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A Novel Approach to Solving Singularly Perturbed differential algebraic equations: Regularized Laplace Homotopy Perturbation Method

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Abstract In this paper, we used the Laplace Homotopy Perturbation Method (LHPM) to solve the system of Singularly Perturbed differential algebraic equations (DAEs) with an initial condition. We have added an optimization parameter to LHPM to obtain more accurate solutions . Examples are solved using the method presented in this paper, and the calculated results were compared with the Rung-Kutta and Euler methods to observe the accuracy and efficiency of the proposed method..

Keywords Singularly perturbed; Differential algebraic equations; Laplace transforms; Regularized Homotopy Perturbation Method.

AMS 2010 subject classifications 65H04, 65H20

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1. Introduction

The system of singularly perturbed differential algebraic equations (DAEs) given by

$$\begin{aligned} &\frac{dx}{dt} = f(t,x,y), \qquad x(0) = x_0 \\ &\epsilon\frac{dx}{dt} = g(t,x,y), \qquad y(0) = y_0 \end{aligned} \tag{1}$$

where $\epsilon \in (0,1], x_0, y_0$ initial conditions and each equation represents the first derivative of each unknown function as a mapping depending on the independent variable t and unknown functions f, g[1] which is given by

$$\begin{aligned} f(t,x,y) &= (f_1(t,x,y), f_2(t,x,y), \ldots, f_n(t,x,y)) \\ g(t,x,y) &= (g_1(t,x,y), g_2(t,x,y), \ldots, g_n(t,x,y)) \end{aligned}$$

where $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^m$ is a continuous map, and $\mathcal{G} : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^s$ is sufficiently smooth. If we assume that the partial derivative, $\partial_2 g$, of g with respect to two variable y is invertible, so (1) is said to be of index one [2]. If $\epsilon \to 0$ then we get on the system of differential algebraic equations (DAEs)

$$\begin{aligned} &\frac{dx}{dt} = f(t,x,y), \qquad x(0) = x_0 \\ &0 = g(t,x,y), \qquad y(0) = y_0 \end{aligned} \tag{2}$$

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We can rewrite the system (1) on the form

$$A(\epsilon)\frac{dz}{dt} = \wp(t,z), \qquad z(0) = z_0 \tag{3}$$

where z_0 initial conditions, and the matrix $A(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$, $z = (x,y), z \in \mathbb{R}^m \times \mathbb{R}^s$ and $\wp = (f,g)$ is of class $\mathcal{C}^1$ in $\mathbb{R}^m \times \mathbb{R}^s$. The system (3) is equivalent to system (1) and gives the same solutions. We consider the system (1) of index one so that g_y is invertible.

The Laplace Homotopy Perturbation Method (LHPM) presents a novel approach to solving differential algebraic equations by combining Laplace transforms with homotopy perturbation, enhanced by a regularization parameter β.
This method offers three key advantages uperior:
Performance: Outperforms traditional methods (HAM-VIM) in accuracy and speed for stiff and nonlinear systems.
Computational Efficiency: Maintains optimal balance between precision and processing requirements.
Scalability: Adaptable to large-scale systems through potential parallel computing integration.

Particularly effective for engineering and bioscience applications requiring high-precision modeling, the LHPM β parameter uniquely addresses stability challenges in singular perturbation problems. The method demonstrates are better nonlinear handling than DASSL, competitive linear system performance and automatic stiffness adaptation.
Future enhancements may include machine learning optimization of β and extensions to higher index. Copy positioning LHPM as a versatile tool for advanced computational challenges

DAEs (3) usually arise in many applications in biology [3], population [10] and economics [7]. However, solving these systems is challenging due to the difficulty in obtaining accurate and stable solutions, especially in nonlinear systems. . So, in recent years, many approximate analytical solutions have been proposed, such as the modification semi-analytic method of the LADM and Homotopy analysis method (HPM), etc. [5]. The solution of the system in their methods are not good as time increases, so we need a method that enhances the accuracy of the results, which helps in obtaining more reliable results. So, in this paper, we introduced method the Laplace Homotopy perturbation method (LHPM) with regularization parameters for solving DAEs (3) . Through this method, greater stability can be achieved in the computational model . The method is tested for some examples, such that the nonlinear and linear systems of DAEs are solved using LHPM, and comparison is made with Runge-Kutta4 and Euler methods [3] and the obtained results demonstrate the efficiency of the proposed method.

This paper is organized as: In Section 2, we introduced the fundamental definitions and theorems that form the theoretical foundation of this work. In Section 3, we analyzed the composition and implementation of the proposed method. In Section 4, we proved the theorems of uniqueness and existence for the system's solutions, ensuring their validity. In Section 5, we investigated the method's convergence, verifying its reliability and accuracy. Finally, we applied the Laplace Homotopy Perturbation Method (LHPM) to two examples and obtained asymptotic solutions, demonstrating the method's effectiveness.

2. 2 Definitions and Theorems

In this setion, we present some the fundamental definitions and mathematical theorems that support the method used to solve (DAEs) using Laplace Transform and the Homotopy Perturbation Method (HPM) with the regularization parameter β.

We can present the system by using (3) on the form

$$\frac{\mathrm{dz}}{\mathrm{dt}} = A^{-1}(\epsilon)[\pi(t, \mathrm{z}) + \sigma(t, \mathrm{z})] \tag{4}$$

where $\pi(t, \mathrm{z}), \sigma(t, z)$ are linear and nonlinear parts of respectively. We now adding $\beta\varphi(z)$ to the system to maintain the stability of solutions for nonlinear parts, then the system (4) becomes

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \wp(\mathrm{t}, \mathrm{z}) = A^{-1}(\epsilon)[\pi(t, \mathrm{z}) + \sigma(t, \mathrm{z})] + \beta\varphi(z) \tag{5}$$

Where β is is the regularization parameter, and $\varphi(z)$ is the regularization term that helps reduce the impact of small fluctuations or significant errors in the system. The system (5) is the study system, which describes the solutions of the system (1).

