STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **14**, July 2025, pp 387–414.
Published online in International Academic Press (www.IAPress.org)

Evaluation of process capability index based on exponential progressively Type-II data with effect from multiple production lines

Zirui Chu [1], Liang Wang [1,2], Yogesh Mani Tripathi [3], Sanku Dey [4]

[1] *School of Mathematics, Yunnan Normal University, Kunming, P.R.China*
[2] *Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Yunnan Normal University, Kunming, P.R. China*
[3] *Department of Mathematics, Indian Institute of Technology Patna, Bihta, India*
[4] *Department of Statistics, St. Anthony's College, Shillong, India*

Abstract Process capability indices have been widely used to assess process performance to drive continuous improvement in quality and productivity, with larger ones being better for life cycle performance indicators. In this paper, an overall process capability index is proposed for multiple production lines. When the lifetime of units follows an exponential distribution and differences in testing facilities are taken into account, the maximum likelihood estimation, uniformly minimum variance unbiased estimation, and generalized estimation for the lifetime performance index were investigated. In order to investigate the advantages of each method, extensive Monte Carlo simulations are carried out. Finally, practical applications of the proposed methods are demonstrated through the analysis of two real-life data sets.

Keywords process capability index, multiple production lines, exponential distribution, generalized confidence interval

AMS 2010 subject classifications 62F10

DOI: 10.19139/soic-2310-5070-2469

1. Introduction

Product quality is a critical issue in both industrial production and economic management, significantly influencing an enterprise's market competitiveness and sustainability. From the consumer's perspective, there is a strong preference for high-quality products; from the viewpoint of the company's production management, the focus is on ensuring a continuous and stable production process, along with a reliable final product. To assess whether the production process meets established quality standards, process capability indices (PCIs) have been developed. These indices are widely used in traditional industries such as automotive manufacturing, semiconductors, and integrated circuit assembly to evaluate whether product quality aligns with specified requirements.[36] In recent years, there has been growing interest in simple numerical indicators that reflect the long-term performance of a product. Thus, research on PCIs has seen a significant increase. Initially, researchers introduced two-sided specification limits to ensure that the useful lifetimes of the product remain within the specified upper and lower limits. If the actual lifetime of a product falls below the lower limit or exceeds the upper limit, the product is deemed nonconforming. Such assessments can be carried out using the following indicators: C_p, C_{pm}, C_{pk}, C_{pmk} et al. For more details on two-sided specification limits PCIs, one may refer to some pioneers' contributions of Jaran et al.[1], Kane[2], Chen et al.[3], Pearn et al.[4], and others. However, for most products, both consumers and companies generally prefer those with longer service lifetimes. Regarding quality characteristics where a larger

*Correspondence to: Liang Wang (Email: liang610112@163.com). School of Mathematics, Yunnan Normal University. 768 Juxian Street, Kunming, Yunnan Province, China (650500).

ISSN 2310-5070 (online) ISSN 2311-004X (print)

value is preferable, one-sided specification limits for PCI $C_L = (\mu - L)/\sigma$ have been developed, here μ, σ, and L represent the process mean, standard deviation and specification lower limit, respectively. The PCI C_L has been widely adopted due to its simplicity and alignment with practical quality characteristics. Numerous authors have discussed it extensively in their studies, for example, Montgomery[5], Guo et al.[6], Wu et al.[18], among others.

In reliability life testing, censoring schemes (CSs) are frequently employed to facilitate the acquisition of failure data. Among all CSs, Type-I and Type-II CSs are the most fundamental. Type-I censoring terminates the test when a predetermined time is reached, while Type-II censoring stops the test when a specified number of failures occur. These schemes are designed to balance the time and cost constraints of testing with the need for accurate and reliable data, allowing the estimation of failure times and rates with relatively small sample sizes and shorter test durations. To enhance test flexibility, progressive Type-I and progressive Type-II censoring schemes (PT-II CS) have been further implemented in life testing and reliability analysis. These schemes permit the removal of the surviving test units at various stages of the experiment, providing greater flexibility in experimental design. Interested readers may refer to the monographs of Lawless[33], Balakrishnan and Cramer[34] as well as extensive references therein. In practice, PT-II CS are most popular for their flexibility and efficiency. It has been implemented as follows: Suppose n independent and identically distributed units are placed on the test with predefined CS $R = (r_1, r_2, \ldots, r_m), m \leq n$ and $\sum_{j=1}^{m}(r_j + 1) = n$. When failure time x_1 occurs, r_1 units are randomly removed from the remaining surviving items. When failure time x_2 occurs, r_2 units are randomly removed from the remaining surviving items. Others are similar. When failure time x_m occurs, all working units are removed and the experiment is stopped. As specifically illustrated in Figure 1.

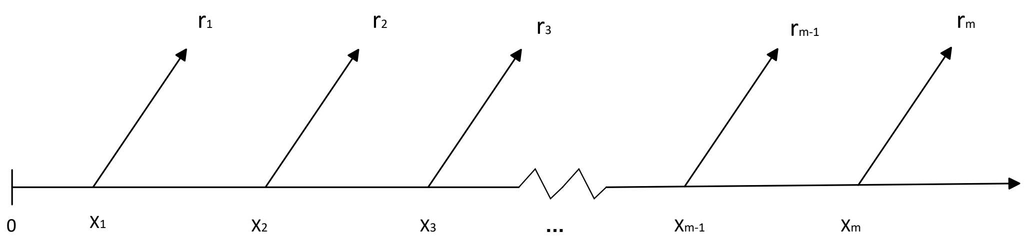

Figure 1. The sketch of the PT-II scenario.

In general, the normal distribution is the most widely used, so quality fluctuation usually follows the normal distribution by default when process stability is assumed. However, in the real production environment, due to the interaction of many factors, the obtained process output data do not always conform to the normal distribution and may sometimes present a skewed distribution. As a result, the calculated PCI may deviate. Unlike many PCIs that assume a normal distribution. Keller et al.[10] point out that the life of electronic components tends to follow an exponential, gamma, and Weibull distribution. EL-Sagheer et al.[12] researched PCI using the Pareto model. Lee et al.[9] indicate that product lifetime distribution is mostly an exponential model. Therefore, to better align with actual production needs, a flexible manufacturing process suitable for multiple production lines has been proposed.

With the advancement of manufacturing, companies are increasingly adopting multiple production lines to produce identical or similar products in parallel. This strategy aims to improve production efficiency, meet market demands, and address production uncertainties. In this context, focusing solely on the process capability of a single production line is no longer sufficient to comprehensively reflect the performance of the entire production system. Therefore, PCIs for multiple production lines have been proposed to provide a holistic assessment of overall performance. From an application perspective, one potential benefit of multiple production lines is that they address the challenge of inspecting all units simultaneously. Inspectors often face difficulties in testing all units at once, making multiple production lines a flexible solution for real-world experimentation. In addition, practical limitations, such as the lack of the equipment necessary to test all units simultaneously, make multiple production lines a more efficient method for testing units. In the study of PCIs, most scholars' research has focused primarily on single production lines. For example, the PCIs proposed by Akdoğan et al. [17] Ahmadi et al. [19] are based on a single production line. Recently, Wu et al. [7] introduced the PCI for multiple production lines. This paper aims to explore a novel approach to constructing a multi-production line PCI, distinct from those presented in other studies. We will leverage the information from multiple production lines to build a comprehensive and robust PCI.

In practical experimental designs, multiple production lines are widely adopted for their efficiency in improving the accuracy of the estimation by using stratification within each line. However, when extending a single production line to multiple production lines in a setting for large-scale product evaluations, differences in different testing facilities (DDTF) emerge as critical sources of bias due to inherent experimental limitations. Specifically, factors such as instrument drift over time, technician-dependent sorting inconsistencies, or shifts in operational protocols across cycles (e.g., modifications in subgroup ranking criteria or uneven allocation of sample units) can disrupt the comparability of rankings and measurements between lines. Therefore, it's necessary to take the DDTF into account in data analysis, otherwise ignoring inter-group variability might lead to biased and inaccurate results. See, for example, some resent contributions of Ahmadi et al.[29], Zhu[30], Wang et al.[22] and references therein.

Some of the potential contributions of this paper include the following: Firstly, based on the established theoretical framework, this study examines the differential effects of PCI on different production lines. To our knowledge, there is a relative paucity of research dedicated to evaluating PCI in the context of multiple production lines under the PT-II CS framework. Secondly, to enrich parameter estimation methods, this paper introduces generalized estimation and uniformly minimum variance unbiased estimation (UMVUE). These methods offer better performance compared to maximum likelihood estimation (MLE), providing more accurate and reliable estimates.

The article is organized as follows. The model will be described by us in Section 2. In Section 3, we present the maximum likelihood estimator and asymptotic confidence interval of unknown parameters and the PCI. Section 4 develops point and interval estimates for generalized estimation and UMVUE under the assumption of an exponential distribution. Numerical studies are presented in Section 5. Finally, Section 6 provides some brief concluding remarks.

2. Model description and likelihood function

In this part, the PCI data description is proposed under multiple production lines PT-II CS and the likelihood function with compact expression is further established.

2.1. Testing strategy and data description

Suppose n identical units are carried out under PT-II CS with k production lines and each line has n_i $(i = 1, 2, \cdots, k)$ units satisfying $\sum_{i=1}^{k} n_i = n$. For each production line, the predefined failure number m_i and CS $R_i = (r_{i1}, r_{i2}, \dots, r_{im_i})$ are provided in advance for $i = 1, 2, \dots, k$. Let x_{ij} be the failure time j-th in the i-th line. In this paper, we assume that the lifetime of products follows the exponential distribution (Exp). Under the PT-II censoring scheme, which arise from environmental factors, material resources, machinery, and experimental conditions, cannot be ignored. These differences are referred to as DDTF. In addition, it is assumed that the parameters of the model λ are different in the k production lines. The same model reflects the common failure mechanism of the production lines. A set of PT-II CS data could be observed as follows.

$$
\begin{array}{ccc}
\text{Lines} & \text{Samples} & \text{Distributions} \\
1 & (x_{11}, r_{11}), (x_{12}, r_{12}), \dots, (x_{1m_1}, r_{1m_1}) & \mathrm{Exp}(\lambda_1) \\
2 & (x_{21}, r_{21}), (x_{22}, r_{22}), \dots, (x_{2m_2}, r_{2m_2}) & \mathrm{Exp}(\lambda_2) \\
\vdots & \vdots \quad\quad \vdots \quad\quad \vdots & \vdots \\
k & (x_{k1}, r_{k1}), (x_{k2}, r_{k2}), \dots, (x_{km_k}, r_{km_k}) & \mathrm{Exp}(\lambda_k)
\end{array} \tag{1}
$$

under the multiple production lines condition, the probability density function (PDF) and cumulative distribution function (CDF) of the exponential distribution in i-th ($i = 1, 2, \ldots, k$) line is

$$f(x; \lambda_i) = \lambda_i e^{-\lambda_i x}, x > 0, \lambda_i > 0, \tag{2}$$

$$F(x; \lambda_i) = 1 - e^{-\lambda_i x}, x > 0, \lambda_i > 0. \tag{3}$$

The PCI is defined by

$$C_L = 1 - L\lambda. \tag{4}$$

2.2. Likelihood function

Let $X_i = (x_{i1}, x_{i2}, \ldots, x_{im_i}), i = 1, 2, \cdots, k$ be PT-II CS data (1) from Exp(λ_i) under i-th production line, then the likelihood function of λ_i can be written by

$$L(\lambda_i) = c_i \prod_{j=1}^{m_i} f(x_{ij})[1 - F(x_{ij})]^{r_{ij}} \propto \lambda_i^{m_i} e^{-\sum_{j=1}^{m_i} \lambda_i x_{ij}(1+r_{ij})}, i = 1, 2, \ldots, k, \tag{5}$$

where c_i is the normalizing constant. Further, the full likelihood function of λ can be expressed as follows

$$L(\lambda) = \prod_{i=1}^{k} L(\lambda_i) = \prod_{i=1}^{k} \prod_{j=1}^{m_i} \lambda_i e^{-\lambda_i x_{ij}(1+r_{ij})}. \tag{6}$$

In addition, since the DDTF cannot be ignored and its equivalent cannot be explicitly obtained. Therefore, we will present the weight coefficient in the following contexts.

3. Classical estimation

In this Section, the MLE and approximate confidence intervals (ACIs) of model parameters and PCI will be presented.

3.1. Maximum likelihood estimation

From (5), the log-likelihood function of λ_i can be rewritten by

$$\ell(\lambda_i) = m_i \ln \lambda_i - \sum_{j=1}^{m_i} \lambda_i x_{ij}(1 + r_{ij}), \tag{7}$$

moreover, the log-likelihood function of λ can be represented as

$$\ell(\lambda) = \sum_{i=1}^{k} m_i \ln \lambda_i - \sum_{i=1}^{k} \sum_{j=1}^{m_i} \lambda_i x_{ij}(1 + r_{ij}). \tag{8}$$

Theorem 1. *For a given CS R_i and predefined failure number $m_i > 0$, the MLE of parameter λ_i is obtained uniquely as follows*

$$\hat{\lambda}_i = \frac{m_i}{\sum_{j=1}^{m_i} x_{ij}(1 + r_{ij})}, i = 1, 2, \ldots, k. \tag{9}$$

Proof. See Appendix A. □

Furthermore, based on the maximum likelihood invariance, the PCI C_L could be further estimated respectively as $\hat{C}_L(\lambda) = C_L(\hat{\lambda})$, where $\hat{\lambda}$ is the maximum likelihood estimator of the population parameter λ that is estimated as

$$\hat{\lambda} = \frac{\sum_{i=1}^{k} \hat{\omega}_i \hat{\lambda}_i}{\sum_{i=1}^{k} \hat{\omega}_i}, \tag{10}$$

where $\hat{\omega}_i$ is the corresponding weight coefficient and $\hat{\omega}_i = 1/var(\hat{\lambda}_i)$. In addition, $var(\hat{\lambda}_i)$ is the observed variance of estimator of λ_i under i-th production line that would be reported later.

