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The notion of radical ideals in MV-algebras and Product MV-algebras

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Abstract In this work, we describe an interpretation of the radical of an ideal in the context of l_u-groups, semi-low l_u-rings and PMV$_f$-algebras. These notions arise naturally from a known concept in the context of MV-algebras and the use of categorical equivalences. The resulting notions in these contexts allow us to propose and prove a version of Hilbert's Nullstellensatz Theorem.

Keywords Radical, Nullstellensatz, MV-algebras, Product MV-algebras, Lattice-ordered groups, Lattice-ordered rings

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1. Introduction

In studying the relationships between MV-algebras and structures from commutative algebra, the concept of the radical of an ideal in the former context holds significant importance as its study can lead to analogies with classical algebraic geometry concepts. In [13], Dubuc and Zilber defined a notion of the radical of an ideal and presented a version of the Nullstellensatz. This work was presented to us by the late Dr. Y. A. Poveda (who passed away in 2021), and he invited us to study this notion in the context of MV-algebras with product.

Several authors have considered a version of the radical of an ideal in the context of MV-algebras and l_u-groups. All these notions have the characteristic of being defined as the intersection of some ideals, and this notion allowed them to prove a Nullstellensatz-like theorem. In [3], Belluce, Di Nola and Lenzi defined the point radical of an ideal for the algebra $A[\overline{x}]$ freely generated by $\overline{x}$ in the variety V_A, where A is an algebra of type F. In [4], the same authors considered the A-point radical of a subset $J \subset F_\mu$, where A is an MV-algebra and F_μ is the free MV-algebra over μ generators. In [10], Di Nola, Lenzi and Vitale presented a version of the Nullstellensatz for the Riesz MV-algebra of Riesz-McNaughton functions from $[0,1]^n$ to $[0,1]$. In [11], they also considered the l-radical of an l-ideal of the free l-group over n generators $FAl(n)$.

In this work, we recall the notions of the radical of an ideal in classical contexts. From the one in MV-algebras, we present a natural notion in the context of l_u-groups via the categorical equivalence between the categories of MV-algebras and l-groups with strong unit given in [12, 8]. Then, we propose an interpretation of the radical of an ideal in other non-classical contexts and establish some Hilbert's Nullstellensatz-like Theorems there.

1.1. Some notation

Through this work, we accord the following: for a ring, we mean a commutative and associative ring with identity 1, such as $\mathbb{R}$ and $\mathbb{C}$. If A and B are sets, the set of functions from B to A will be denoted as A^B, and we will consider $0 \in A$. In this context, we define for $S \subset A^B$ the **zero set** generated by S as $\mathcal{Z}(S) := \{b \in B \mid f(b) =$

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$0\} = \bigcap_{f\in S} f^{-1}(0)$, and for $X \subset B$, we define the set $\mathcal{I}(X) := \{f \in A^B \mid f(x) = 0, \forall x \in B\}$. Thus, we have the following functions:

$$\mathcal{Z} :\ \mathcal{P}(A^B) \longrightarrow \mathcal{P}(B) \qquad \mathcal{I} :\ \mathcal{P}(B) \longrightarrow \mathcal{P}(A^B)$$

When $S = \{f\}$ and $X = \{a\}$, we will write $\mathcal{Z}(f)$ and $\mathcal{I}(a) = I_a$, instead of $\mathcal{Z}(\{f\})$ and $\mathcal{I}(\{a\})$, respectively. In classical algebraic geometry, these functions exhibit the relation between some algebraic structures and some geometric objects, like the one-to-one correspondence between prime ideals in some rings of polynomials and irreducible algebraic sets [17, §1.7, Corollary 2], and we will see this relation for new structures. Even without the algebraic and geometric structures, these functions have some immediate properties, summarized in the following lemma.

Lemma 1
The functions $\mathcal{Z}$ and $\mathcal{I}$ defined above satisfy the following:

1. If $\{S_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{P}(A^B)$, then $\mathcal{Z}\left(\bigcup_{\alpha\in\Gamma} S_\alpha\right) = \bigcap_{\alpha\in\Gamma} \mathcal{Z}(S_\alpha)$.
2. $\mathcal{Z}(0) = B$ and for any constant function $c \neq 0$ in A^B, $\mathcal{Z}(c) = \emptyset$.
3. $\mathcal{I}(\emptyset) = A^B$ and $\mathcal{I}(B) = 0$.
4. If $S \subset T \subset A^B$ and $X \subset Y \subset B$, then $\mathcal{Z}(T) \subset \mathcal{Z}(S)$ and $\mathcal{I}(Y) \subset \mathcal{I}(X)$.
5. If $S \subset A^B$ and $X \subset B$, then $S \subset \mathcal{I}(\mathcal{Z}(S))$ and $X \subset \mathcal{Z}(\mathcal{I}(X))$.
6. If $S \subset A^B$ and $X \subset B$, then $\mathcal{Z}(S) = \mathcal{Z}(\mathcal{I}(\mathcal{Z}(S)))$ and $\mathcal{I}(X) = \mathcal{I}(\mathcal{Z}(\mathcal{I}(X)))$.

These properties show the following antitone Galois connection: $S \subset \mathcal{I}(X)$ if and only if $X \subset \mathcal{Z}(S)$.

1.2. Rings

The interpretation of certain notions in algebraic geometry within other contexts begins with the properties of specific rings. The study of radical ideals in this context is of particular interest, and we examine it from two perspectives: radicals and real radicals.

Example 1
For a ring R, consider the ring of polynomials in one variable $R[x]$ or several variables $R[x_1, \cdots, x_n]$. If $R \subset \mathbb{C}$ and $a \in R$, then I_a denotes the ideal of polynomials $p \in R[x]$ such that $p(a) = 0$; i.e. the maximal ideal generated by $x - a$. Not all ideals are of this form in $\mathbb{R}[x]$, for instance, the ideal generated by $x^2 + 1$ is a maximal ideal.

The notion of the radical of an ideal is intimately related to the operations of the algebraic structure where ideals are defined [1, 17, 5]. We recall the definition of the radical of an ideal in a ring.

