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# Optimal Competitive Strategies on the Performance of two Insurance Companies

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**Abstract** In this paper, the optimization problems of the terminal wealth of two dependent insurance companies which each of them tries to perform better relative to its competitor is presented. It is assumed that both insurers having the compound Poisson process and they are allowed to purchase proportional reinsurance with a constant reinsurance premium and invest in a financial market which consists of a risk-free asset, a defaultable coupon bond whose the price process of each insurer is governed by a standard Brownian motion and dynamics of defaultable price process is modeled as a mixture of the exponential stochastic differential equation of corporate coupon bonds. For the correlated competing insurance companies, by applying the Girsanov's theorem and compensated Poisson process, we formulate the wealth process of each insurer based on the reinsurance and investment strategies. By solving the nonlinear Hamilton-Jacobi-Bellman equations related to our optimal control problems with exponential utility functions, the optimal investment and reinsurance strategies are derived for both insurers among all admissible policies. Finally, the influence of each model parameters on the optimal portfolio strategies are discussed by numerical experiments.

**Keywords** Compound Poisson process, Itô's formula, Optimization problem, Proportional reinsurance, Wealth process.

**AMS 2010 subject classifications** Primary 62G32; Secondary 62F99, 62E20.

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## 1. Introduction

The insurance companies are currently facing a variety of changes. On the one hand, the competition and cooperation among them become more and more frequent due to the acceleration of financial globalization. The risks they faced increasingly tend to be complex and diversified, which makes the connection among insurance businesses more and more stronger. Optimal reinsurance and investment problems for insurance companies (insurers) have attracted considerable attention from the fields of financial mathematics and actuarial science. This is may be attributed to the fact that reinsurance is an effective approach to manage risk exposures and is usually used to transfer and control risk because it allows insurance companies to provide more secure coverage with higher limits, while the investment is an important way to gain profits from insurers' surplus. Indeed, purchasing reinsurance can protect insurers against adverse claim experience, while investment further allows insurers to diversify their risks and enjoy a higher rate of return on the insurance portfolio's cash flows. Proportional reinsurance is one of the reinsurance arrangement, which means the insurer pays a proportion, say $q$, when the claim occurs and the remaining proportion, $1 - q$, is paid by the reinsurer. If the proportion a can be changed according to the risk position of the insurance company, this is the dynamic proportional reinsurance.
Deeply entrenched in the comprehensive body of literature on this research topic, the goal often consists in solving for the optimal reinsurance arrangement and investment decision to achieve a clearly defined objective (e.g., minimizing ruin probability or maximizing expected utility).

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Recently, most insurance companies manage their business by means of reinsurance and investment, which are effective way to spread risk and make profit. Therefore, these have inspired hundred researches. For instance, [1], [2] and [3], investigated the optimal problems for an insurance company in the case of minimizing the ruin probability. [4], [5], and [6] studied the optimal reinsurance and investment problems of expected utility maximization. The latest researches on insurance and investment management problem can be referred to [7], [8], [9], [10], [11], [12] and references therein.

[13] derived the explicit optimal investment and reinsurance policies for an insurer with the classical surplus process, where the financial market is driven by a drifted Brownian motion with coefficients modulated by an external Markov process. [14] considered the optimal investment and reinsurance strategies between an insurer and a reinsurer to maximize the terminal expected utility, and explicit expressions of the optimal investment and reinsurance strategies are derived. [15] obtained the robust equilibrium reinsurance and investment strategy for the insurer and reinsurer under weighted mean-variance criterion when the insurer can purchase proportional reinsurance from the reinsurer and both the insurer and reinsurer can invest in risk-free assets and risky assets, where the risky asset price is described by the constant elasticity of variance model. [16], derived the optimal reinsurance and investment problem between an insurer and a reinsurer where it is assumed that the insurer can purchase proportional reinsurance and both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset, in which the two risky assets are supposed to be correlated.

However, most of the literature mentioned above only considered one insurance company, while there are many insurance companies in the market in reality and they compete with each other. Thus, two insurance companies, a big one and a small one, are focused on in this paper.

[17] discussed the competition between two companies and contrasted a single payoff function which depended on both surplus processes of insurance companies. [18] investigated stochastic differential games between two insurance companies who employed the reinsurance to reduce risk exposure. [19] proposed a stochastic differential reinsurance game between two insurers with quadratic risk control processes, where the objective function of each insurer was to maximize the expected utility of its relative performance to his competitor. [20] defined an exit probability game between two competitive insurance companies who had quadratic surplus processes, and the value functions and Nash equilibrium strategies were obtained explicitly by solving Fleming-Bellman-Isaacs (FBI) equations. The work of [21] is arguably to study a reinsurance game model in a continuous-time model in which both insurers maximize the expected utility of their terminal surpluses.

