having minimum cardinality is called the local multiset basis and its cardinality is called the local multiset dimension of G, denoted by $md_l(G)$. In our paper, we determined the establish bounds of local multiset dimension of graph resulting comb product of two connected graphs.

Keywords Multiset, local resolving set, local multiset dimension, comb product, connected graphs

AMS 2010 subject classifications 05C12

DOI: 10.19139/soic-2310-5070-2431

1. Introduction

One of the topics of distance in graphs is resolving set problems. This topic has many application in science and technology namely the application of resolving set problems in networks is one of the describe navigation robots, chemistry structure, and computer sciences. The application of metric dimension in networks is one of the describe navigation robots. Each place is called the vertex and the connection between vertex is called edges. The minimum numbers of the robots required to locate each and the vertex of a some networks is called as resolving set problems, for more detail this application in [1].

All graphs G are a simple and connected graph. We have the vertex set and edge set, respectively are $V(G)$ and $E(G)$. The distance of u and v and denoted by $d(u, v)$ is the length of a shortest path of the vertices u to v. For the set $W = \{s_1, s_2, ..., s_k\} \subset V(G)$. The vertex representations of the vertex x to the set W is an ordered k-tuple, $r(x|W) = (d(x, s_1), d(x, s_2), ..., d(x, s_k))$. The set W is called the resolving set of G if every vertices of G has different vertex representations. The resolving set having minimum cardinality is called basis and its cardinality is called metric dimension of G and denoted by $dim(G)$. Okamoto et al [8] introduced the new variant of resolving set problems which is called local resolving set problems. In his paper its concept is called local multiset dimension of graphs G. The set W is called a local resolving set if $\forall xy \in E(G), r(xW) \neq r(y|W)$. The local resolving set having minimum cardinality is called local basis and its cardinality is called local metric dimension of G and denoted by $ldim(G)$.

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

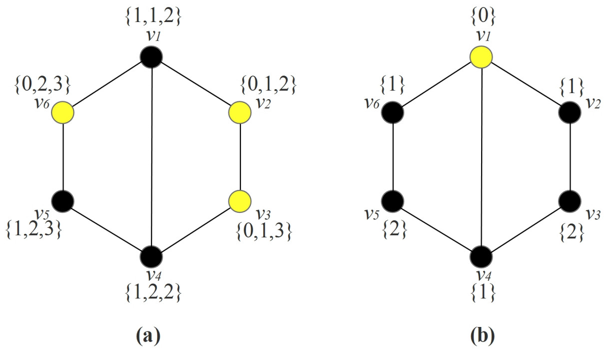

Figure 1. (a) A graph with multiset dimension 3; (b) A graph with local multiset dimension 1

Simanjuntak et al. [2] defined multiset dimension of graphs G. Suppose the set $W = \{s_1, s_2, \ldots, s_k\} \subset V(G)$, the vertex representations of a vertex $x \in V(G)$ to the set W is the multiset, $r_m(x|W) = \{d(x, s_1), d(x, s_2), \ldots, d(x, s_k)\}$ where $d(x, s_i)$ is the length of a shortest path of the vertex x and the vertex in W together with their multiplicities. The set W is called an m-resolving set if $\forall xy \in E(G), r_m(xW) \neq r_m(y|W)$. If G has an m-resolving set, then an m-resolving set having minimum cardinality is called a multiset basis and it's cardinality is called the multiset dimension of graphs G and denoted by $md(G)$; otherwise we say that G has an infinite multiset dimension and we write $md(G) = \infty$. Alfarisi et al. [3] determined of multiset dimension problems of almost hypercube graphs. Alfarisi et al. [4] defined a new notion based on the multiset dimension of G, namely a local multiset dimension. Suppose the set $W = \{s_1, s_2, \ldots, s_k\} \subset V(G)$, the vertex representations of a vertex $x \in V(G)$ to the set W is $r_m(x|W) = \{d(x, s_1), d(x, s_2), \ldots, d(x, s_k)\}$. The set W is called a local m-resolving set of G if $r_m(vW) \neq r_m(u|W)$ for $uv \in E(G)$. The local m-resolving set having minimum cardinality is called the local multiset basis and its cardinality is called the local multiset dimension and denoted by $md_l(G)$: otherwise we say that G has an infinite local multiset dimension and we write $md_l(G) = \infty$.