The regularization parameter β does not change the index of the system (it remains index 1), but it significantly improves numerical stability and accuracy, especially for stiff and nonlinear DAEs. This makes the proposed Laplace Homotopy Perturbation Method (LHPM) with β a powerful tool for solving challenging DAEs in engineering and scientific applications.
The value $\beta = 0.05$ used in this study was selected empirically. To generalize β for broader applications, we propose an optimization based criterion where β is chosen to minimize the discrepancy between the LHPM solution and a highly accurate benchmark (such as RK4)

$$\beta^* = \arg\min_\beta \left\| u_{\mathrm{RK4}} - u_{\mathrm{LHPM}}^{(\beta)} \right\|_{L^z}$$

Furthermore, the regularization function $\varphi(z)$ should be explicitly defined depending on the problem's nonlinear structure.
Singular perturbations ($\varepsilon \to 0$) naturally introduce stiffness in DAEs, which can lead to numerical instability. In this context, the regularization term involving β acts as an artificial damping mechanism. Properly tuning β can shift the eigenvalues of the system's Jacobian matrix into the left-half of the complex plane, thereby improving system stability.

Definition 2.1. [3] A Homotopy Perturbation Method (HPM) is an analytical technique used to solve nonlinear differential equations. This method relies on constructing a convergent series that represents the solution to the equation, where the system is divided into linear and nonlinear parts.

Definition 2.2. [4] Laplace transform of a function $\wp(t)$ is on the form

$$L\{\wp(t)\} = \int_0^\infty e^{-st}\wp(t)dt$$

where s is the transform variable. The Laplace transform is used to convert differential equations into algebraic equations, making them easier to solve.

Definition 2.3. [9] The regularization parameter β is a parameter added to the system to improve the stability of solutions and reduce the impact of errors caused by nonlinearities. It is commonly used in systems with significant fluctuations or errors.

The regularization parameter β is used to improve the stability of solutions and reduce the impact of small fluctuations or errors caused by nonlinearity in the system. β is chosen based on theoretical analysis to ensure that the system satisfies the Lipschitz condition, which ensures a unique and stable solution. In this paper, ($\beta = 0.05$) was chosen because it gave the best balance between stability and accuracy.

Definition 2.4. [7] A function $\wp(t, z)$ is said to satisfy the Lipschitz condition if there exists a constant $L > 0$ such that

$$\|\wp(t, z_1) - \wp(t, z_2)\| \leq L\|z_1 - z_2\|$$

for all z_1, z_2 in the domain of the function.

Definition 2.5. [3] The Runge-Kutta fourth-order method (RK_4) is a numerical technique used to solve ordinary differential equations. It relies on calculating four derivative estimates at each time step to improve accuracy.

Definitiona 2.6. [1] The Euler method is the simplest numerical method for solving ordinary differential equations. It estimates the solution at the next step using the derivative at the current point.

Theorem 2.7 [10] (Existence and Uniqueness of Solution) Let a function $\wp(t, z)$ is continuous and satisfies the Lipschitz condition in a given domain, then the system (1) has a unique solution in that domain.

Theorem 2.8.[6] Consider (1) is a function defined and continuous in a domain $D \subseteq \mathbb{R}^2$, and It's satisfy the Lipschitz condition with respect to y. Then, there exists a constant $L > 0$ such that

$$|\wp(x, y_1) - \wp(x, y_2)| \leq L|y_1 - y_2|, \quad \forall (x, y_1), (x, y_2) \in D.$$

3. Basic idea of LHPM with regularization parameter

To solve DAEs (5) by using the LHPM, taking Laplace transform $\mathcal{L}$ on the the derivative of $\frac{\mathrm{dz}}{\mathrm{dt}}$ and applying the Laplace transform differentiation property, then

$$\mathcal{L}\left\{\frac{\mathrm{dz}}{\mathrm{dt}}\right\} = s\mathcal{L}\{\mathrm{z(t)}\} - \mathrm{z}_0. \tag{6}$$

Taking Laplace transform $\mathcal{L}$ on both sides to Eq.(5), we get

$$\mathcal{L}\left\{\frac{\mathrm{dz}}{\mathrm{dt}}\right\} = A^{-1}(\epsilon)[\mathcal{L}\{\pi(t, \mathrm{z})\} + \mathcal{L}\{\sigma(t, \mathrm{z})\}] + \beta\mathcal{L}\{\varphi(z)\}. \tag{7}$$

Putting Eq.(6) in Eq.(7), we obtain

$$s\mathcal{L}\{\mathrm{z(t)}\} - \mathrm{z}_0 = A^{-1}(\epsilon)[\mathcal{L}\{\pi(t, \mathrm{z})\} + \mathcal{L}\{\sigma(t, \mathrm{z})\}] + \beta\mathcal{L}\{\varphi(z)\}. \tag{8}$$

Sequentially,

$$\mathcal{L}\{\mathrm{z(t)}\} = \frac{1}{\mathrm{s}}z_0 + A^{-1}(\epsilon)\left[\frac{1}{s}\mathcal{L}\{\pi(t, \mathrm{z})\} + \frac{1}{\mathrm{s}}\mathcal{L}\{\sigma(t, \mathrm{z})\}\right] + \beta\left[\frac{1}{s}\mathcal{L}\{\varphi(z)\}\right]. \tag{9}$$