Definition 1
If I is an ideal of a ring R, then the **radical of** I is defined as:

$$\text{Rad}(I) := \{x \in R \mid x^n \in I,\ \text{for some } n \geq 0\}.$$

For rings, the real radical of an ideal also plays an important role in the context of real closed fields [5].

Definition 2
If I is an ideal of a ring R, then the **real radical of** I is defined as:

$$\sqrt[R]{I} := \left\{x \in R \mid x^{2n} + b_1^2 + \cdots + b_p^2 \in I,\ \text{for some } n \geq 1 \text{ and } b_1, \cdots, b_p \in R\right\}$$

Remark 1
An ideal I of R is called **real** if for every sequence $a_1, \cdots, a_p \in R$ such that $a_1^2 + \cdots + a_p^2 \in I$, then $a_1, \cdots, a_p \in I$ [5, 4.1.3].

As we recall below, the real algebraic context provides an important feature in terms of an order relation. An order relation is also considered for MV-algebras, and several results highlight its significance. This sense of order is naturally defined for l_u-groups and l_u-rings, so the results over the real and MV-algebraic contexts allow us to find analogous results on them.

1.3. *MV-algebras and Product MV-algebras*

Many valued algebras (MV-algebras) were defined by Chang [6, 7] as the algebraic systems that correspond to many valued propositional calculus. Many works have been done around the theory of MV-algebras and for us, it will be of interest the theory of MV-algebras with a product (PMV-algebras) and its varieties. It is known the categorical relation of MV-algebras and PMV-algebras with l_u-groups and l_u-rings, respectively, and that is why we are interested in these structures and, in particular, in the notions of ideals on them.

We will recall the notions of MV-algebras that are of interest in this work, which can be found in several references, from which we recommend [8, 19, 16].

Definition 3
An **MV-algebra** is a $(0,1,2)$-type algebra $\langle A, 0, ^*, \oplus \rangle$ such that:

1. $a \oplus b = b \oplus a$,
2. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
3. $a \oplus 0 = a$,
4. $(a^*)^* = a$,
5. $a \oplus 0^* = 0^*$,
6. $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Additionally, the following auxiliary elements are defined in every MV-algebra:

$$1 := 0^*, \quad a \odot b := (a^* \oplus b^*)^*, \quad a \ominus b := a \odot b^* = (a^* \oplus b)^*$$

In each MV-algebra A, a **natural partial order** is defined by the following relation:

$$a \leq b \text{ if and only if } a^* \oplus b = 1$$

When the order is total, we say the MV-algebra is an **MV-chain**.

Example 2
1. The real interval $[0,1]$ is the **standard MV-algebra**, where operations are defined as follows: $a^* := 1 - a$ and $a \oplus b := \min\{1, a+b\}$. The standard MV-algebra generates the variety of MV-algebras [8, 8.1.1].
2. For a non-empty set X, the set of functions from X to $[0,1]$, denoted by $[0,1]^X$, is an MV-algebra, where operations are defined point-wise. For instance, $[0,1]^{\mathbb{R}}$ is an MV-algebra. Furthermore, the subset $\mathrm{Cont}(\mathbb{R}, [0,1])$ of functions from $\mathbb{R}$ to $[0,1]$ that are continuous with respect to the natural topology on $[0,1]$ is a sub MV-algebra of $[0,1]^{\mathbb{R}}$.
3. A McNaugthon fuction in one variable is a piecewise function from $[0,1]$ to $[0,1]$ that is continuous with respect to the natural topology of $[0,1]$ and defined by finite linear polynomials with integer coefficients. The collection of all McNaughton functions is denoted by $\mathcal{M}_1$ and is a sub MV-algebra of $\mathrm{Cont}([0,1],[0,1])$. This MV-algebra plays an important role since it is the **Free MV-algebra** over one generator [8, 9.1.5]. The MV-algebra $\mathcal{M}_n$ of functions McNaughton functions from $[0,1]^n$ to $[0,1]$ is defined similarly [8, 3.1.6].

Definition 4
Given A, B MV-algebras, a function $h : A \to B$ is an **MV-algebra homomorphism** if:

$$h(a \oplus_A b) = h(a) \oplus_B h(b), \quad h(a^{*_A}) = h(a)^{*_B}$$

When operations and MV-algebras are clear, we just write $\oplus$ and *.

Definition 5
A subset I of an MV-algebra A is an **ideal** if:

1. $0 \in I$,
2. If $a, b \in I$, then $a \oplus b \in I$,

3. If $a \in I, b \in A$ and $b \leq a$, then $b \in I$.

An ideal I of A is called **proper** if $I \neq A$. Proper ideals are **prime** if either $a \ominus b \in I$ or $b \ominus a \in I$ for each pair $a, b \in A$. A proper ideal is called **maximal** if no proper ideal of A strictly contains I.

We denote by $\text{Id}(A)$, $\text{Spec}(A)$ and $\text{Max}(A)$ the collections of all ideals, prime ideals, and maximal ideals of an MV-algebra A, respectively. We will use the same notation for ideals in other structures. The standard MV-algebra $[0,1]$ has only the ideals: $\{0\}$ and itself. However, there are some non-trivial ideals in other MV-algebras, some of which are trivial or have no sense for rings of polynomials.

Example 3
Consider the MV-algebras $A = \text{Cont}([0,1],[0,1])$ and $\mathcal{M}_1$:

1. If $a \in [0,1]$, then I_a is defined as before is a maximal ideal. If $a \in [0,1]\backslash\mathbb{Q}$, functions from $\mathcal{M}_1$ have to equal zero over an interval $[a-\epsilon, a+\epsilon]$ for some $\epsilon > 0$.
2. For $a \in [0,1]$, the ideal I_{a^+} is defined to be the set of functions f such that for each one, there exists ϵ_f for which f is zero on $[a, a+\epsilon_f)$. Similarly, the ideal I_{a^-} is defined. Observe that if $a \in [0,1]\backslash\mathbb{Q}$, then in $\mathcal{M}_1$ holds $I_a = I_{a^+} = I_{a^-}$.
3. For $a, b \in [0,1]$ and $a < b$, the ideal $I_{[a,b]}$ is the set of functions that are equal to zero over $[a,b]$.