Now, using (4), (9) and (10) the index C_L of MLE can be obtained as

$$\hat{C}_L = 1 - L\frac{\sum_{i=1}^{k} \hat{\omega}_i \hat{\lambda}_i}{\sum_{i=1}^{k} \hat{\omega}_i}, \tag{11}$$

where $\hat{\omega}_i$ is the corresponding weight and the same as defined above.

3.2. *Approximate confidence interval*

In this subsection, the ACIs of unknown parameters are constructed by using asymptotic theory. Suppose $v = (v_1, v_2, \dots, v_k)$ with $v_i = \lambda_i$, $i = 1, 2, \dots, k$, by differentiating from (6) twice with parameters $\lambda_1, \lambda_2, \dots, \lambda_k$, the second derivative of $\ell(v) = \ell(\lambda_1, \lambda_2, \dots, \lambda_k)$ could be obtained. Thus, the expected Fisher information matrix is given by

$$I(\hat{v}) = E\left[-\frac{\partial^2 \ell}{\partial v_i \partial v_j}\right]_{\lambda=\hat{\lambda}} = \begin{pmatrix} \frac{m_1}{\hat{\lambda}_1^2} & & & \\ & \frac{m_2}{\hat{\lambda}_2^2} & & \\ & & \ddots & \\ & & & \frac{m_k}{\hat{\lambda}_k^2} \end{pmatrix}. \tag{12}$$

Theorem 2. *Under mild regularity conditions, the asymptotic distribution of MLE $\hat{v}$ is $\hat{v} - v \longrightarrow N(0, I^{-1}(\hat{v}))$, where $I^{-1}(\hat{v})$ is the inverse of the observed Fisher information matrix, i.e.*

$$I^{-1}(\hat{v}) = \begin{pmatrix} var(\hat{\lambda}_1) & cov(\hat{\lambda}_1, \hat{\lambda}_2) & \cdots & cov(\hat{\lambda}_1, \hat{\lambda}_k) \\ cov(\hat{\lambda}_2, \hat{\lambda}_1) & var(\hat{\lambda}_2) & \cdots & cov(\hat{\lambda}_2, \hat{\lambda}_k) \\ \vdots & \vdots & \ddots & \vdots \\ cov(\hat{\lambda}_k, \hat{\lambda}_1) & cov(\hat{\lambda}_k, \hat{\lambda}_2) & \cdots & var(\hat{\lambda}_k) \end{pmatrix} = \begin{pmatrix} \frac{\hat{\lambda}_1^2}{m_1} & & & \\ & \frac{\hat{\lambda}_2^2}{m_2} & & \\ & & \ddots & \\ & & & \frac{\hat{\lambda}_k^2}{m_k} \end{pmatrix}. \tag{13}$$

Proof. Using the multivariate asymptotic normality of the maximum likelihood estimation, the result could be established directly. □

For arbitrary $0 < \xi < 1$, $100(1-\xi)\%$ ACI of λ_i can be constructed as

$$\left(\hat{\lambda}_i - z_{\xi/2}\sqrt{var(\hat{\lambda}_i)}, \hat{\lambda}_i + z_{\xi/2}\sqrt{var(\hat{\lambda}_i)}\right), i = 1, 2, \dots, k, \tag{14}$$

where z_ξ is upper $100\xi\%$ percentile of standard normal distribution.

Furthermore, in order to determine confidence intervals for the population of $g(v)$, delta technique and asymptotic distribution theory are used here.

Theorem 3. *Let $g(v)$ be arbitrary continuous function of parameter v, $\hat{g}(v) = g(\hat{v})$ be the MLE of $g(v)$, then the approximate confidence interval of $g(v)$ can be constructed as*

$$\left(g(\hat{v}) - z_{\xi/2}\sqrt{var(g(\hat{v}))}, g(\hat{v}) + z_{\xi/2}\sqrt{var(g(\hat{v}))}\right), \tag{15}$$

where $var(\hat{g}(v)) = \Delta g^\top(\hat{v}) I^{-1}(\hat{v}) \Delta g(\hat{v})$ and $\Delta g(\hat{v}) = \left(\frac{\partial g(v)}{\partial \lambda_1}, \frac{\partial g(v)}{\partial \lambda_2}, \cdots, \frac{\partial g(v)}{\partial \lambda_k}\right)^\top \Big|_{\lambda_i = \hat{\lambda}_i}$.

Proof. Using **Theorem 2** and applying the delta method, the asymptotic distribution of $g(\hat{v})$ could be obtained consequently, and the details are omitted for concision. □

Obviously, when the function $g(v)$ refers to λ and PCI, the relevant ACI can be established directly. Details are omitted for simplicity and to save space.

4. Pivot quantities based on generalized inference

In this Section, we will inference the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda, C_L$ using generalized estimation and UMVUE.

4.1. Generalized inference

Although classical estimation methods have long been established in the statistical domain, recent developments have introduced generalized estimation. Generalized estimation offers several advantages over classical methods, particularly in complex scenarios where standard techniques often fail to provide satisfactory results, necessitating the use of asymptotic approximations. The robustness and flexibility of generalized pivot estimation (GPE) and generalized confidence intervals (GCI) have been widely recognized and applied in various fields, including regression analysis, analysis of variance, mixed models, and growth curve analyses. These applications demonstrate the superiority of generalized estimation in handling complex statistical problems and ensuring reliable and repeatable results. Recent research has further expanded the scope and applicability of generalized estimation. Studies by Tsui and Weerahandi [32] and Weerahandi [14] have laid the foundational work, and subsequent research has built upon these contributions to address a broader range of statistical challenges.

Theorem 4. *The quantity*

$$U_i = 2\lambda_i \sum_{j=1}^{m_i} x_{ij}(1 + r_{ij}) \sim \chi^2(2m_i), i = 1, 2, \ldots, k. \tag{16}$$

Proof. See Appendix B. □

From **Theorem 4**, it follows that for a given $u_i \sim \chi^2(2m_i)$, the equation

$$\check{\lambda}_i = \frac{u_i}{2\sum_{j=1}^{m_i} x_{ij}(1 + r_{ij})} \tag{17}$$

has a unique solution with respect to λ_i, denoted as $\check{\lambda}_i$.

By repeating the calculation of Equation (17) B times, then calculate their average to get $\check{\lambda}_i$, and $var(\check{\lambda}_i)$ can be calculated based on the B values of $\check{\lambda}_i$. Therefore, the pivotal quantities based generalized estimator for λ and overall PCI C_L can be expressed as

$$\check{\lambda} = \frac{\sum_{i=1}^{k} \check{\omega}_i \check{\lambda}_i}{\sum_{i=1}^{k} \check{\omega}_i}, \tag{18}$$

$$\check{C}_L = 1 - \check{\lambda} L, \tag{19}$$

where $\check{\omega}_i = 1/var(\check{\lambda}_i)$.

In **Algorithm 1**, the process of calculating GPE and GCI using generalized estimation is introduced in detail. It is crucial to perform N iterations to verify the accuracy of the code and ensure its stability at different functional levels. This iterative process helps establish the robustness of the estimators and ensures that the results are reliable and repeatable.

Algorithm 1 Generalized Estimation

Step 1 Generate a sample u_i from the $\chi^2(2m_i)$ distribution, and obtain an observation of $\check{\lambda}_i$ from the equation $\check{\lambda}_i = \frac{u_i}{2\sum_{j=1}^{m_i} x_{ij}(1+r_{ij})}, i = 1, 2, \ldots, k$.

Step 2 Repeat Step 1 B times and obtain their average to get $\check{\lambda}_i$.

Step 3 Based on the B values of $\check{\lambda}_i$, calculate var$(\check{\lambda}_i)$, and then use Equations (18) and (19) to compute $\check{\lambda}_i$ and $\check{C}_L$.

Step 4 Repeat Steps 1-3 N times, obtaining N values of $\check{\lambda}_i$. Sort these values in ascending order, denoted by $\check{\lambda}_i(1), \check{\lambda}_i(2), \ldots, \check{\lambda}_i(N)$. The mean of the above N values is $\check{\lambda}_i$. The calculation of $\check{\lambda}$ and $\check{C}_L$ follows the same procedure.

Step 5 Based on $\check{\lambda}_i(1), \check{\lambda}_i(2), \ldots, \check{\lambda}_i(N)$ and for $0 < \xi < 1$, a series of $100(1-\xi)\%$ confidence intervals of parameters can be expressed as

$$(p[h], p[h + B - \lfloor B\xi + 1 \rfloor]), \quad h = 1, 2, \ldots, \lfloor N\xi \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the ceiling function, and p refers to $\check{\lambda}_i, \check{\lambda}, \check{C}_L$. The $100(1-\xi)\%$ confidence interval for p can be selected as the $h^\star$-th one satisfying

$$p[h^\star + N - \lfloor N\xi + 1 \rfloor] - p[h^\star] = \min_{h=1}^{\lfloor N\xi \rfloor} (p[h + N - \lfloor N\xi + 1 \rfloor] - p[h]).$$

4.2. Uniformly minimum variance unbiased estimation

UMVUE holds significant advantages over MLE due to its ability to be unbiased and have a minimum variance. These properties ensure that in multiple independent replications, the UMVUE not only approaches the true value of the parameter without systematic bias, but also has the smallest variance among all unbiased estimators, thereby providing more stable and precise estimation results. Consequently, using UMVUE is driven by the pursuit of accuracy and reliability of the estimation, especially in statistical analysis scenarios, where it is crucial to ensure that the estimator is both unbiased and efficient. This subsection will introduce UMVUE and its confidence interval (UCI) in multiple production lines.

Theorem 5. *The UMVUE of λ_i denoted as $\tilde{\lambda}_i$ can be expressed as*

$$\tilde{\lambda}_i = \frac{m_i - 1}{\sum_{j=1}^{m_i} x_{ij}(1 + r_{ij})}. \tag{20}$$

Proof. See Appendix C. □

Using (20), we get the UMVUE of the λ is

$$\tilde{\lambda} = \frac{\sum_{i=1}^{k} \tilde{\omega}_i \tilde{\lambda}_i}{\sum_{i=1}^{k} \tilde{\omega}_i}, \tag{21}$$

now, by (4) and (21), the index $\tilde{C_L}$ can be obtained as

$$\tilde{C_L} = 1 - L\tilde{\lambda}, \tag{22}$$

where $\tilde{\omega}_i$ is the corresponding weight coefficient and $\tilde{\omega}_i = 1/var(\tilde{\lambda}_i)$. In addition, $var(\tilde{\lambda}_i)$ is the observed variance of the estimate of $\tilde{\lambda}_i$ in the i-th production line.

From **Theorem 4**, we get $P(\chi^2_{\xi/2}(2m_i) < 2\lambda_i\sum_{j=1}^{m_i} x_{ij}(1+r_{ij}) < \chi^2_{1-\xi/2}(2m_i))$. Thus the $100(1-\xi)\%$ UCI of $\widetilde{\lambda}_i, i = 1, 2, \ldots, k$ is obtained that

$$\left(\frac{\chi^2_{\xi/2}(2m_i)}{2\sum_{j=1}^{m_i} x_{ij}(1+r_{ij})}, \frac{\chi^2_{1-\xi/2}(2m_i)}{2\sum_{j=1}^{m_i} x_{ij}(1+r_{ij})}\right), \tag{23}$$

where $\chi^2_{\xi/2}(2m_i)$ and $\chi^2_{1-\xi/2}(2m_i)$ are the upper $100(\xi/2)\%$ and $100(1-\xi/2)\%$ percentile for the chi-square distribution with the degree of freedom $2m_i$ for $i = 1, 2, \ldots, k$.

Using **Theorem 3**, the $100(1-\xi)\%$ percent confidence intervals of $\tilde{\lambda}$ and $\tilde{C_L}$ can be constructed as

$$\left(g(\tilde{\lambda}) - z_{\xi/2}\sqrt{var(g(\tilde{\lambda}))}, g(\tilde{\lambda}) + z_{\xi/2}\sqrt{var(g(\tilde{\lambda}))}\right), \tag{24}$$

where $var(\tilde{g}(\lambda)) = \Delta g^\top(\tilde{\lambda})I^{-1}(\tilde{\lambda})\Delta g(\tilde{\lambda})$ and $\Delta g(\tilde{\lambda}) = \left(\frac{\partial g(\lambda)}{\partial \lambda_1}, \frac{\partial g(\lambda)}{\partial \lambda_2}, \ldots, \frac{\partial g(\lambda)}{\partial \lambda_k}\right)^\top\Big|_{\lambda=\tilde{\lambda}}$. Meanwhile, z_ξ is upper $100\xi\%$ percentile of standard normal distribution.

5. Numerical illustration

In this Section, Monte Carlo simulations are performed to compare the performance of the proposed MLE, generalized estimation and UMVUE. Meanwhile, we present two real data examples to illustrate the proposed model for the exponential distribution.

5.1. Simulation Studies

To test the efficiency of different estimates, various criteria are used for comparison, such as the absolute bias (AB) and mean square error (MSE) of the criteria, while the corresponding interval estimation results are evaluated by the average length (AL) and the coverage probability (CP) in consequence. In the simulation study, different total sample sizes n_i, effective sample sizes m_i and censoring schemes are considered for different test equipment for $i = 1, 2, \cdots, k$. The censoring schemes are presented below.

(I) $r_1 = r_2 = \ldots = r_{m_i-1} = 0, r_{m_i} = n_i - m_i$;
(II) $r_1 = n_i - m_i, r_2 = r_3 = \ldots = r_{m_i} = 0$;
(III) $r_1 = \lfloor (n_i - m_i)/2 \rfloor, r_2 = \ldots = r_{m_i-1} = 0, r_{m_i} = n_i - m_i - \lfloor (n_i - m_i)/2 \rfloor$.