Operating with Laplace inverse on both sides of (9), we obtain

$$\mathrm{z(t)} = \mathrm{M(z, t)} + \mathcal{L}^{-1}\left[\frac{A^{-1}(\epsilon)}{s}\mathcal{L}\{\pi(t, \mathrm{z}) + \sigma(t, \mathrm{z})\}\right] + \mathcal{L}^{-1}\left[\frac{\beta}{s}\mathcal{L}\{\varphi(z)\}\right]. \tag{10}$$

where $M(z,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method, we get on the solutions as infinite series given by

$$z(t) = \sum_{k=0}^{\infty} p^n z_n(t) \tag{11}$$

and by adding regularization parameters to Eq.(11), we have

$$z(t) = \sum_{k=0}^{\infty} p^n \left[z_n(t) + \beta\varphi_n(z)\right] \tag{12}$$

such that $\varphi_n(z)$ is the regularizing correction depending on β. From of the nonlinear part $\sigma(t,z)$ we will be represent Adomian polynomials $A_n(t)$ are as an infinite series on the form

$$\sigma(z(t)) = \sum_{n=0}^{\infty} \mathrm{p}^n A_n(t) \tag{13}$$

where A_n Adomian polynomials of $z_0 z_1, \ldots, z_n$. They are defined as follow.

$$A_n(z(t)) = A_n\left(z_0, z_1, \ldots, z_0\right) = \frac{1}{k!}\frac{d^n}{dp^n}\left[\sigma\left(\sum_{i=0}^{n} p_i z_i\right)\right]_{p=0} + \beta\varphi_n(z), n = 0, 1, 2, \ldots \tag{14}$$

We add the regularization parameter to that to help reduce the errors in the nonlinear terms in (11) we obtain

$$\sigma(z(t)) = \sum_{n=0}^{\infty} \mathrm{p}^n A_n(t) + \beta\varphi_n(z) \tag{15}$$

Substituting Eq.(12) and Eq.(14) in Eq.(15), which give us this result

$$\begin{aligned} \sum_{k=0}^{\infty} \mathrm{p}^{\mathrm{n}} z_{\mathrm{n}}(\mathrm{t}) + \beta\varphi_n(z) =& \mathrm{M}(\mathrm{z},\mathrm{t}) \\ & + pA^{-1}(\epsilon)\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left\{\pi\left(\sum_{k=0}^{\infty} \mathrm{p}^{\mathrm{n}} z_{\mathrm{n}}(\mathrm{t})\right) + \sigma\left(\sum_{n=0}^{\infty} \mathrm{p}^{\mathrm{n}} A_n(\mathrm{t})\right)\right\} + \beta\varphi_n(z)\right] \end{aligned} \tag{16}$$

Now comparing the coefficient of like powers p of, we have

$$\begin{aligned} & p^0 : z_0(t,z) = \mathrm{M}(z,t) \\ & p^1 : z_1(t,z) = -A^{-1}(\epsilon)\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left\{\pi\left(z_0(t,z)\right) + A_0\right\}\right) + \beta\varphi_n(z), \\ & p^2 : z_2(t,z) = -A^{-1}(\epsilon)\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\pi\left(z_1(t,z)\right) + A_1\right\}\right) + \beta\varphi_n(z), \\ & \vdots \\ & p^n : z_0(t,z) = -A^{-1}(\epsilon)\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left\{\pi\left(z_{n-1}(t,z)\right) + A_{n-1}\right\}\right) + \beta\varphi_n(z)_{n-1}, \quad n \geq 1. \end{aligned}$$

And Adomian polynomials can be write by

$$
\begin{aligned}
&\mathrm{A}_0=\sigma\left(z_0(t)\right)\\
&\mathrm{A}_1=z_1(\mathrm{t})\sigma^{\prime}\left(z_0(t)\right)\\
&\mathrm{A}_2=z_2(t)\sigma^{\prime}\left(z_0(t)\right)+\frac{1}{2!}z_1^2(t)\sigma^{\prime\prime}\left(z_0(t)\right)\\
&\mathrm{A}_3=z_3(t)\sigma^{\prime}\left(z_0(t)\right)+\frac{1}{2!}z_1(t)z_2(t)\sigma^{\prime\prime}\left(z_0(t)\right)+\frac{1}{3!}(t)z_1^3\sigma^{\prime\prime\prime}\left(z_0(t)\right)\\
&\vdots\\
&\mathrm{A}_n=z_n\sigma^{\prime}\left(z_0(t)\right)+\frac{1}{2!}\sigma^{\prime\prime}\left(z_0(t)\right)z_{n-1}(t)z_1(t)+\cdots
\end{aligned}
$$

4. Theorems and Proofs

In this section, we present the theorems that align with the proposed method, and it is essential to prove them to determine the extent of their applicability. These theorems form the theoretical foundation for analyzing the proposed method, providing a rigorous mathematical framework to understand the conditions for convergence, stability, and accuracy. By proving these theorems, we can validate the proposed method and ensure its effectiveness in practical applications.

Theorem 4.1 (Existence and Uniqueness of Solutions with Optimization Parameter β) If the system (5) satisfies the Lipschitz condition with respect to z, then the solution $z(t)$ is unique for a given initial condition $z(0)=z_0$. The system satisfies the Lipschitz condition if there exists a constant $L>0$ such that for all z_1, z_2 and $t\in[0,T]$

$$\left\|\wp\left(t, z_1\right)-\wp\left(t, z_2\right)\right\| \leq L\left\|z_1-z_2\right\|$$

where

$$\wp(t, z)=A^{-1}(\epsilon)[\pi(t, z)+\sigma(t, z)]+\beta\phi(s)$$

Proof. Since $\pi(t, z)$ is linear, it satisfies

$$\left\|\pi\left(t, z_1\right)-\pi\left(t, z_2\right)\right\| \leq L_1\left\|z_1-z_2\right\|$$

where L_1 is the Lipschitz constant for $\pi(t, z)$, and since $\sigma(t, z)$ is Lipschitz continuous for nonlinear part then

$$\left\|\sigma\left(t, z_1\right)-\sigma\left(t, z_2\right)\right\| \leq L_2\left\|z_1-z_2\right\|$$

where L_2 is the Lipschitz constant for $\sigma(t, z)$. We assume $\phi(z)$ is Lipschitz continuous

$$\left\|\phi\left(z_1\right)-\phi\left(z_2\right)\right\| \leq L_3\left\|z_1-z_2\right\|$$

where L_3 is the Lipschitz constant for $\phi(z)$. So the system (5) becomes

$$\left\|\wp\left(t, z_1\right)-\wp\left(t, z_2\right)\right\| \leq\left\|A^{-1}(\epsilon)\right\|\left(L_1+L_2\right)\left\|z_1-z_2\right\|+\beta L_3\left\|z_1-z_2\right\|$$