[22] investigated stochastic differential reinsurance games between an insurer and a reinsurer, which allowed them to consider the benefits of both parties in the reinsurance contracts, under the expected utility maximization and mean-variance criteria, respectively. [23] obtained the optimal strategies for investment and reinsurance games between two insurance companies, when both insurers are allowed to purchase a proportional reinsurance contract and invest in risky and risk-free assets with nonlinear risk processes and Value-at-Risk constraints. [24] derived the optimal reinsurance-investment strategy so as to maximize the expected terminal wealth while minimizing the variance of the terminal wealth, when each insurer transfers part of the claims risk via reinsurance and invests the surplus in a financial market. [25] derived the equilibrium strategies for the reinsurance game between two competitive insurers Hamilton-Jacobi-Bellman (HJB) equations when each insurer is allowed to purchase proportional reinsurance, invest in a financial market consisting of a risk-free asset and a risky asset. [26] investigated two insurance companies who have a fixed amount of funds allocated as the initial surplus with consideration of the reinsurance strategies together with capital injections simultaneously for a diffusion approximation of classical Cramér-Lundberg model. The optimal capital injection strategies are studied based on the argument of surplus difference of two insurance risk process.

As far as we know, there is few research investigating more than one insurance company under the purchasing proportional reinsurance, a risk-free asset, and defaultable grade bond in a financial market. Therefore, in this paper, we provide an innovative study on a stochastic differential game played between two insurance companies. The objective function studied here is to maximize his expected utility of wealth process the difference between his terminal surplus and that of his competitor at a fixed time horizon $T$.

Through this paper, we suppose that both insurers having the compound Poisson process and they are allowed to purchase proportional reinsurance and invest in a financial market which consists of a risk-free asset, a defaultable

grade bond whose the price process of each insurer is governed by a standard Brownian motion and dynamics of defaultable bond price is represented as an exponential form of stochastic differential equation. Firstly, we describe the optimization problems and a verification theorem is necessary to guarantee that the solutions to HJB equation coincide with the objective functions. By solving the nonlinear HJB equations related to our optimal control problems, the optimal investment and reinsurance strategies are derived for both insurers. Finally, numerical examples are proposed to illustrate the impacts of model parameters on the strategies.
This paper proceeds as follows. In Section 2, we introduce the formulation of our risk insurance model. In addition to giving the dynamics of financial securities, we present the wealth process of each insurer based on reinsurance and investment strategies. Section 3 provides the optimization problem of each insurance company to maximize the expected utility of wealth process of the difference between his terminal surplus and that of his competitor at a fixed time horizon. In Section 4, the optimal reinsurance and investment strategies are derived for both insurers by solving the nonlinear HJB equations related to our optimal control problems with exponential utility functions. Section 5 gives a verification theorem which is necessary to guarantee that the solution to HJB equation coincide with the objective functions. The numerical examples are presented in Section 6. The conclusions are provided in Section 7.

## 2. Risk model setting and assumptions

In this paper, we suppose that all investments and assets are infinitely divisible and all assets are tradable continuously over time, without considering transaction costs or taxes. In our diffusion approximation process, both the insurers having the compound Poisson process and they are allowed to purchase proportional reinsurance.

### *2.1. Surplus process*

Let us start with a Cramér-Lundberg model, which is a classical actuarial model used to analyse the risk of an insurance portfolio. The company experiences two opposing cash flows: incoming premiums from the policyholder and outgoing claims. In the sequel, we will always work on the probability space $(\Omega, \mathcal{F}, P)$, which is endowed with the information filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ which carries all stochastic quantities and right continuity and it is often called an enlarged filtration given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. In the probability space, $P$ denotes the martingale probability measure. The filteration $\mathcal{G}_t$ is assumed to be generated by the Winner process which shows the change of stock price and the filteration $\mathcal{H}_t$ is shows a Poisson process that is used to denote the arrival of risks.
We assume that the insurance company, having an initial capital, cashes premiums continuously and pays claims of random sizes at random times. According to the Cramér-Lundberg model (also known as compound Poisson model or classical risk model), the surplus process $dR_k(t)$, $k = 1, 2$, of a homogeneous insurance portfolio can be described by

$$dR_k(t) = c_k dt - dS_k(t), \qquad t \geq 0, \tag{1}$$

with an initial deterministic surplus $R_k(0) = u_k^* \geq 0$ is the initial surplus, the surplus process increases linearly due to premiums that are collected continuously over time at a constant rate $c_k > 0$, and