Note: There exist a variety of censoring schemes in practice; to maintain simplicity, this paper considers only the three most commonly used methods.

Monte Carlo simulations are widely used to evaluate and compare the performance of different methods under uncertainty. However, their reliability can be affected by several limitations. One key issue is computational intensity, as accurate results often require a large number of iterations, leading to high time and resource costs. The validity of the results also depends heavily on the accuracy of input assumptions and probability distributions; oversimplified or incorrect inputs may yield misleading conclusions. Additionally, random sampling variability can introduce instability, especially with fewer iterations, causing differences in results across simulation runs. To mitigate these issues, we performed simulation runs $N = 5000$ for each comparison, ensuring sufficient stability and reliability in the estimated performance differences among methods.

Furthermore, to account for the DDTF effect present among the i-th production line, each equipped with multiple facilities, a stochastic component is introduced. Specifically, random noise ε_i is added to the PT-II CS data set originating from the i-th testing facility. These noises are assumed to follow a normal distribution characterized by a mean is 0 and variance is 0.01, denoted as $\varepsilon_i \sim N(0, 0.01)$. In addition, it also uses the gamma model with the shape parameter of 1 and the scale parameter of 0.1 as random noise to naturally capture line-to-line variability, denoted as $\varepsilon_i \sim Gamma(1, 0.1)$. In this study, we use the three methods that are MLE, UMVUE and generalized estimation for point and interval estimation of parameters and PCI. And the internal circulation of generalized estimation is $B = 10000$ times. The criterion measures of point estimation including AB and MSE, interval estimation including

AL and CP of the 95% confidence intervals have been calculated for both parameters and PCI. To keep the Monte Carlo simulation simple, we take $n_1 = n_2 = \ldots = n_k$, $m_1 = m_2 = \ldots = m_k$ and $R_1 = R_2 = \ldots = R_k$. Moreover, we suppose the specification lower limit $L = 0.2$. The corresponding outcomes are systematically presented in Tables 1-9. The random noise in Tables 1-6 and Table 9 follows normal distribution, while the random noise in Tables 7-8 follows gamma distribution. The number of lines in Tables 1-2, 7-8, 9 are $k = 2$, Tables 3-4 are $k = 3$, Tables 5-6 are $k = 5$, respectively.

From the results tabulated in Tables 1-9, following conclusions could be observed as:

- As the effective sample size n and the predefined failure number m increase in UMVUE, MLE and GPE, the criteria quantities AB and MSE of the point estimation decrease. This indicates that these estimation methods have consistent properties and are satisfactory in the designed scenarios.
- For fixed sample size n and predefined failure number m, the UMVUE has the lowest MSE and AB compared to other methods, indicating its superior performance, with GPQ being the second best.
- In the context of interval estimation, the analysis of the experimental data shows that the CP of each method is approximately 95%. Meanwhile, the GCI has the shortest AL. Therefore, GCI performs best for interval estimation.
- The interval lengths of ACI, UCI and GCI decrease with increase in sample size n and predefined failure number m.

In summary, when dealing with an exponential model under multiple production lines conditions where DDTF are significant and cannot be overlooked. Meanwhile, UMVUE performs well in point estimation, while GCI shows superior performance in interval estimation.

5.2. Data analysis

In this part, two real-life data sets are implemented to illustrate the applications of the proposed methods.

Example one (electrical insulation data sets) We utilize the data sets originally reported by Lawless [31] to demonstrate the practical application of the proposed model. The original data set I included the failure times of two types of electrical insulation subjected to increasing voltage stress. Data are provided in Table 10 with sample sizes $n_1 = n_2 = 12$.

Before proceeding, we first verify whether the exponential model can adequately fit these data sets. Based on the complete data presented in Table 10, the MLEs of λ_1 and λ_2 were computed, resulting in Kolmogorov-Smirnov (KS) distances with p-values of 0.6699 and 0.6921 for the electrical insulation datasets. From the test result, it can be seen that the exponential distribution can be used as an appropriate model to adapt to these actual data. Additionally, Figure 2 provides the empirical cumulative distribution obtained through theoretical distribution plots (CDF plots), probability-probability plots (P-P plots), and quantile-quantile plots (Q-Q plots). These visual plots indicate that the exponential distribution provides a reasonable fit for the real-life data. Thus, we could use Exp(λ_1) and Exp(λ_2) to fit the electrical insulation datasets. In addition, the DDTF effect appears, which affects the inference results of different estimates and is reflected in the different values of parameters λ_1 and λ_2. Two sets of production lines PT-II CS data with $m_1 = m_2 = 8$ in the electrical insulation data sets were generated based on the raw data presented in Table 10 and similar CS provided in the simulation study, and detailed samples are provided in Table 11.

Using PT-II CS data presented in Table 11, the classical likelihood, minimum variance unbiased and generalized estimates were calculated, considering the impact of variations in testing facilities. With the generalized estimation's inner loop set to $B = 10000$. The corresponding point estimates and the estimated standard error (ESE) are provided in Table 12, while the interval estimates for AL, with a significance level of 95%, are given in Table 13. The trace plots of the generalized method of $\lambda_1, \lambda_2, \lambda, C_L$ in the electrical insulation data sets are pretend in Figure 3. In these figures, the solid green line represents the mean of the results obtained from B iterations of the generalized method, while the dashed red lines indicate the upper and lower limits of the interval.

The tabular results presented in Table 12 demonstrate that, based on the correlation of ESE, UMVUE outperforms both MLE and GPE in terms of model-related parameters and PCI of the point estimates. Examining the AL in Table 13, it is evident that GCI provides the most optimal intervals among the three methods. The results further indicate

Table 1. ABs and MSEs for parameters $\lambda = 2$ with $k = 2$.

				MLE		UMVUE		GPE	
n	m	CS	par.	MSE	AB	MSE	AB	MSE	AB
20	14	I	λ_1	0.4054	0.4620	0.3340	0.4379	0.3899	0.4570
			λ_2	0.4142	0.4694	0.3604	0.4570	0.3925	0.4617
			λ	0.1555	0.3108	0.0807	0.1954	0.1549	0.3089
			C_L	0.0062	0.0621	0.0032	0.0390	0.0061	0.0610
		II	λ_1	0.4457	0.4839	0.3619	0.4609	0.4107	0.4771
			λ_2	0.4269	0.4811	0.3329	0.4378	0.3922	0.4626
			λ	0.2469	0.3694	0.0811	0.1962	0.1514	0.3132
			C_L	0.0098	0.0738	0.0032	0.0392	0.0060	0.0626
		III	λ_1	0.4467	0.4814	0.3620	0.4539	0.3929	0.4600
			λ_2	0.4832	0.4997	0.3699	0.4661	0.4306	0.4929
			λ	0.1902	0.3296	0.0825	0.2082	0.1550	0.3094
			C_L	0.0076	0.0659	0.0033	0.0416	0.0062	0.0618
	18	I	λ_1	0.3257	0.4267	0.2772	0.4073	0.3054	0.4173
			λ_2	0.3426	0.4384	0.2918	0.4151	0.3221	0.4252
			λ	0.1246	0.3061	0.0729	0.1953	0.1215	0.2713
			C_L	0.0059	0.0612	0.0029	0.0384	0.0048	0.0542
		II	λ_1	0.3720	0.4516	0.3053	0.4256	0.3390	0.4369
			λ_2	0.2891	0.4002	0.2401	0.3835	0.2849	0.3955
			λ	0.2076	0.3434	0.0701	0.1952	0.1195	0.2732
			C_L	0.0083	0.0686	0.0028	0.0384	0.0047	0.0546
		III	λ_1	0.3091	0.4099	0.2580	0.3882	0.2874	0.4055
			λ_2	0.3568	0.4409	0.2815	0.4055	0.3155	0.4217
			λ	0.1475	0.2908	0.0644	0.1863	0.1219	0.2730
			C_L	0.0059	0.0581	0.0025	0.0372	0.0048	0.0546
40	25	I	λ_1	0.1945	0.3345	0.1654	0.3215	0.1825	0.3287
			λ_2	0.1977	0.3408	0.1742	0.3254	0.1871	0.3301
			λ	0.1008	0.2459	0.0416	0.1412	0.0831	0.2328
			C_L	0.0040	0.0491	0.0017	0.0280	0.0032	0.0431
		II	λ_1	0.1945	0.3335	0.1649	0.3168	0.1737	0.3254
			λ_2	0.1854	0.3225	0.1689	0.3165	0.1727	0.3185
			λ	0.0942	0.2370	0.0423	0.1423	0.0804	0.2364
			C_L	0.0037	0.0474	0.0016	0.0294	0.0033	0.0462
		III	λ_1	0.2114	0.3462	0.1759	0.3261	0.1963	0.3375
			λ_2	0.1891	0.3301	0.1673	0.3189	0.1759	0.3199
			λ	0.1010	0.2427	0.0431	0.1463	0.0855	0.2323
			C_L	0.0040	0.0485	0.0017	0.0292	0.0034	0.0434
	30	I	λ_1	0.1581	0.3040	0.1397	0.2932	0.1524	0.2974
			λ_2	0.1594	0.3060	0.1395	0.2932	0.1565	0.3022
			λ	0.0796	0.2206	0.0348	0.1315	0.0723	0.2169
			C_L	0.0031	0.0441	0.0013	0.0263	0.0028	0.0430
		II	λ_1	0.1428	0.2900	0.1355	0.2848	0.1402	0.2888
			λ_2	0.1631	0.3103	0.1563	0.3100	0.1578	0.3063
			λ	0.0772	0.2155	0.0362	0.1371	0.0704	0.2140
			C_L	0.0030	0.0431	0.0014	0.0274	0.0028	0.0427
		III	λ_1	0.1443	0.2905	0.1246	0.2754	0.1357	0.2840
			λ_2	0.1418	0.2883	0.1277	0.2800	0.1352	0.2814
			λ	0.0725	0.2157	0.0425	0.1394	0.0720	0.2251
			C_L	0.0029	0.0431	0.0016	0.0278	0.0024	0.0420

Table 2. Length and coverage probability for parameter $\lambda = 2$ with 95% confidence intervals and $k = 2$.

				ACI		UCI		GCI	
n	m	CS	par.	Length	CP	Length	CP	Length	CP
20	14	I	λ_1	2.2247	0.9434	2.2226	0.9492	2.1686	0.9542
			λ_2	2.3028	0.9506	2.2990	0.9520	2.2445	0.9552
			λ	1.6151	0.9364	1.5122	0.9448	1.4836	0.9462
			C_L	0.3230	0.9364	0.3024	0.9462	0.2967	0.9468
		II	λ_1	2.5063	0.9380	2.3361	0.9506	2.2929	0.9486
			λ_2	2.5224	0.9434	2.2585	0.9452	2.2200	0.9526
			λ	1.6761	0.9532	1.6521	0.9558	1.5154	0.9602
			C_L	0.3352	0.9532	0.3304	0.9558	0.3030	0.9602
		III	λ_1	2.3239	0.9470	2.3193	0.9498	2.2728	0.9514
			λ_2	2.4301	0.9468	2.4019	0.9550	2.3481	0.9552
			λ	1.6930	0.9586	1.5915	0.9622	1.5619	0.9642
			C_L	0.3386	0.9586	0.3183	0.9622	0.3123	0.9642
	18	I	λ_1	2.1662	0.9432	2.0582	0.9494	2.0272	0.9546
			λ_2	2.2010	0.9386	2.0856	0.9420	2.0508	0.9426
			λ	1.5008	0.9454	1.4912	0.9626	1.3966	0.9704
			C_L	0.3001	0.9454	0.2982	0.9626	0.2793	0.9704
		II	λ_1	2.2982	0.9450	2.1555	0.9482	2.1164	0.9524
			λ_2	2.1105	0.9440	1.9448	0.9408	1.9102	0.9524
			λ	1.4795	0.9486	1.4779	0.9642	1.3793	0.9672
			C_L	0.2959	0.9486	0.2955	0.9642	0.2758	0.9672
		III	λ_1	2.0229	0.9586	1.9935	0.9502	1.9732	0.9500
			λ_2	2.1046	0.9476	2.0980	0.9514	2.0568	0.9552
			λ	1.4829	0.9578	1.3985	0.9654	1.3861	0.9688
			C_L	0.2965	0.9578	0.2797	0.9654	0.2772	0.9688
40	25	I	λ_1	1.6897	0.9434	1.6451	0.9460	1.6038	0.9506
			λ_2	1.7183	0.9450	1.6280	0.9446	1.5938	0.9466
			λ	1.1840	0.9456	1.1663	0.9516	1.0205	0.9530
			C_L	0.2368	0.9456	0.2326	0.9516	0.2141	0.9530
		II	λ_1	1.6775	0.9424	1.5969	0.9472	1.5806	0.9496
			λ_2	1.6073	0.9410	1.5809	0.9476	1.5522	0.9516
			λ	1.1529	0.9330	1.1413	0.9434	1.0940	0.9446
			C_L	0.2305	0.9330	0.2282	0.9434	0.2188	0.9446
		III	λ_1	1.8210	0.9464	1.6390	0.9466	1.6288	0.9456
			λ_2	1.7080	0.9432	1.5931	0.9498	1.5635	0.9524
			λ	1.1747	0.9434	1.1603	0.9658	1.1080	0.9662
			C_L	0.2349	0.9434	0.2320	0.9658	0.2216	0.9662
	30	I	λ_1	1.5270	0.9464	1.4480	0.9464	1.4318	0.9488
			λ_2	1.5876	0.9490	1.4791	0.9496	1.4690	0.9458
			λ	1.0659	0.9356	1.0515	0.9498	1.0126	0.9540
			C_L	0.2131	0.9356	0.2103	0.9498	0.2025	0.9540
		II	λ_1	1.4665	0.9446	1.4195	0.9456	1.4023	0.9508
			λ_2	1.5600	0.9490	1.5357	0.9502	1.5114	0.9528
			λ	1.0709	0.9422	1.0532	0.9526	1.0167	0.9558
			C_L	0.2141	0.9422	0.2106	0.9526	0.2033	0.9558
		III	λ_1	1.4659	0.9458	1.3977	0.9544	1.3786	0.9524
			λ_2	1.4508	0.9488	1.4014	0.9492	1.3854	0.9496
			λ	1.0155	0.8860	1.0015	0.9284	0.9703	0.9264
			C_L	0.2031	0.8860	0.2003	0.9284	0.1940	0.9264

Table 3. ABs and MSEs for parameter $\lambda = 1$ with $k = 3$.