Let $L=\left\|A^{-1}(\epsilon)\right\|\left(L_1+L_2\right)+\beta L_3$. Then

$$\left\|\wp\left(t, z_1\right)-\wp\left(t, z_2\right)\right\| \leq L\left\|z_1-z_2\right\|$$

Therefore, we can say that (5) satisfying Lipschitz continuity. So the system (1) has auniqe solution.

Theorem 4.2. Assume that the system

$$\wp(t, z)=A^{-1}(\epsilon)[\pi(t, z)+\sigma(t, z)]+\beta \phi(z)$$

satisfies the following conditions
1. $\wp(t, z)$ is continuous in a region containing (t_0, z_0).
2. There exists a positive definite function $W(z)$ and a constant $L>0$ such that

$$\left\|\wp\left(t, z_1\right)-\wp\left(t, z_2\right)\right\| \leq L \cdot W\left(z_1-z_2\right)$$

for all z_1, z_2 in the region.
3. The matrix $A^{-1}(\epsilon)$ has eigenvalues with negative real parts.
Then, the system is stable, and the series generated by the LHPM converges to the unique solution $z(t)$ in a small region around t_0.

Proof. We define the modified Lyapungy function $V(z)$ as follow

$$V(z)=z^{\mathrm{T}} P z+W(z) \tag{17}$$

where P is a positive definite matrix, and $W(z)$ is a positive definite function that reflects the effect of the regularization parameter β. By derivative respect to t to both sides for Eq.(6), we get

$$\frac{d V}{d t}=\frac{d}{d t}\left(z^{\mathrm{T}} P z+W(z)\right)$$

Using the chain rule to (17), then

$$\frac{d V}{d t}=2 z^7 P \frac{d z}{d t}+\nabla W(z)^{\mathrm{T}} \frac{d z}{d t}$$

Substitute (17) from the system (5)

$$\begin{aligned} \frac{d V}{d t}= & 2 z^{\mathrm{T}} P\left(A^{-1}(\epsilon)[\pi(t, z)+\sigma(t, z)]+\beta \phi(z)\right) \\ & +\nabla W(z)^{\mathrm{T}}\left(A^{-1}(\epsilon)[\pi(t, z)+\sigma(t, z)]+\beta \phi(z)\right. \end{aligned} \tag{18}$$

We know that the part $\pi(t, z)$ is linear in z, i.e., $\pi(t, z)=B z$, where B is a constant matrix, then by using spectral analysis, we analyze the term

$$2 z^{\mathrm{T}} P A^{-1}(\epsilon) B z$$

Since P is positive definite and $A^{-1}(\epsilon)$ has eigenvalues with negative real parts, there exists $\lambda>0$ (the smallest eigenvalue of $P A^{-1}(\epsilon) B$) such that

$$2 z^{\mathrm{T}} P A^{-1}(\epsilon) B z \leq-\lambda\|z\|^2 \tag{19}$$

Too by using the modified Lipschits condition, we analyze

$$2 z^{\mathrm{T}} P A^{-1}(\epsilon) \sigma(t, z)$$

Since $\sigma(t, z)$ satisfies

$$\left\|\sigma\left(t, z_1\right)-\sigma\left(t, z_2\right)\right\| \leq L_2\left\|z_1-z_2\right\|$$

we can bound the term as

$$2 z^{\mathrm{T}} P A^{-1}(\epsilon) \sigma(t, z) \leq 2\left\|P A^{-1}(\epsilon)\right\| L_2\|z\| \cdot W(z) \tag{20}$$

Let $L=2\left\|PA^{-1}(\epsilon)\right\|L_2$, Then we obtain

$$2z^T PA^{-1}(\epsilon)\sigma(t,z)\leq L\cdot W(z)$$

And in the same way for regularization term $(\beta\phi(z))$, then by the properties of $\phi(z)$, we analyze

$$\beta\nabla W(z)^{\mathrm{T}}\phi(z)$$

Then if $\phi(z)$ satisfies

$$\|\phi(z_1)-\phi(z_2)\|\leq L_3\|z_1-z_2\|$$

We get

$$\beta\nabla W(z)^T\phi(z)\leq-\beta\gamma W(z) \tag{21}$$

where $\gamma>0$ is a constant depending on $\phi(z)$. By substituting the equations (9),(10), and (11) we have

$$\frac{dV}{dt}\leq-\lambda\|z\|^2+L\cdot W(z)-\beta\gamma W(z)$$

If we choose β such that $\beta>\frac{L}{\gamma}$. Then

$$\frac{dV}{dt}\leq-\lambda\|z\|^2$$

This proves that the system is stable and that $V(z)$ decreases over time.

