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Bayesian Premium Estimators for NXLindley Model Under Different Loss Functions

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Abstract The conditional distribution of (X|θ) is regarded as the NXLindley distribution. This study is centered on the estimation of the Bayesian premium using the symmetric squared error loss function and the asymmetric Linex loss function, employing the extension of Jeffreys as non-informative priors and Gamma prior as informative priors. Owing to its complexity and lack of linearity, we rely on a numerical approximation for establishing the Bayesian premium. The Lindley approximation and the Markov chain Monte Carlo method (MCMC) are employed for obtaining Bayes estimates. A simulation and comparison study between Lindley's approximation and MCMC methodunder different loss functions with several minor sample sizes is presented.

Keywords Bayesian premium, NXLindley distribution, Gamma distribution, extension of Jeffreys distribution, MCMC, loss function, Linex.

AMS 2025 subject classifications 62P05, 62.1

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1. Introduction

Insurance is a legally binding arrangement in which a person or organization (insured) transfers the potential financial risk to an insurance firm (insurer) in return for regular payments (premiums). The insurer takes on the financial obligation to reimburse the insured for any financial losses that may occur due to certain unforeseen circumstances. When determining insurance premiums, there are numerous approaches available, including Generalized Linear Models (GLM) introduced by Nelder and Wedderburn [33] and developed by De Jong [13], credibility theory (see for more details Norberg [34]), and the bonus-malus system (Lemaire [26]). The credibility theory will be employed in this research. The credibility theory serves as a rating methodology in the field of actuarial science. It is considered a quantitative approach that allows insurers to use experience rating, which involves adjusting future rates according to past experiences. The Bayesian approach is a statistical framework that integrates previous information (prior distribution) into the examination of data. Within the insurance field, this refers to the process of merging past data (collective experience) with individual policyholder information in order to calculate insurance rates. Our study centered on the Bayesian premium estimator, a well-established technique in credibility theory (see Buhlmann and Gisler [11], Robert [36], Basu [9], Mollaie [30], Ahmed [2]). This estimator was constructed by Bailey [8]. We used the NXLindley distribution as the claim distribution, it is a new distribution that is used in the insurance. In this work we aim to employ the NXLindley in purpose to determin the Bayesian premium estimators. The NXLindley distribution is characterized by a combination of the conventional exponential θ and Gamma (2,θ) distributions. It is often used in the fields of wait time studies, lifetime

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testing, reliability modeling, and traditional statistical theory. Various features of the Lindley distribution have been studied by Ghitany [19]. Currently, there is a considerable number of studies that focus on combines of the Lindley distribution, including those by Sankaran [39], Zeghdoudi and Nedjar [46], Shanker [40] among others.

Krishna and Kumar [23] employed both the maximum likelihood and Bayesian techniques in their latest research. However, they failed to evaluate the whole data set while evaluating various loss functions. The work conducted by Sajid Ali and al. [4] aims to examine the influence of various loss functions on Bayes estimates and posterior risk estimations for the Lindley distribution. Metiri [29] offers a comprehensive elucidation on the modeling of posterior distributions for the Lindley distribution by applying Linex loss functions. The calculation is performed for both non-informative and informative priors. Estimating the Bayesian premium through the squared error loss function, which is symmetric, and the linear-exponential (Linex) loss function (see also Ganji [17] and Nassar [32]), which is asymmetric, will be Sadoun's main emphasis [38]. Attoui and Sadoun [7] emphasize calculating the Bayesian premium by using three distinct loss functions: squared error (which exhibits symmetry), Linex, and entropy (both of which display asymmetry). Two distinct priors were employed in the present study. First, the extension of Jeffreys prior. It is a well-known non-informative prior that is utilized in Bayesian statistics due to its appealing invariance characteristic (see, for example Arora [6], Ahmed [2]). The last prior, known as the Gamma prior, is a fundamental component of Bayesian analysis when working with strictly positive parameters. Because of its mathematical tractability, versatility in storing many types of prior beliefs, and, often, conjugacy with popular likelihood models (see, for reference Abubakar [1] and Hasan [20]). In this work, we'll also use two distinct methods to establish the bayesian premium estimator. The Lindley's approxximation introduced by [28] and the Markov Chain Monte Carlo (MCMC) method proposed by Tierry [43]. We suggest using the Lindley and MCMC methods in the Bayesian framework using three different loss functions: squared, linex, and entropy. A simulation analysis is conducted to evaluate the performance of Lindley and MCMC estimates and to compare their efficacy across varying sample sizes

The organization of this paper is as follows: The first section is devoted to the introduction of the NXLindley distribution, while the next section demonstrates the calculation of Bayesian premiums using three distinct loss functions. The first subsection applies the squared error loss function, whereas the second subsection utilizes the linex loss function, and finally, the third subsection implements the entropy loss function. Each subsection employs two distinct priors: the extension of Jeffrey's prior and the Gamma prior. The last subsection outlines the process of determining the hyperparameters. The final part showcases a simulation analysis using Lindley's Approximation and the MCMC approach across various sample sizes. A presentation of the findings and a comparison between Lindley and MCMC over several loss functions will be provided. Finally, the concluding section of the work is presented.

2. About NXLindley distribution

Suppose we have a set of lifetimes, $x_1, x_2, ..., x_n$, that are independently and identically distributed according to the NXLindley distribution where the parameter θ is unknown. Here is the probability density function:

$$f_{NXL}(x;\theta) = \frac{\theta}{2}\left(1+\theta x\right)\exp\left(-\theta x\right),\ x,\ \theta > 0, \tag{1}$$

This distribution may be expressed as a combination of an exponential distribution with parameter (θ) and gamma distribution with parameters $(2, \theta)$.

What follows is the cumulative distribution function (c.d.f.) that represents it:

$$F_{NXL}(x;\theta) = 1 - \left(\frac{1}{2}\theta x + 1\right)e^{-x\theta}, \tag{2}$$

An alternative distribution that nearly resembles (1) is the widely recognized exponential distribution, which is defined as

$$f\left(x;\theta\right) = \theta e^{-\theta x}, \quad x > 0, \theta > 0. \tag{3}$$

Several writers have demonstrated that (1) offers a more accurate model when it comes to analyzing waiting times and survival times data compared to the exponential distribution.

Nevertheless, despite the widespread use of the exponential distribution, the NXLindley distribution described by equation (1) has been disregarded in actuarial literature and several practical fields.

For the NXLindley distribution, the related expectation and variance are

$$\begin{aligned} \mu\left(\theta\right) &= E\left[x \mid \theta\right] = \frac{3}{2\theta}, \\ \sigma^2 \ \left(\theta\right) &= var\left[x \mid \theta\right] = \frac{7}{4\theta^2}. \end{aligned} \tag{4}$$

$x_1, x_2 ..., x_n$ is a random sample obtained from NXLindley distribution, its likelihood function is:

$$L\left(x,\ \theta\right) = \left(\frac{\theta}{2}\right)^n \prod_{i=1}^{n} \left(1 + \theta x_i\right) e^{-\theta \sum_{i=1}^{n} x_i},\ x, \theta > 0 \tag{5}$$

The likelihood function's Logarithm is as follows:

$$\log L\left(x,\ \theta\right) = n \log \frac{\theta}{2} + \sum_{i=1}^{n} \log(1 + \theta x_i) - \theta \sum_{i=1}^{n} x_i. \tag{6}$$

3. Computing Bayesian premiums

In order to derive Bayesian premium estimators, we are employing the assumption that $\pi(\theta)$ is the density function of θ, which takes real values.

In this part, we consider that the distribution of θ is established and that the NXLindley is the conditional distribution of $X_n \mid \theta$. Our interest is about the posterior distribution of θ which is $f\left(\theta \mid x\right)$. We intend to obtain the Bayesian premium $P_{\bullet}^{B}$ with the already-defined priors and loss functions.

3.1. Bayesian premium estimators under squared error loss function

The authors [10, 25, 27] introduced the squared error loss function to establishing the theory of least squares. It is characterized by

$$L(\hat{\theta}, \theta) = \left(\hat{\theta} - \theta\right)^2, \tag{7}$$

Within the actuarial study, it is common to express

$$L(P_{\text{SELF}}^{B}, \mu\left(\theta\right)) = \left(P_{\text{SELF}}^{B} - \mu\left(\theta\right)\right)^2, \tag{8}$$

The estimator of $\mu\left(\theta\right)$ is the Bayesian premium denoted by $\mathrm{P}_{\text{SELF}}^{B}$, The selection should be made in order that minimizes the posterior expectation of the squared error loss function.

$$E\left[L(\hat{\theta}, \theta)\right] = \int_0^{\infty} L(\hat{\theta}, \theta) f\left(\theta \mid x\right) d\theta = \int_0^{\infty} \left(\mathrm{P}_{\text{SELF}}^{B} - \mu\left(\theta\right)\right) f\left(\theta \mid x\right) d\theta, \tag{9}$$

Here

$$\begin{aligned} \mathrm{P}_{\text{SELF}}^{B} &= E\left[\mu\left(\theta\right) \mid x\right]. \\ &= \int_0^{\infty} \mu\left(\theta\right) f\left(\theta \mid x\right) d\theta, \end{aligned} \tag{10}$$

where the individual premium is as follows:

$$\mu\left(\theta\right) = E\left[x \mid \theta\right] = \frac{3}{2\theta}. \tag{11}$$

3.1.1. Posterior distribution using the extension of Jeffreys prior The Bayesian technique utilizes both the data that is accessible and one's past understanding of the parameters. The non-informative prior may be used in a Bayesian approach when one does not have any previous information concerning the parameter.