				MLE		UMVUE		GPE	
n	m	CS	parameter	MSE	AB	MSE	AB	MSE	AB
15	8	I	λ_1	0.9044	0.6579	0.6831	0.6034	0.8985	0.6484
			λ_2	0.9747	0.6641	0.6594	0.5983	0.8690	0.6375
			λ_3	1.1565	0.7332	0.7642	0.6453	0.9047	0.6683
			λ	0.4191	0.4764	0.0669	0.1887	0.2131	0.3889
			C_L	0.0167	0.0952	0.0026	0.0377	0.0085	0.0777
		II	λ_1	0.8766	0.6273	0.5731	0.5533	0.7646	0.6056
			λ_2	0.9397	0.6712	0.6930	0.5962	0.8729	0.6422
			λ_3	1.0500	0.7135	0.7515	0.6306	0.9986	0.6780
			λ	0.3717	0.4430	0.0622	0.1816	0.2249	0.4049
			C_L	0.0148	0.0886	0.0024	0.0363	0.0089	0.0809
		III	λ_1	0.8219	0.6207	0.5681	0.5590	0.7584	0.6100
			λ_2	0.8755	0.6487	0.6653	0.5974	0.8864	0.6399
			λ_3	0.9057	0.6565	0.6559	0.5808	0.8058	0.6192
			λ	0.3080	0.4089	0.0768	0.2061	0.2589	0.3996
			C_L	0.0123	0.0817	0.0030	0.0412	0.0103	0.0079
15	12	I	λ_1	0.5313	0.5348	0.4429	0.4959	0.5287	0.5245
			λ_2	0.5353	0.5198	0.4104	0.4783	0.4678	0.4953
			λ_3	0.5386	0.5280	0.4083	0.4776	0.4356	0.4857
			λ	0.2164	0.3404	0.0409	0.1438	0.1387	0.3107
			C_L	0.0081	0.0600	0.0016	0.0287	0.0055	0.0621
		II	λ_1	0.5697	0.5389	0.4333	0.4922	0.4842	0.5011
			λ_2	0.5075	0.5140	0.4037	0.4863	0.4400	0.4850
			λ_3	0.5642	0.5400	0.4251	0.4886	0.4946	0.5187
			λ	0.2215	0.3524	0.0389	0.1411	0.1372	0.3074
			C_L	0.0088	0.0706	0.0015	0.0282	0.0054	0.0614
		III	λ_1	0.5667	0.5308	0.4078	0.4840	0.4845	0.5020
			λ_2	0.5355	0.5414	0.4401	0.4805	0.4816	0.5033
			λ_3	0.6234	0.5612	0.4503	0.5090	0.5281	0.5260
			λ	0.1853	0.3246	0.0444	0.1611	0.1226	0.2859
			C_L	0.0110	0.0653	0.0017	0.0322	0.0049	0.0535
30	12	I	λ_1	0.4477	0.4799	0.3345	0.4389	0.3969	0.4609
			λ_2	0.5333	0.5100	0.4057	0.4686	0.4610	0.4940
			λ_3	0.4639	0.4796	0.3695	0.4523	0.4123	0.4647
			λ	0.1761	0.3140	0.0402	0.1407	0.1372	0.3096
			C_L	0.0070	0.0599	0.0022	0.0261	0.0054	0.0599
		II	λ_1	0.5215	0.5199	0.3940	0.4689	0.4730	0.5002
			λ_2	0.4925	0.5010	0.3638	0.4561	0.4353	0.4795
			λ_3	0.5574	0.5329	0.4245	0.4796	0.4882	0.5128
			λ	0.1992	0.3359	0.0382	0.1405	0.1446	0.2981
			C_L	0.0079	0.0671	0.0014	0.0303	0.0053	0.0611
		III	λ_1	0.4350	0.4801	0.3810	0.4569	0.3977	0.4659
			λ_2	0.4981	0.5079	0.4049	0.4777	0.4666	0.4973
			λ_3	0.4763	0.4921	0.3521	0.4508	0.4388	0.4882
			λ	0.1519	0.2975	0.0437	0.1606	0.1180	0.2811
			C_L	0.0060	0.0595	0.0016	0.0315	0.0048	0.0522

Table 4. Length and coverage probability for parameter $\lambda = 1$ with 95% confidence intervals and $k = 3$.

n	m	CS	parameter	ACI Length	ACI CP	UCI Length	UCI CP	GCI Length	GCI CP
15	8	I	λ_1	3.5764	0.9384	3.1726	0.9456	3.0931	0.9458
			λ_2	3.9633	0.9480	3.1379	0.9490	3.0386	0.9550
			λ_3	3.9869	0.9380	3.3460	0.9486	3.2483	0.9554
			λ	2.2154	0.9116	1.9736	0.9574	1.8590	0.9790
			C_L	0.4430	0.9116	0.3947	0.9574	0.3718	0.9790
		II	λ_1	3.4922	0.9406	2.9824	0.9510	2.8984	0.9512
			λ_2	3.5806	0.9368	3.2224	0.9520	3.0961	0.9532
			λ_3	3.7981	0.9320	3.3121	0.9410	3.1971	0.9464
			λ	2.1804	0.8976	1.9200	0.9590	1.8145	0.9800
			C_L	0.4360	0.8976	0.3840	0.9590	0.3629	0.9800
		III	λ_1	3.4585	0.9368	2.9581	0.9484	2.8879	0.9494
			λ_2	3.4447	0.9320	3.1200	0.9460	3.0508	0.9514
			λ_3	3.4719	0.9290	3.0113	0.9422	2.9332	0.9492
			λ	2.0661	0.8544	1.8495	0.9564	1.7555	0.9636
			C_L	0.4132	0.8544	0.3699	0.9564	0.3511	0.9636
15	12	I	λ_1	2.7430	0.9376	2.5700	0.9478	2.5147	0.9530
			λ_2	2.7563	0.9424	2.4215	0.9476	2.3746	0.9476
			λ_3	2.7621	0.9360	2.4625	0.9440	2.3914	0.9486
			λ	1.7931	0.8832	1.4954	0.9534	1.4885	0.9740
			C_L	0.3586	0.8832	0.2990	0.9534	0.2977	0.9740
		II	λ_1	2.9248	0.9374	2.5213	0.9432	2.4506	0.9532
			λ_2	2.6851	0.9422	2.4241	0.9460	2.3634	0.9490
			λ_3	2.8517	0.9406	2.5263	0.9474	2.4906	0.9500
			λ	1.8128	0.9228	1.5082	0.9462	1.4941	0.9864
			C_L	0.3625	0.9228	0.3016	0.9462	0.2988	0.9864
		III	λ_1	2.7680	0.9414	2.5388	0.9518	2.4538	0.9526
			λ_2	2.8585	0.9420	2.4857	0.9470	2.4081	0.9516
			λ_3	2.9420	0.9476	2.5956	0.9478	2.5360	0.9474
			λ	1.8305	0.8588	1.5531	0.9578	1.5262	0.9758
			C_L	0.3661	0.8588	0.3106	0.9578	0.3106	0.9758
30	12	I	λ_1	2.4911	0.9334	2.2728	0.9472	2.2325	0.9474
			λ_2	2.7410	0.9386	2.3964	0.9498	2.3396	0.9494
			λ_3	2.5583	0.9396	2.2855	0.9518	2.2251	0.9536
			λ	1.7443	0.9244	1.4454	0.9480	1.4407	0.9882
			C_L	0.3488	0.9244	0.2890	0.9480	0.2881	0.9882
		II	λ_1	2.7158	0.9334	2.4662	0.9436	2.4123	0.9504
			λ_2	2.6833	0.9386	2.3658	0.9424	2.2994	0.9496
			λ_3	2.7698	0.9336	2.5080	0.9530	2.4592	0.9508
			λ	1.8002	0.9136	1.4840	0.9544	1.4846	0.9850
			C_L	0.3600	0.9136	0.2968	0.9544	0.2969	0.9850
		III	λ_1	2.5554	0.9370	2.3167	0.9532	2.2560	0.9536
			λ_2	2.7057	0.9406	2.4285	0.9470	2.3637	0.9528
			λ_3	2.7187	0.9310	2.3416	0.9488	2.3083	0.9496
			λ	1.7219	0.9310	1.4160	0.9546	1.4144	0.9928
			C_L	0.3443	0.9310	0.2832	0.9546	0.2828	0.9928

Table 5. ABs and MSEs for parameter $\lambda = 0.8$ with $k = 5$.

				MLE		UMVUE		GPE	
n	m	CS	parameter	MSE	AB	MSE	AB	MSE	AB
10	8	I	λ_1	0.1477	0.2607	0.0922	0.2226	0.1342	0.2491
			λ_2	0.1658	0.2754	0.1107	0.2438	0.1475	0.2653
			λ_3	0.1388	0.2595	0.1030	0.2381	0.1205	0.2422
			λ_4	0.2197	0.3237	0.1561	0.2874	0.2055	0.3047
			λ_5	0.2319	0.3245	0.1499	0.2856	0.1915	0.2996
			λ	0.0597	0.1897	0.0079	0.0696	0.0296	0.1486
			C_L	0.0024	0.0379	0.0003	0.0139	0.0012	0.0297
		II	λ_1	0.1471	0.2623	0.1089	0.2410	0.1465	0.2653
			λ_2	0.1694	0.2796	0.1144	0.2513	0.1545	0.2745
			λ_3	0.1798	0.2910	0.1258	0.2610	0.1690	0.2796
			λ_4	0.1476	0.2586	0.0947	0.2254	0.1207	0.2419
			λ_5	0.1636	0.2806	0.1175	0.2487	0.1443	0.2652
			λ	0.0481	0.1652	0.0047	0.0529	0.0362	0.1571
			C_L	0.0019	0.0330	0.0002	0.0106	0.0014	0.0334
		III	λ_1	0.1559	0.2708	0.1098	0.2426	0.1457	0.2637
			λ_2	0.1316	0.2510	0.0924	0.2221	0.1213	0.2411
			λ_3	0.1358	0.2508	0.1010	0.2292	0.1355	0.2477
			λ_4	0.1376	0.2528	0.0961	0.2262	0.1169	0.2368
			λ_5	0.1153	0.2371	0.0819	0.2084	0.1122	0.2289
			λ	0.0280	0.1264	0.0088	0.0761	0.0542	0.2146
			C_L	0.0011	0.0253	0.0004	0.0152	0.0022	0.0429
40	30	I	λ_1	0.0241	0.1193	0.0218	0.1150	0.0228	0.1163
			λ_2	0.0199	0.1080	0.0177	0.1039	0.0187	0.1050
			λ_3	0.0264	0.1250	0.0239	0.1211	0.0262	0.1250
			λ_4	0.0163	0.0985	0.0147	0.0951	0.0161	0.0981
			λ_5	0.0165	0.1090	0.0162	0.0996	0.0163	0.1010
			λ	0.0379	0.1753	0.0101	0.0876	0.0216	0.1360
			C_L	0.0013	0.0351	0.0004	0.0175	0.0009	0.0272
		II	λ_1	0.0162	0.0975	0.0151	0.0940	0.0155	0.0958
			λ_2	0.0229	0.1159	0.0189	0.1027	0.0220	0.1140
			λ_3	0.0223	0.1160	0.0202	0.1112	0.0225	0.1159
			λ_4	0.0347	0.1430	0.0306	0.1378	0.0322	0.1387
			λ_5	0.0288	0.1300	0.0266	0.1255	0.0269	0.1267
			λ	0.0147	0.0951	0.0067	0.0678	0.0117	0.0936
			C_L	0.0009	0.0210	0.0003	0.0136	0.0005	0.0187
		III	λ_1	0.0416	0.1569	0.0382	0.1530	0.0408	0.1536
			λ_2	0.0199	0.1065	0.0164	0.1000	0.0194	0.1059
			λ_3	0.0232	0.1167	0.0212	0.1142	0.0230	0.1151
			λ_4	0.0265	0.1243	0.0232	0.1193	0.0253	0.1218
			λ_5	0.0129	0.0865	0.0114	0.0836	0.0126	0.0863
			λ	0.0169	0.0693	0.0141	0.0957	0.0185	0.0524
			C_L	0.0008	0.0239	0.0006	0.0191	0.0007	0.0244

Table 6. Length and coverage probability for parameter $\lambda = 0.8$ with 95% confidence intervals and $k = 5$.