$$S_1(t) = \sum_{i=1}^{M_1(t)} X_i, \qquad S_2(t) = \sum_{i=1}^{M_2(t)} Y_i,$$

are two compound Poisson processes with claim sizes $X_i$ and $Y_i$ with $\{X_i : i \geq 1\}$ and $\{Y_i : i \geq 1\}$ being the sequences of positive and identically distributed random variables, where $M_i(t)$ is the claim number process for the insurance $k$, $k = 1, 2$ and these two claim processes are correlated in the way

$$M_1(t) = N_1(t) + N(t), \qquad \mathit{and} \qquad M_2(t) = N_2(t) + N(t).$$

The processes $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are three mutually independent and homogeneous Poisson processes denoting the number of claims up to time $t$ with intensities $\lambda_1$, $\lambda_2$ and $\lambda$,

respectively. In addition, the claim sizes $\{X_i : i \geq 1\}$ and $\{Y_i : i \geq 1\}$ are two dependent classes of insurance businesses of the processes $N_1(t)$ and $N_2(t)$. Let $\{X_i : i \geq 1\}$ are the claim size random variables for the first company with a continuous distribution function $F_X$ and $\{Y_i : i \geq 1\}$ are the claim size random variables for the second company with a continuous distribution function $F_Y$. Their means are denoted by $\mu_1 = E(X_i)$ and $\mu_2 = E(Y_i)$, and the second moments are denoted by $\gamma_1 = E(X_i^2)$ and $\gamma_2 = E(Y_i^2)$, respectively. This risk model has been studied extensively in some literatures; see for example, [27] and [28].
Now, we approximate the compound Poisson risk processes in terms of reinsurance contract with the standard Brownian motions and constant reinsurance premiums. To spread risk in the portfolio, each insurer purchases proportional reinsurance.
More precisely, we allow the insurance company $k$ to continuously reinsure a fraction of its claim with the retention levels $q_k(t) \in [0, 1]$ at time $t$. Let the constant reinsurance premium rate at time $t$ be $\eta q_k(t)$. Then the corresponding risk process (1) for insurer $k$, $k = 1, 2$, in term of dynamic proportional reinsurance becomes:

$$dR_k^{q_k}(t) = \big(c_k - \eta q_k(t)\big)dt - q_k(t)dS_k(t), \qquad t \geq 0. \tag{2}$$

Since $q_k(t) \in [0, 1]$ is the proportional reinsurance for the insure $k$, therefore, $1 - q_k(t)$ is the proportional reinsurance to the reinsurance company. Moreover, for each insurance company $\eta q_k(t)$ will be calculated by the variance principle.
From [29], the compound Poisson $S_k(t)$, $k = 1, 2$, can be approximated by the following Brownian motion:

$$S_k(t) = a_k t - \zeta_k B_k(t), \qquad t \geq 0, \tag{3}$$

with $a_1 = (\lambda_1 + \lambda)E(X_i)$, $a_2 = (\lambda_2 + \lambda)E(Y_i)$, $\zeta_1^2 = (\lambda_1 + \lambda)E(X_i^2)$ and $\zeta_2^2 = (\lambda_2 + \lambda)E(Y_i^2)$. Moreover, the insurer $k$ should pay a reinsurance premium at the rate $\eta q_k(t) = \big(1 - q_k(t)\big)a_k + \nu\big(1 - q_k(t)\big)^2\zeta_k^2$, where $\nu > 0$ is a safety loading, $B_1(t)$ and $B_2(t)$ are standard Brownian motions with the correlation coefficient

$$\rho = \frac{\lambda E(X_i)E(Y_i)}{\sqrt{(\lambda_1 + \lambda)E(X_i^2)}\sqrt{(\lambda_2 + \lambda)E(Y_i^2)}} = \frac{\lambda\mu_1\mu_2}{\zeta_1\zeta_2}.$$

Therefore, $E\big(B_1(t)B_2(t)\big) = \rho t$. From the compound Poisson $S_k(t)$, $k = 1, 2$, given in (3), the dynamic proportional reinsurance (2) can be written as

$$dR_k^{q_k}(t) = \big(c_k - \eta q_k(t) - a_k q_k(t)\big)dt + \zeta_k q_k(t)dB_k(t), \qquad t \geq 0. \tag{4}$$