Due to the lack of knowledge about the parameters, we aim to adopt Jeffreys' prior information extension. where the square root of the Fisher information determinant is Jeffreys' prior.

Jeffrey prior is determined by assuming $\pi(\theta) = \sqrt{I(\theta)}$, where

$$\begin{aligned} I(\theta) &= -nE\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right] = \frac{1+\theta}{\theta^2}, \\ \pi(\theta) &= k......., \text{ with } k \text{ constant.} \end{aligned}$$

For the parameter θ, the Jeffreys distribution extension is considered to be a non-informative prior. It was suggested by [28] and [5] and is provided as follows:

$$\begin{aligned} I(\theta) &= \frac{1+\theta}{\theta^2}, \\ \pi(\theta) &= [I(\theta)]^c = k\left[\frac{1+\theta}{\theta^2}\right]^c, \ \theta, c > 0, \ k \text{ is a constant.} \end{aligned} \tag{12}$$

The constant k in the Jeffreys prior (or its extension) represents a normalizing constant. It ensures that the function $\pi(\theta)$ is a valid probability density function, meaning it integrates to 1 over the parameter space:

$$\int_0^\infty \pi(\theta) = 1.$$

By incorporating the likelihood function of the NXlindley distribution with equation , for provided the data $(x_1, x_2..., x_n)$, we can compute the posterior distribution of parameter θ in the following manner:

$$\begin{aligned} f(\theta \mid x) &= \frac{\Pi_{i=1}^n L(x_i, \theta)\pi(\theta)}{\int_0^\infty \Pi_{i=1}^n L(x_i, \theta)\pi(\theta)\, d\theta} \qquad (13) \\ &= \frac{\theta^{(n-2c)}(1+\theta)^c \prod_{i=1}^n (1+\theta x_i) e^{-\theta\sum_{i=1}^n x_i}}{\int_0^\infty \theta^{(n-2c)}(1+\theta)^c \prod_{i=1}^n (1+\theta x_i) e^{-\theta\sum_{i=1}^n x_i} d\theta}, \theta > 0 \end{aligned}$$

The individual premium $\mu(\theta)$ is

$$\mu(\theta) = \frac{3}{2\theta}.$$

The Bayesian premium estimator may be obtained by inserting the posterior distribution (14) into equation (10) using the squared error loss function, as described below:

$$\begin{aligned} \mathrm{P}_{\mathrm{SELF}}^B &= \int_0^\infty \mu(\theta) f(\theta \mid x)\, d\theta \\ &= \frac{\frac{3}{2}\int_0^\infty \theta^{(n-2c-1)}(1+\theta)^c \prod_{i=1}^n (1+\theta x_i) e^{-\theta\sum_{i=1}^n x_i} d\theta}{\int_0^\infty \theta^{(n-2c)}(1+\theta)^c \prod_{i=1}^n (1+\theta x_i) e^{-\theta\sum_{i=1}^n x_i} d\theta}, \quad \theta > 0 \end{aligned} \tag{14}$$

It is important to mention that the posterior distribution $f(\theta \mid x)$ is expressed as a ratio and does not offer a credibility formula. It includes an integration in the denominator and can not be simplified into a closed form. Therefore, calculating the posterior expectation to determine the Bayesian premium of θ will be laborious. Lindley's (1980) [27] approximation technique represents one of the most straightforward methods suggested for estimating the ratio of integrals in the provided form. This approach considers the ratio of the integrals as an entirety and produces an only numerical result. Therefore, we suggest using Lindley's (1980) [27] estimate to get the Bayesian premium of θ. Several writers employed this approach to derive the Bayes estimators for certain distributions. For additional references, please consult publications such as [21, 22].

If n is suitably big, as stated by Lindley (1980), each ratio of the integral in the provided form

$$\begin{aligned} I(x) &= E[h(\theta)] \\ &= \frac{\int_\theta h(\theta)\exp[L(\theta,x)+g(\theta)]\,d\theta}{\int_\theta \exp[L(\theta,x)+g(\theta)]\,d\theta}, \quad \theta>0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} h(\theta) &= \text{function of } \theta \text{solely}, \\ L(\theta,x) &= \text{log of the likelihood}, \\ g(\theta) &= \text{log of the prior of } \theta. \end{aligned}$$

Therefore,

$$I(x) = h\left(\hat\theta\right) + 0.5\left[\left(\hat h_{\theta\theta} + 2\hat h_\theta \hat p_\theta\right)\hat\sigma_{\theta\theta}\right] + 0.5\left[\left(\hat h_\theta \hat\sigma_{\theta\theta}\right)\left(\hat L_{\theta\theta\theta}\hat\sigma_{\theta\theta}\right)\right] \tag{16}$$

$$\begin{array}{ccc} \hat h_\theta = \frac{\partial h(\hat\theta)}{\partial\hat\theta}, & \hat h_{\theta\theta} = \frac{\partial^2 h(\hat\theta)}{\partial\hat\theta^2}, & \hat p_\theta = \frac{\partial\, g(\hat\theta)}{\partial\hat\theta}, \\ \hat L_{\theta\theta} = \frac{\partial^2 L(\hat\theta)}{\partial\hat\theta^2}, & \hat\sigma_{\theta\theta} = -\frac{1}{\hat L_{\theta\theta}}, & \hat L_{\theta\theta\theta} = \frac{\partial^3 L(\hat\theta)}{\partial\hat\theta^3}. \end{array} \tag{17}$$

Upon replacing the value of $f(\theta \mid x)$, the expression can then be articulated as:

$$\mathrm{P}^B_{\text{SELF}} = E[\mu(\Theta) \mid x] = \frac{\int_\theta \mu(\theta)\exp[L(\theta,x)+g(\theta)]\,d\theta}{\int_\theta \exp[L(\theta,x)+g(\theta)]\,d\theta}, \quad \theta>0 \tag{18}$$

where

$$h(\theta) = \mu(\theta) = \frac{3}{2\theta}, \tag{19}$$

$$L(\theta,x) = n\log\frac{\theta}{2} + \sum_{i=1}^n \log(1+\theta x_i) - \theta\sum_{i=1}^n x_i, \tag{20}$$

$$g(\theta) = c\left[\log(1+\theta) - 2\log(\theta)\right], \tag{21}$$

It could be clearly confirmed that

$$\begin{array}{ccc} \hat h_\theta = -\frac{3}{2\theta^2}, & \hat h_{\theta\theta} = \frac{3}{\theta^3}, & \hat p_\theta = c\left(-\frac{1}{\theta(1+\theta)}\right), \\ \hat L_{\theta\theta} = -\frac{n}{\theta^2} - \sum\limits_{i=1}^n \frac{x_i^2}{(1+x_i\theta)^2}, & \hat\sigma_{\theta\theta} = \frac{1}{\frac{n}{\theta^2}+\sum\limits_{i=1}^n \frac{x_i^2}{(1+x_i\theta)^2}}, & \hat L_{\theta\theta\theta} = \frac{2n}{\theta^3} + \sum\limits_{i=1}^n \frac{2x_i^3}{(1+x_i\theta)^3}, \end{array}$$

Thereafter, we obtain

$$\begin{aligned} & E\left[\mu\left(\theta\right)|x\right] \\ = & -\frac{3}{2\hat{\theta}^2}+0.5\left[\left(\left(\frac{3}{\hat{\theta}^3}+\frac{3c}{\hat{\theta}^2}\left(\frac{1}{\hat{\theta}\left(1+\hat{\theta}\right)}\right)\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right]\right. \\ & -0.5\left[\left(\frac{3}{2\hat{\theta}^2\left(\frac{n}{\hat{\theta}^2}+\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}\right)}\right)\right. \\ & \left.\left(\left(\frac{2n}{\hat{\theta}^3}+\sum_{i=1}^{n}\frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right)\right]. \end{aligned} \tag{22}$$

Finally, we obtain

$$\begin{aligned} \mathrm{P}^B_{\mathrm{SELF}} = & \frac{3}{2\hat{\theta}}+0.5\left[\left(\frac{\left(1+\hat{\theta}+c\right)}{\left(1+\hat{\theta}\right)\hat{\theta}^3}\right)-\left(\frac{\frac{n}{\hat{\theta}^3}+\sum_{i=1}^{n}\frac{x_i^3}{\left(1+x_i\hat{\theta}\right)^3}}{\hat{\theta}^2\left(\frac{n}{\hat{\theta}^2}+\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}\right)}\right)\right] \\ & *\left(\frac{3}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^{n}\frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right) \end{aligned} \tag{23}$$

3.1.2. Posterior distribution using the Gamma prior (G) The gamma prior distribution (G) with parameters α and β is provided as

$$\pi\left(\theta\right)=\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\theta^{\alpha-1}e^{-\beta\theta};\alpha,\beta,\theta>0 \tag{24}$$

The following are first two moments of $G(\alpha,\beta)$:

$$E\left(\theta\right)=\frac{\alpha}{\beta},$$

and

$$Var\left(\theta\right)=\frac{\alpha}{\beta^2},$$

For the parameter θ, the posterior distribution, provided the data $(x_1,x_2...,x_n)$, may be articulated by integrating the likelihood of the NXLindley distribution with the Gamma prior (G).

$$\begin{aligned} f\left(\theta\mid x\right) &= \frac{\Pi_{i=1}^{n}L\left(x_i\mid\theta\right)\pi\left(\theta\right)}{\int_0^{\infty}\Pi_{i=1}^{n}L\left(x_i\mid\theta\right)\pi\left(\theta\right)d\theta} \\ &= \frac{\theta^{n+\alpha-1}\Pi_{i=1}^{n}\left(1+\theta x_i\right)e^{-\theta\left(\sum_{i=1}^{n}x_i+\beta\right)}}{\int_0^{\infty}\theta^{n+\alpha-1}\Pi_{i=1}^{n}\left(1+\theta x_i\right)e^{-\theta\left(\sum_{i=1}^{n}x_i+\beta\right)}d\theta}. \end{aligned} \tag{25}$$