				ACI		UCI		GCI	
n	m	CS	parameter	Length	CP	Length	CP	Length	CP
10	8	I	λ_1	1.4224	0.9358	1.1902	0.9490	1.1599	0.9460
			λ_2	1.5523	0.9416	1.3027	0.9492	1.2599	0.9518
			λ_3	1.3747	0.9282	1.2497	0.9498	1.1955	0.9542
			λ_4	1.7473	0.9390	1.5076	0.9452	1.4350	0.9456
			λ_5	1.7722	0.9332	1.4938	0.9466	1.4283	0.9496
			λ	0.7288	0.8858	0.6408	0.9046	0.6227	0.9912
			C_L	0.1458	0.8858	0.1282	0.9046	0.1245	0.9912
		II	λ_1	1.4827	0.9392	1.2651	0.9436	1.2246	0.9506
			λ_2	1.5593	0.9416	1.3209	0.9414	1.2753	0.9512
			λ_3	1.6209	0.9392	1.3542	0.9414	1.3051	0.9440
			λ_4	1.3931	0.9344	1.1875	0.9474	1.1505	0.9500
			λ_5	1.4700	0.9320	1.3146	0.9466	1.2555	0.9534
			λ	0.7024	0.8274	0.6158	0.9248	0.5983	0.9874
			C_L	0.1232	0.8274	0.1405	0.9248	0.1197	0.9874
		III	λ_1	1.4948	0.9376	1.2781	0.9420	1.2286	0.9512
			λ_2	1.3147	0.9288	1.1883	0.9500	1.1429	0.9526
			λ_3	1.3927	0.9364	1.1915	0.9368	1.1509	0.9474
			λ_4	1.3974	0.9360	1.1788	0.9410	1.1224	0.9478
			λ_5	1.2456	0.9304	1.0964	0.9412	1.0547	0.9514
			λ	0.6463	0.9482	0.5658	0.9612	0.5475	0.9654
			C_L	0.1293	0.9482	0.1132	0.9612	0.1095	0.9654
40	30	I	λ_1	0.6107	0.9472	0.5796	0.9510	0.5678	0.9536
			λ_2	0.5489	0.9456	0.5229	0.9488	0.5083	0.9490
			λ_3	0.6360	0.9398	0.6082	0.9466	0.5937	0.9504
			λ_4	0.4912	0.9450	0.4781	0.9462	0.4706	0.9518
			λ_5	0.4953	0.9438	0.4914	0.9480	0.4810	0.9482
			λ	0.3263	0.8466	0.2559	0.8738	0.2408	0.8682
			C_L	0.0653	0.8466	0.0512	0.8738	0.0482	0.8682
		II	λ_1	0.4970	0.9450	0.4720	0.9460	0.4618	0.9480
			λ_2	0.5829	0.9396	0.5553	0.9420	0.5437	0.9572
			λ_3	0.5746	0.9454	0.5607	0.9500	0.5524	0.9592
			λ_4	0.7319	0.9422	0.6881	0.9480	0.6755	0.9506
			λ_5	0.6518	0.9412	0.6320	0.9466	0.6160	0.9512
			λ	0.3508	0.9222	0.2799	0.9328	0.2588	0.9550
			C_L	0.0702	0.9222	0.0560	0.9328	0.0518	0.9550
		III	λ_1	0.7861	0.9452	0.7583	0.9458	0.7458	0.9530
			λ_2	0.5486	0.9466	0.5111	0.9470	0.5030	0.9522
			λ_3	0.6022	0.9414	0.5778	0.9436	0.5668	0.9482
			λ_4	0.6520	0.9420	0.6052	0.9458	0.5939	0.9568
			λ_5	0.4373	0.9454	0.4222	0.9456	0.4151	0.9516
			λ	0.3397	0.8280	0.2791	0.8588	0.2480	0.9098
			C_L	0.0679	0.8280	0.0558	0.8588	0.0496	0.9098

Table 7. ABs and MSEs for parameter $\lambda = 1.5$ with $k = 2$.

				MLE		UMVUE		GPE	
n	m	CS	parameter	MSE	AB	MSE	AB	MSE	AB
8	4	I	λ_1	2.4944	0.8778	1.1598	0.6892	1.8160	0.7893
			λ_2	2.1756	0.8515	1.1213	0.6764	1.6983	0.7830
			λ	1.2781	0.7094	0.1833	0.2967	0.3201	0.4520
			C_L	0.0511	0.1419	0.0073	0.0593	0.0128	0.0904
		II	λ_1	2.9986	0.8967	1.2260	0.6898	2.0953	0.7994
			λ_2	2.4832	0.9012	0.9879	0.6916	2.1051	0.8532
			λ	1.5344	0.7631	0.1864	0.2936	0.3391	0.4544
			C_L	0.0614	0.1526	0.0075	0.0587	0.0136	0.0908
		III	λ_1	2.1923	0.8681	1.2254	0.6812	1.9678	0.8001
			λ_2	2.4287	0.9018	1.3633	0.7167	1.7113	0.8006
			λ	1.3145	0.7415	0.1660	0.2937	0.3285	0.4592
			C_L	0.0526	0.1483	0.0066	0.0587	0.0131	0.0918
	6	I	λ_1	0.9266	0.6340	0.5932	0.5510	0.8850	0.6155
			λ_2	1.0730	0.6613	0.6305	0.5667	0.9303	0.6291
			λ	0.6041	0.5393	0.1115	0.2325	0.2303	0.3773
			C_L	0.0242	0.1079	0.0045	0.0465	0.0092	0.0755
		II	λ_1	1.0161	0.6344	0.5917	0.5415	0.8191	0.5856
			λ_2	1.1777	0.6966	0.6775	0.6023	1.0975	0.6718
			λ	0.6768	0.5612	0.1234	0.2513	0.2292	0.3688
			C_L	0.0271	0.1122	0.0049	0.0503	0.0092	0.0738
		III	λ_1	0.9431	0.6283	0.6129	0.5538	0.8611	0.6072
			λ_2	1.0559	0.6496	0.5452	0.5333	0.8158	0.5981
			λ	0.5725	0.5204	0.1091	0.2263	0.2178	0.3707
			C_L	0.0229	0.1041	0.0044	0.0453	0.0087	0.0741
18	10	I	λ_1	0.4409	0.4621	0.3130	0.4179	0.3680	0.4384
			λ_2	0.4605	0.4744	0.3288	0.4296	0.4191	0.4547
			λ	0.2951	0.3955	0.0688	0.1903	0.1305	0.2792
			C_L	0.0118	0.0791	0.0028	0.0381	0.0052	0.0558
		II	λ_1	0.3850	0.4353	0.2826	0.4022	0.3440	0.4189
			λ_2	0.4935	0.4497	0.3237	0.4201	0.4025	0.4461
			λ	0.2804	0.3506	0.0689	0.1836	0.1246	0.2784
			C_L	0.0592	0.0701	0.0028	0.0367	0.0050	0.0557
		III	λ_1	0.4380	0.4590	0.2994	0.4111	0.3694	0.4345
			λ_2	0.5103	0.5006	0.3637	0.4568	0.4322	0.4745
			λ	0.3311	0.4183	0.0846	0.2142	0.1266	0.2742
			C_L	0.0132	0.0837	0.0034	0.0428	0.0051	0.0549
	14	I	λ_1	0.2285	0.3531	0.1780	0.3275	0.2138	0.3447
			λ_2	0.2445	0.3644	0.1908	0.3350	0.2241	0.3502
			λ	0.1330	0.2732	0.0420	0.1440	0.0875	0.2392
			C_L	0.0053	0.0546	0.0017	0.0288	0.0035	0.0478
		II	λ_1	0.2691	0.3811	0.2257	0.3592	0.2461	0.3717
			λ_2	0.2741	0.3835	0.2204	0.3646	0.2736	0.3803
			λ	0.1930	0.3285	0.0619	0.1848	0.0959	0.2395
			C_L	0.0077	0.0657	0.0025	0.0370	0.0038	0.0479
		III	λ_1	0.2832	0.3824	0.2231	0.3566	0.2556	0.3765
			λ_2	0.2851	0.3928	0.2378	0.3724	0.2853	0.3866
			λ	0.2066	0.3408	0.0647	0.1891	0.0978	0.2419
			C_L	0.0083	0.0682	0.0026	0.0378	0.0039	0.0484

Table 8. Length and coverage probability for parameter $\lambda = 1.5$ with 95% confidence intervals and $k = 2$.

				ACI		UCI		GCI	
n	m	CS	parameter	Length	CP	Length	CP	Length	CP
8	4	I	λ_1	5.2631	0.9368	3.9114	0.9444	3.6529	0.9496
			λ_2	5.7206	0.9488	3.8198	0.9498	3.6512	0.9514
			λ	3.0163	0.8872	2.2895	0.9120	1.9434	0.9754
			C_L	0.6033	0.8872	0.4579	0.9120	0.3887	0.9754
		II	λ_1	6.0902	0.9434	3.9432	0.9468	3.6775	0.9498
			λ_2	5.8048	0.9430	3.9812	0.9452	3.8268	0.9512
			λ	3.1102	0.9016	2.3321	0.9210	2.0065	0.9796
			C_L	0.6220	0.9016	0.4664	0.9210	0.4013	0.9796
		III	λ_1	5.6171	0.9426	3.9056	0.9458	3.6667	0.9512
			λ_2	5.9253	0.9444	4.0386	0.9458	3.7008	0.9512
			λ	3.0778	0.9000	2.3043	0.9120	1.9888	0.9802
			C_L	0.6156	0.9000	0.4609	0.9120	0.3978	0.9802
	6	I	λ_1	3.5681	0.9332	2.9565	0.9464	2.8403	0.9474
			λ_2	3.7691	0.9366	3.0478	0.9410	2.8844	0.9470
			λ	2.2549	0.9340	1.9537	0.9434	1.7278	0.9752
			C_L	0.4510	0.9340	0.3907	0.9434	0.3456	0.9752
		II	λ_1	3.5446	0.9330	2.9156	0.9422	2.7464	0.9462
			λ_2	4.3117	0.9436	3.2654	0.9444	3.1132	0.9534
			λ	2.3220	0.9472	1.9916	0.9498	1.7832	0.9760
			C_L	0.4644	0.9472	0.3983	0.9498	0.3566	0.9760
		III	λ_1	3.5460	0.9384	2.9513	0.9412	2.8149	0.9492
			λ_2	3.8953	0.9414	2.8857	0.9512	2.7917	0.9526
			λ	2.2253	0.9316	1.9105	0.9336	1.6827	0.9682
			C_L	0.4451	0.9316	0.3821	0.9336	0.3365	0.9682
18	10	I	λ_1	2.4408	0.9342	2.1683	0.9486	2.0952	0.9508
			λ_2	2.5841	0.9406	2.2233	0.9410	2.1536	0.9484
			λ	1.6182	0.9562	1.5328	0.9618	1.3915	0.9634
			C_L	0.3236	0.9562	0.3066	0.9618	0.2783	0.9634
		II	λ_1	2.3735	0.9418	2.0624	0.9490	2.0092	0.9542
			λ_2	2.4742	0.9426	2.1802	0.9428	2.1074	0.9528
			λ	1.5536	0.9450	1.4830	0.9468	1.3425	0.9620
			C_L	0.3107	0.9450	0.2966	0.9468	0.2685	0.9620
		III	λ_1	2.5179	0.9416	2.1440	0.9438	2.0609	0.9494
			λ_2	2.6546	0.9332	2.3830	0.9482	2.2698	0.9498
			λ	1.6493	0.9500	1.5557	0.9658	1.4384	0.9716
			C_L	0.3299	0.9500	0.3111	0.9658	0.2877	0.9716
	14	I	λ_1	1.8426	0.9434	1.6696	0.9480	1.6386	0.9500
			λ_2	1.8605	0.9352	1.7082	0.9422	1.6672	0.9484
			λ	1.2392	0.9420	1.2109	0.9510	1.1087	0.9566
			C_L	0.2478	0.9420	0.2422	0.9510	0.2217	0.9566
		II	λ_1	1.9684	0.9386	1.8133	0.9466	1.7502	0.9492
			λ_2	1.9870	0.9402	1.8414	0.9428	1.7989	0.9510
			λ	1.3247	0.9446	1.2992	0.9592	1.1991	0.9684
			C_L	0.2649	0.9446	0.2598	0.9592	0.2398	0.9684
		III	λ_1	1.9691	0.9352	1.8189	0.9444	1.7603	0.9494
			λ_2	2.0290	0.9380	1.8744	0.9486	1.8230	0.9492
			λ	1.3364	0.9418	1.3102	0.9662	1.2171	0.9734
			C_L	0.2673	0.9418	0.2620	0.9662	0.2434	0.9734

Table 9. MSEs, ABs, Length and coverage probability for parameter $\lambda = 0.5$ with 95% confidence intervals and $k = 2$.