We illustrate this concept in Figure 2. In this case, the resolving set is $W = \{v_2, v_3, v_6\}$, shown in Figure 2 (a). The multiset dimension is $md(G) = 3$. The representations of $v \in V(G)$ with respect to W are all distinct (please list all the representations). For the local multiset dimension, we only need to make sure the adjacent vertices having distinct representations. Thus we could have the local resolving set $W = \{v_1\}$, shown in Figure 2 (b). Thus, the local multiset dimension is $\mu_l(G) = 1$.

$$
\begin{array}{lll}
r(v_1|\Pi) = \{0\}, & r(v_2|\Pi) = \{1\}, & r(v_3|\Pi) = \{2\} \\
r(v_4|\Pi) = \{1\}, & r(v_5|\Pi) = \{2\}, & r(v_6|\Pi) = \{1\}
\end{array}
$$

We have some results on the local multiset dimension of some known graphs namely path, star, tree, and cycle and also the local multiset dimension of graph operations namely, cartesian product [4], m-shadow graph [7]. Adawiyah et al. [6] also studied local multiset dimension of unicyclic graphs. There are some results which used for proving the other results as follows.

Lemma 1.1
[10] Let G be a connected graphs and $W \subset V(G)$. If W contains a resolving set of G, then W is a resolving set of G.

Proposition 1.2
[11] A graph is bipartite if and only if it contains no odd cycle.

Theorem 1.3
[5] The local multiset dimension of G is one if and only if G is a bipartite graph.

Theorem 1.4
[5] If T is tree graph with order n, then $md_l(T) = 1$.

Proposition 1.5
[3] Let K_n be a complete graph with $n \geq 3$, then $md_l(K_n) = \infty$.

Definition 1.6
[9] Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by $G \triangleright_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identify the i-th copy of H at the vertex o with the i-th vertex of G. More detail definition of comb product.

2. Section

In this section, the results of the study of local multiset dimensions on graphs resulting from the comb product of two connected graphs are presented. First, the properties of the graph resulting from the comb product are presented because the comb product relatively maintains the structure of the operating parent graph. The properties of the graph resulting from the comb product are given in Theorem 2.1 as follows.

Theorem 2.1
Let G and H be a connected graph. Graph $G \triangleright H$ is a bipartite graph if and only if G and H is a bipartite graph.

Proof. Let G and H be bipartite graph with $|V(G)| = n$ and $|V(H)| = m$. Since G and H is a bipartite graph, then we can choose the set $A, A' \subset V(G)$ and $B, B' \subset V(H)$ with
$A = \{v_i | (v_i, v_j) \notin E(G), i \neq j; i, j = 1, 2, 3, 4, \ldots, s\}$
$A' = \{v_i | (v_i, v_j) \notin E(G), i \neq j; i, j = s+1, s+2, s+3, s+4, \ldots, s+r = n\}$
such that $A \cap A'^{=}$ and $A \cup A' = V(G)$.
$B = \{u_i | (u_i, u_j) \notin E(G), i \neq j; i, j = 1, 2, 3, 4, \ldots, k\}$
$B' = \{u_i | (u_i, u_j) \notin E(G), i \neq j; i, j = k+1, k+2, k+3, k+4, \ldots, k+l = m\}$
such that $B \cap B' =$ dan $B \cup B' = V(H)$.

Let $H_i, i = 1, 2, 3, 4, \ldots, n$ be a copy of H and $V(G \triangleright H) = \{u_{i,j}; i = 1, 2, 3, 4, \ldots, nandj = 1, 2, 3, 4, \ldots, m\}$ where $u_{i,1}, i = 1, 2, 3, \ldots, n$ is a terminal vertex $G \triangleright H$, then $|G \triangleright H| = nm$. Selected the set
$C = B_1 \cup B_2 \cup B_3 \cup B_4 \cup \cdots \cup B_s \cup B'_{s+1} \cup B'_{s+2} \cup B'_{s+3} \cup B'_{s+4} \cup \cdots \cup B'_{s+r}$, so $|C| = ks + lr$
$C' = B'_1 \cup B'_2 \cup B'_3 \cup B'_4 \cup \ldots'_s \cup B_{s+1} \cup B_{s+2} \cup B_{s+3} \cup B_{s+4} \cup \cdots \cup B_{s+r}$, so $|C| = ls + kr$
Since $A \cap A' =$ and $B \cap B' =$ then $C' =$ and $|C \cup C'| = |C| + |C'| = ks + lr + ls + kr = (k+l)s + (k+l)r = (k+l)(s+r) = mn$, so $C' = V(G \triangleright H)$.