The Bayes' estimator for the parameter θ is determined by inserting the posterior distribution (25) into equation (10) applying the squared error loss function, as below:

$$\begin{aligned} \mathrm{P}_{\mathrm{SELF}}^{B} &= E\left[\mu\left(\theta\right) \mid x\right] = \int_{0}^{\infty} \mu\left(\theta\right) f\left(\theta \mid x\right) d\theta \\ &= \frac{3}{2}\frac{\int_{0}^{\infty} \theta^{n+\alpha-2}\Pi_{i=1}^{n}\left(1+\theta x_i\right) e^{-\theta\left(\sum_{i=1}^{n} x_i+\beta\right)} d\theta}{\int_{0}^{\infty} \theta^{n+\alpha-1}\Pi_{i=1}^{n}\left(1+\theta x_i\right) e^{-\theta\left(\sum_{i=1}^{n} x_i+\beta\right)} d\theta}, \quad \theta > 0 \end{aligned} \tag{26}$$

Implementing the aforementioned process, we get

$$g\left(\theta\right) = \beta \ln \alpha - \ln \Gamma\left(\beta\right) + \left(\beta - 1\right)\ln\theta - \alpha\theta, \tag{27}$$

$$\begin{array}{ccc} \hat{h}_{\theta} = -\frac{3}{2\theta^2}, & \hat{h}_{\theta\theta} = \frac{3}{\theta^3}, & \hat{p}_{\theta} = \frac{\beta-\alpha\theta-1}{\theta}, \\ \hat{L}_{\theta\theta} = -\frac{n}{\theta^2} - \sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\theta)^2}, & \hat{\sigma}_{\theta\theta} = \frac{1}{\frac{n}{\theta^2}+\sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\theta)^2}}, & \hat{L}_{\theta\theta\theta} = \frac{2n}{\theta^3} + \sum\limits_{i=1}^{n}\frac{2x_i^3}{(1+x_i\theta)^3}, \end{array}$$

Upon simplifying, we find

$$\begin{aligned} E\left[\mu\left(\theta\right)|x\right] &= \frac{3}{2\hat{\theta}} + 0.5\left[\left(\frac{3}{\theta^3} + 2\left(-\frac{3}{2\theta^2}\right)\left(\frac{\beta-\alpha\theta-1}{\theta}\right)\right)\left(\frac{1}{\frac{n}{\theta^2}+\sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\theta)^2}}\right)\right] \\ &\quad +0.5\left[\left(-\frac{3}{2\theta^2}\right)\left(\frac{1}{\frac{n}{\theta^2}+\sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\theta)^2}}\right)^2\left(\frac{2n}{\theta^3}+\sum_{i=1}^{n}\frac{2x_i^3}{(1+x_i\theta)^3}\right)\right] \end{aligned} \tag{28}$$

Finally, we obtain

$$\mathrm{P}_{\mathrm{SELF}}^{B} = \frac{3}{2\hat{\theta}} + 0.5\left[\left(\frac{2-\beta+\alpha\hat{\theta}}{\hat{\theta}^3}\right) - \left(\frac{\frac{n}{\hat{\theta}^3}+\sum\limits_{i=1}^{n}\frac{x_i^3}{(1+x_i\hat{\theta})^3}}{\hat{\theta}^2\left(\frac{n}{\hat{\theta}^2}+\sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\hat{\theta})^2}\right)}\right)\right]\left(\frac{3}{\frac{n}{\hat{\theta}^2}+\sum\limits_{i=1}^{n}\frac{x_i^2}{(1+x_i\hat{\theta})^2}}\right) \tag{29}$$

3.2. Bayesian premium estimators under linex loss function

The linex loss function, also known as the linear-exponential loss function, has been proposed firstly by Varian [44], then by several authors [9, 29, 31, 35, 37, 41] including . This asymmetric loss function demonstrates a linear increasing on one extremity of zero and an exponential rise on the other part.

It can be represented as:

$$L(\hat{\theta},\theta) = \exp\left(a\left(\hat{\theta}-\theta\right)\right) - a\left(\hat{\theta}-\theta\right) - 1, \qquad a \neq 0 \tag{30}$$

The sign of the shape parameter a and its size denote the orientation and degree of asymmetry, respectively. If a exceeds 0, the overestimation is more significant than the underestimate, and conversely. As the value nears zero, the Linex loss converges closely to the squared error loss, rendering it almost symmetrical.

The linex loss function equation for the posterior expectation is:

$$E\left[L(\hat{\theta},\theta)\right] \propto \exp(a\hat{\theta})E\left[\exp(-a\theta)\right] - a\left(\hat{\theta}-E(\theta)\right) - 1,$$

$\hat{\theta}$ the estimate of θ that minimizes the formula aforementioned under the linex loss is determined by the outcome of Zellner [47] as follows

$$\hat{\theta} = -\frac{1}{a} \ln \left[E \left[e^{-a\theta} \right] \right]. \tag{31}$$

Our objective is to ascertain the Bayesian premium estimator $\mathrm{P}_{\mathrm{LIN}}^{B}$, which matches the value that reduces the previously described equation. This value is provided by:

$$\mathrm{P}_{\mathrm{LIN}}^{B} = -\frac{1}{a} \ln \left[E \left[e^{-a\mu(\theta)} \right] \right], \tag{32}$$

when the expectation $E\left[e^{-a\mu(\theta)}\right]$ is both existent and finite (refere to Calabria [12]).

A set of loss functions was discovered by Thomson and Basu [42], denoted as $L(\Delta)$, where Δ represents the estimate error $\left(\hat{\theta} - \theta\right)$,in a way that

- $L(0) = 0$.
- $L(\Delta) > (<) L(-\Delta) > 0$ for all $\Delta > 0$.
- $L(\cdot)$ is twice differentiable with $L'(0) = 0$ and $L''(\Delta) > 0$ for all $\Delta \neq 0$.
- $0 < L'(\Delta) > (<) - L'(-\Delta) > 0$ for all $\Delta > 0$.

3.2.1. Posterior distribution using the extension of Jeffreys prior The estimator of Bayes for θ generated from the linex loss function is

$$\mathrm{P}_{\mathrm{LIN}}^{B} = -\frac{1}{a} \ln E \left[e^{-a\mu(\theta)} \mid x \right].$$

$$\begin{aligned}
E\left[e^{-a\mu(\theta)} \mid x\right] &= \int_0^\infty e^{-a\,\mu(\theta)} f(\theta \mid x)\, d\theta \\
&= \frac{\int_0^\infty \theta^{(n-2c)} (1+\theta)^c \prod_{i=1}^{n} (1+\theta x_i)\, e^{-\left(\theta \sum_{i=1}^{n} x_i + a\,\mu(\theta)\right)} d\theta}{\int_0^\infty \theta^{(n-2c)} (1+\theta)^c \prod_{i=1}^{n} (1+\theta x_i)\, d\theta} \\
&= \frac{\int_\theta h(\theta) \exp\left[L(\theta, x) + g(\theta)\right] d\theta}{\int_\theta \exp\left[L(\theta, x) + g(\theta)\right] d\theta}, \quad \theta > 0
\end{aligned} \tag{33}$$

By replicating the aforementioned procedure, we get

$$h(\theta) = e^{-a\,\mu(\hat{\theta})}, \tag{34}$$

The functions $L(\theta, x)$ and $g(\theta)$ correspond to the ones provided in equations (20) and (21).
Thereafter, we obtain

$$\hat{h}_\theta = -a\mu'(\theta)\, e^{-a\,\mu(\theta)} = \frac{3a}{2\theta^2} e^{-\frac{3a}{2\theta}}, \qquad \hat{h}_{\theta\theta} = -a\left(\mu''(\theta) - a\mu'^2(\theta)\right) e^{-a\,\mu(\theta)} = -a\left(\frac{12\theta - 9a}{4\theta^4}\right) e^{-\frac{3a}{2\theta}}, \qquad \hat{p}_\theta = c\left(-\frac{1}{\theta(1+\theta)}\right),$$

$$\hat{L}_{\theta\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}, \qquad \hat{\sigma}_{\theta\theta} = \frac{1}{\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}}, \qquad \hat{L}_{\theta\theta\theta} = \frac{2n}{\theta^3} + \sum_{i=1}^{n} \frac{2x_i^3}{(1+x_i\theta)^3},$$

where $\mu(\theta)$ is the individual premium,

$$\mu'(\theta) = \frac{\partial \mu(\theta)}{\partial \theta} = \left(\frac{3}{2\theta}\right)' = \left(-\frac{3}{2\theta^2}\right), \tag{35}$$

$$\mu^{''}(\theta) = \frac{\partial^2 \mu(\theta)}{\partial \theta^2} = \left(\frac{3}{\theta^3}\right)^{''}, \tag{36}$$

with

$$\begin{aligned} E\left[e^{-a\mu(\theta)} \mid x\right] = e^{-a\mu(\hat{\theta})} + 0.5 \Bigg[\Bigg(-a\left(\frac{12\hat{\theta}-9a}{4\hat{\theta}^4}\right) e^{-\frac{3a}{2\hat{\theta}}} \\ +2\left(\frac{3a}{2\hat{\theta}^2} e^{-\frac{3a}{2\theta}}\right)\left(\frac{-c}{\hat{\theta}\left(1+\hat{\theta}\right)}\right)\Bigg)\left(\frac{1}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^n \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\Bigg] \\ +0.5\left[\left(\frac{3a}{2\hat{\theta}^2} e^{-\frac{3a}{2\hat{\theta}}}\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^n \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)^2 \right. \\ \left. \left(\frac{2n}{\hat{\theta}^3}+\sum_{i=1}^n \frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}\right)\right] \end{aligned} \tag{37}$$