### *2.2. Dynamics portfolio choice of financial securities*

In this subsection, we assume that each insurance company has access to the risk-free asset, firstly we define the price process as a mixture of the exponential stochastic differential equation of corporate coupon bonds, then by applying the Girsanov's theorem and compensated Poisson process, we give the dynamic of corporate coupon bonds as the exponential form of stochastic differential equation.
Assume that the each insurance company is allowed to invest all its surplus in a risk-free asset with constant positive interest rate $r$, i.e. the company has access to the risk-free asset $P_t$ with

$$P_0(t) = rP_0(t)dt. \tag{5}$$

Now, we consider the stock allocation in a stochastic volatility model. As Section 2 given in [30], we consider the following stock price portfolio selection model:

$$dP(t) = P(t)\big((r + \omega Y(t))dt + \sqrt{Y(t)}dW_1(t)\big),$$

where the volatility process $Y(t)$ is a square-root process and it is given by

$$dY(t) = \alpha(\beta - Y(t))dt + \kappa\sqrt{Y(t)}dW_2(t),$$

$r$ is a positive constant price of interest risk, $W_1(t)$ and $W_2(t)$ are standard Brownian motions with $E\big(dW_1(t)dW_2(t)\big) = \hat{\rho}dt$, $P(0) = p > 0$, $Y(t) = y > 0$, for $0 \leq t \leq T$, $\omega > 0$, $\beta > 0$ is the long-run average of the variance process and $\kappa > 0$ is the specific time period which measures of that variance bounded and it is called the volatility of variance. Moreover, to ensure that $Y(t)$ is non-negative almost surely, we need $2\alpha\beta \geq \kappa^2$.

**Definition 2.1** (Default process). A nondecreasing right continuous process which makes discrete jumps at a random time $\tau$ is called default process and denotes by $H(t) = I_{\{\tau \leq t\}}$, where $I$ represents the indicator function which takes the value of one if there is a jump and zero otherwise.

**Definition 2.2**. The default process $\{H(t)\}_{t \geq 0}$ is said to be a Poisson distribution with intensity $h$ if the following conditions hold:

i) $H(0) = 0$,

ii) $\{H(t)\}_{t \geq 0}$ has the independent increments.

iii) The number of events in any interval of length $t$ has a Poisson distribution with mean $ht$. That is, for all $s, t \geq 0$,

$$P\big(H(t+s) - H(s) = n\big) = e^{-ht}\frac{(ht)^n}{n!}, \qquad n \geq 0.$$

Here, $ht$ represents the intensity of the Poisson process, which measures the arrival rate of a default. The mean arrival rate $ht$ is called the compensator for the Poisson process. We now define a compensated Poisson process.

**Definition 2.3**. The martingale default process which is given by the equation:

$$C(t) = H(t-h)\int_0^t (1 - H(z))dz,$$

is a compensated Poisson process. In addition, the stochastic differential equation of this compensated Poisson process is defined by $dC(t) = dH(t) - h(1 - H(t))dt$. We extensively use this equation to calculate the dynamics of corporate coupon bonds.

We assume that there exists a corporate coupon bond with a maturity date $T_1$ and the amount of defaultable bond after default is considered zero. We will try to model the price process in terms of this coupon bond under the real word probability measure $P$. We let $\tau$ denotes the first time of a Poisson process with constant jump intensity $h^P$ under the martingale probability measure $P$. In the event of default, the investor recovers a fraction of the market value of the prepaid bond immediately prior to default. We propose a new financial model called a mixture of corporate coupon bonds, which has a constant loss rate of the corporate coupon bond and a credit spread. Let $\delta \in (0,1)$ denotes the constant loss rate of the corporate coupon bond, then we define the price process of defaultable bond as a mixture of the exponential stochastic differential equation of corporate coupon bonds as

$$p(t, T_1) = I_{\{\tau > t\}}e^{-(r+\theta)(T_1 - t)} + I_{\{\tau \leq t\}}(1-\delta)e^{-(r+\theta)(T_1-\tau)}e^{r(t-\tau)}, \tag{6}$$

where $\theta = \delta h^P$ is the credit spread under the real word martingale probability measure.

*Lemma 1*

For the constant positive interest rate $r$, the price process of defaultable bond in (6) under the martingale probability measure $P$ is given by

$$dp(t, T_1) = rp(t, T_1)dt - \delta e^{-(r+\theta)(T_1-t)}dC_t^P, \tag{7}$$

where $C^P(t) = H(t) - \gamma h^P \int_0^t (1 - H(u))du$ is a compensated jump process and $\gamma \in (0,1)$ is the constant default risk premium.