Take any two different vertices x, y in C, there are three possibilities, including 1) $x \in B_i$ and $y \in B_j, i, j \leq s$, 2) $x \in B'_{s+i}$ and $y \in B'_{s+j}, i, j \leq s$, and 3) $x \in B_i$ and $y \in B'_{s+j}, i, j \leq s$.
Case 1. $x \in B_i$ and $y \in B_j, i, j \leq s$
Since H is a bipartite graph, then $(x, y) \notin E(G \triangleright H)$.
Case 2. $x \in B'_{s+i}$ dan $y \in B'_{s+j}, i, j \leq s$
Since H is a bipartite graph, then $(x, y) \notin E(G \triangleright H)$.
Case 3. $x \in B_i$ dan $y \in B'_{s+j}, i, j \leq s$
Since x and y are on different terminal, then $(x, y) \notin E(G \triangleright H)$.

In the same way, it can be proved that any two different vertices x, y in C', occur $(x, y) \notin E(G \triangleright H)$. Thus, $G \triangleright H$ is a bipartite graph.

On the other hand, suppose that the graph G or H is not bipartite, without loss of the generality of the proof, let's say that H is a bipartite graph and G is a tripartite graph. The connection of the vertices in G does not change in $G \triangleright H$, so $G \triangleright H$ is a tripartite graph.

In the following, we will present the characterization of the graph resulting from the comb product so that the graph resulting from the comb product is a path graph.

Theorem 2.2
Suppose G and H are a connected graphs with order at least 2 and o are terminal vertex. The graph $G \triangleright H$ is a path graph if and only if G is a path graph with order 2 and H is a path graph where o is a multiset basis element of H.

Proof. Let $G \triangleright H$ be a path graph and suppose G is a path graph with order at least 2 or H is a path graph with o not a multiset basis element of H. For G is a path graph with order at least 2, let $a_1, a_2, a_3, a_4, \dots, a_n, n > 2$ be their path and H be a path graph with o does not multiset basis element of H. Whatever the sticking point is on H, it means that $G \triangleright H$ is not a path graph, it is a contradict. For H is a path graph with o not a multiset basis element of H, then the order of H is at least 3. As a result, $G \triangleright H$ is not a path graph, a contradiction.

On the other hand, suppose G is a path graph of order 2 and H is a path graph where o is a multiset basis element of H, it is clear that $G \triangleright H$ is a path graph.

In the following, the characterization of the local multiset dimensions on the graph resulting from the comb product with the local multiset dimensions of the main graph is presented, namely a bipartite graph.

Lemma 2.3
Let G and H be a nontrivial connected graph. $md_l(G) = md_l(H) = 1$ if and only if $md_l(G \triangleright H) = 1$.

Proof. Let $md_l(G) = md_l(H) = 1$, based on Theorem 1.3 that G and H is a bipartite graph. Based on Theorem 2.1 obtained $G \triangleright H$ is a bipartite graph. Since $G \triangleright H$ is a bipartite graph, based on Theorem 1.3 obtained $md_l(G \triangleright H) = 1$.

On the other hand, let $md_l(G \triangleright H) = 1$, based on Theorem 1.3 that $G \triangleright H$ is a bipartite graph. Since G and H is a bipartite graph, then based on Theorem 1.3 obtained $md_l(G) = md_l(H) = 1$.

In Susilowati (2016), the local metric dimensions of the comb product graph are presented with the local metric dimensions of the main graph as follows.

Theorem 2.4
Let G and H be a connected graph. If H is a bipartite graph, then $dim_l(G \triangleright H) = dim_l(G)$

Characterization of local multiset dimensions on the graph resulting from comb product with the local multiset dimensions of the main graph and the relationship of local multiset dimensions on the graph resulting from comb product with the local metric dimensions of the graph resulting from the comb product. For more details, it can be seen as follows:

Lemma 2.5
Let G and H is a nontrivial connected graph. If $md_l(G) \geq 2$ and $md_l(H) = 1$, then $dim_l(G) \leq md_l(G \triangleright H) \leq md_l(G)$.