As some MV-algebras are collections of functions, they are related to rings of polynomials over a ring or a field, so there is a natural sense of developing an algebraic geometry theory for MV-algebras. To develop this idea, it is important to have a sort of geometric relation between ideals and the sets where those functions are defined. We recall the definition of the radical of an ideal in the context of MV-algebras. This definition will be extended to other structures, like MV-algebras with product, following the idea of having a geometric relation with the ideals on the structures.

Definition 6
If I is an ideal of an MV-algebra A, the **radical of** I is defined as:

$$\sqrt{I} := \{a \in A \mid na \ominus a \in I, \text{ for some } n \geq 0\}.$$

Remark 2
The set $\sqrt{I}$ is named the **infradical of** I, although it is known that it coincides with the intersection of maximal ideals containing I, which is called the radical of I [13, 3.2, 3.7 and 3.12].

The class of MV-algebras with product was defined in [14]. Since then, different classes and relations among them have been defined [15, 9]. The interest in these structures is their categorical relation with l-rings, as will be recalled below. As usual, we will use the notation xy for the product $x \cdot y$ and consider this operation to be commutative.

Definition 7
An **MV-algebra with product** $\langle P, 0, ^*, \oplus, \cdot\rangle$ is an MV-algebra $\langle P, 0, ^*, \oplus\rangle$ with a binary operation $\cdot$ such that $(P, \cdot)$ is a semigroup.

We consider products to be commutative.

Definition 8
Given P, Q MV-algebras with product, a function $h : P \to Q$ is an **MV-algebra with product homomorphism** if h is an MV-algebra homomorphism of the subjacent MV-algebra and for all $x, y \in P$, it holds $h(xy) = h(x)h(y)$.

Definition 9
A **PMV-algebra** P, is an MV-algebra with product P such that for all $x, y, z \in P$:

$$x \odot y = 0 \text{ implies } xz \odot yz = 0 \text{ and } z(x \oplus y) = xz \oplus yz$$

Definition 10
A **PMV$_f$-algebra** is an MV-algebra with product P such that:

1. $x0 = 0$,
2. $(x(y \oplus z)) \ominus (xy \oplus xz) = 0$,
3. $(xy \ominus xz) \ominus (x(y \ominus z)) = 0$,
4. $xy \leq x \wedge y$,
5. $x(y \ominus c) = xy \ominus xz$.

As we recall below, the category of PMV$_f$-algebras is equivalent to the category of semi-low l_u-rings, a class of ordered rings where the product of elements is bounded by their infimum.

Example 4
The standard MV-algebra with the usual multiplication is a PMV$_f$-algebra. So is the MV-algebra Cont$([0,1]^n, [0,1])$. In general, for a set X, the set $[0,1]^X$ of functions from X to $[0,1]$ with operations defined pointwise is a PMV$_f$-algebra. A last example is the set $F[x_1, \cdots, x_n]$ of continuous functions from $[0,1]^n$ to $[0,1]$ defined piecewise by finite polynomials with integer coefficients [9, 5.4].

An MV-algebra with a product that satisfies 1. - 3. in Definition 10 is called an MVW-rig [15]. It is worth noticing that all PMV$_f$-algebras are PMV-algebras [9, 3.8].

As mentioned before, it is of interest to consider the notion of ideals in the structures we are discussing. The following is the definition in the context of PMV$_f$-algebras.

Definition 11
A subset I of a PMV$_f$-algebra P is an **ideal of** P if it is an ideal of the subjacent MV-algebra P and it holds that $ab \in I$ for all $a, b \in I$.

1.4. l-groups and l-rings

l-groups and l-rings are algebraic structures equipped with a lattice structure. Some of their varieties are categorically equivalent to certain classes of MV-algebras and product MV-algebras. This lattice structure will allow us to characterize their ideals and adapt the definition of their radicals and, finally, establish a series of Nullstellensatz theorems.

We are interested in the categorical equivalences described in [12, 9], for this we recall the definitions of l_u-groups, l_u-rings and their varieties, as well as other basic elements. Of course, [2] is a suggested source for a detailed treatment of l-groups and l-rings.

Definition 12
A group $(G, +)$, is an l**-group** if it has a partial order $\leq$ for which $(G, +, \leq)$ is both a poset and a lattice, where the order is compatible with the group operation. If there exists $u \in G$ such that for any $x \in G$ it holds $|x| \leq nu$ for some integer $n \geq 1$, then G is called an l_u**-group** and u is called a strong unit. When the order is total, we say the l_u-group is a l_u**-chain**.

Definition 13
Given (G, u) and (H, v) l-groups with strong units, a function $f : G \to H$ is an l_u**-group homomorphism** if f is a group homomorphism for the subjacent groups G and H, and $h(u) = v$.

For an element x in an l-group, its absolute value is defined as $|x| = x \vee (-x)$. This notion will be used in the following definition and also plays a role in important proofs later on.

Definition 14
A normal subgroup I of an l-group G is called an l**-ideal** if $x \in I, y \in G, |y| \leq |x|$ implies $y \in I$.

Definition 15
A ring $(R, +, \cdot)$ is a l**-ring** if it has a partial order $\leq$ such that $(R, +, \leq)$ is an l-group and $0 \leq x, y$ implies $0 \leq xy$.

Definition 16
Given (R, u) and (S, v) l-rings with strong units, a function $f : G \to H$ is a l_u**-ring homomorphism** if f is an l_u-group homomorphism between the subjacent l_u-groups (R, u) and (S, v), and for all $x, y \in R$, then $f(xy) = f(x)f(y)$.

Example 5
1. $(\mathbb{R},+)$ with the usual addition is an l_u-group for any $u>0$, for instance $u=1$. Even more, $(\mathbb{R},+,\cdot)$ with the usual multiplication is a l_u-ring.
2. For a topological space X, the set Cont$(X,[0,1])$ with pointwise addition is an l_u-group and with pointwise multiplication, it is also an l_u-ring, where any positive function is a strong unit.
3. The set $\mathcal{M}_{\mathbb{R}n}$ of continuous functions from $[0,1]^n$ to $\mathbb{R}$ defined piecewise by finite linear polynomials with integer coefficients, the set $F_{\mathbb{R}}[x_1,\cdots,x_n]$ of continuous functions from $[0,1]^n$ to $\mathbb{R}$ defined piecewise by finite polynomials with integer coefficients and Cont$([0,1],\mathbb{R})$ with usual addition (and multiplication) are l_u-groups (and l_u-rings), where any $u>0$ is a strong unit.