Finally, we obtain

$$\begin{aligned} P_{\text{LIN}}^B = -\frac{1}{a}\ln\Bigg[e^{-a\frac{3}{2\hat{\theta}}} + \Bigg(\frac{\left((3a-4\hat{\theta})(1+\hat{\theta})-4\hat{\theta}c\right)}{4\hat{\theta}^2(1+\hat{\theta})} \\ + \frac{\frac{2n}{\hat{\theta}^3}+\sum_{i=1}^n \frac{2x_i^3}{(1+x_i\hat{\theta})^3}}{2\left(\frac{n}{\hat{\theta}^2}+\sum_{i=1}^n \frac{x_i^2}{(1+x_i\hat{\theta})^2}\right)}\Bigg) \\ \times \left(\frac{\frac{3a}{2\hat{\theta}^2} e^{-\frac{3a}{2\hat{\theta}}}}{\frac{n}{\hat{\theta}^2}+\sum_{i=1}^n \frac{x_i^2}{(1+x_i\hat{\theta})^2}}\right)\Bigg] \end{aligned} \tag{38}$$

3.2.2. Posterior distribution using the Gamma prior (G) The Bayesian premium estimator, when relying on the Gamma prior and the linex loss function, can be formulated as:

$$\mathrm{P}_{\text{LIN}}^B = -\frac{1}{a}\ln E\left[e^{-a\mu(\theta)} \mid x\right].$$

Where

$$\begin{aligned} E\left[e^{-a\mu(\theta)} \mid x\right] &= \int_0^\infty e^{-a\mu(\theta)} f(\theta \mid x)\, d\theta \\ &= \frac{\int_0^\infty \theta^{n+\alpha-1}\Pi_{i=1}^n (1+\theta x_i)\, e^{-\left(\theta\left(\sum_{i=1}^n x_i+\beta\right)+a\mu(\Theta)\right)}}{\int_0^\infty \theta^{n+\alpha-1}\Pi_{i=1}^n (1+\theta x_i)\, e^{-\theta\left(\sum_{i=1}^n x_i+\beta\right)} d\theta} d\theta. \end{aligned} \tag{39}$$

By repeating the previous processes, we obtain
$h(\theta) = e^{-a\mu(\theta)}$, $L(\theta, x)$ and $g(\theta)$ have the same values as those provided in equation (17).

$$\hat{h}_\theta = -a\mu^{'}(\theta)\, e^{-a\,\mu(\theta)}, \quad \hat{h}_{\theta\theta} = -a\left(\mu^{''}(\theta) - a\mu^{'2}(\theta)\right) e^{-a\,\mu(\theta)}, \quad \hat{p}_\theta = \frac{\beta-\alpha\theta-1}{\theta},$$
$$= \frac{3a}{2\theta^2} e^{-\frac{3a}{2\theta}} \qquad = -a\left(\frac{12\theta-9a}{4\theta^4}\right) e^{-\frac{3a}{2\theta}}$$
$$\hat{L}_{\theta\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}, \quad \hat{\sigma}_{\theta\theta} = \frac{1}{\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}}, \quad \hat{L}_{\theta\theta\theta} = \frac{2n}{\theta^3} + \sum_{i=1}^{n} \frac{2x_i^3}{(1+x_i\theta)^3},$$

Therefore, we have

$$E\left[e^{-a\mu(\theta)} \mid x\right] = -\frac{1}{a}\ln\left[e^{-a\frac{3}{2\hat{\theta}}} + \left(\frac{(3a-4\hat{\theta})(1+\hat{\theta}) - 4\hat{\theta}c}{4\hat{\theta}^2(1+\hat{\theta})} + \frac{\frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{2x_i^3}{(1+x_i\hat{\theta})^3}}{2\left(\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\hat{\theta})^2}\right)} \times \left(\frac{\frac{3a}{2\hat{\theta}^2} e^{-\frac{3a}{2\hat{\theta}}}}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\hat{\theta})^2}}\right)\right]\right. \tag{40}$$

Finaly, we obtain

$$P_{\text{LIN}}^{B} = -\frac{1}{a}\ln\left[e^{-a\frac{3}{2\hat{\theta}}} + \left(\frac{3a-4\hat{\theta}+4\hat{\theta}(\beta-\alpha\hat{\theta}-1)}{4\hat{\theta}^2} + \frac{\frac{2n}{\hat{\theta}^3} + \sum\limits_{i=1}^{n} \frac{2x_i^3}{(1+x_i\hat{\theta})^3}}{2\left(\frac{n}{\hat{\theta}^2} + \sum\limits_{i=1}^{n} \frac{x_i^2}{(1+x_i\hat{\theta})^2}\right)}\right) \times \left(\frac{\frac{3a}{2\hat{\theta}^2} e^{-\frac{3a}{2\hat{\theta}}}}{\frac{n}{\hat{\theta}^2} + \sum\limits_{i=1}^{n} \frac{x_i^2}{(1+x_i\hat{\theta})^2}}\right)\right] \tag{41}$$

3.3. *Bayesian premium estimators under entropy loss function*

Using the ratio $\frac{\hat{\theta}}{\theta}$ to measure the loss is frequently more accurate in many real-life instances. The General Entropy Loss function, sometimes called Stein loss, was proposed in this setting by Calabria and Pulcini [12]. Both under- and over-estimation are rewarded by this asymmetrical loss function. The entropy loss function may be represented in the form of:

$$L(\theta,\hat{\theta})\left(\frac{\hat{\theta}}{\theta}\right)^q - qln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \qquad q \neq 0, \tag{42}$$

The minimal value is found when $\hat{\theta} = \theta$. The loss parameter q reflects the magnitude of asymmetry. The loss parameter q enables the customization of the loss function's shape. When $q > 0$, the overestimation has a greater impact than underestimation, and vice-versa. It is important to observe that when q is equal to -1, the Bayesian premium is the same as the Bayesian premium under the SELF. The Bayesian premium estimator can be derived using this loss function as follows:

$$\hat{\theta} = \left[E_\theta\left[\mu(\theta)^{-q}\,|X\right]\right]^{-\frac{1}{q}}. \tag{43}$$

Given that the expectation $\left[E_\theta\left[\mu(\theta)^{-q}\,|X\right]\right]$ exits and is finite.

In the following parts of this study, we will consider the initial configuration where q=1, as used in references Dey et al. [15] and Dey and Liu [16]. Therefore,

$$P_{ENT}^{B} = \left[E_\theta\left[\mu(\theta)^{-1}\,|X\right]\right]^{-1}. \tag{44}$$

3.3.1. Posterior distribution using the extension of Jeffreys prior The Bayesian premium estimator has been computed using the entropy loss function, As outlined below

$$\mathrm{P}^{B}_{\mathrm{ENT}} = \left(E\left[\mu(\theta)^{-1} \mid x\right]\right)^{-1}. \tag{45}$$

where

$$\begin{aligned} E\left[\mu(\theta)^{-1} \mid x\right] &= \int_0^{\infty} \mu(\theta)^{-1} f(\theta \mid x)\, d\theta \\ E\left[\mu(\theta)^{-1} \mid x\right] &= \frac{\int_0^{\infty} \theta^{(n-2c+1)} (1+\theta)^c \prod_{i=1}^{n} (1+\theta x_i)\, e^{-\theta \sum_{i=1}^{n} x_i}}{\int_0^{\infty} \theta^{(n-2c)} (1+\theta)^c \prod_{i=1}^{n} (1+\theta x_i)\, e^{-\theta \sum_{i=1}^{n} x_i} d\theta} \\ &= \frac{\int_\theta h(\theta) \exp\left[L(\theta, x) + g(\theta)\right] d\theta}{\int_\theta \exp\left[L(\theta, x) + g(\theta)\right] d\theta}, \quad \theta > 0 \end{aligned} \tag{46}$$

and

$$h(\theta) = \mu(\theta)^{-1} = \frac{1}{\mu(\theta)} = \frac{2\theta}{3}, \tag{47}$$

The functions $L(\theta, x)$ and $g(\theta)$ are equivalent to the ones stated in equations (20) and (21).

$$\hat{h}_\theta = \tfrac{2}{3}, \qquad \hat{h}_{\theta\theta} = 0, \qquad \hat{p}_\theta = -\tfrac{c}{\theta(1+\theta)},$$
$$\hat{L}_{\theta\theta} = -\tfrac{n}{\theta^2} - \sum_{i=1}^{n} \tfrac{x_i^2}{(1+x_i\theta)^2}, \quad \hat{\sigma}_{\theta\theta} = \frac{1}{\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}}, \quad \hat{L}_{\theta\theta\theta} = \tfrac{2n}{\theta^3} + \sum_{i=1}^{n} \tfrac{2x_i^3}{(1+x_i\theta)^3},$$

subsequently, we obtain

$$\begin{aligned} E\left[\mu(\theta)^{-1} \mid x\right] &= \mu\left(\hat{\theta}\right)^{-1} + 0.5\left[\left(2\left(\frac{2}{3}\right)\left(-\frac{c}{\hat{\theta}\left(1+\hat{\theta}\right)}\right)\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right] \\ &\quad + 0.5\left[\left(\frac{2}{3}\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)^2 \left(\frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}\right)\right] \end{aligned} \tag{48}$$