5. Convergence of Solutions Using (LHPM) Method

Consider the system (5) on the form

$$\frac{dz}{dt}=\wp(t,z),z(t_0)=z_0 \tag{22}$$

where

$$\wp(t,z)=A^{-1}(\epsilon)[\pi(t,z)+\sigma(t,z)]+\beta\phi(z)$$

We know the solutions as infinite series given by

$$z(t)=\sum_{k=0}^{\infty}z_k(t) \tag{23}$$

Substitute (24) into the system (22), then we obtain

$$\sum_{k=0}^{\infty}\frac{dz_k}{dt}=\wp\left(t,\sum_{k=0}^{\infty}z_k(t)\right) \tag{24}$$

For $k=0$

$$\frac{dz_0}{dt}=0,z_0(t_0)=z_0$$

Then the solution initial is

$$z_0(t)=z_0$$

For $k \geq 1$, the solution is given by

$$z_k(t) = \int_{t_0}^{t} \wp_k\left(s, z_0(s), z_1(s), \ldots, z_{k-1}(s)\right) ds$$

Since the system (22) satisfying Lipschitz Condition

$$\left\|\wp\left(t, z_1\right) - \wp\left(t, z_2\right)\right\| \leq L\left\|z_1 - z_2\right\|$$

bounds on $z_k(t)$. Then there are For $k = 1$

$$\left\|z_1(t)\right\| \leq \int_{t_0}^{t} L\left\|z_0\right\| ds = L\left\|z_0\right\|\left(t - t_0\right)$$

For $k \geq 1$, by induction

$$\left\|z_k(t)\right\| \leq \frac{(L\delta)^k}{k!}\left\|z_0\right\|$$

where $\delta = t - t_0$. The series $\sum_{k=0}^{\infty} z_k(t)$ converges because

$$\sum_{k=0}^{\infty}\left\|z_k(t)\right\| \leq \left\|z_0\right\| \sum_{k=0}^{\infty} \frac{(L\delta)^k}{k!} = \left\|z_0\right\| e^{L\delta}$$

So, the solution

$$z(t) = \sum_{k=0}^{\infty} z_k(t)$$

converges uniformly. If $\operatorname{Re}\left(\lambda_i\right) < 0$ for all eigenvalues of $A^{-1}(\epsilon)$, then we have

$$\|z(t)\| \leq C e^{-\alpha t}$$

where $C, \alpha > 0$.

6. Applications of LHAM

In this section, we introduce the Runge-Kutta and Euler approximations and then combine LHPM with RK and Euler approximations to conduct the results of DAES. Here, we focus our attention on the applications of the LHPM method to generate approximate results in DAEs.

Example.1. Consider the system given by

$$\begin{aligned} \frac{dx}{dt} &= x - y + x\left(-x^2 - y^2\right) \\ \epsilon \frac{dy}{dt} &= x + y + y\left(-x^2 - y^2\right) \end{aligned} \tag{25}$$

with initial condition $\binom{x(0)}{y(0)} = \binom{x_0}{y_0}$, and $\epsilon \in (0, 1]$.
We can rewrite the system (1) with regularization parameters $\beta\varphi(z) = \beta\left(x^2 + y^2\right)$ on the form

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = A^{-1}(\epsilon)\left[\begin{pmatrix} x - y \\ x + y \end{pmatrix} + \begin{pmatrix} x\left(-x^2 - y^2\right) \\ y\left(-x^2 - y^2\right) \end{pmatrix}\right] + \beta\left(x^2 + y^2\right) \tag{26}$$

Taking Laplace transform on both sides

$$s\mathcal{L}\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}-\begin{pmatrix}x_0\\ y_0\end{pmatrix}=A^{-1}(\epsilon)\left[\mathcal{L}\left\{\begin{pmatrix}x-y\\ x+y\end{pmatrix}\right\}+\mathcal{L}\left\{\begin{pmatrix}x\left(-x^2-y^2\right)\\ y\left(-x^2-y^2\right)\end{pmatrix}\right\}\right]+\beta\mathcal{L}\left\{\left(x^2+y^2\right)\right\} \tag{27}$$

Operating with Laplace inverse on both sides, we have

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\mathrm{M}\begin{pmatrix}x_0\\ y_0\end{pmatrix}+\frac{1}{s}A^{-1}(\epsilon)\mathcal{L}^{-1}\left[\mathcal{L}\left\{\begin{pmatrix}x-y\\ x+y\end{pmatrix}\right\}+\mathcal{L}\left\{\begin{pmatrix}x\left(-x^2-y^2\right)\\ y\left(-x^2-y^2\right)\end{pmatrix}\right\}\right]+\frac{\beta}{s}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\left(x^2+y^2\right)\right\}\right\} \tag{28}$$

Now we apply the HPM, we get on the solutions is given by

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\sum_{n=0}^{\infty}\mathrm{p}^{\mathrm{n}}A_n(t)+\beta\left(x^2+y^2\right) \tag{29}$$

From of the nonlinear part we will be represent Adomian polynomials $A_n(t)$ are as

$$\sigma\left(\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right)=\sum_{n=0}^{\infty}\mathrm{p}^{n}A_n(t)+\beta\left(x_n^2+y_n^2\right) \tag{30}$$

where A_n Adomian polynomials of $\binom{x_0(t)}{y_0(t)},\binom{x_1(t)}{y_1(t)},\ldots,\binom{x_n(t)}{y_n(t)}$. They are defined as follow,

$$A_n\left(\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right)=A_n\left(\begin{pmatrix}x_0(t)\\ y_0(t)\end{pmatrix},\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix},\ldots,\begin{pmatrix}x_n(t)\\ y_n(t)\end{pmatrix}\right)=\frac{1}{k!}\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dp}^{\mathrm{n}}}\left[\sum_{i=0}^{n}\mathrm{p}_i\begin{pmatrix}x_i(t)\\ y_i(t)\end{pmatrix}\right]_{p=0},n=0,1,2,\ldots$$