Definition 17
An l-ideal I of the underlying l-group of an l-ring R is called an L**-ideal** if in addition $x\in I, y\in R$ implies $xy\in I$.

An l-ring R is called low if for all $x,y\geq 0$, then $xy\leq x\wedge y$. The following definition captures the same idea for elements in its positive segment bounded by u [9, 6.8].

Definition 18
An l_u-ring R is called **semi-low** if for all $x,y\in[0,u]$ it holds that $xy\leq x\wedge y$.

Example 6
The l_u-ring $(\mathbb{R},+,\cdot)$ is semi-low for any $0<u\leq 1$. Other semi-low l_u-rings are $\mathcal{M}_{\mathbb{R}n}$, $F_{\mathbb{R}}[x_1,\cdots,x_n]$ and Cont$([0,1],\mathbb{R})$, for any strong unit $0<u\leq 1$.

1.5. Categorical equivalences

The following categorical equivalences will allow us to identify some properties of the algebraic structures that lead to establishing a natural notion of the radical of an ideal. Both of the cited equivalences follow the construction of C.C. Chang in [7] where he relates each MV-chain A with an l_u-chain A^*. In this work, we denote by $\mathcal{MV},\mathcal{PMV}_f,\mathcal{LG}_u$ and $\mathcal{LR}_u$ the categories of MV-algebras, PMV$_f$-algebras, l_u-groups and semi-low l_u-rings, respectively.

Theorem 1 ([12, 3.2 and 3.3])
The categories $\mathcal{MV}$ and $\mathcal{LG}_u$ are equivalent.

Functors in the equivalence are defined as follows:

$$\mathcal{MV}\xrightarrow{(-)^*}\mathcal{LG}_u \qquad\qquad \mathcal{LG}_u\xrightarrow{\Gamma}\mathcal{MV}$$
$$A\longmapsto A^* \qquad\qquad G\longmapsto\Gamma(G,u)$$

Where A^* is a sub l_u-group of the subdirect product of the family of l_u-chains $\{(A/P)^*_{P\in Spec(A)}\}$. $\Gamma(G,u)$ is the segment $[0,u]$ which is an MV-algebra with $x\oplus y:=u\wedge(x+y)$ and $x^*=u-x$. When A and G are totally ordered, the functors are the ones defined in [7]. For instance, to the MV-algebra $A=[0,1]$ corresponds the l_u-group $[0,1]^*=\mathbb{Z}\times[0,1)\cong\mathbb{R}$, where the order is given by $(m,x)\leq(n,y)$ if and only if either $m<n$ or $m=n$ and $x\leq y$ [7, Lemma 5]. In the other direction, $\Gamma(\mathbb{R},1)=[0,1]$ and $\Gamma(\mathcal{M}_{\mathbb{R}n})=\mathcal{M}_n$.

Theorem 2 ([9, 8.20])
The categories $\mathcal{PMV}_f$ and $\mathcal{LR}_u$ are equivalent.

Functors in the equivalence are defined below:

$$\mathcal{PMV}_f\xrightarrow{(-)^\#}\mathcal{LR}_u \qquad\qquad \mathcal{LR}_u\xrightarrow{\Gamma}\mathcal{PMV}_f$$
$$P\longmapsto P^\# \qquad\qquad R\longmapsto\Gamma(R,u)$$

Where $P^\#$ is a sub l_u-ring generated in the l_u-ring $\{(P/Q)^\#_{Q\in Spec(P)}\}$, where the latter is the product of a family of (chain semi-low) l_u-rings. $\Gamma(R,u)$ is the segment $[0,u]$ which is a PMV$_f$-algebra with the inherited product. For instance $\Gamma(F_{\mathbb{R}}[x_1,\cdots,x_n])=F[x_1,\cdots,x_n]$.

These functors have an important application in terms of ideals. In Proposition 4, we recall this for the functor $(-)^*$, which shows a one-to-one correspondence between the ideals of an MV-algebra and the l-ideals of its associated l_u-group. A similar result is obtained between ideals of PMV$_f$-algebras and l_u-rings. This application is the key feature of this work; from this correspondence, we can transfer the notion of radical ideals from MV-algebras to l_u-groups.

2. The interpretations of the radical of an ideal

2.1. Notions of the radical in non-classical contexts

The notion of the radical of an ideal in the contexts of rings and MV-algebras is characterized by a similar property: they are the intersection of some ideals of the algebra, as summarized below. These definitions are valid for proper ideals. The new definitions of the radical of an ideal proposed here are also for proper ideals.

Proposition 1 ([1, 1.14])
If I is an ideal of a ring R, then Rad(I) is the intersection of the prime ideals that contain I.

Proposition 2 ([5, 4.1.7])
If I is an ideal of a ring R, then $\sqrt[R]{I}$ is the intersection of all real prime ideals containing I.

Proposition 3 ([13, 3.12])
If I is an ideal of an MV-algebra A, then $\sqrt{I}$ is the intersection of all maximal ideals containing I.

We want an interpretation for the radical of an ideal of an MV-algebra in the corresponding l_u-group via the categorical equivalence. The following proposition shows that it can be well described by a similar structure to that defined in MV-algebras.

Proposition 4
Let A be an MV-algebra, $G = A^*$ its associated l_u-group and I an ideal of A. Then, the associated l-ideal in G of the radical of I is the intersection of the maximal l-ideals of G containing I^*.