Finally, we get

$$\mathrm{P}^{B}_{\mathrm{ENT}} = \left[\frac{2\hat{\theta}}{3} + \left[-\frac{2c}{3\hat{\theta}\left(1+\hat{\theta}\right)} + \frac{\frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}}{3\left(\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}\right)}\right]\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right]^{-1}. \tag{49}$$

3.3.2. Posterior distribution using the Gamma prior (G) The Bayes estimator which matches to the entropy loss function for the parameter θ is:

$$\mathrm{P}^{B}_{\mathrm{Ent}} = E\left[\mu(\theta)^{-1} \mid x\right]^{-1}.$$

With

$$
\begin{aligned}
E\left[\mu(\theta)^{-1} \mid x\right] &= \int_0^{\infty} \mu(\theta)^{-1} f(\theta \mid x)\, d\theta \\
&= \frac{2}{3} \frac{\int_0^{\infty} \theta^{n+\alpha} \Pi_{i=1}^{n} (1+\theta x_i)\, e^{-\theta\left(\sum_{i=1}^{n} x_i+\beta\right)} d\theta}{\int_0^{\infty} \theta^{n+\alpha-1} \Pi_{i=1}^{n} (1+\theta x_i)\, e^{-\theta\left(\sum_{i=1}^{n} x_i+\beta\right)} d\theta}.
\end{aligned} \tag{50}
$$

By executing the similar process explained earlier, we can determine $h(\theta) = \frac{2\theta}{3}$, $L(\theta, x)$ and $g(\theta)$ are identical to the ones specified in (17).

$$
\begin{array}{lll}
\hat{h}_{\theta} = \frac{2}{3}, & \hat{h}_{\theta\theta} = 0, & \hat{p}_{\theta} = \frac{\beta-\alpha\theta-1}{\theta}, \\
\hat{L}_{\theta\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}, & \hat{\sigma}_{\theta\theta} = \frac{1}{\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{x_i^2}{(1+x_i\theta)^2}}, & \hat{L}_{\theta\theta\theta} = \frac{2n}{\theta^3} + \sum_{i=1}^{n} \frac{2x_i^3}{(1+x_i\theta)^3},
\end{array}
$$

Furthermore,

$$
\begin{aligned}
E\left[\mu(\theta)^{-1} \mid x\right] &= \mu\left(\hat{\theta}\right)^{-1} \\
&+0.5\left[\left(2\left(\frac{2}{3}\right)\left(\frac{\beta-\alpha\hat{\theta}-1}{\hat{\theta}}\right)\right)\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right] \\
&+0.5\left[\left(\frac{2}{3}\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)^2\right)\left(\frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}\right)\right],
\end{aligned} \tag{51}
$$

Finally, we find

$$
\mathrm{P}_{\mathrm{ENT}}^{B} = \left[\frac{2\hat{\theta}}{3} + \left[\frac{2\left(\beta-\alpha\hat{\theta}-1\right)}{3\hat{\theta}} + \frac{\frac{2n}{\hat{\theta}^3} + \sum_{i=1}^{n} \frac{2x_i^3}{\left(1+x_i\hat{\theta}\right)^3}}{3\left(\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}\right)}\right]\left(\frac{1}{\frac{n}{\hat{\theta}^2} + \sum_{i=1}^{n} \frac{x_i^2}{\left(1+x_i\hat{\theta}\right)^2}}\right)\right]^{-1}. \tag{52}
$$

3.3.3. Elicitation of hyperparameters Elicitation, as stated in Garthwaite [18], refers to the process of articulating an individual's knowledge and convictions on some or several unknown information into a probability distribution associated with the quantities. In Bayesian statistics, it often occurs as a technique for defining the prior distribution of undefined variables in a statistical model. The job is challenging as it requires the initial identification of the previous distribution and then determining its hyperparameters.

This paper focuses on the methodology suggested by Ahn et al. [3]. In order to assess the hyperparameters α and β of the gamma prior, our approach relies on the bootstrap technique. We follow the same methods outlined in Ali et al. [4].

4. Numerical simulation

The following part provides a Monte Carlo simulation analysis to assess different techniques of estimation. The comparison is based on the mean square errors (MSEs) of the approaches.

$$MSE\left(\hat{\mathrm{P}}_{\bullet}^{B}\right)=\frac{\sum_{i=1}^{N}\left(\hat{\mathrm{P}}_{\bullet}^{B}-\mu\left(\theta\right)\right)^{2}}{N}. \tag{53}$$

N represents the number of replications. We obtained 100000 sets with sampling sizes of $n = 10$ 20, 40, 60, 80, 100, and 200 to reflect minor, medium, and major sample sizes according to the NXLindley distribution for the Lindley approximation. For the MCMC, we used samples of sizes: $n = 10, 20, 40$. The samples were created using two distinct quantities of θ ($\theta = 0.5$ and 1.5), and two different values of $\mu(\theta) = (1$ and $3)$

For comparing the Bayesian premium estimators generated in the previous section, employing three alternative loss functions and two priors. The values of the extended Jeffreys constants, which are ($c = 0.5$ and 2), get picked, the hyperparameters α and β for the Gamma prior are selected as $\alpha = 1.5$ and $\beta = (0.25$ and $0.5)$, using the linex loss function with the value of $a = (-0.1$ and $0.1)$.

4.1. Bayesian premium estimators using the Lindley's Approximation

The following subsection gives the findings of the Bayesian premium estimators using Lindley's approximation, including the Extension of Jeffreys prior and the Gamma prior. We established the Bayesian premium estimators employing three different types of loss functions: squared error, linex, and entropy.
Tables 1 to 4 illustrate the estimators generated from the Extension of Jeffreys prior and Gamma prior under the squared error loss function for Lindley's approximation.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n	**Extension of Jeffreys Prior**			
10	0.4835323(0.0002711852)	3.512331(0.1699568)	1.514138(0.0001998885)	1.068303(0.005999099)
20	0.4932283(4.58557$e-05$)	3.193638(0.02324089)	1.507463(5.569699$e-05$)	1.034782(0.001548874)
40	0.4961524(1.480373$e-05$)	3.099704(0.005842976)	1.50339(1.148992$e-05$)	1.017563(0.0003935584)
60	0.4972557(7.531059$e-06$)	3.067608(0.002606247)	1.502548(6.492539$e-06$)	1.011736(0.0001758575)
80	0.4978973(4.421449$e-06$)	3.050979(0.001467623)	1.501809(3.272417$e-06$)	1.008818(9.918924$e-05$)
100	0.4983448(2.739825$e-06$)	3.040586(0.0009376598)	1.501315(1.72943$e-06$)	1.007059(6.358418$e-05$)
200	0.4991665(6.947511$e-07$)	3.020321(0.0002344444)	1.501243(4.745561$e-07$)	1.003536(1.59479$e-05$)

Table 1 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(c = 2)$.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n	**Extension of Jeffreys Prior**			
10	0.5227481(0.0005174742)	2.967656(0.009671296)	1.577059(0.005938113)	0.9664091(0.0003597153)
20	0.5112765(0.0001271587)	2.982814(0.002460675)	1.563864(0.004078674)	0.9497712(7.824924$e-05$)
40	0.5057253(3.277897$e-05$)	2.991207(0.0006204117)	1.530999(0.0009609667)	0.9741882(2.211073$e-05$)
60	0.5037361(1.395874$e-05$)	2.994087(0.0002764415)	1.521496(0.000462097)	0.9826434(1.023451$e-05$)
80	0.5027527 (7.577559$e-06$)	2.995527(0.0001556786)	1.51611(0.0002595383)	0.9869133(5.87784$e-06$)
100	0.5022901(5.2447$e-06$)	2.996423(9.971324$e-05$)	1.512855(0.0001652585)	0.9895039(3.80782$e-06$)
200	0.5011186(1.251363$e-06$)	2.998205(2.49648$e-05$)	1.506354(4.037763$e-05$)	0.994721(9.756861$e-07$)

Table 2 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(c = 0.5)$.

n	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
	Gamma Prior			
10	0.4773312(0.0005138734)	3.596016(0.2060253)	1.251941(0.06153321)	1.498169(0.09404119)
20	0.4886524(0.0001287677)	3.295182(0.05100805)	1.375921(0.0153955)	1.238489(0.02284702)
40	0.4943478(3.194773e-05)	3.147043(0.01270224)	1.440589(0.003529637)	1.117004(0.005658656)
60	0.496235(1.417553e − 05)	3.097872(0.00563815)	1.459253(0.001660355)	1.077535(0.002509028)
80	0.4971925(7.882005e-06)	3.073357(0.003169554)	1.4703(0.0008820651)	1.058029(0.001409974)
100	0.4977436(5.091558e − 06)	3.058655(0.002027775)	1.476399(0.0005570282)	1.046313(0.0009017926)
200	0.4988731(1.269854e − 06)	3.029298(0.0005065872)	1.488067(0.0001423893)	1.023088(0.0002252134)

Table 3 - Results for Bayesian premium estimators and respective MSE's under squared error loss function ($\alpha = 1.5, \beta = 0.25$).

n	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
	Gamma Prior			
10	0.4875309(0.0001554785)	3.454012(0.1402666)	1.284337(0.04651041)	1.441072(0.07632448)
20	0.4934956(4.230658e-05)	3.226589(0.03509958)	1.390808(0.01192291)	1.213412(0.0188966)
40	0.4966737(1.106397e-05)	3.113177(0.00878083)	1.448427(0.002659779)	1.105278(0.004719965)
60	0.4978594(4.581982e − 06)	3.075415(0.003903633)	1.464013(0.001295059)	1.069916(0.002098266)
80	0.498422(2.490136e-06)	3.05656(0.002196156)	1.473655(0.0006940731)	1.052314(0.001180471)
100	0.4986851(1.728994e − 06)	3.045248(0.001405683)	1.478914(0.0004446401)	1.041794(0.0007556087)
200	0.499354(4.173548e − 07)	3.022622(0.000351491)	1.489233(0.0001159314)	1.020853(0.0001889755)

Table 4 - Results for Bayesian premium estimators and respective MSE's under squared error loss function ($\alpha = 1.5, \beta = 0.5$).