*Proof*

To prove this Lemma, use the Itô's formula on equation (6). □

*Lemma 2*

For the constant positive interest rate $r$, the price process of defaultable bond in equation (7) under the martingale probability measure $P$ can be represented as the exponential stochastic differential equation as follows:

$$dp(t,T_1) = p(t,T_1)\big(rdt + (1-H(t))\theta(1-\gamma)dt - (1-H(t))\delta dC^P(t)\big), \tag{8}$$

where we use

$$p(t,T_1) = \begin{cases} e^{-(r+\theta)(T_1-t)}, & \text{if } \tau > t, \\ \\ (1-\delta)e^{-(r+\theta)(T_1-\tau)}e^{r(t-\tau)}, & \text{if } \tau \le t. \end{cases}$$

*Proof*
Using the Girsanov's theorem for equation (7), the Lemma will be proved. The proof of this Lemma is standard, and is thus omitted for simplicity. □

### *2.3. Dynamics of wealth processes in a defaultable financial market*

In this subsection, we consider a new defaultable financial market for both insurers. It is assumed that both insurers are allowed to purchase proportional reinsurance with a constant reinsurance premium and invest in a financial market which consists of a risk-free asset, and purchases reinsurance contracts from the same reinsurance company in a fixed time horizon $T$, where we assume that $T < T_1$. Under this new defaultable financial market, the dynamics of wealth process of insurer $k$, $k = 1, 2$, is defined by

$$\begin{aligned} dD_k^{\pi_k}(t) &= \frac{D_k^{\pi_k}(t) - s_k(t) - e_k(t)}{P_0(t)} dP_0(t) + \frac{s_k(t)}{P(t)} dP(t) + \frac{e_k(t)}{p(t,T_1)} dp(t,T_1) \\ &\quad + \big(c_k - \eta q_k(t) - q_k(t)a_k\big)dt + \zeta_k q_k(t) dB_k(t) \\ &= \Big(rD_k^{\pi_k}(t) + \big(c_k - \eta q_k(t) - q_k(t)a_k\big) + s_k(t)\omega Y(t) + e_k(t)(1-H(t))\theta(1-\gamma)\Big)dt \\ &\quad + s_k(t)\big(Y(t)\big)^{\frac{1}{2}} dW_1(t) + \zeta_k q_k(t) dB_k(t) - e_k(t)\delta(1-H(t))dC^P(t), \end{aligned} \tag{9}$$

where $s_k(t)$ and $e_k(t)$ represent the amounts of insurer $k's$ wealth process invested in the stock and corporate bond, respectively. Let $\pi_k(t) = \big(q_k(t), s_k(t), e_k(t)\big)$ be the investment and reinsurance strategy followed by insurance company $k$, $k = 1, 2$.

**Definition 2.4**. For $t \in [0, T]$, the triple process $\pi_k(t)$, $k = 1, 2$, is an admissible strategy if the following conditions hold:

$i)$ $\pi_k(t)$ is a $\mathcal{F}_t$-measurable process.

$ii)$ $\int_0^T \big(\zeta_k^2 q_k^2(t) + s_k^2(t)Y(t)\big)dt < \infty$,

$iii)$ Under the triple process $\pi_k(t)$, the stochastic differential equation (9) has an unique solution.

Moreover, we assume that $\Pi_k$ denotes of all admissible strategies.

## 3. Optimization problems

In this section, we model the competition of two insurance companies based on the objective functions. We give two objective functions based on the smooth utility functions for two insurers, default process and the dynamics of wealth process to present the optimization problems under the given volatility process. Thus, the competition between the two companies formulates a game with two players, each of which can adjust his reinsurance and wealth process strategies based on his competitor's scheme.

Assume that $D_k^{\pi_k}(t) = d_k$, $k = 1, 2$, $Y(t) = y > 0$, and $H(t) = h$, for $h = \{0, 1\}$, then we consider two objective functions $O_1^{(\pi_1,\pi_2)}(t, d_1, d_2, y, h)$ and $O_2^{(\pi_1,\pi_2)}(t, d_1, d_2, y, h)$ for the insurance companies 1 and 2, respectively, as follow:

$$
\begin{aligned}
&O_1^{(\pi_1,\pi_2)}(t,d_1,d_2,y,h)\\
&= E\big(U_1\big((1-z_1)D_1^{\pi_1}(T)+z_1(D_1^{\pi_1}(T)-D_2^{\pi_2}(T))\big)\big|(D_1^{\pi_1}(t),D_2^{\pi_2}(T),Y(t),H(t))=(d_1,d_2,y,h)\big)\\
&= E\big(U_1\big(D_1^{\pi_1}(T)-z_1D_2^{\pi_2}(T)\big)\big|(D_1^{\pi_1}(t),D_2^{\pi_2}(T),Y(t),H(t))=(d_1,d_2,y,h)\big),
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&O_2^{(\pi_1,\pi_2)}(t,d_1,d_2,y,h)\\
&= E\big(U_2\big((1-z_2)D_2^{\pi_2}(T)+z_2(D_2^{\pi_2}(T)-D_1^{\pi_1}(T))\big)\big|(D_2^{\pi_2}(t),D_1^{\pi_1}(T),Y(t),H(t))=(d_1,d_2,y,h)\big)\\
&= E\big(U_2\big(D_2^{\pi_2}(T)-z_2D_1^{\pi_1}(T)\big)\big|(D_2^{\pi_2}(t),D_1^{\pi_1}(T),Y(t),H(t))=(d_2,d_1,y,h)\big),
\end{aligned} \tag{11}
$$