Proof
The functors Γ and $(-)^*$ between the categories $\mathcal{LG}_u$ and $\mathcal{MV}$ define an order-isomorphism between the ideals of an MV-algebra A and the l-ideals of the associated l_u-group G as follows: if I is an ideal of A, then $I^* := \{x \in G \mid |x| \wedge u \in I\}$ is an l-ideal of G and the inverse is given by $\Gamma(J) := J \cap [0, u]$ [8, 7.2.2]. In particular, they define a restricted bijection that preserves inclusion between Max (A) and Max (G). Thus, for each $M \in$ Max (A), $I \subset M$ if and only if $I^* \subset M^*$, where $M^* \in$ Max (G), so we have:

$$(\text{Rad}\,(I))^* = \left(\bigcap_{I \subset M \in \text{Max}(A)} M \right)^* = \bigcap_{I^* \subset M^* \in \text{Max}(G)} M^*$$

□

From the last result, we justify the following natural interpretation of the radical of an l-ideal of an l_u-group.

Definition 19
If G is an l_u-group and J is an l-ideal of G, the l**-radical** of J is defined as the intersection of all maximal l-ideals of G containing J.

Inspired by this definition and the equivalent notions of the radical of an ideal shown in this section, we extend this definition to l_u-rings.

Definition 20
If R is a l_u-ring and J is an l-ideal of R, the L**-radical** of J is defined as the intersection of the maximal l-ideals of R containing J.

Remark 3
This is the definition of the L-radical of an ideal given in [2, XVII-§3], which shows that the application of the functors in the equivalences has been adequate.

Now, we will define the radical of an ideal in PMV$_f$-algebras and in semi-low-l_u-rings. The l_u-ring Cont$(X, \mathbb{R})$ is a semi-low l_u-ring with $0 < u \leq 1$. In turn, the MV-algebra Cont$(X, [0, 1])$ is a PMV$_f$ algebra with the usual product of $[0, 1]$. For this, we consider the categorical equivalence between the categories $\mathcal{LR}_u$ and $\mathcal{PMV}_f$ given in [9]. For instance, the semi-low l_u-ring Cont$(X, \mathbb{R})$ is associated to the PMV$_f$-algebra Cont$(X, [0, 1])$ [9, 8.11]. From the equivalence, we obtain a notion of the radical of an ideal in the context of PMV$_f$-algebras. This is possible since there is an order preserving bijection between maximal ideals of a semi-low l_u-ring R and the PMV$_f$-algebra $\Gamma(R, u)$, so the proof of the next Proposition is similar to the one given in Proposition 4.

Proposition 5
Let R be a semi-low l_u-ring, $P = \Gamma(R, u)$ its associated PMV$_f$-algebra and J an l-ideal of R. The related ideal in P of the l-radical of J is the intersection of all maximal ideals of P containing $\Gamma(J)$.

The last definition is of the radical of an ideal in the context of PMV$_f$-algebras.

Definition 21
If P is a PMV$_f$-algebra and I is an ideal of A, the **radical** of I is defined as the intersection of all maximal ideals of P containing I.

We will use the notation Rad$\,(I)$ for the radical of an ideal in l_u-groups, l_u-rings, MV-algebras and PMV$_f$-algebras. Also, we will be careful when referring to the algebraic structure.

2.2. Properties of the radical of an ideal

The radical of an ideal satisfies some special properties both in classical and non-classical structures. We first prove some of them for l_u-groups.

Proposition 6
Consider G, G' l_u-groups, J an l-ideal of G and $\varphi : G' \longrightarrow G$ a surjective l_u-group homomorphism, then:

1. $J \subset$ Rad$\,(J)$.
2. Rad$\,(J)$ is an l-ideal.
3. Rad$\,(J)$ is an l-radical ideal, i.e., Rad$\,($Rad$\,(J)) =$ Rad$\,(J)$.
4. Rad$\,(J)$ is the smallest l-radical ideal containing J.
5. $\varphi^{-1}\,($Rad$\,(J)) =$ Rad$\,\left(\varphi^{-1}(J)\right)$.

Proof

1. By definition, $J \subset M$ for all maximal l-ideals of G containing J, therefore $J \subset \bigcap_{J \subset M \in \text{Max}(G)} M =$ Rad$\,(J)$.
2. Follows since the intersection of an arbitrary collection of l-ideals is an l-ideal.
3. If Rad$\,(J) \subset M$, then $J \subset M$ from 1., therefore $\{M \in$ Max$\,(G) \mid$ Rad$\,(J) \subset M\} \subset \{M \in$ Max$\,(G) \mid J \subset M\}$, from where Rad$\,($Rad$\,(J)) \subset$ Rad$\,(J)$. The other inclusion follows from 1. and 2.
4. Follows from 3.
5. It is direct to see that if J and M are l-ideals of G, then $J \subset M$ if and only if $\varphi^{-1}(J) \subset \varphi^{-1}(M)$. Also, $M \in$ Max$\,(G)$ if and only if $\varphi^{-1}(M) \in$ Max$\,(G')$. Indeed, if $M \in$ Max$\,(G)$ and $\varphi^{-1}(M) \subset K$ for some l-ideal K of G', then $M \subset \varphi(K)$, so either $\varphi(K) = M$ or $\varphi(K) = G$, which implies either $K = \varphi^{-1}(M)$ or $K = G'$. In the other direction, if $\varphi^{-1}(M) \in$ Max$\,(G')$ and $M \subset K$ for some l-ideal K of G, then $\varphi^{-1}(M) \subset \varphi^{-1}(K)$, thus either $\varphi^{-1}(K) = \varphi^{-1}(M)$ or $\varphi^{-1}(K) = G'$, therefore either $K = M$ or $K = G$, since φ is surjective. Finally, $x \in \varphi^{-1}($Rad$\,(J))$ means $\varphi(x) \in$ Rad$\,(J)$, which is equivalent to say that $\varphi(x) \in M$ for all $M \in$ Max$\,(G)$ such that $J \subset M$, which in turn means $x \in \varphi^{-1}(M)$ for all $\varphi^{-1}(M) \in$ Max$\,(G')$ such that $\varphi^{-1}(J) \subset \varphi^{-1}(M)$, that is, $x \in$ Rad$\,\left(\varphi^{-1}(J)\right)$.