Tables 5 to 12 display the estimators derived from the Extension of Jeffreys prior and Gamma prior based on the linex loss function for Lindley's approximation.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.340677	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.10213
	Extension of Jeffreys Prior			
10	0.5234258(0.000548769)	1.350725(0.0003385998)	1.579565(0.00633063)	1.10194(6.564526e − 06)
20	0.5116397(0.0001354824)	1.350381(9.00504e-05)	1.542506(0.001806722)	1.103521(1.767326e − 06)
40	0.5057642(3.322552e-05)	1.35014(1.042375e − 05)	1.520938(0.0004384189)	1.10434(4.591111e − 07)
60	0.5038102(1.451735e − 05)	1.350047(0.002308709)	1.514288(0.0002041337)	1.104615(2.066776e − 07)
80	0.5027408(7.511852e-06)	1.349999(5.893516e-06)	1.510484(0.0001099212)	1.104754(1.170139e − 07)
100	0.5022648(5.129366e − 06)	1.349974(9.518336e-07)	1.507974 (6.358912e − 05)	1.104837(7.517411e − 08)
200	0.5011326(1.28276e − 06)	1.349917(0.0002070445)	1.504174(1.742599e − 05)	1.105003(1.89378e − 08)

Table 5 - Results for Bayesian premium estimators and respective MSE's under linex loss function ($a = -0.1, c = 0.5$).

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.340677	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.10213
	Extension of Jeffreys Prior			
10	0.4846985(0.0002341367)	1.426004(0.003913913)	1.513678(0.0001870977)	1.113242(8.23836e − 05)
20	0.4918753(6.601106e-05)	1.387145(0.0009385692)	1.50658 (4.329774e − 05)	1.109281(2.130989e − 05)
40	0.4960839(1.533621e-05)	1.368323(0.0002300122)	1.503372(1.136707e − 05)	1.107246(5.420421e − 06)
60	0.4972634(7.489252e − 06)	1.362128(0.0001015539)	1.502147(4.608217e − 06)	1.106559(2.423066e − 06)
80	0.4980174(3.930751e-06)	1.359046(5.693606e-05)	1.501765(3.114811e − 06)	1.106214(1.366912e − 06)
100	0.4983757(2.63832e − 06)	1.3572(3.636675e-05)	1.501396(1.94836e − 06)	1.106006(8.763464e − 07)
200	0.4991961(6.462584e − 07)	1.353522 (9.055936e − 06)	1.500719(5.168467e − 07)	1.105589(2.198494e − 07)

Table 6 - Results for Bayesian premium estimators and respective MSE's under linex loss function ($a = -0.1, c = 2$).

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7473835	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n	**Extension of Jeffreys Prior**			
10	$0.5229563(0.0005269912)$	$0.7461779(1.931323e-05)$	$1.585997(0.007395425)$	$0.9082912(1.76485e-06)$
20	$0.5107494(0.0001155497)$	$0.7436543(4.505885e-06)$	$1.592878\,(0.008626334)$	$0.540309\,(0.2030772)$
40	$0.5056686(3.213342e-05)$	$0.7422794(1.083046e-06)$	$1.56385\,(0.004076807)$	$0.7753505\,(0.0462984)$
60	$0.503831(1.467687e-05)$	$0.7418039(4.746252e-07)$	$1.547647\,(0.002270225)$	$0.853155\,(0.018833)$
80	$0.5028864(8.331114e-06)$	$0.7415591(2.651642e-07)$	$1.540735\,(0.00165936)$	$0.8912051\,(0.009823848)$
100	$0.5023177(5.371635e-06)$	$0.7414124(1.689732e-07)$	$1.532977\,(0.001087472)$	$0.9135371\,(0.005888664)$
200	$0.5011547(1.333225e-06)$	$0.7411172(4.187509e-08)$	$1.518373\,(0.0003375681)$	$0.9576038\,(0.00106074)$

Table 7- Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = 0.1, c = 0.5)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n	**Extension of Jeffreys Prior**			
10	$0.4845611(0.000238361)$	$0.7066194(0.0007194278)$	$1.51244(0.0001547453)$	$0.8990811(4.336733e-05)$
20	$0.49203(6.352055e-05)$	$0.7239091(0.0001763053)$	$1.506653(4.425719e-05)$	$0.9019087(1.117169e-05)$
40	$0.4959665(1.626927e-05)$	$0.7324065(4.365025e-05)$	$1.503373(1.137652e-05)$	$0.9033602(2.835232e-06)$
60	$0.4973231(7.165757e-06)$	$0.7352212(1.933681e-05)$	$1.502197(4.824642e-06)$	$0.9038498(1.266448e-06)$
80	$0.4980498(3.803131e-06)$	$0.7366243(1.085934e-05)$	$1.501726(2.979063e-06)$	$0.9040961(7.141571e-07)$
100	$0.4983739(2.6441e-06)$	$0.7374641(6.943356e-06)$	$1.501352(1.829137e-06)$	$0.9042435(4.577602e-07)$
200	$0.4991898(6.564994e-07)$	$0.7391432(1.732455e-06)$	$1.5007(4.894533e-07)$	$0.90454(1.147858e-07)$

Table 8- Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = 0.1, c = 2)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n	**Gamma Prior**			
10	$0.4764064(0.0005566586)$	$0.7007082(0.0008795655)$	$1.249536(0.0627324)$	$0.8610468(0.0007081658)$
20	$0.4883269(0.0001362609)$	$0.7207796(0.0002223624)$	$1.384926(0.01324199)$	$0.883679(0.000176789)$
40	$0.4945346(2.987079e-05)$	$0.7308075(5.59162e-05)$	$1.437277(0.003934199)$	$0.8944031(4.431893e-05)$
60	$0.4962796(1.384113e-05)$	$0.7341422(2.490283e-05)$	$1.459831(0.001613558)$	$0.8979162(1.972262e-05)$
80	$0.4971124(8.337986e-06)$	$0.7358128(1.402189e-05)$	$1.469973(0.0009016318)$	$0.8996561(1.110305e-05)$
100	$0.4977545(5.042117e-06)$	$0.7368136(8.979563e-06)$	$1.475557(0.0005974384)$	$0.9006975(7.109445e-06)$
200	$0.4988793(1.256042e-06)$	$0.7388157(2.247674e-06)$	$1.487949(0.0001452262)$	$0.9027727(1.779215e-06)$

Table 9 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.25, a = 0.1)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n	**Gamma Prior**			
10	$0.4880092(0.0001437801)$	$0.7108031(0.0005832452)$	$1.267933(0.05385528)$	$0.8660409(0.0005761741)$
20	$0.4935779(4.124355e-05)$	$0.7257783(0.0001475661)$	$1.393023(0.011444)$	$0.8859038(0.0001461343)$
40	$0.4967236(1.073508e-05)$	$0.7332858(3.712102e-05)$	$1.445846(0.00293263)$	$0.8954562(3.689125e-05)$
60	$0.4978509(4.618738e-06)$	$0.7357935(1.653252e-05)$	$1.465(0.001224976)$	$0.8986061(1.645305e-05)$
80	$0.4984135(2.516889e-06)$	$0.7370493(9.309202e-06)$	$1.473281(0.0007138912)$	$0.9001679(9.272451e-06)$
100	$0.4987021(1.684552e-06)$	$0.7378023(5.961621e-06)$	$1.478705(0.000453468)$	$0.9011062(5.94082e-06)$
200	$0.4993573(4.131121e-07)$	$0.7393098(1.49227e-06)$	$1.489293(0.0001146325)$	$0.9029747(1.488624e-06)$

Table 10 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.5, a = 0.1)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n		**Gamma Prior**		
10	0.477807(0.0004925288)	1.437901(0.004720624)	1.238892(0.06817758)	1.161987(0.001249502)
20	0.4884908(0.0001324626)	1.393118(0.001145945)	1.373003(0.01612826)	1.132028(0.0002949387)
40	0.4944598(3.069406$e-05$)	1.371309(0.0002826258)	1.439935(0.003607748)	1.118302(7.221786$e-05$)
60	0.4963446(1.336191$e-05$)	1.364119(0.0001250639)	1.459794(0.001616542)	1.113859(3.190111$e-05$)
80	0.4972605(7.505107$e-06$)	1.36054(7.019873$e-05$)	1.471054(0.0008378609)	1.111664(1.789351$e-05$)
100	0.4976774(5.394324$e-06$)	1.358398(4.487109$e-05$)	1.476014(0.0005753392)	1.110356(1.143313$e-05$)
200	0.4988669(1.283989e-06)	1.354121(1.118942e-05)	1.488009(0.0001437767)	1.107752(2.849109e-06)

Table 11 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.25, a = -0.1)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.349859	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.105171
n		**Gamma Prior**		
10	0.4864083(0.0001847334)	1.417827(0.003272332)	1.268124(0.05376633)	1.155306(0.001009876)
20	0.493783(3.865139$e-05$)	1.38363(0.0008103851)	1.392761(0.0115002)	1.129216(0.0002441183)
40	0.496805 (1.020827$e-05$)	1.366692(0.0002017034)	1.445535(0.002966413)	1.116988(6.034451$e-05$)
60	0.4979062(4.384169$e-06$)	1.361069 (8.95213$e-05$)	1.464772(0.001241031)	1.113007(2.673721$e-05$)
80	0.4983731(2.646799$e-06$)	1.358263(5.032156$e-05$)	1.473307(0.0007125122)	1.111033(1.50185$e-05$)
100	0.4987039(1.679945$e-06$)	1.35658(3.219289$e-05$)	1.478633(0.0004565613)	1.109852(9.603911$e-06$)
200	0.4993512(4.209343e-07)	1.353218 (8.041872e-06)	1.489536(0.0001094878)	1.107505(2.397319e-06)

Table 12 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.5, a = -0.1)$.