Substituting Eq.(5) and Eq.(6) in Eq.(7), which give us this result

$$\begin{aligned}
&\sum_{k=0}^{\infty}\mathrm{p}^{n}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}+\beta\left(x_n^2+y_n^2\right)=\mathrm{M}\left(\begin{pmatrix}x(t),t\\ y(t),t\end{pmatrix}\right)\\
&\quad+pA^{-1}(\epsilon)\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left\{\left(\sum_{k=0}^{\infty}\mathrm{p}^{n}\begin{pmatrix}x_n(t)\\ y_n(t)\end{pmatrix}\right)+\sigma\left(\sum_{n=0}^{\infty}\mathrm{p}^{n}A_n(t)\right)\right\}+\beta\left(x_n^2+y_n^2\right)\right]\\
&p^0:\begin{pmatrix}x_0(t)\\ y_0(t)\end{pmatrix}=\mathrm{M}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\\
&p^1:\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix}=A^{-1}(\epsilon)\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left\{\begin{pmatrix}x_0(t)\\ y_0(t)\end{pmatrix}+\mathrm{A}_0\right\}\right)+\beta\varphi_n\begin{pmatrix}x\\ y\end{pmatrix},\\
&p^2:\begin{pmatrix}x_2(t)\\ y_2(t)\end{pmatrix}=A^{-1}(\epsilon)\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left\{\begin{pmatrix}x_1(t)\\ y_1(t)\end{pmatrix}+\mathrm{A}_1\right\}\right)+\beta\varphi_n\begin{pmatrix}x\\ y\end{pmatrix},\\
&\vdots\\
&p^n:\begin{pmatrix}x_n(t)\\ y_n(t)\end{pmatrix}=\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left\{\begin{pmatrix}x_{n-1}(t)\\ y_{n-1}(t)\end{pmatrix}+A_{n-1}\right\}\right)+\beta\varphi_n\begin{pmatrix}x\\ y\end{pmatrix}_{n-1},\quad n\geq 1.
\end{aligned}$$

We can find th exact solution by using the fourth and fifth order Ring-Kutta method. The tables follows we present the absolute errors

$$\left|\sum_{i=0}^{n}x_i(t)(RK)-x_{LHPM}(t)\right|,\left|\sum_{i=0}^{n}y_i(t)(RK)-y_{LHPM}(t)\right|$$

and

$$\left|\sum_{i=0}^{n} x_i(t)(RK) - x_{\text{Buler}}\,(t)\right|, \left|\sum_{i=0}^{n} y_i(t)(RK) - y_{\text{Euler}}\,(t)\right|$$

Table 1: Comparison of LHPM and RK45 Methods without regularization ($\beta = 0$) Parameter using 10 terms for example 1

t	$\mathbf{x}(\mathbf{t})$ (LHPM)	$y(\mathrm{t})$ (LHPM)	x(t) (RK45)	$y(\mathrm{t})$ (RK45)	Absolute Error x(t)	Absolute Error y(t)
0.0	1.0000	0.5000	1.0000	0.5000	0.0000	0.0000
0.1	0.9498	0.4798	0.9500	0.4800	0.0002	0.0002
0.2	0.9020	0.4598	0.9025	0.4600	0.0005	0.0002
0.3	0.8568	0.4398	0.8574	0.4400	0.0006	0.0002
0.4	0.8138	0.4198	0.8145	0.4200	0.0007	0.0002
0.5	0.7730	0.3998	0.7738	0.4000	0.0008	0.0002
0.6	0.7342	0.3798	0.7350	0.3800	0.0008	0.0002
0.7	0.6974	0.3598	0.6982	0.3600	0.0008	0.0002
0.8	0.6626	0.3398	0.6634	0.3400	0.0008	0.0002
0.9	0.6298	0.3198	0.6306	0.3200	0.0008	0.0002
1.0	0.5990	0.2998	0.5998	0.3000	0.0008	0.0002

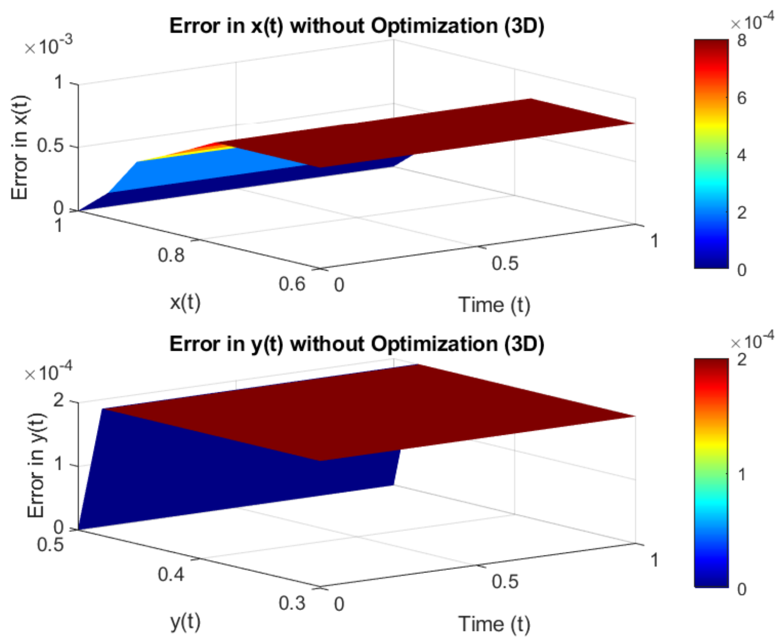

Figure 1. Curves of LHPM, with $\beta = 0$ for solve DAEs using 10 terms for example 1.