□

Remark 4
For a family of l_u-groups $\{G_l \mid l \in L\}$, the radical of l-ideals of their product l_u-group $G = \prod_{l \in L} G_l$ cannot be characterized by a property like $\text{Rad}\,(I) = \prod_{l \in L} \text{Rad}\,(I_l)$. Although it is possible to show that if $M \in \text{Max}\,(G)$, then $M_l \in \text{Max}\,(G_l)$, where M_l denotes the image of M under the projection map $\pi_l : G \longrightarrow G_l$, the converse is not always true. In the latter case, if for each $l \in L$, M_l is a maximal l-ideal of each G_l and $M = \prod_{l \in L} M_l \subset J$ for some l-ideal J of G, then $M_l \subset J_l$ for each $l \in L$, which implies either $J_l = M_l$ or $J_l = G_l$ for each $l \in L$, from which it cannot be concluded that $J = M$ or $J = G$.

Without major differences in the last proof, we have the following.

Proposition 7
Properties in Proposition 6 are satisfied by L-ideals and ideals of l_u-rings, MV-algebras and PMV$_f$-algebras.

3. Nullstellensatz - like theorems

3.1. Nullstellensatz in non-classical contexts

In the following, we will consider algebraic structures whose elements are functions. The notions of ideal in these structures have been already defined. Moreover, the functions $\mathcal{Z}$ and $\mathcal{I}$ defined at the beginning of the paper have a significant role here. If S is a subset of the algebraic structure, then the ideal generated by S is defined as the smallest ideal of the structure containing S and it is denoted by $\langle S \rangle$. For ideals in l_u-groups, l_u-rings, MV-algebras, and PMF$_f$-algebras, the following result follows from definitions.

Lemma 2
If $I = \langle S \rangle$, then I satisfies the same properties as S in Lemma 1.

Hilbert's Nullstellensatz is stated in terms of the radical of an ideal: it is equal to the generated ideal of the zero set of the ideal, for ideals in the ring of polynomials with coefficients in a field. We recall the classical version for algebraically closed fields.

Theorem 3 ([17, §1.7])
Let k be an algebraically closed field and I an ideal of the ring $k[x_1, \cdots, x_n]$, then $\text{Rad}(I) = \mathcal{I}\,(\mathcal{Z}\,(I))$.

A version of this theorem is known in the context of real fields. A field F is called real if it can be ordered [5, 1.1.9]. A real field F is called real closed if it has no non-trivial real algebraic extensions [5, 1.2.1]. As should be expected, the field $\mathbb{R}$ is an example of a real closed field.

Theorem 4 ([5, 4.1.4 and 4.1.8])
If I is an ideal of the ring $R[x_1, \cdots, x_n]$, where R is a real closed field, then $\sqrt[R]{I} = \mathcal{I}\,(\mathcal{Z}\,(I))$.

Remark 5
Theorem 4 is stated as "$I = \mathcal{I}\,(\mathcal{Z}\,(I))$ if and only if I is real", which is equivalent to the presented version.

In the real algebraic context, there is a lack of the classical sense of a closed field, but the order of the field leads to establishing a version of the Nullstellensatz, i.e., to characterize the ideals generated by their zero sets. In the context of MV-algebras, there is also a lack of sense of closed fields, but again, the lattice structure of the algebra plays an important role to characterize ideals generated by their zero sets. In this context, the Theorem is stated in terms of Cont$(X, [0,1])$, the MV-algebra of continuous functions from a topological space X to $[0,1]$.

Theorem 5 ([13, 3.10])
Let X be a compact topological space, A a subalgebra of Cont$(X, [0,1])$ and I an ideal of A. Then, $\text{Rad}(I) = \mathcal{I}\,(\mathcal{Z}\,(I))$.

The following corollary highlights the role of free MV-algebras in this context.

Corollary 1 ([13, 3.11])
If I is an ideal of the free MV-algebra $\mathcal{M}_n$, then $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$.

We present a version of Hilbert's Nullstellensatz Theorem in three different contexts, each of which has a similar statement as the original theorem. To start, the following proposition is a version of the Weak Nullstellensatz, which states that the zero set of a proper ideal in a structure of functions is non-empty. Proving this in the proposed context is easier than the proof of the classical version for the ring of polynomials with coefficients in an algebraically closed field, since the functions are continuous over a compact topological space. From now on, consider X a compact topological space and denote by Cont$(X, \mathbb{R})$ the subset of continuous functions of $\mathbb{R}^X$.

Proposition 8
For the l_u-group Cont$(X, \mathbb{R})$ and J a proper l-ideal, then, $\mathcal{Z}(J) \neq \emptyset$.

Proof
First, observe that if $f \in J$, then $\mathcal{Z}(f) \neq \emptyset$, since otherwise $u \leq m|f| \in J$ for some $m \geq 1$, so J would not be proper. Now, by contradiction, suppose $\bigcap_{f \in J} \mathcal{Z}(f) = \mathcal{Z}(J) = \emptyset$. Taking the complement leads to $X = \bigcup_{f \in J} \mathcal{Z}(f)'$ and using the fact that functions in J are continuous over the compact set X, then the open cover contains a finite subcover such that $X = \bigcup_{i=1}^{n} \mathcal{Z}(f_i)'$, from which $\bigcap_{i=1}^{n} \mathcal{Z}(f_i) = \emptyset$. Now, define $f = |f_1| + \cdots + |f_n| \in J$, so $\mathcal{Z}(f) = \emptyset$, which is a contradiction. □

Since Cont$(X, \mathbb{R})$ is also an l_u-ring, the following is immediate.

Corollary 2
Consider Cont$(X, \mathbb{R})$ as a l_u-ring with u a strong unit and J a proper l-ideal. Then, $\mathcal{Z}(J) \neq \emptyset$.

It is worth noticing that for Cont$(X, \mathbb{R})$, both as an l_u-group and an l_u-ring, maximal l-ideals and L-ideals are of the form $J_a := \{f \in \text{Cont}(X, \mathbb{R}) \mid f(a) = 0\}$.

Lemma 3
Let J be a maximal l-ideal of the l_u-group Cont$(X, \mathbb{R})$. Then, $J = J_a$ for some $a \in X$.