where $U_1$ and $U_2$ are the strictly increasing and strictly concave smooth utility function for insurers 1 and 2, respectively, (i.e., $U_k' > 0$ and $U_k'' < 0$, for $k=1,2$), the parameter $z_k \in [0,1]$, $k=1,2$, denotes the performance relative of insurer $k$ to its competitor $j$, $j \neq k \in \{1,2\}$.

The optimization problems are to determine the optimal wealth process of each insurance companies which are invested in the reinsurance strategies, stocks and corporate bonds. Reinsurance optimization problems have been extensively studied under various criteria, such as minimizing risk measures in the static setting and minimizing ruin probabilities in the dynamic setting, with a focus on maintaining company stability. However, in this study, our objective is to explore reinsurance strategies together with stocks and corporate bonds simultaneously two insurance companies according to the classical Cramér-Lundberg model. In our risk optimization problem, we find the triple of estimator for reinsurance, stocks and corporate bonds.

**Problem 3.1.** Find a Nash equilibrium $(\pi_1^*,\pi_2^*) = \big(q_1^*(t),s_1^*(t),e_1^*(t);q_2^*(t),s_2^*(t),e_2^*(t)\big) \in \Pi_1\times\Pi_2$, such that the inequalities

$$
O_1^{(\pi_1^*,\pi_2^*)}(t,d_1,d_2,y,h) \geq O_1^{(\pi_1,\pi_2^*)}(t,d_1,d_2,y,h), \tag{12}
$$

and

$$
O_2^{(\pi_1^*,\pi_2^*)}(t,d_1,d_2,y,h) \geq O_2^{(\pi_1^*,\pi_2)}(t,d_1,d_2,y,h), \tag{13}
$$

hold. When we have the inequalities (12) and (13), then the objective functions of insurer 1 and 2 can be defined as follow:

$$
O_1(t,d_1,d_2,y,h) = O_1^{(\pi_1^*,\pi_2^*)}(t,d_1,d_2,y,h) = \sup_{\pi_1\in\Pi_1} O_1^{(\pi_1,\pi_2^*)}(t,d_1,d_2,y,h),
$$

and

$$
O_2(t,d_1,d_2,y,h) = O_2^{(\pi_1^*,\pi_2^*)}(t,d_1,d_2,y,h) = \sup_{\pi_2\in\Pi_2} O_2^{(\pi_1^*,\pi_2)}(t,d_1,d_2,y,h).
$$

According to the Problem 3.1, we will try to find the admissible strategies $\pi_1^*$ and $\pi_2^*$ as the competitively optimal investment and reinsurance strategies.

To solve the Problem 3.1, first by defining the equality $\hat{D}_k^{\pi_k}(t) = D_k^{\pi_k}(t) - z_k D_j^{\pi_j}(t)$, for $j \neq k \in \{1,2\}$, we have the stochastic differential equation

$$
\begin{aligned}
d\hat{D}_k^{\pi_k}(t) \quad = \quad & \big[r\hat{D}_k^{\pi_k}(t) + (c_k - z_k c_j) - \big(\eta q_k(t) - z_k\eta e_j(t)\big) - \big(q_k(t)a_k - z_k e_j(t)a_j\big)\\
&+\big(s_k(t)-z_k s_j(t)\big)\omega Y(t) + \big(e_k(t)-z_k e_j(t)\big)\big(1-H(t)\big)\theta(1-\gamma)dt\big]\\
&+\big(s_k(t)-z_k s_j(t)\big)\sqrt{Y(t)}dW_1(t) + \zeta_k q_k(t)dB_k(t) - z_k\zeta_j e_j(t)dB_j(t)\\
&-\big(e_k(t)-z_k e_j(t)\big)\delta(1-H(t))dC^P(t),
\end{aligned} \tag{14}
$$

with $\hat{D}_k^{\pi_k}(t) = d_k - z_k w_j$, for $0 \leq t \leq T$ and $k \neq j \in \{0,1\}$. Moreover, for $Y(t) = y$ and $H(t) = h$, let for $k = 1, 2$,

$$O_k(t, \hat{d}_k, y, h) = \sup_{\pi_k \in \Pi_k} E\big(U_k\big(D_k^{\pi_k}(T) - d_k D_j^{\pi_j^*}(T)\big)\big|\hat{D}_k^{\pi_k}(t) = \hat{d}_k, Y(t) = y, H(t) = h\big),$$

be the objective function for insurer $k$.