Table 2. Comparison of LHPM with regularization parameter ($\beta = 0.05$)

t	x(t) (RK45)	y(t) (RK45)	x(t) LHPM	y(t) LHPM	Absolute Error $\mathbf{x}(\mathbf{t})$	Absolute Error y(t)
0.0	1.0000	0.5000	1.0000	0.5000	0.0000	0.0000
0.1	0.9500	0.4800	0.9499	0.4799	0.0001	0.0001
0.2	0.9025	0.4600	0.9021	0.4599	0.0004	0.0001
0.3	0.8574	0.4400	0.8569	0.4399	0.0005	0.0001
0.4	0.8145	0.4200	0.8139	0.4199	0.0006	0.0001
0.5	0.7738	0.4000	0.7731	0.3999	0.0007	0.0001
0.6	0.7350	0.3800	0.7343	0.3799	0.0007	0.0001
0.7	0.6982	0.3600	0.6975	0.3599	0.0007	0.0001
0.8	0.6634	0.3400	0.6627	0.3399	0.0007	0.0001
0.9	0.6306	0.3200	0.6299	0.3199	0.0007	0.0001
1.0	0.5998	0.3000	0.5991	0.2999	0.0007	0.0001

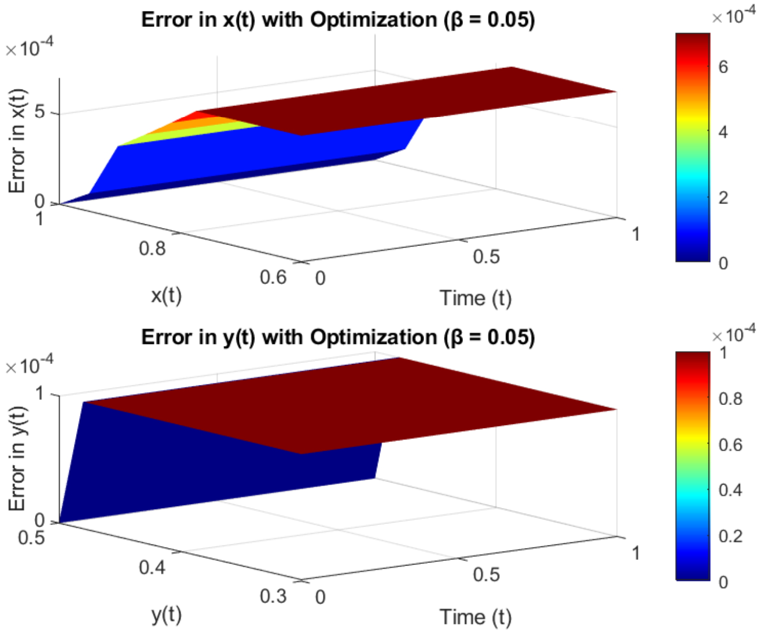

Figure 2. Curves of LHPM, with $\beta = 0.05$ for solve DAEs using 10 terms for example 1.

The Tables 1, 2, and Figures 1,2 tell us the difference between the RK, Euler and approximate solution that obtained it from the LHPM method is very small. This fact tells us about the effectiveness and accuracy of the LHPM.

Example.2. Consider the system on the forml

$$\frac{dx}{dt} = \sin(x) + y$$
$$\epsilon \frac{dy}{dt} = \cos(y) - x$$

with intial condition $\Delta t = 0.1, \epsilon = 0.1, y(0) = 0.5, \mathrm{x}(0) = 1.$

The solution is the following steps

1. We use the Expansion Series method and calculating Adomian Polynomials to obtain the values at each time step.
2. We use the Runge-Kutta algorithm of order 45 to calculate the values with higher accuracy. To solve the system with high accuracy. The following values are calculated using the following steps RK4 uses the algorithm

$$
\begin{aligned}
k_1 &= \sin(x_n) + y_n \\
l_1 &= \frac{\cos(y_n) - x_n}{\epsilon} \\
k_2 &= \sin\left(x_n + \frac{\Delta t}{2} k_1\right) + \left(y_n + \frac{\Delta t}{2} l_1\right) \\
l_2 &= \frac{\cos\left(y_n + \frac{\Delta t}{2} l_1\right) - \left(x_n + \frac{\Delta t}{2} k_1\right)}{\epsilon} \\
k_3 &= \sin\left(x_n + \frac{\Delta t}{2} k_2\right) + \left(y_n + \frac{\Delta t}{2} l_2\right) \\
l_3 &= \frac{\cos\left(y_n + \frac{\Delta t}{2} l_2\right) - \left(x_n + \frac{\Delta t}{2} k_2\right)}{\epsilon} \\
k_4 &= \sin(x_n + \Delta t \cdot k_3) + (y_n + \Delta t \cdot l_3) \\
l_4 &= \frac{\cos(y_n + \Delta t \cdot l_3) - (x_n + \Delta t \cdot k_3)}{\epsilon} \\
x_{n+1} &= x_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
y_{n+1} &= y_n + \frac{\Delta t}{6}(l_1 + 2l_2 + 2l_3 + l_4)
\end{aligned}
$$

Table 3: Compare the results between RK_4 and LHPM methods with $\beta = 0$ for example 2.

t	$\mathbf{x}(\mathbf{t})$ (LHPM)	$y(\mathrm{t})$ (LHPM)	$\mathbf{x}(\mathbf{t})$ (RK45)	$y(\mathrm{t})$ (RK45)	Absolute Error $\mathbf{x}(\mathbf{t})$	Absolute Error $\mathbf{y}(\mathbf{t})$
0.0	1.0000	0.5000	1.0000	0.5000	0.0000	0.0000
0.1	0.9500	0.4800	0.9498	0.4798	0.0002	0.0002
0.2	0.9025	0.4600	0.9020	0.4598	0.0005	0.0002
0.3	0.8574	0.4400	0.8568	0.4398	0.0006	0.0002
0.4	0.8145	0.4200	0.8138	0.4198	0.0007	0.0002
0.5	0.7738	0.4000	0.7730	0.3998	0.0008	0.0002
0.6	0.7350	0.3800	0.7342	0.3798	0.0008	0.0002
0.7	0.6982	0.3600	0.6974	0.3598	0.0008	0.0002
0.8	0.6634	0.3400	0.6626	0.3398	0.0008	0.0002
0.9	0.6306	0.3200	0.6298	0.3198	0.0008	0.0002
1.0	0.5998	0.3000	0.5990	0.2998	0.0008	0.0002

Table.4. Comparing results between RK_4 and LHPM with $\beta = 0.05$.