						MLE		ACI		UMVUE		UCI		GPQ		GCI	
n_1	n_2	m_1	m_2	CS	Par.	MSE	AB	Length	CP	MSE	AB	Length	CP	MSE	AB	Length	CP
10	8	4	6	I	λ_1	0.4750	0.4077	2.4407	0.9364	0.2513	0.3353	1.9182	0.9476	0.7734	0.4018	1.8384	0.9516
					λ_2	0.1192	0.2243	1.2709	0.9352	0.0760	0.1977	1.0568	0.9474	0.1192	0.2195	1.0149	0.9494
					λ	0.1393	0.2610	0.9739	0.9662	0.0342	0.1372	0.8487	0.9824	0.0428	0.1450	0.8397	0.9844
					C_L	0.0056	0.0522	0.1948	0.9662	0.0014	0.0274	0.1797	0.9824	0.0017	0.0290	0.1679	0.9844
				II	λ_1	0.2662	0.3042	2.0530	0.9488	0.1248	0.2372	1.3534	0.9494	0.2409	0.2826	1.2914	0.9486
					λ_2	0.1513	0.2534	1.4207	0.9386	0.1042	0.2230	1.1951	0.9512	0.1313	0.2385	1.1228	0.9518
					λ	0.1422	0.2638	0.9727	0.9556	0.0213	0.1052	0.7956	0.9772	0.0345	0.1358	0.6835	0.9730
					C_L	0.0057	0.0528	0.1945	0.9556	0.0009	0.0210	0.1591	0.9772	0.0014	0.0272	0.1367	0.9730
				III	λ_1	0.3632	0.3110	2.2596	0.9478	0.1384	0.2418	1.3945	0.9544	0.2871	0.2914	1.3281	0.9556
					λ_2	0.2985	0.3489	2.0185	0.9418	0.1770	0.2985	1.5960	0.9478	0.2431	0.3210	1.5219	0.9484
					λ	0.2934	0.3971	1.1656	0.9422	0.0534	0.1744	0.9552	0.9612	0.0632	0.1749	0.8036	0.9836
					C_L	0.0117	0.0794	0.2331	0.9422	0.0021	0.0349	0.1910	0.9612	0.0025	0.0350	0.1607	0.9836
21	17	9	5	I	λ_1	0.0230	0.1124	0.5757	0.9398	0.0204	0.1087	0.5452	0.9470	0.0218	0.1102	0.5305	0.9448
					λ_2	0.1564	0.2476	1.4777	0.9400	0.1032	0.2143	1.1808	0.9442	0.1457	0.2427	1.1112	0.9466
					λ	0.0263	0.1183	0.5155	0.9574	0.0078	0.0663	0.5045	0.9648	0.0133	0.0896	0.4649	0.9716
					C_L	0.0011	0.0237	0.1031	0.9574	0.0003	0.0133	0.1009	0.9648	0.0005	0.0179	0.0930	0.9716
				II	λ_1	0.0226	0.1112	0.5677	0.9380	0.0197	0.1089	0.5437	0.9422	0.0225	0.1107	0.5298	0.9490
					λ_2	0.1974	0.2769	1.6276	0.9360	0.1043	0.2269	1.2638	0.9424	0.1689	0.2571	1.1891	0.9510
					λ	0.0274	0.1217	0.5168	0.9568	0.0099	0.0754	0.5109	0.9648	0.0140	0.0901	0.4769	0.9712
					C_L	0.0011	0.0243	0.1034	0.9568	0.0004	0.0151	0.1022	0.9648	0.0006	0.0180	0.0954	0.9712
				III	λ_1	0.0209	0.1077	0.5421	0.9358	0.0174	0.1008	0.5068	0.9478	0.0189	0.1021	0.4936	0.9486
					λ_2	0.4600	0.4197	2.4706	0.9396	0.2635	0.3441	1.9053	0.9422	0.4134	0.3860	1.7823	0.9482
					λ	0.0231	0.1115	0.5149	0.9602	0.0138	0.0771	0.5022	0.9646	0.0150	0.0916	0.4970	0.9684
					C_L	0.0009	0.0223	0.1030	0.9602	0.0011	0.0135	0.1004	0.9646	0.0006	0.0183	0.0994	0.9684

Table 10. Electrical insulation data sets of the failure times.

Lines	Data											
line I	0.185	0.217	0.351	0.405	0.423	0.487	0.794	0.860	1.219	1.471	1.502	2.193
line II	0.123	0.218	0.244	0.286	0.432	0.469	0.707	0.753	0.955	0.981	1.386	1.519

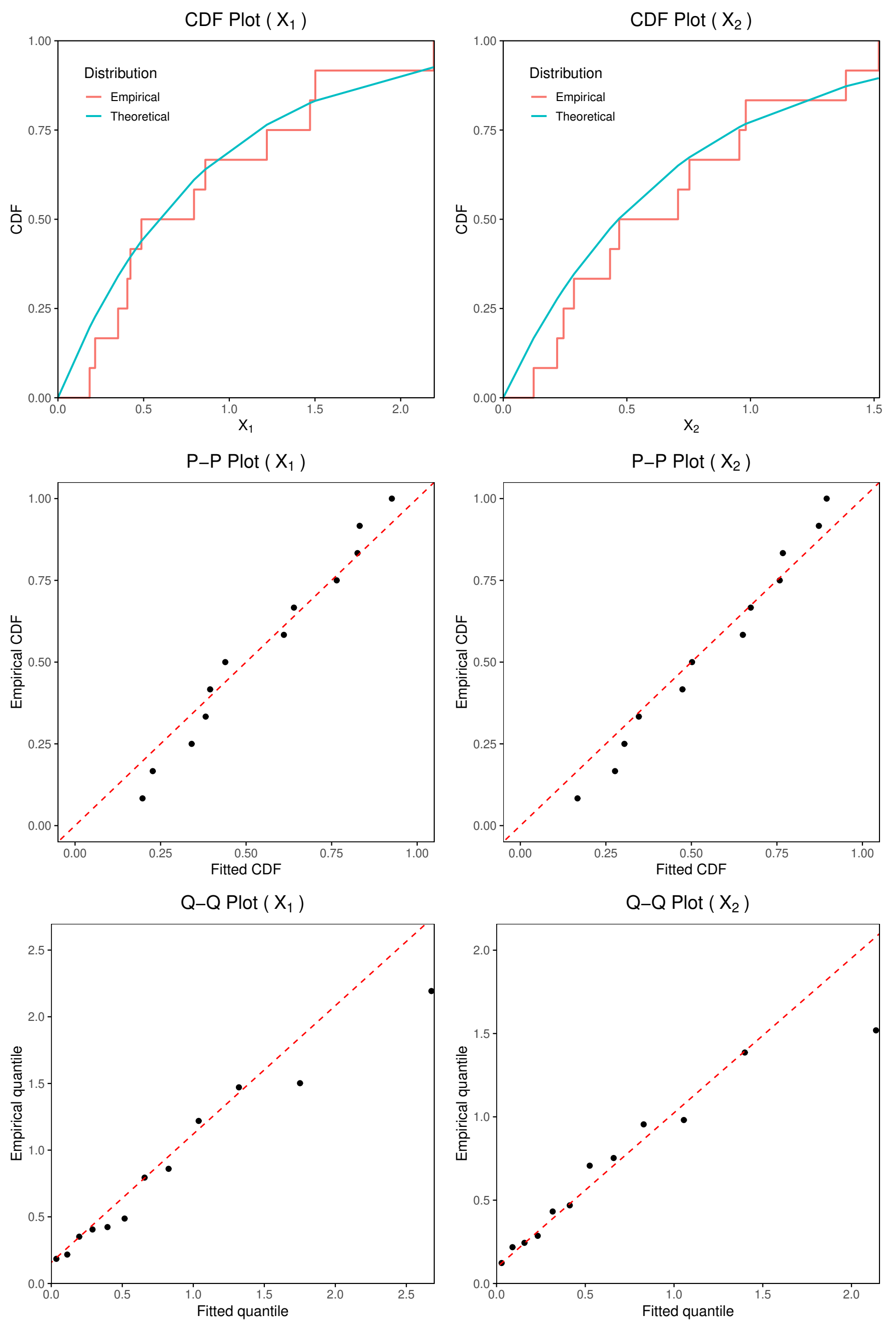

Figure 2. Empirical distribution and fitted Exp models, P-P and Q-Q plots for X_1 and X_2 under the electrical insulation data sets.

Table 11. PT-II CS of electrical insulation data sets with designed censoring scenarios.

CS	Types	Data							
I	X_1	0.185	0.217	0.351	0.405	0.423	0.487	0.794	0.860
	X_2	0.123	0.218	0.244	0.286	0.432	0.469	0.707	0.753
II	X_1	0.423	0.487	0.794	0.860	1.219	1.471	1.502	2.193
	X_2	0.432	0.469	0.707	0.753	0.955	0.981	1.386	1.519
III	X_1	0.351	0.405	0.423	0.487	0.794	0.860	1.219	1.471
	X_2	0.244	0.286	0.432	0.469	0.707	0.753	0.955	0.981

Table 12. Point estimates on $L = 0.3$ under the electrical insulation data sets.

CS	par.	MLE	ESE	UMVUE	ESE	GPE	ESE
I	λ_1	1.1170	0.1560	0.9774	0.1194	1.0751	0.1476
	λ_2	1.2812	0.2052	1.1211	0.1571	1.2380	0.2005
	λ	1.1879	0.0886	1.0394	0.0678	1.1513	0.0733
	C_L	0.6436	0.0080	0.6882	0.0061	0.6846	0.0074
II	λ_1	1.7929	0.4018	1.5688	0.3076	1.7323	0.3922
	λ_2	2.1482	0.5769	1.8797	0.4417	2.0653	0.5689
	λ	1.9388	0.2368	1.6964	0.1813	1.8173	0.2232
	C_L	0.4184	0.0213	0.4911	0.0163	0.4848	0.0208
III	λ_1	1.3765	0.2368	1.2044	0.1813	1.3224	0.2297
	λ_2	1.6051	0.3221	1.4045	0.2466	1.5476	0.3163
	λ	1.4734	0.1365	1.2892	0.1045	1.3994	0.1299
	C_L	0.5580	0.0123	0.6132	0.0094	0.6101	0.0115

Table 13. Interval estimates on $L = 0.3$ under the electrical insulation data sets.

CS	par.	ACI[length]	UCI[length]	GCI[length]
I	λ_1	(0.3430, 1.8910)[1.5481]	(0.4822, 2.0138)[1.5315]	(0.4284, 1.9123)[1.4840]
	λ_2	(0.3934, 2.1691)[1.7757]	(0.5531, 2.3098)[1.7567]	(0.4938, 2.1662)[1.6724]
	λ	(0.6045, 1.7714)[1.1669]	(0.5110, 1.5679)[1.0569]	(0.6457, 1.6906)[1.0450]
	C_L	(0.4686, 0.8187)[0.3501]	(0.4796, 0.8167)[0.3371]	(0.4928, 0.8253)[0.3325]
II	λ_1	(0.5505, 3.0353)[2.4848]	(0.7741, 3.2323)[2.4583]	(0.6893, 3.0696)[2.3803]
	λ_2	(0.6596, 3.6368)[2.9772]	(0.9275, 3.8729)[2.9454]	(0.7825, 3.6322)[2.8497]
	λ	(0.9850, 2.8926)[1.9077]	(0.9960, 2.8769)[1.8809]	(0.8510, 2.6856)[1.8346]
	C_L	(0.1322, 0.7045)[0.5723]	(0.1509, 0.7012)[0.5503]	(0.1643, 0.7047)[0.5404]
III	λ_1	(0.4226, 2.3303)[1.9076]	(0.5943, 2.4815)[1.8873]	(0.5013, 2.3496)[1.8483]
	λ_2	(0.4929, 2.7174)[2.2246]	(0.6930, 2.8938)[2.2008]	(0.6381, 2.7819)[2.1439]
	λ	(0.7493, 2.1974)[1.4481]	(0.7575, 2.0209)[1.2634]	(0.7010, 1.8973)[1.1964]
	C_L	(0.3408, 0.7752)[0.4344]	(0.3573, 0.7727)[0.4154]	(0.3708, 0.7797)[0.4089]

that the performance of generalized estimation is satisfactory, surpassing both MLE and UMVUE in the context of interval estimation.

Example two (steel specimen data sets) The data sets contains the observed lifetimes of steel specimens tested at kinds of subtly different stress levels[27]. To validate the theoretical findings, we conducted tests using three lines of samples subjected to similarly high stress levels. These lines were designated as X_1, X_2 and X_3. The failure data for these lines comprised $n_1 = 24, n_2 = n_3 = 20$ observations each, as detailed in Table 14. The p-value with the KS test of the steel specimen data sets are calculated as 0.7096, 0.6067 and 0.5776. Furthermore, Figure 4 illustrates the empirical distribution in conjunction with the fitted exponential distribution, as well as CDF plots, P-P plots, and Q-Q plots. These graphical representations suggest a strong alignment between the exponential distribution and the empirical data. Thus, Exp(λ_1), Exp(λ_2) and Exp(λ_3) could fit the lifetimes of the steel specimens tested in Table 14, respectively.

Based on the original data presented in Table 14 and the analogous CS employed in the simulation studies, three lines of PT-II data were generated, each comprising $m_1 = m_2 = m_3 = 16$ observations. The detailed samples are subsequently provided in Table 15.

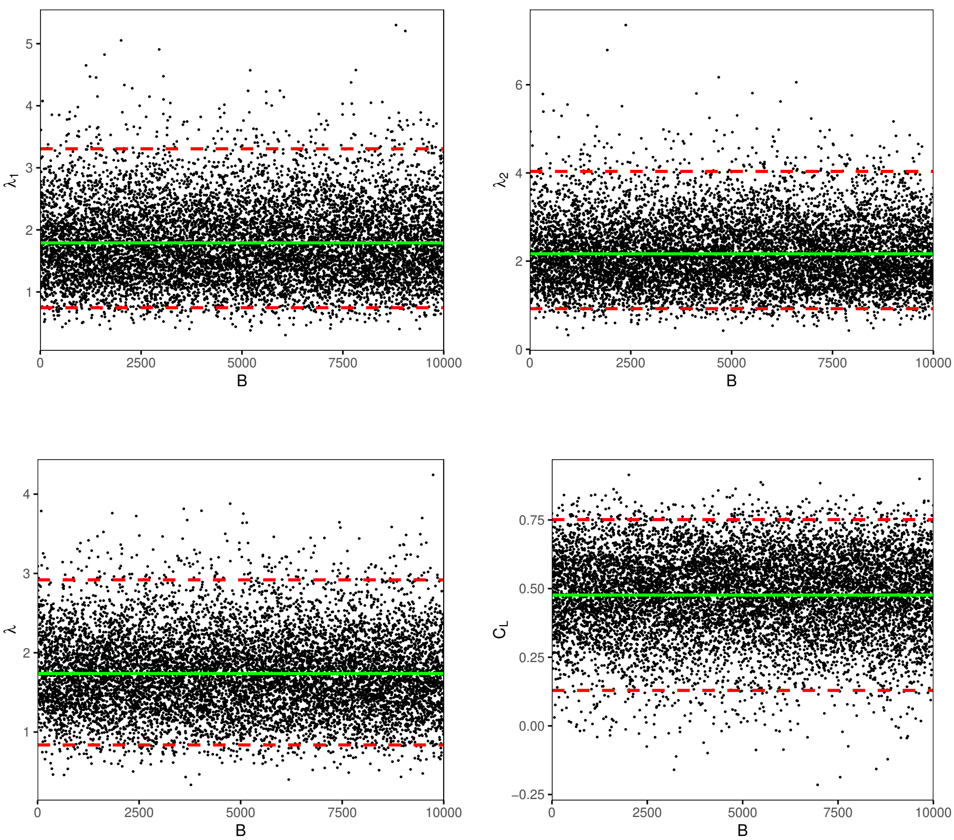

Figure 3. Trace plots of generalized method with 95% credible intervals of $\lambda_1, \lambda_2, \lambda, C_L$ under the electrical insulation data sets.