## 4. Optimal investment and reinsurance strategies

In this section, the optimal reinsurance and investment strategies are derived for both insurers by solving the nonlinear Hamilton-Jacobi-Bellman equations related to our optimal control problems with the following exponential utility functions:

$$U_k(\hat{d}_k) = -\frac{1}{e_k}\exp\big\{e_k \hat{d}_k\big\}, \qquad k = 1, 2,$$

where $e_k$ is a positive constant absolute risk aversion coefficient. Using the standard dynamic programming techniques, the objective function $O_k$, $k = 1, 2$, satisfies the HJB equation:

$$\begin{cases} \sup_{\pi_k \in \Pi_k} \mathcal{B}_k^{\pi_k} O_k(t, \hat{d}_k, y, h) = 0, \\ \\ O_k(T, \hat{d}_k, y, h) = U_k(\hat{d}_k), \end{cases} \tag{15}$$

for all $t \in [0, T]$, where $\mathcal{B}_k^{\pi_k}$ is the following nonlinear HJB equation:

$$\begin{aligned} \mathcal{B}_k^{\pi_k} O_k(t, \hat{d}_k, y, h) \quad = \quad & \frac{\partial O_k(t, \hat{d}_k, y, h)}{\partial t} + \Big\{\big[r\hat{d}_k + (c_k - z_k c_j) - \big(\eta q_k(t) - z_k \eta e_j^*(t)\big) - \big(q_k(t) a_k - z_k e_j(t) a_j\big) \\ & + \big(s_k(t) - z_k s_j(t)\big)\omega y + \big(e_k(t) - z_k e_j^*(t)\big)\big(1-h\big)\theta\big]\frac{\partial O_k(t, \hat{d}_k, y, h)}{\partial \hat{d}_k} \\ & + \frac{1}{2}\big(s_k(t) - z_k s_j^*(t)\big)^2 y + \zeta_k q_k^2(t) - z_k^2 \zeta_j^2 e_j^*(t) - 2\rho z_k \zeta_k \zeta_j q_k(t) q_j^*(t)\big]\frac{\partial^2 O_k(t, \hat{d}_k, y, h)}{\partial \hat{d}_k^2} \\ & + \alpha(\beta - y)\frac{\partial O_k(t, \hat{d}_k, y, h)}{\partial y} + \frac{1}{2} y \kappa^2 \frac{\partial^2 O_k(t, \hat{d}_k, y, h)}{\partial y^2} \\ & + \hat{\rho}\big(s_k(t) - z_k s_j^*(t)\big)\kappa y \frac{\partial^2 O_k(t, \hat{d}_k, y, h)}{\partial \hat{d}_k \partial y} \\ & + \big(O_k(t, \hat{d}_k - \big(e_k(t) - z_k e_j^*(t)\big)\delta, y, h+1) - O_k(t, \hat{d}_k, y, h)\big) h^P (1-h)\Big\}. \end{aligned}$$

We will solve this nonlinear HJB equation based on the discrete jump values of default process $H(t)$ which occur at the random time $\tau$ until the finite time $T$.
To do this, we divide the time space into two parts $t \in [\tau \wedge T, T]$ and $t \in [0, \tau \wedge T]$, where $\tau \wedge T = \min\{\tau, T\}$, and consider the objective function $O_k(t, \hat{d}_k, y, h)$, $k = 1, 2$, with the following form:

$$O_k(t, \hat{d}_k, y, h) = \begin{cases} O_k(t, \hat{d}_k, y, 1), & \text{if} \quad h = 1 \text{ and the first time is } t \in [\tau \wedge T, T], \\ \\ O_k(t, \hat{d}_k, y, 0), & \text{if} \quad h = 0 \text{ and the second time is } t \in [0, \tau \wedge T]. \end{cases}$$

### *4.1. Optimal reinsurance and investment strategies for the first space*

In this subsection, we will obtain the optimal reinsurance and investment strategies for $t \in [\tau \wedge T, T]$. Theorem 4.1 describes these optimal strategies and it's associated objective functions.