Tables 13 to 16 provide the estimators employing the Extension of Jeffreys prior and Gamma prior according to the entropy loss function for Lindley's approximation.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Extension of Jeffreys Prior**		
10	0.524601(0.0006052115)	0.3267611(0.0004733034)	1.581939(0.00671398)	0.9749019(0.006521949)
20	0.5113818(0.0001295456)	0.3299238(0.0001208535)	1.540121 (0.001609673)	0.9875611(0.001615864)
40	0.5056486(3.190658$e-05$)	0.3315963 (3.054306$e-05$)	1.521075(0.0004441445)	0.9937925(0.0004017421)
60	0.5037868 (1.433969$e-05$)	0.3321662(1.362192$e-05$)	1.513673(0.0001869462)	0.995874(0.0001783387)
80	0.5027848 (7.755119$e-06$)	0.3324557(7.675692$e-06$)	1.510073(0.0001014618)	0.9969152(0.0001002438)
100	0.5022894(5.241257$e-06$)	0.3326297(4.917492$e-06$)	1.508237(6.784772$e-05$)	0.9975316(6.412331$e-05$)
200	0.5011362(1.290914$e-06$)	0.3329804(1.231994$e-06$)	1.504132(1.707605$e-05$)	0.9987677 (1.601559$e-05$)

Table 13 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(c = 0.5)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Extension of Jeffreys Prior**		
10	0.4842605(0.0002477312)	0.3342202(0.0001356229)	1.514258(0.000203282)	0.9906165(0.0003459564)
20	0.4842605(0.0002477312)	0.3342202(0.0001356229)	1.512323(0.0001518645)	0.9906292(0.0003465013)
20	0.4919346(6.505098$e-05$)	0.3336655(3.251462$e-05$)	1.5082(6.724252$e-05$)	0.9949601(9.276045$e-05$)
40	0.4960483(1.561633$e-05$)	0.3334612(7.928529$e-06$)	1.503582(1.282716$e-05$)	0.9973845(2.406921$e-05$)
60	0.4972876(7.357179$e-06$)	0.3334102(3.493034$e-06$)	1.502309(5.332319$e-06$)	0.9982328(1.083415$e-05$)
80	0.4979548(4.182982$e-06$)	0.3333882(1.956083$e-06$)	1.501671(2.793115$e-06$)	0.9986703(6.13257$e-06$)
100	0.4984163(2.508079$e-06$)	0.3333759(1.248484$e-06$)	1.501334(1.779681$e-06$)	0.9989298 (3.938576$e-06$)
200	0.4991919(6.529538$e-07$)	0.3333532(3.103964$e-07$)	1.500665(4.42412$e-07$)	0.9994614(9.923874$e-07$)

Table 14 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(c = 2)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Gamma Prior**		
10	$0.4778117(0.0004923187)$	$0.3333899(0.0002290632)$	$1.267014(0.05428261)$	$0.9740615(0.01811707)$
20	$0.4885741(0.0001305504)$	$0.3337516(6.317058e-05)$	$1.377522(0.01500084)$	$1.002278(0.006823875)$
40	$0.4943295(3.215502e-05)$	$0.3336328(1.656132e-05)$	$1.440443(0.003547061)$	$1.005437(0.002072244)$
60	$0.4962204(1.428515e-05)$	$0.3335579(7.478605e-06)$	$1.460586(0.001553429)$	$1.004617(0.0009811488)$
80	$0.4970906(8.464477e-06)$	$0.3335087(4.239667e-06)$	$1.469959(0.0009024484)$	$1.0038(0.0005694102)$
100	$0.4977242(5.179422e-06)$	$0.3334775(2.726196e-06)$	$1.476222(0.0005653917)$	$1.003233(0.00037133)$
200	$0.4988728(1.270622e-06)$	$0.3334093(6.879754e-07)$	$1.487962(0.0001449226)$	$1.001802(9.636205e-05)$

Table 15 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(\alpha = 1.5, \beta = 0.25)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Gamma Prior**		
10	$0.4870515(0.0001676628)$	$0.3327538(6.454778e-05)$	$1.295304(0.0419003)$	$0.9790416(0.01449312)$
20	$0.493616\ (4.075536e-05)$	$0.333129(1.678899e-05)$	$1.388574(0.01241583)$	$1.002447(0.00533492)$
40	$0.4967763(1.039243e-05)$	$0.3332507(4.271883e-06)$	$1.448062(0.002697588)$	$1.004697(0.001602072)$
60	$0.4979058(4.385561e-06)$	$0.3332834(1.909366e-06)$	$1.465035(0.001222576)$	$1.003963(0.0007560392)$
80	$0.4983887(2.596357e-06)$	$0.3332963(1.076691e-06)$	$1.473874(0.000682583)$	$1.003314(0.0004381472)$
100	$0.4987086(1.667825e-06)$	$0.3333047(6.901988e-07)$	$1.478903(0.0004450766)$	$1.002788(0.0002854179)$
200	$0.499344(4.303491e-07)$	$0.3333194(1.73078e-07)$	$1.489589(0.0001083962)$	$1.00154(7.391885e-05)$

Table 16 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(\alpha = 1.5, \beta = 0.5)$.

4.1.1. Discussion In this subsection we calculated the bayesian premium estimators using the Lindley's approximation under three different loss functions(squared error, linex and entropy).
We observed that for the squared error loss function, the mean squared errors using the extension of Jeffreys prior are the lowest for θ and $\mu(\theta)$.
We noticed that for the linex loss function, the mean squared errors using the extension of Jeffreys prior in the over-estimation and under-estimation $(a = -0.1$ and $a = 0.1)$ are the smallest for θ and $\mu(\theta)$.
For the entropy loss function, we found that the mean squared errors for θ and $\mu(\theta)$ are the least when we use the extension of Jeffreys prior.

4.2. Bayesian premium estimators using the MCMC method

The next subsection gives the outcomes of the Bayesian premium estimators using MCMC with the extension of Jeffreys prior and the Gamma prior. We computed the Bayesian premium estimators using three distinct loss functions: squared error, linex, and entropy.
Tables 17 to 20 exhibit the estimators derived from the Extension of Jeffreys prior and Gamma prior under the squared error loss function for the MCMC.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n		**Extension of Jeffreys Prior**		
10	$0.4896553(0.0001070134)$	$3.338001(0.0740001)$	$1.508674(7.524142e-05)$	$1.046095(0.002724131)$
20	$0.4946209(2.893439e-05)$	$3.167264(0.01813138)$	$1.504505(2.02987e-05)$	$1.02333(0.0006958431)$
40	$0.496796(1.026556e-05)$	$3.099958(0.006475829)$	$1.502683(7.200888e-06)$	$1.014069(0.0002526872)$

Table 17 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(c = 2)$.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n		**Extension of Jeffreys Prior**		
10	$0.5151525(0.0002295977)$	$2.97774\ (0.004351839)$	$1.556494(0.003191541)$	$0.9773069(0.0001658494)$
20	$0.507558(5.712319e-05)$	$2.988333(0.00109986)$	$1.530768(0.0009466999)$	$0.9883924(4.289657e-05)$
40	$0.5043525(1.8944e-05)$	$2.992896(0.0003976158)$	$1.516766(0.0002810825)$	$0.9929973(1.566535e-05)$

Table 18 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(c = 0.5)$.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n	**Gamma Prior**			
10	$0.4846465(0.0002357286)$	$3.394863(0.09096334)$	$1.335011(0.0272213)$	$1.322195(0.0409342)$
20	$0.4925003(5.624589e-05)$	$3.196302(0.02261077)$	$1.157015(0.00667591)$	$1.157015(0.01008807)$
40	$0.495434(2.084841e-05)$	$3.117512(0.008122969)$	$1.452756(0.00223203)$	$1.09329(0.003616503)$

Table 19 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(\alpha = 1.5, \beta = 0.25)$.

	$\theta = 0.5$	$\mu(\theta) = 3$	$\theta = 1.5$	$\mu(\theta) = 1$
n	**G.P**			
10	$0.4916804(6.921591e-05)$	$3.302231(0.06237005)$	$1.361338(0.01922724)$	$1.287754(0.03367933)$
20	$0.4956886(1.858812e-05)$	$3.150931(0.01560585)$	$1.42971(0.00494064)$	$1.141022(0.008391726)$
40	$0.4974394(6.556568e-06)$	$3.090525(0.005620652)$	$1.457448(0.001810659)$	$1.084(0.003020992)$

Table 20 - Results for Bayesian premium estimators and respective MSE's under squared error loss function $(\alpha = 1.5, \beta = 0.5)$.