Proof
It is direct to see that ideals of the form J_a are maximal. Now, if J is a maximal ideal and $a, b \in \mathcal{Z}(J)$, then $J \subset J_a \cap J_b$, so $J_a = J_b = J$. □

In the below Proposition and Corollaries, we present the announced version of Hilbert's Nullstellensatz theorems for l_u-groups, l_u-rings and PMV$_f$-algebras.

Proposition 9 (l_u-groups Nullstellensatz)
Let J be a proper ideal of the l_u-group Cont$(X, \mathbb{R})$. Then, Rad$(J) = \mathcal{I}(\mathcal{Z}(J))$.

Proof
Observe that $J \subset J_a$ if and only if $a \in \mathcal{Z}(J)$. Indeed, if $J \subset J_a$, by Lemmas 1 and 2, $a \in \mathcal{Z}(J)$. On the other hand, if $a \in \mathcal{Z}(J)$, then $f \in J_a$ for all $f \in J$, so $J \subset J_a$. Then, the following holds:

$$\text{Rad}(J) = \bigcap_{J \subset J_a} J_a = \bigcap_{a \in \mathcal{Z}(J)} J_a = \mathcal{I}(\mathcal{Z}(J))$$

□

Corollary 3 (l_u-rings Nullstellensatz)
Let J be a proper ideal of the l_u-ring Cont$(X, \mathbb{R})$. Then, Rad$(J) = \mathcal{I}(\mathcal{Z}(J))$.

It also holds that maximal ideals of the PMV$_f$-algebra Cont$(X,[0,1])$ are of the form J_a, for some $a \in X$. Then, we get the following version of the Nullstellensatz in the context of PMV$_f$-algebras.

Proposition 10 (PMV$_f$-algebras Nullstellensatz)
Let I be a proper ideal of the PMV$_f$-algebra Cont$(X,[0,1])$, then Rad $(I) = \mathcal{I}(\mathcal{Z}(I))$.

Example 7
1. Consider $X = [0,1]$, the l_u-group $G = \text{Cont}(X, \mathbb{R})$, the l_u-ring $R = \text{Cont}(X, \mathbb{R})$ and the PMV$_f$-algebra $P = \text{Cont}(X,[0,1])$. For $a \in [0,1]$, the radical of maximal l-ideals of G of the form J_a (or L-ideals J_a of R, or ideals I_a of P) are the same ideals, i.e. Rad$(J_a) = J_a$ (similar in R and P), so these ideals should be called **radical ideals**, as in classical definitions.
2. Similar results as the presented could be obtained for the l_u-groups, l_u-rings and PMV$_f$-algebras of continuous functions defined piecewise by polynomials with integer coefficients, namely: the free MV-algebra $\mathcal{M}_n$, the l_u-group $\mathcal{M}_n^* = \mathcal{M}_{\mathbb{R}n}$, the l_u-ring $F[x_1, \cdots, x_n]^\# = F_{\mathbb{R}}[x_1, \cdots, x_n]$ and the PMV$_f$-algebra $F[x_1, \cdots, x_n]$. These are the structures we aim to study from an algebraic-geometric perspective, starting from classical concepts, as their elements are those most closely related to traditional contexts.

All the notions presented regarding radical ideals, even in classical contexts, are characterized by the Nullstellensatz theorems through a common feature: they are ideals that are also generated by their zero sets. This property reveals a deeper interpretation from a geometric perspective, although it is not explored in this work. However, the properties that are lost from classical and real algebraic geometry are compensated for by the lattice structure underlying the four algebraic structures presented here. This also opens avenues for further exploration of adaptations of concepts from classical contexts to these non-classical algebraic structures.

3.2. A comparison among the theorems

Consider the function $f(x) = x^2 + 1$, for which $\mathcal{Z}(f) = \{-i, i\}$ over the polynomial ring $\mathbb{C}[x]$ and $\mathcal{Z}(f) = \emptyset$ over the polynomial ring $\mathbb{R}[x]$. In the polynomial ring $\mathbb{C}[x]$, $\mathcal{I}(\mathcal{Z}(f)) = \langle (x+i)(x-i) \rangle = I_i \cap I_{-i}$. If $f' \in \mathbb{C}[x]$ and $f' \in \langle f \rangle$, then it is possible to show that both $x - i$ and $x + i$ are factors of f', which implies $\langle (x+i)(x-i) \rangle = \langle f \rangle$ and from Theorem 3, we have that $\langle f \rangle$ is a radical ideal although not a maximal ideal.

In $\mathbb{R}[x]$, $\langle f \rangle$ is a maximal ideal, then it is also a radical ideal. Observe that $\mathcal{I}(\mathcal{Z}(f)) = \mathbb{R}[x] \neq \langle f \rangle$, however Theorem 3 does not apply since $\mathbb{R}$ is not algebraically closed. Additionally, Theorem 4 implies $\langle f \rangle$ is not a real radical ideal.

In the l_u-group $G = \text{Cont}([0,1], \mathbb{R})$, $\mathcal{I}(\mathcal{Z}(f)) = G$ and $\langle f \rangle = G$ since $1 \leq f$, so Theorem 9 implies that $\langle f \rangle$ is an l-radical ideal. The same follows from Corollary 3 in the l_u-ring $R = \text{Cont}(X, \mathbb{R})$.

In the MV-algebra $A = \text{Cont}([0,1],[0,1])$, $f = 1$, then $\langle f \rangle = A$ and $\mathcal{I}(\mathcal{Z}(f)) = A$, therefore by Theorem 5, $\langle f \rangle$ is a radical ideal, although not a maximal ideal. The same holds in the PMV$_f$-algebra $P = \text{Cont}([0,1],[0,1])$ from Proposition 10.

Consider the function $g(x) = (x-1)^2$, for which $\mathcal{Z}(g) = \{1\}$ and in every structure $\mathcal{I}(\mathcal{Z}(g)) = I_1$. Denote by $I = \langle g \rangle$ the ideal generated in each of the following structures. In $\mathbb{C}[x]$, $\mathcal{I}(\mathcal{Z}(I)) = \langle x - 1 \rangle \supsetneqq I$, so from Theorem 3, I is not a radical ideal. From Theorem 4, it is neither a real radical ideal. In the l_u-group G, since there is no integer $n \geq 1$ such that $1 - x \leq ng$, we have that $I \subsetneqq I_1$ and from Theorem 9, I is not an l-radical ideal. With a similar argument and Theorem 5, we conclude that I is not a radical ideal in the MV-algebra A.