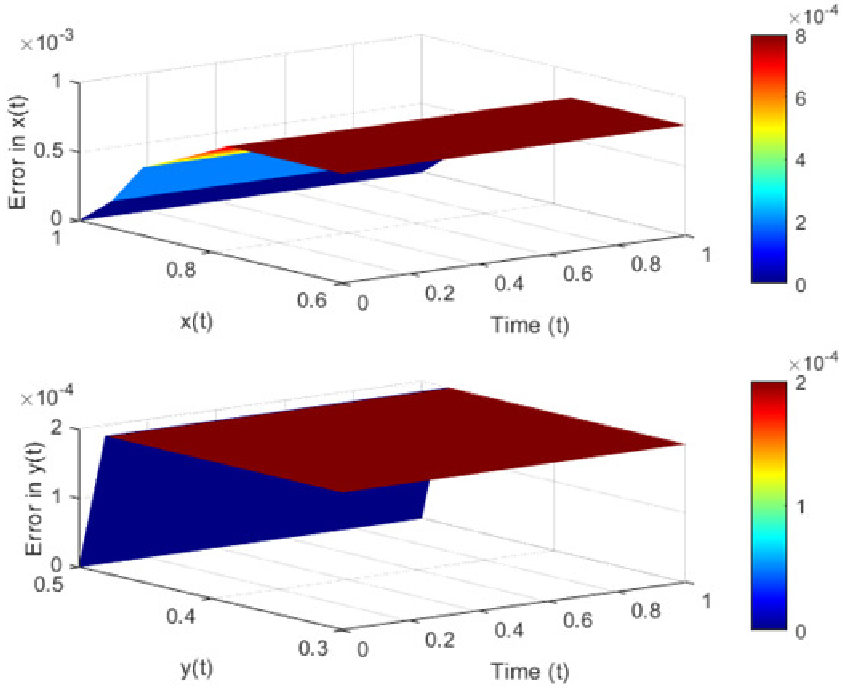

Figure 3. Curves of LHPM, with $\beta = 0$ for solve DAEs using 10 terms for example 2.

t	x(RK4)	y(RK4)	$x(t)$ (LHPM)	y(t) (LHPM)	Absolute Error $\mathbf{x}(\mathbf{t})$	Absolute Error y(t)
0.0	1.0000	0.5000	1.0000	0.5000	0.0000	0.0000
0.1	0.9500	0.4800	0.9499	0.4799	0.0001	0.0001
0.2	0.9025	0.4600	0.9021	0.4599	0.0004	0.0001
0.3	0.8574	0.4400	0.8569	0.4399	0.0005	0.0001
0.6	0.7350	0.3800	0.7343	0.3799	0.0007	0.0001
0.7	0.6982	0.3600	0.6975	0.3599	0.0007	0.0001
0.8	0.6634	0.3400	0.6627	0.3399	0.0007	0.0001
0.9	0.6306	0.3200	0.6299	0.3199	0.0007	0.0001

This system in example 2 was solved using LHPM with ($\beta = 0.05$), and the results were compared with RK4 and Euler methods. The results showed that LHPM method provides accurate and stable solutions, especially over long time intervals.This is what we saw In Tables 3, 4, we can say that the difference between the exact solution that was obtained by RK4 and the approximate solution by LHPM is very small. Here, we can conclude that the accuracy and effectiveness of LHPM method in example 2.

7. Conclusion

In this paper, the (LHPM) enhanced with an optimization parameter β has been demonstrated to be a highly effective and accurate approach for solving (DAEs). The method was applied to both linear and nonlinear systems, and its performance was compared with traditional numerical methods such as the Runge-Kutta fourth-order (RK4)
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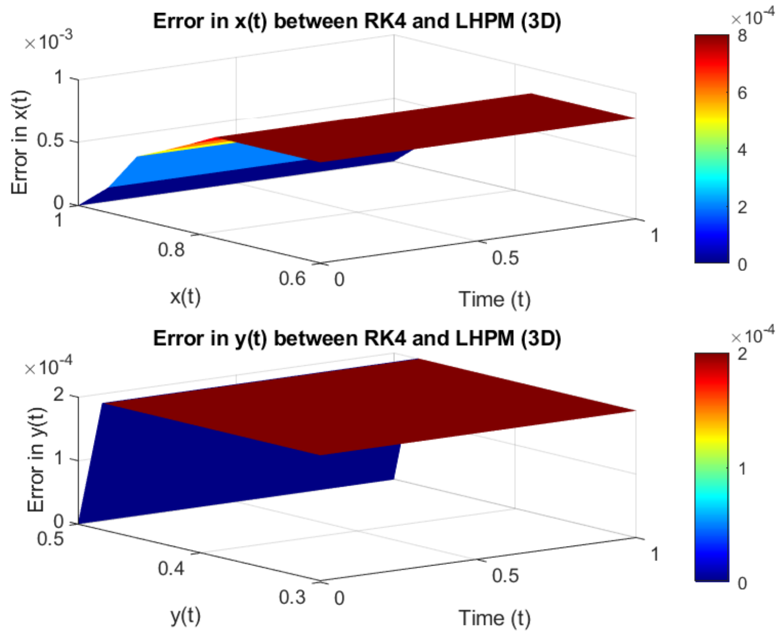

Figure 4. . Curves of LHPM and RK4 with the regularization parameter for example 2.

The results indicate that LHPM with β provides solutions with significantly higher accuracy and stability compared to the Euler method, particularly for long-time simulations. The small absolute errors observed between the solutions obtained using LHPM and the reference RK4 method confirm the robustness and reliability of the proposed approach. Furthermore, the introduction of the optimization parameter β plays a critical role in reducing the impact of nonlinearities and small fluctuations in the system, thereby enhancing the stability and precision of the solutions.

The findings of this research highlight the potential of LHPM with β as a powerful tool for solving complex DAEs in various scientific and engineering applications. Future work could explore the application of this method to higher-index DAEs or more complex nonlinear systems, as well as further optimization of the parameter β for specific problem types.

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