Tables 21 to 28 provide the estimators obtained from the Extension of Jeffreys prior and Gamma prior under the linex loss function for the MCMC.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.349859	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.105171
n	**Extension of Jeffreys Prior**			
10	$0.516306(0.0002658847)$	$1.350543\ (0.0001569165)$	$0.5162137(0.0002628831)$	$1.3505(0.0001567723)$
20	$0.5079553(6.328756e-05)$	$1.350219(4.084764e-05)$	$0.5076865(5.9082e-05)$	$1.350223(4.085136e-05)$
40	$0.5043432(1.886302e-05)$	$1.350081(1.494849e-05)$	$0.5043899(1.927085e-05)$	$1.350085(1.494907e-05)$

Table 21 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = -0.1, c = 0.5)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.349859	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.105171
n	**Extension of Jeffreys Prior**			
10	$0.4896553(0.0001070134)$	$1.399923(0.001691685)$	$1.508842(7.818304e-05)$	$1.110621(3.745688e-05)$
20	$0.4946209(2.893439e-05)$	$1.374557(0.0004116294)$	$1.504939(2.43919e-05)$	$1.107928(9.580735e-06)$
40	$0.496796(1.026556e-05)$	$1.3646(0.000146622)$	$1.502595(6.735737e-06)$	$1.106835(3.481135e-06)$

Table 22 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = -0.1, c = 2)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7473835	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
n	**Extension of Jeffreys Prior**			
10	$0.5151508(0.0002295476)$	$0.7445204(8.213055e-06)$	$1.550572(0.002557508)$	$0.9071799(7.974028e-07)$
20	$0.5074546(5.557122e-05)$	$0.7427434(1.952559e-06)$	$1.527981(0.0007829533)$	$0.9060313(2.023644e-07)$
40	$0.5045797(2.097356e-05)$	$0.7419949(6.87447e-07)$	$1.516909(0.0002858982)$	$0.90556(7.31901e-08)$

Table 23- Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = 0.1, c = 0.5)$.

	$\theta = 0.5$	$\mu(\theta)$ =0.7408182	$\theta = 1.5$	$\mu(\theta)$ =0.9048374
n	**Extension of Jeffreys Prior**			
10	$0.4897668(0.0001047185)$	$0.7181851(0.0003155175)$	$1.5109(0.0001188145)$	$0.9009556(1.966216e-05)$
20	$0.4946606(2.850903e-05)$	$0.7295811(7.785533e-05)$	$1.504626(2.140279e-05)$	$0.902874(5.015209e-06)$
40	$0.4967859(1.033038e-05)$	$0.7340972(2.788099e-05)$	$1.502467(6.084506e-06)$	$0.9036538()$

Table 24- Results for Bayesian premium estimators and respective MSE's under linex loss function $(a = 0.1, c = 2)$.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
	Gamma Prior			
10	$0.4848249(0.0002302825)$	$0.7140865(0.0003938422)$	$1.33198(0.02823073)$	$0.8763185(0.0003141514)$
20	$0.492712(5.311482e-05)$	$0.7274623(9.921528e-05)$	$1.420636(0.006298626)$	$0.8908662(7.869497e-05)$
40	$0.4954269(2.091359e-05)$	$0.7328086(3.583029e-05)$	$1.451732(0.00232978)$	$0.8965172(2.838407e-05)$

Table 25 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.25, a = 0.1)$.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.7408182	$\theta = 1.5$	$\mathbf{h}(\theta)$ =0.9048374
	Gamma Prior			
10	$0.4912795(7.604696e-05)$	$0.7207697(0.0002613064)$	$1.351776(0.02197038)$	$0.8794095(0.000258336)$
20	$0.4956623(1.881524e-05)$	$0.7307805(6.585626e-05)$	$1.431295(0.004720318)$	$0.8922924(6.53633e-05)$
40	$0.4974063(6.72706e-06)$	$0.7347889(2.378739e-05)$	$1.457273(0.001825605)$	$0.8973484(2.365918e-05)$

Table 26 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.5, a = 0.1)$.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.349859	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.105171
	Gamma Prior			
10	$0.4848696(0.0002289281)$	$1.407867(0.00205661)$	$1.342126(0.02492418)$	$1.141607(0.000533207)$
20	$0.4924763(5.660596e-05)$	$1.378535(0.0005046634)$	$1.419616(0.006461559)$	$1.122801(0.0001292122)$
40	$0.495491(2.033088e-05)$	$1.366992(0.0001804088)$	$1.451226(0.00237888)$	$1.115625(4.604474e-05)$

Table 27 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.25, a = -0.1)$.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =1.349859	$\theta = 1.5$	$\mathbf{h}(\theta)$ =1.105171
	Gamma Prior			
10	$0.4913738(7.441056e-05)$	$1.394969(0.001444854)$	$1.367683(0.01750776)$	$1.137654(0.000438189)$
20	$0.4957107(1.83979e-05)$	$1.372326(0.0003590967)$	$1.429883(0.004916452)$	$1.121013(0.0001076436)$
40	$0.4973896(6.814382e-06)$	$1.363314(0.0001289759)$	$1.457786(0.001781987)$	$1.114593(3.854739e-05)$

Table 28 - Results for Bayesian premium estimators and respective MSE's under linex loss function $(\alpha = 1.5, \beta = 0.5, a = -0.1)$.

Tables 29 to 32 show the estimators generated from the Extension of Jeffreys prior and Gamma prior applying the entropy loss function for the MCMC.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
	Extension of Jeffreys Prior			
10	$0.5145649(0.0002121376)$	$0.3288554(0.0002135414)$	$1.558266(0.003394881)$	$0.9833908(0.0028841)$
20	$0.5074062(5.48517e-05)$	$0.3310329(5.411951e-05)$	$1.526415(0.0006977578)$	$0.9917268(0.0007156718)$
40	$0.5046447(2.157316e-05)$	$0.3319376(1.958881e-05)$	1.516633 (0.0002766521)	$0.9950381(0.0002569429)$

Table 29 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(c = 0.5)$.

n	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
	Extension of Jeffreys Prior			
10	$0.4895724(0.0001087344)$	$0.3338278(5.867139e-05)$	$1.510352(0.0001071668)$	$0.9934398(0.0001613839)$
20	$0.4946268(2.887134e-05)$	$0.333518(1.421421e-05)$	$1.504083(1.667265e-05)$	$0.996555(4.223259e-05)$
40	$0.4968033(1.021895e-05)$	$0.3334303(5.048067e-06)$	$1.502798(7.830792e-06)$	$0.9978908(1.551805e-05)$

Table 30 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(c = 2)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Gamma Prior**		
10	$0.4848591(0.000229248)$	$0.3337086(0.0001087034)$	$1.337087(0.02654052)$	$0.9960216(0.01061999)$
20	$0.4921304(6.193039e-05)$	$0.3336965(2.898458e-05)$	$1.419077(0.006548589)$	$1.005307(0.003455542)$
40	$0.4955499(1.980323e-05)$	$0.3335927(1.070174e-05)$	$1.452542(0.002252239)$	$1.00502(0.001377488)$

Table 31 - Results for Bayesian premium estimators and respective MSE's under entropy loss function $(\alpha = 1.5, \beta = 0.5)$.

	$\theta = 0.5$	$\mathbf{h}(\theta)$ =0.3333333	$\theta = 1.5$	$\mathbf{h}(\theta) = 1$
n		**Gamma Prior**		
10	$0.4911833(7.773383e-05)$	$0.3330322(2.949327e-05)$	$1.363551(0.01861828)$	$0.9972141(0.00836051)$
20	$0.4956453(1.896319e-05)$	$0.3332153(7.552057e-06)$	$1.433505(0.004421625)$	$1.004762(0.002681691)$
40	$0.4973806(6.861481e-06)$	$0.3332719(2.743708e-06)$	$1.457422(0.001812895)$	1.004385 (0.001063085)

Table 32- Results for Bayesian premium estimators and respective MSE's under entropy loss function $(\alpha = 1.5, \beta = 0.5)$.

4.3. General discussion

In this numerical application we calculated the bayesian premium estimators using two different priors: the Extension of Jeffreys prior and the Gamma prior under several loss functions: squared error, linex and entropy loss function. We employed two different methods to determin the bayesian premium estimators: the Lindley's approximation and the MCMC method.
As per the simulation outcomes, we observed that when using the Lindley's approximation, the mean squared errors using the extension of Jeffreys prior under the linex loss function are the smallest, and the estimators are the most precise.
Comparing the results of the Lindleys approximation and the MCMC method, we remark that the mean squared errors of the MCMC are the tiniest compared to those of the Lindleys approximation for the small sample sizes. It means that the bayesian premium estimators of the MCMC method are the most accurate.

It is worth mentioning that as θ grows, the value of $\mu(\theta)$ drops and the Bayesian premium estimator approaches $\mu(\theta)$.

Furthermore, the outcomes of the extension of Jeffreys prior has greater accuracy compared to the gamma prior.
Based on the preceding explanation, it can be concluded that the Bayes process described in the present paper is favored for use.

5. Conclusion and Perspectives

In this paper, we established Bayesian premium estimators using Bayesian inference techniques due to the unknown risk parameter for a policyholder. By establishing a prior distribution, we can statistically define the risk structure of the whole rating category. Practically, the choice of this previous distribution is based on subjective assessments or extrapolated from past information related to the relevant group.

This research emphasizes the Bayesian estimation difficulty implementing the NXLindley distribution as a conditional distribution. The effectiveness of Bayesian premium estimators is impacted by both the prior distribution and the assumed loss function. Several researchers have commonly employed the symmetric squared error loss function. Nevertheless, in reality, the loss function frequently exhibits asymmetry.

This analysis used the extension of Jeffreys prior and the Lindley's approximation under squared error, linex and entropy loss functions employing the Lindley's approximation and the MCMC method.

Through numerical simulation, it appears to indicate that Bayesian premiums demonstrate consistency and confirm the criterion for convergence to the individual premium.

We notice that the MCMC's mean squared errors are the smallest when compared to the Lindleys approximation's for minor sample sizes.

In additional research, we may examine the log-Lindley, inverse Lindley, and gamma-Lindley distributions as conditional distributions rather of the NXLindley distribution. Furthermore, this task may be expanded by using censored data.

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