Table 14. Steel specimen data sets of the failure times.

Lines	Data											
line I	0.206	0.231	0.283	0.370	0.413	0.474	0.523	0.597	0.605	0.619	0.727	0.815
	0.935	1.056	1.144	1.336	1.580	1.786	1.826	1.943	2.214	3.107	4.510	6.297
line II	0.196	0.227	0.250	0.271	0.308	0.347	0.393	0.475	0.548	0.669	0.799	0.879
	0.975	1.154	1.388	1.705	2.073	2.211	2.925	4.257				
line III	0.166	0.184	0.241	0.251	0.273	0.312	0.371	0.418	0.493	0.562	0.683	0.760
	0.830	0.981	1.306	1.463	1.842	1.867	2.220	2.978				

According to Table 15, the corresponding point estimates and the ESE are provided in Table 16, while the interval estimates for AL with a confidence interval level of 95% are given in Table 17. Figure 5 displays a generalized estimation trace plot in the steel specimen data sets.

6. Conclusions

The PCI is widely used in practice to measure the stability of production processes and the consistency of product quality. For companies, a low PCI usually means that the production process has large variations and the product quality is not yet at the desired level. In this case, companies should strengthen quality control and

Table 15. PT-II CS of steel specimen data sets with designed censoring scenarios.

CS	Types	Data							
I	X_1	0.206	0.231	0.283	0.370	0.413	0.474	0.523	0.597
		0.605	0.619	0.727	0.815	0.935	1.056	1.144	1.336
	X_2	0.196	0.227	0.250	0.271	0.308	0.347	0.393	0.475
		0.548	0.669	0.799	0.879	0.975	1.154	1.388	1.705
	X_3	0.166	0.184	0.241	0.251	0.273	0.312	0.371	0.418
		0.493	0.562	0.683	0.760	0.830	0.981	1.306	1.463
II	X_1	0.605	0.619	0.727	0.815	0.935	1.056	1.144	1.336
		1.580	1.786	1.826	1.943	2.214	3.107	4.510	6.297
	X_2	0.308	0.347	0.393	0.475	0.548	0.669	0.799	0.879
		0.975	1.154	1.388	1.705	2.073	2.211	2.925	4.257
	X_3	0.273	0.312	0.371	0.418	0.493	0.562	0.683	0.760
		0.830	0.981	1.306	1.463	1.842	1.867	2.220	2.978
III	X_1	0.413	0.474	0.523	0.597	0.605	0.619	0.727	0.815
		0.935	1.056	1.144	1.336	1.580	1.786	1.826	1.943
	X_2	0.250	0.271	0.308	0.347	0.393	0.475	0.548	0.669
		0.799	0.879	0.975	1.154	1.388	1.705	2.073	2.211
	X_3	0.241	0.251	0.273	0.312	0.371	0.418	0.493	0.562
		0.683	0.760	0.830	0.981	1.306	1.463	1.842	1.867

Table 16. Point estimates on $L = 0.2$ under the steel specimen data sets.

CS	par.	MLE	ESE	UMVUE	ESE	GPQ	ESE
I	λ_1	0.5565	0.0794	1.0088	0.0636	0.6458	0.0756
	λ_2	1.0038	0.0630	0.8500	0.0452	0.9651	0.0607
	λ_3	1.0790	0.0728	1.0023	0.0628	1.0592	0.0724
	λ	0.7322	0.2036	0.9415	0.0186	0.7650	0.0196
	C_L	0.8536	0.0010	0.8117	0.0007	0.8470	0.0008
II	λ_1	0.6253	0.0274	0.4964	0.0154	0.6359	0.0265
	λ_2	1.1221	0.0787	0.9380	0.0550	0.6991	0.0611
	λ_3	0.8186	0.0419	1.0760	0.0324	0.7371	0.0360
	λ	0.7663	0.0244	0.6619	0.0103	0.6420	0.0114
	C_L	0.8467	0.0005	0.8676	0.0004	0.8716	0.0005
III	λ_1	0.8043	0.0404	0.9202	0.0339	0.7259	0.0348
	λ_2	0.7510	0.0852	1.1487	0.0525	1.0854	0.0750
	λ_3	0.7725	0.0873	0.9051	0.0512	1.1409	0.0843
	λ	0.7747	0.0404	0.9692	0.0198	0.8555	0.0242
	C_L	0.8451	0.0010	0.8062	0.0008	0.8289	0.0009

Table 17. Interval estimates on $L = 0.2$ under the steel specimen data sets.

CS	par.	ACI[length]	UCI[length]	GCI[length]
I	λ_1	(0.2182, 0.8592)[0.6410]	(0.2151, 0.8464)[0.6313]	(0.3410, 0.9707)[0.6296]
	λ_2	(0.5119, 1.4956)[0.9837]	(0.5182, 1.4719)[0.9537]	(0.5430, 1.4878)[0.9447]
	λ_3	(0.5503, 1.6077)[1.0574]	(0.6111, 1.6531)[1.0420]	(0.5835, 1.6110)[1.0275]
	λ	(0.4891, 1.0417)[0.5527]	(0.6743, 1.2087)[0.5345]	(0.5148, 0.9496)[0.4348]
	C_L	(0.7917, 0.9022)[0.1105]	(0.7583, 0.8651)[0.1069]	(0.8101, 0.8970)[0.0870]
II	λ_1	(0.2889, 0.9317)[0.6428]	(0.2927, 0.9288)[0.6361]	(0.3488, 0.9725)[0.6237]
	λ_2	(0.5723, 1.6718)[1.0996]	(0.5719, 1.5471)[0.9752]	(0.3787, 1.0527)[0.6740]
	λ_3	(0.4175, 1.2196)[0.8022]	(0.6561, 1.4048)[0.7487]	(0.4003, 1.1265)[0.7262]
	λ	(0.3952, 0.8613)[0.4661]	(0.4028, 0.8610)[0.4581]	(0.5437, 0.9889)[0.4452]
	C_L	(0.8077, 0.9110)[0.1032]	(0.8078, 0.9074)[0.0996]	(0.8022, 0.8913)[0.0890]
III	λ_1	(0.4102, 1.1984)[0.7882]	(0.5610, 1.3177)[0.7567]	(0.3876, 1.1084)[0.7208]
	λ_2	(0.5830, 1.7189)[1.1359]	(0.6004, 1.7147)[1.1143]	(0.5978, 1.6490)[1.0512]
	λ_3	(0.3840, 1.5097)[1.1258]	(0.5518, 1.5928)[1.0410]	(0.6307, 1.6503)[1.0196]
	λ	(0.5537, 1.1626)[0.6089]	(0.6935, 1.2448)[0.5513]	(0.5554, 0.9939)[0.4385]
	C_L	(0.7675, 0.8893)[0.1218]	(0.7510, 0.8613)[0.1103]	(0.8012, 0.8889)[0.0877]

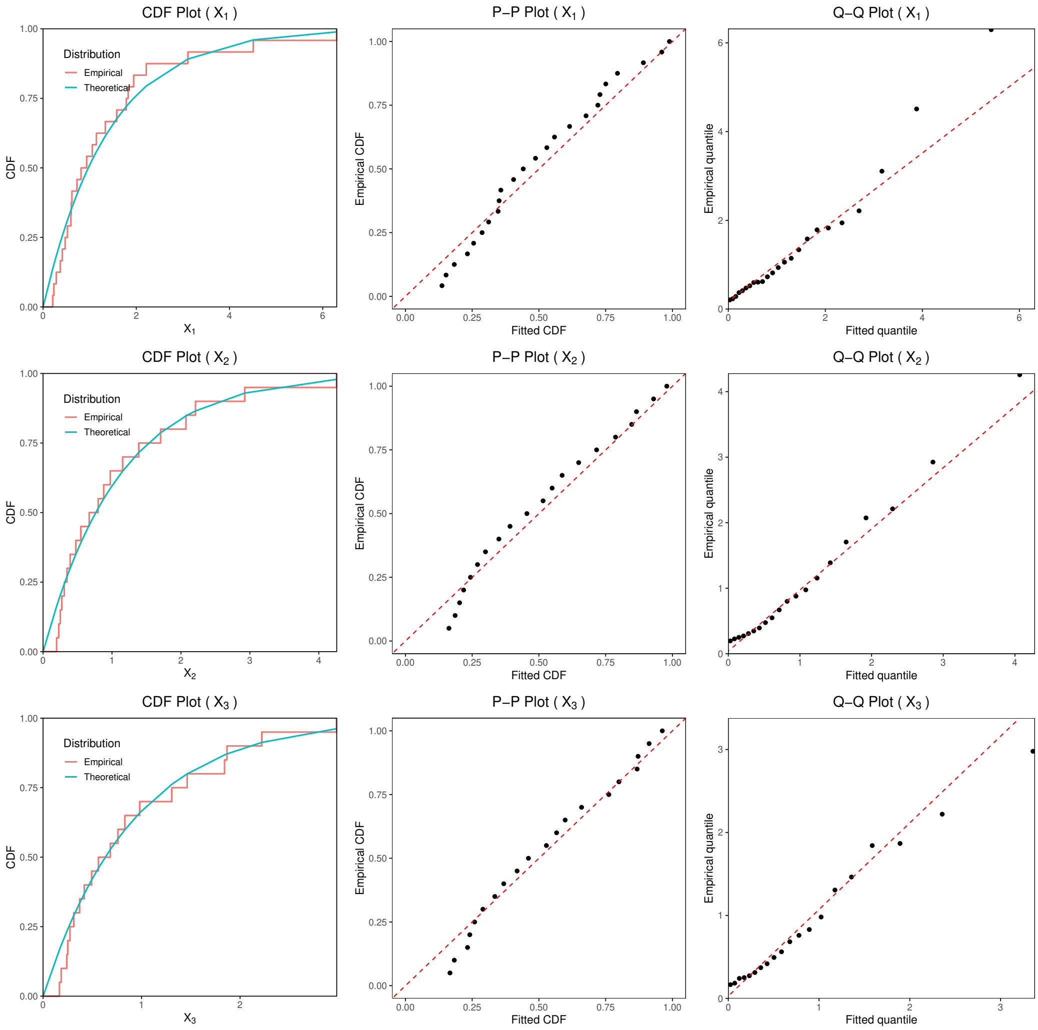

Figure 4. Empirical distribution and fitted Exp models, P-P and Q-Q plots for X_1, X_2 and X_3 under the steel specimen data sets.

improve production methods to enhance overall product quality. However, if the PCI is too high, it can mean that the quality of the product exceeds the actual needs or design requirements. This can lead to wasted resources and slower production. In such cases, companies can consider slightly relaxing quality control standards while still meeting customer needs—to speed up production and reduce costs. By adjusting quality management strategies with care, companies can improve efficiency while maintaining acceptable quality levels.

Building upon this practical context, this paper proposes a product life performance index for multiple production lines under the PT-II CS condition, assuming that product lifetimes follow an exponential distribution. Comparison parameter and index estimations using MLE, GPE, and UMVUE. The simulation results show that UMVUE excels in point estimation, GCI is superior in interval estimation, and MLE performs least favorably among the methods evaluated. Although GPE achieves higher estimation accuracy, especially in interval inference, it comes at the cost of increased computational time. Therefore, investigating the trade-off between accuracy and computational efficiency, particularly in large-scale applications, remains an important direction for both theoretical research and practical implementation. This study is based on the assumption of exponentially distributed data. However, in practical applications, there are datasets that do not fit the exponential distribution well. Conducting simulations