Proof. Based on Theorem 1.3 that if $md_l(H) = 1$, then H is a bipartite graph. Based on Lemma 2.3 and Theorem 2.4 obtained that $md_l(G \triangleright H) \geq dim_l(G \triangleright H) = dim_l(G)$.

Theorem 2.6
Let T be a tree with ordo $n \geq 2$ and T_1, T_2 be a subtree of T with $V(T_1) \cup V(T_2) = V(T)$ and $V(T_1) \cap V(T_2) =$, then

$$md_l(T \triangleright C_3) \geq \begin{cases} n, & \text{for } T_1 \neq T_2 \\ n+1, & \text{for } T_1 = T_2 \end{cases}$$

Proof. Let T be a tree graph with order $n \geq 2$. Vertex set of C_3, $V(C_3) = \{v_j, j = 1, 2, 3\}$ and vertex set of tree graph, $V(T) = \{v_i, i = 1, 2, \dots, n\}$. Based on definition of comb product that $V(T \triangleright C_3) = \{v_{i,j}, j = 1, 2, 3$ and $i = 1, 2, \dots, n\}$. Graph $T \triangleright C_3$ has n copies subgraph C_3, denoted by $(C_3)_i$ with $i = 1, 2, 3, \dots, n$. The vertex $v_{i,1}$ for $i = 1, 2, 3, \dots, n$ is terminal vertex and the vertex $v_{i,j}, j \neq 1$ is a vertex in subgraph $(C_3)_i$ for $i = 1, 2, 3, \dots, n$.

Case 1. $T_1 \neq T_2$
Take $W = \{v_{i,2}; i = 1, 2, 3, \dots, n\}$, it will be shown that the vertex representation of $T \triangleright C_3$ is different.

1. The vertex representation $v_{k,1}, v_{l,1} \in V(T \triangleright C_3)$. Since $T_1 \neq T_2$, then for all two vertices $v_{k,1}v_{l,1} \in E(T \triangleright C_3)$ occur $d(v_{k,1}, v) \neq d(v_{l,1}, v)$ with $v \in V(T \triangleright C_3)$. Thus, $d(v_{k,1}, v_{s,2}) \neq d(v_{l,1}, v_{s,2})$, then $r_m(v_{k,1}|W) \neq r_m(v_{l,1}|W)$.
2. The vertex representation $v_{k,1}, v_{k,3} \in V(T \triangleright C_3)$. It is known that $d(v_{k,3}, v_{s,2}) = d(v_{k,3}, v_{k,1}) + d(v_{k,1}, v_{s,2})$, then $d(v_{k,3}, v_{s,2}) \neq d(v_{k,1}, v_{s,2})$. Thus, $r_m(v_{k,1}|W) \neq r_m(v_{k,3}|W)$.

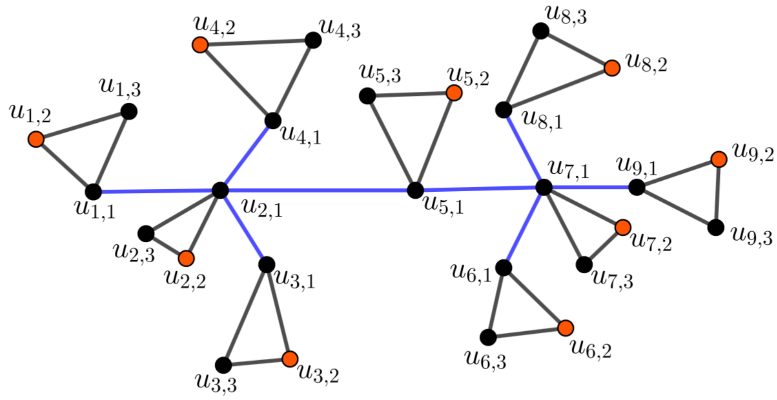

Figure 2. Local multisen dimension of $T \triangleright C_3$ is 9

Based on point 1) and 2) obtained that the vertex representation of $T \triangleright C_3$ is different.