Now, consider $h(x) = \sin(x)$ restricted to $X = [0,1]$. Its zero set is $\mathcal{Z}(h) = \{0\}$, so $\mathcal{I}(\mathcal{Z}(I)) = I_0$, for $I = \langle h \rangle$. In the l_u-group G we have that $h \leq x$ and that $x \leq 2h$, therefore $I = \langle x \rangle$, and from Theorem 9 we have that I is an l-radical ideal. The same can be obtained for the MV-algebra A.

The examples above illustrate that the search for radical ideals in different structures yields similar approximations, such as the characterization of their zero sets. However, they also reveal algebraic aspects that differentiate them significantly, including order properties and natural bounds within the structures.

4. Conclusions

The study of the notion of radical ideals can be interpreted in various contexts beyond traditional ring theory, where algebraically closed fields may not be well defined. Analogous results in classical algebraic geometry have been established in real algebraic geometry and algebraic structures like MV-algebras, where the order of the structure plays an important role. Inspired by some results in terms of ideals in ordered structures, we showed that a natural definition of radical ideals for product MV-algebras, l_u-groups and l_u-rings can be formulated in terms of the intersection of maximal ideals.

Building on this foundational understanding of the radical of an ideal, we proposed and proved several theorems reminiscent of the Nullstellensatz in various contexts. These results indicate that the study of algebraic geometry is significant for MV-algebras, both with and without a product. Furthermore, the connection with the connection with l_u-groups and l_u-rings offers an insightful perspective, allowing certain problems from one structure in terms of the other. This interchangeability may provide opportunities to solve those problems and subsequently translate the solutions back into their original context.

The results presented in this work represent the initial steps toward future research. From one perspective, we could investigate the concept of radical ideals in contexts where the domain of the functions has weaker properties, like being non-compact. This exploration would broaden the study of algebraic geometry concerning MV-algebras and their varieties, allowing for the examination of other examples, such as free MV-algebras, finite MV-algebras, and perfect MV-algebras. Other characterizations for radical ideals can be explored, in particular, some in terms of the elements and not in terms of the maximal spectrum of the algebras, such as the known ones in classical contexts.

Additionally, while developing this work, it is essential to explore the geometric connections of the theory in terms of algebraic sets and irreducible algebraic sets. Some results related to this idea for MV-algebras can be found in works like [3].

Finally, the study of concepts from algebraic geometry could be further pursued within the structures presented here, thereby strengthening the current theory. This investigation should focus on coordinate rings and polynomial maps. Relevant studies of this for MV-algebras can also be found in works like [3].

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REFERENCES

1. Atiyah, M. F. and Macdonald, I. G., *Introduction to commutative algebra* Addison-Wesley Publishing, 1969
2. Birkhoff, G., *Lattice theory* American Mathematical Society, vol. XXV, 1973
3. Belluce, L. P., and Di Nola, A. and Lenzi, G., *Algebraic geometry for MV-algebras* The Journal of Symbolic Logic, vol. 79, no. 4, pp. 1061–1091, 2014
4. Belluce, L. P., and Di Nola, A. and Lenzi, G., *On generalizing the Nullstellensatz for MV algebras* Journal of Logic and Computation, vol. 25, no. 3, pp. 701–717, 2015
5. Bochnak, J. and Coste, M. and Roy, M. F., *Real Algebraic Geometry* Springer Berlin, Heidelberg, 1998
6. Chang, C. C., *Algebraic analysis of many valued logics* Transactions of the American Mathematical Society, vol. 88, no. 2, pp. 467–490, 1958
7. Chang, C. C., *A new proof of the completeness of the Łukasiewicz axioms* Transactions of the American Mathematical Society, vol. 93, no. 1, pp. 74–80, 1959
8. Cignoli, R. L. O. and D'Ottaviano, I. M. L. and Mundici, D., *Algebraic Foundations of Many-Valued Reasoning* Springer Dordrecht, 2000
9. Cruz, L. J. and Poveda, Y. A., *Categorical Equivalence Between PMV$_f$-Product Algebras and Semi-Low f_u-Rings* Studia Logica, vol. 107, no. 6, pp. 1135–1158, 2019
10. Di Nola, A. and Lenzi, G. and Vitale, G., *Riesz–McNaughton functions and Riesz MV-algebras of nonlinear functions* Fuzzy Sets and Systems, vol. 311, pp. 1–14, 2017
11. Di Nola, A. and Lenzi, G. and Vitale, G., *Algebraic geometry for l-groups* Algebra Universalis, vol. 79, no. 3, art. 64, 2018

12. Dubuc, E. J. and Poveda, Y. A., *On the equivalence between MV-algebras and l-groups with strong unit* Studia Logica, vol. 103, pp. 807–814, 2015
13. Dubuc, E. J. and Zilber, J., *Some remarks on infinitesimals in MV-algebras* Journal of Multiple-Valued Logic and Soft Computing, vol. 29, no. 6, pp. 647–656, 2017
14. Dvurečenskij, A. and Di Nola, A., *Product MV-algebras* Multiple-Valued Logic, vol. 6, pp. 193–215, 2001
15. Estrada, A. and Poveda, Y. A., *MVW-rigs and product MV-algebras* Journal of Applied Non-Classical Logics, vol. 29, no. 1, pp. 78–96, 2019
16. Estrada, A. and Poveda, Y. A. and Serrano, H., *Introducción a las MV-álgebras* Universidad Tecnológica de Pereira, 2024
17. Fulton, W., *Algebraic Curves: An Introduction to Algebraic Geometry* Addison-Wesley Publishing Company, 2008
18. Mundici, D., *Interpretation of AF C∗-algebras in Łukasiewicz sentential calculus* Journal of Functional Analysis, vol. 65, no. 1, pp. 15–63, 1986
19. Mundici, D., *Recent Developments of Many-valued Logic* IX Congreso Dr. Antonio Monteiro, 2007