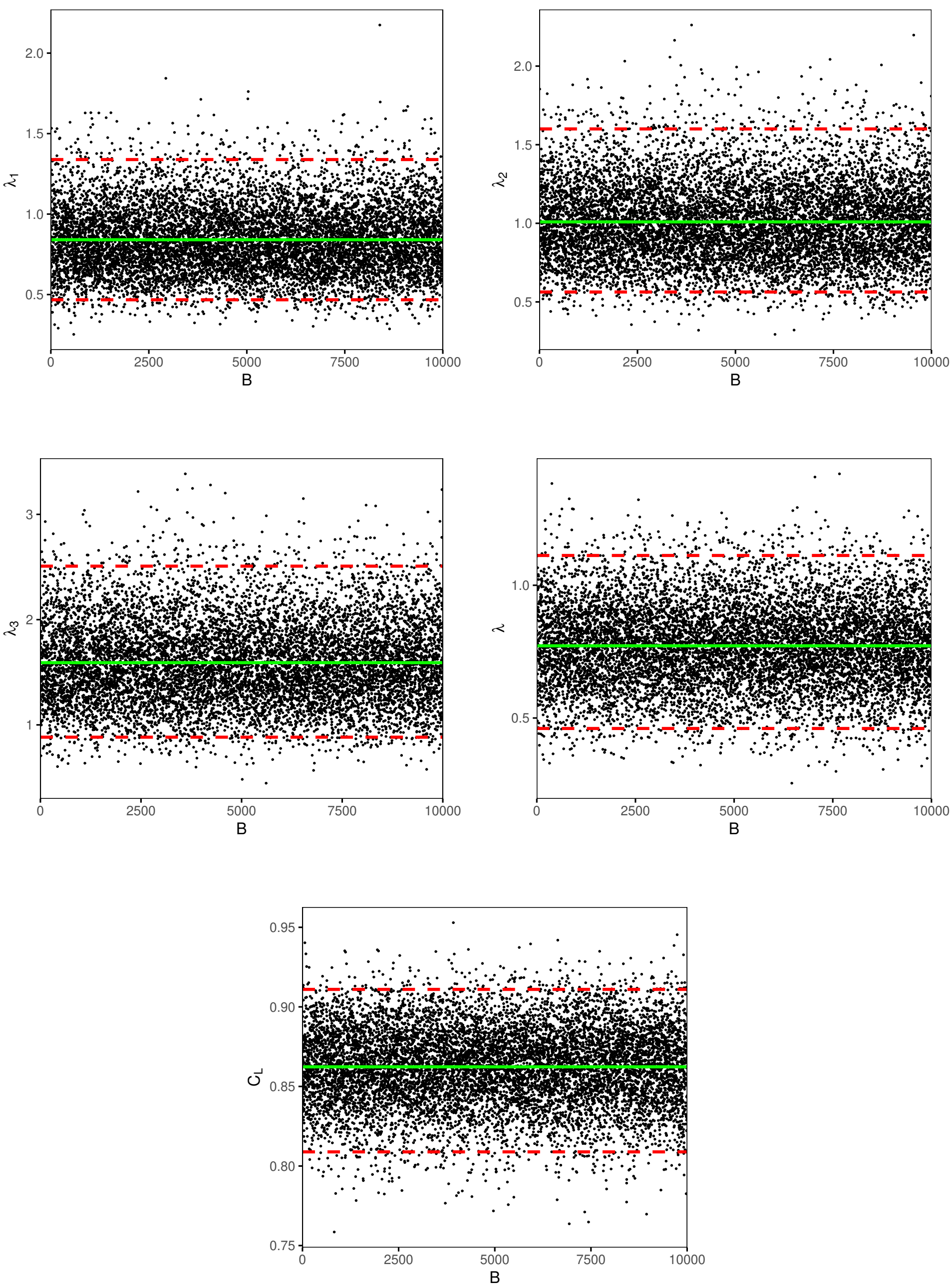

Figure 5. Trace plots of generalized method with 95% credible intervals of $\lambda_1, \lambda_2, \lambda_3, \lambda, C_L$ under the steel specimen data sets.

under misspecified distributions (e.g. Weibull or log-normal distributions) to evaluate the robustness of the method could be a promising direction for future research. Although estimation issues are discussed within the context of the exponential model in multiple production lines PT-II CS, the findings can be extended to other distributions, such as Burr XII, Pareto and Lomax, with appropriate adjustments. For future research, considering the influence of varying effects across multiple product lines, it will be of interest to address the optimal design problem of synthesizing a single product line PCI into an overall PCI, a topic that will be explored in forthcoming studies.

Appendix

A. Proof of Theorem 1

By taking derivatives of $\ell(\lambda_i)$ with respect to parameter λ_i in (7), the maximum likelihood estimator of $\lambda_i, i = 1, 2, \ldots, k$ could be obtained via following likelihood equation as

$$\frac{\partial \ell(\lambda_1)}{\partial \lambda_1} = 0, \frac{\partial \ell(\lambda_2)}{\partial \lambda_2} = 0, \ldots, \frac{\partial \ell(\lambda_k)}{\partial \lambda_k} = 0,$$

then the maximum likelihood estimator of λ_i denoted by $\hat{\lambda}_i$ can be derived. Taking the second derivative of (7) then we get

$$\frac{\partial \ell^2(\lambda_i)}{\partial \lambda_i^2} = -\frac{m_i}{\lambda_i^2} < 0, i = 1, 2, \cdots, k,$$

it can be shown that the log-likelihood function is strictly concave with respect to λ_i, as its second derivative is less than zero. This implies that the log-likelihood function has at most one maximum point. Therefore, $\hat{\lambda}_i$ is unique.

B. Proof of Theorem 4

Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ be the multiple production lines sample from $Exp(\lambda_i)$ with sample size n_i, m_i and CS $R_i = (r_{i1}, r_{i2}, \ldots, r_{im_i})$. Let T_{i1} be the total failure time of all products in the first time period, T_{i2} for the second time period, so on and so forth, T_{im_i} for the last time period. The total test time is $T_i = \sum_{j=1}^{m_i} T_{ij} = \sum_{j=1}^{m_i} x_{ij}(1 + r_{ij})$. It is observed that

$$\begin{aligned}
&T_{i1} = n_i x_{i1} \\
&T_{i2} = (n_i - r_{i1} - 1)(x_{i2} - x_{i1}) \\
&\vdots \\
&T_{im_i} = [n_i - \textstyle\sum_{j=1}^{m_i-1}(r_{im_i} + 1)](x_{im_1} - x_{im_{i-1}}).
\end{aligned}$$

Thus, we find that $T_{i1}, T_{i2}, \ldots, T_{im_i}$ are independent and identically distributed as the standard exponential distribution. Therefore, it is directly conducted that the quantities

$$U_i = 2\check{\lambda}_i T_i \sim \chi^2(2m_i), i = 1, 2, \ldots, k,$$

further, it is found that

$$\check{\lambda}_i = \frac{U_i}{2T_i}, i = 1, 2, \ldots, k.$$

Therefore, the assertion is shown.

C. Proof of Theorem 5

Known by the invariance of maximum likelihood, we have $\hat{C}_{L(i)} = 1 - \frac{m_i L}{T_i}$. From (4), (9) and **Theorem 4**, we get

$$E(\hat{\lambda}_i) = \frac{m_i}{m_i - 1}\lambda_i \neq \hat{\lambda}_i,$$

so $\hat{\lambda}_i$ is biased estimators of λ_i. Therefore, it is modified as

$$\tilde{\lambda}_i = \frac{m_i - 1}{T_i},$$

moreover,

$$E(\tilde{\lambda}_i) = \lambda_i.$$

Therefore, $\tilde{\lambda}_i$ is the unbiased and uniform estimate of λ_i. We easily show that $\tilde{\lambda}_i$ is the UMVUE of λ_i.

Acknowledgement

This work of Liang Wang was supported by the National Natural Science Foundation of China (No. 12061091), the Yunnan Fundamental Research Projects (No. 202401AT070116) and the Yunnan Key Laboratory of Modern Analytical Mathematics and Applications (No. 202302AN360007). This work of Zirui Chu was supported by the Yunnan Provincial Department of Education Science Research Fund Project (No. 2025Y0328).

REFERENCES

[1] J. M. Juran, *Juran's quality control handbook*, McGraw-Hill, pp. 11–13, 1974.

[2] V. E. Kane, *Process capability indices*, Journal of Quality Technology, vol. 18, no. 1, pp. 41–52, 1986.

[3] K. L. Chen, S. W. Cheng, and F. A. Spiring, *A new measure of process capability: Cpm*, Journal of Quality Technology, vol. 20, no. 2, pp. 162–175, 1988.

[4] W. L. Pearn, S. Kotz, and N. L. Johnson, *Distributional and inferential properties of process capability indices*, Journal of Quality Technology, vol. 24, pp. 216–231, 1992.

[5] D. C. Montgomery, *Introduction to statistical quality control*, John Wiley and Sons, Inc., New York, USA, 1985.

[6] B. C. Guo, et al., *Generalized confidence intervals of quantile-based process capability indices for inverse Gaussian distribution*, Quality Technology and Quantitative Management, pp. 405–417, 2023.

[7] S. F. Wu and P. T. Huang, *The testing procedure for the overall lifetime performance index of Rayleigh products in multiple production lines based on the progressive Type-I interval censored sample*, Symmetry, vol. 16, no. 2, pp. 195, 2024.

[8] J. W. Wu, H. M. Lee, and C. L. Lei, *Computational procedure for assessing lifetime performance index of products for a one-parameter exponential lifetime model with the upper record values*, Journal of Engineering Manufacture, vol. 222, no. 12, pp. 1729–1739, 2008.

[9] W. C. Lee, J. W. Wu, and C. W. Hong, *Assessing the lifetime performance index of products with the exponential distribution under progressively Type-II right censored samples*, Journal of Computational and Applied Mathematics, vol. 231, no. 2, pp. 648–656, 2009.

[10] G. Keller, B. Warrack, and H. Bartel, *Statistics for management and economics*, Belmont and California Duxbury Press, 1994.

[11] M. Ahmadi, M. Doostparast, and J. Ahmadi, *Block censoring scheme with two-parameter exponential distribution*, Journal of Statistical Computation and Simulation, vol. 88, no. 7, pp. 1229–1251, 2018.

[12] R. M. EL-Sagheer, M. El-Morshedy, L. A. Al-Essa, et al., *The process capability index of Pareto model under progressive Type-II censoring: Various Bayesian and Bootstrap algorithms for asymmetric data*, Symmetry, vol. 15, no. 4, pp. 879, 2023.

[13] H. M. Lee, J. W. Wu, C. L. Lei, et al., *Implementing lifetime performance index of products with two-parameter exponential distribution*, International Journal of Systems Science, vol. 42, no. 8, pp. 1–17, 2010.

[14] S. Weerahandi, *Generalized confidence intervals*, Exact statistical methods for data analysis, pp. 143–168, 1995.

[15] W. Chankham, S. A. Niwitpong, and S. Niwitpong, *The simultaneous confidence interval for the ratios of the coefficients of variation of multiple inverse Gaussian distributions and its application to PM 2.5 data*, Symmetry, vol. 16, no. 3, 2024.

[16] M. La-ongkaew, S. A. Niwitpong, and S. Niwitpong, *Simultaneous confidence intervals for all pairwise differences between means of Weibull distributions*, Symmetry, vol. 15, no. 12, 2023.

[17] Y. Akdoğan, *On the confidence intervals of process capability index Cpm based on a progressive Type-II censored sample*, Quality and Reliability Engineering International, vol. 38, no. 5, pp. 2845–2861, 2022.

[18] S. F. Wu and K. Y. Chiang, *Assessment of the overall lifetime performance index of Weibull products in multiple production lines*, Mathematics, vol. 12, no. 4, pp. 514, 2024.

[19] M. V. Ahmadi, J. Ahmadi, and M. Abdi, *Evaluating the lifetime performance index of products based on generalized order statistics from two-parameter exponential model*, International Journal of System Assurance Engineering and Management, vol. 10, no. 2, pp. 251–275, 2019.

[20] M. H. de Almeida, P. L. Ramos, G. S. Rao, et al., *Objective Bayesian inference for the capability index of the gamma distribution*, Quality and Reliability Engineering International, vol. 37, no. 5, pp. 2235–2247, 2021.

[21] P. L. Ramos, M. H. Almeida, F. Louzada, et al., *Objective Bayesian inference for the capability index of the Weibull distribution and its generalization*, Computers and Industrial Engineering, vol. 167, 2022.

[22] L. Wang, S. J. Wu, H. Lin, et al., *Inference for block progressive censored competing risks data from an inverted exponentiated exponential model*, Quality and Reliability Engineering International, vol. 39, no. 7, pp. 2736–2764, 2023.

[23] M. V. Ahmadi, M. Doostparast, and J. Ahmadi, *Block censoring scheme with two-parameter exponential distribution*, Journal of Statistical Computation and Simulation, vol. 88, no. 7, pp. 1229–1251, 2018.

[24] H. M. Lee, J. W. Wu, and C. L. Lei, *Assessing the lifetime performance index of exponential products with step-stress accelerated life-testing data*, IEEE Transactions on Reliability, vol. 62, no. 1, pp. 296–304, 2013.

[25] J. W. Wu, C. W. Hong, and W. C. Lee, *Computational procedure of lifetime performance index of products for the Burr XII distribution with upper record values*, Applied Mathematics and Computation, vol. 227, pp. 701–716, 2014.

[26] J. Shaabani and A. A. Jafari, *Inference on the lifetime performance index of gamma distribution: Point and interval estimation*, Communications in Statistics-Simulation and Computation, vol. 53, no. 3, pp. 1368–1386, 2024.

[27] A. C. Kimber, *Exploratory data analysis for possibly censored data from skewed distributions*, Journal of the Royal Statistical Society. Series C-Applied Statistics, vol. 39, pp. 21–30, 1990.

[28] C. Piao and Z. S. Ye, *A systematic look at the gamma process capability indices*, European Journal of Operational Research, vol. 265, no. 2, pp. 589–597, 2018.

[29] M. V. Ahmadi, M. Doostparast, and J. Ahmadi, *Statistical inference for the lifetime performance index based on generalized order statistics from exponential distribution*, International Journal of Systems Science, vol. 46, no. 6, pp. 1094–1107, 2015.

[30] T. Zhu, *Reliability estimation for two-parameter Weibull distribution under block censoring*, Reliability Engineering and System Safety, vol. 203, 2020.

[31] J. Lawless, *Statistical models and methods for lifetime data*, John Wiley and Sons, Inc., 2003.

[32] K. Tsui and S. Weerahandi, *Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters*, Journal of the American Statistical Association, vol. 84, no. 406, pp. 602–607, 1989.

[33] J. F. Lawless, *Statistical models and methods for lifetime data*, John Wiley and Sons, Inc., 2011.

[34] N. Balakrishnan and E. Cramer, *The art of progressive censoring. Statistics for industry and technology*, Springer, 2014.

[35] J. Liu, L. Wang, Y. M. Tripathi, and Y. Lio, *Inference of constant-stress model of Fréchet distribution under a maximum ranked set sampling with unequal samples*, Axioms, vol. 13, 394, 2024.

[36] H. M. Lee, W. C. Lee, C. L. Lei, J. W. Wu, *Computational procedure of assessing lifetime performance index of Weibull lifetime products with the upper record values*, Mathematics and Computers in Simulation, vol. 81(6), 1177-1189, 2011.