Furthermore, it will be proven that $md_l(T \triangleright C_3) \geq n$. Take $P \subset V(T \triangleright C_3)$ with $|P| = n-1$. Since $u_{k,2}, u_{k,3} \notin P$, then $d(v_{k,2}, v_{k,1}) = d(v_{k,3}, v_{k,1})$ and $u_{k,2} - u_{k,3}$, then $r_m(v_{k,2}|W) = d(v_{k,3}|W)$. Since $r_m(v_{k,2}|W) = d(v_{k,3}|W)$, then P is not local m-resolving set. Based on the description above, it is obtained that the cardinality of the local m-resolving set, namely $md_l(T \triangleright C_3) = n$.

Case 2. $T_1 = T_2$
Take $W = \{v_{i,2}; i = 1, 2, 3, \ldots, n\} \cup \{v_{n,1}\}$, it will be shown that the vertex representation of $T \triangleright C_3$ is different.

1. The vertex representation $v_{k,1}, v_{l,1} \in V(T \triangleright C_3)$. It is known that $d(v_{k,1}, v_{n,1}) \neq d(v_{l,1}, v_{n,1})$ and $d(v_{k,1}, v_{s,2}) \neq d(v_{l,1}, v_{s,2})$, then $r_m(v_{k,1}|W) \neq r_m(v_{l,1}|W)$.
2. The vertex representation $v_{k,1}, v_{k,3} \in V(T \triangleright C_3)$. It is known that $d(v_{k,3}, v_{n,1}) = d(v_{k,3}, v_{k,1}) + d(v_{k,1}, v_{n,1})$, then $d(v_{k,3}, v_{n,1}) \neq d(v_{k,1}, v_{n,1})$. Thus, $r_m(v_{k,1}|W) \neq r_m(v_{k,3}|W)$.
3. The vertex representation $v_{n,1}, v_{n,2} \in V(T \triangleright C_3)$. It is known that $d(v_{n,2}, v_{s,2}) = d(v_{n,2}, v_{n,1}) + d(v_{n,1}, v_{s,2})$, then $d(v_{n,2}, v_{s,2}) \neq d(v_{n,1}, v_{s,2})$. Thus, $r_m(v_{n,2}|W) \neq r_m(v_{n,1}|W)$.

Based on point 1), 2) and 3) obtained that the vertex representation of $T \triangleright C_3$ is different.

Furthermore, it will be proven that $md_l(T \triangleright C_3) \geq n+1$. Take $P \subset V(T \triangleright C_3)$ with$|P| = n$. There are two possibilities as follows.

a For $u_{k,2}, u_{k,3} \neq P$
Since $d(v_{k,2}, v_{k,1}) = d(v_{k,3}, v_{k,1})$ dan $u_{k,2} - u_{k,3}$, then $r_m(v_{k,2}|W) = d(v_{k,3}|W)$. Since $r_m(v_{k,2}|W) = d(v_{k,3}|W)$, then P is not local m-resolving set.

b For $u_{i,2} \in P$ or $u_{i,3} \in P; 1 \leq i \leq n$.
Since $T_1 = T_2$, for $v_{i,1} - v_{k,1}$, $v_{i,1}, v_{k,1} \in V(T \triangleright C_3) - P$ with $i \neq k$. Since $v_{i,1}v_{k,1} \in E(T), v_{i,1} \in V(T_1)$ and $v_{k,1} \in V(T_2)$ such that $\{d(v_{i,1}, u_{i,2}); u_{i,2} \in P\} = \{d(v_{k,1}, u_{i,2}); u_{i,2} \in P\}$ occur $r_m(v_{i,1}P) = r_m(v_{k,1}P)$. Since $r_m(v_{i,1}P) = r_m(v_{k,1}P)$, then P is not local m-resolving set.

Based on the description above, it is obtained that the cardinality of the local m-resolving set, namely $md_l(T \triangleright C_3) = n+1$.

3. Conclusion

We have characterized the local multiset dimension of comb product with the main graph is bipartite graph.

Acknowledgment

We gratefully acknowledge the support from University of Jember for the support provided through Pasca Doktor Grant No.: 2827/UN25.3.1/LT/2025.

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