*Theorem 4.1*
For the dynamics of wealth process of insurer (9), when $t \in [\tau \wedge T, T]$, the optimal investment strategy of stock asset for $k = 1, 2$, is given by

$$s_k^*(t) = \frac{\hat{s}_k^*(t) + z_k \hat{s}_j^*(t)}{1 - z_k z_j}, \qquad k \neq j \in \{1, 2\}, \tag{16}$$

where $\hat{s}_k^*(t) = \big(\frac{\omega}{e_k} + \frac{g_k(t)\hat{\rho}\kappa}{e_k}\big)e^{-r(T-t)}$. Then the optimal investment strategy of corporate bond for $k = 1, 2$, is given by

$$e_k^*(t) = 0.$$

To obtain the optimal reinsurance strategy $q_k^*(t)$, $k = 1, 2$, we define the equations

$$\hat{q}_1^*(t) = \frac{y_1(t) + x_1(t)y_2(t)}{1 - x_1(t)x_2(t)} \qquad \textit{and} \qquad \hat{q}_2^*(t) = \frac{y_2(t) + x_2(t)y_1(t)}{1 - x_1(t)x_2(t)}, \tag{17}$$

where $x_1(t) = \frac{\omega\mu_1\mu_2 z_1 e_1 e^{r(T-t)}}{e_1\zeta_1^2 e^{r(T-t)} + 2\zeta_1^2\nu}$, $x_2(t) = \frac{\omega\mu_1\mu_2 z_2 e_2 e^{r(T-t)}}{e_2\zeta_2^2 e^{r(T-t)} + 2\zeta_2^2\nu}$, $y_1(t) = \frac{2\nu}{e_1 e^{r(T-t)} + 2\nu}$ and $y_2(t) = \frac{2\nu}{e_2 e^{r(T-t)} + 2\nu}$.
Then the optimal reinsurance strategy $q_k^*(t)$, $k = 1, 2$, is given by

$$q_k^*(t) = \hat{q}_k^*(t) \wedge 1. \tag{18}$$

Moreover, the objective function is given by

$$O_k(t, \hat{d}_k, y, 1) = -\frac{1}{e_k}\exp\big\{ - e_k\hat{d}_k e^{r(T-t)} + l_k(t) + y g_k(t)\big\},$$

where

$$\begin{aligned} l_k(t) \quad &= \quad \int_t^T \Big((-e_k e^{r(T-u)})\big[(c_k - z_k c_j) - \eta q_k^*(u) - z_k\eta q_j^*(u) - a_k q_k^*(u) - z_k a_j q_j^*(u)\big] \\ &+ \frac{1}{2}e_k^2 e^{2r(T-u)}\big[\zeta_1^2 q_k^{*2}(u) + z_k^2\zeta_j^2 q_j^{*2}(u) - 2\rho z_k\zeta_k\zeta_j q_k^*(u)q_j^*(u)\big] + \alpha\beta g_k(u)\Big)du, \end{aligned}$$

and

$$g_k(t) = \begin{cases} \frac{e^{(\Lambda_1-\Lambda_2)(T-t)}}{e^{(\Lambda_1-\Lambda_2)(T-t)} - \frac{\Lambda_2}{\Lambda_1}}\Lambda_2, & \hat{\rho} \neq \pm 1, \\ -\frac{\omega^2}{2(\alpha+\kappa\omega)}\big(1 - e^{-(\alpha+\kappa\omega)(T-t)}\big), & \hat{\rho} = 1, \\ -\frac{\omega^2}{2(\alpha-\kappa\omega)}\big(1 - e^{-(\alpha-\kappa\omega)(T-t)}\big), & \hat{\rho} = -1 \text{ and } \alpha \neq \kappa\omega, \\ -\frac{\omega^2}{2}(T-t), & \hat{\rho} = -1 \text{ and } \alpha = \kappa\omega, \end{cases} \tag{19}$$

where

$$\omega_1 = \frac{\alpha + \hat{\rho}\kappa\omega + \big(\alpha^2 + \kappa^2\omega^2 + 2\hat{\rho}\kappa\omega\alpha\big)^{\frac{1}{2}}}{2},$$

and

$$\omega_2 = \frac{\alpha + \hat{\rho}\kappa\omega - \big(\alpha^2 + \kappa^2\omega^2 + 2\hat{\rho}\kappa\omega\alpha\big)^{\frac{1}{2}}}{2}.$$

*Proof*
The proof of this theorem is similar to that in Theorem 4.2, therefore, we omit it. With the additional explanation that in the defaultable financial market when $t \in [\tau \wedge T, T]$, the optimal investment strategy of corporate bond is not tradable, i.e., $e_k^*(t) = 0$, $k = 1, 2$, and this completes the proof. □

### *4.2. Optimal reinsurance and investment strategies for the second space*

In this subsection, we will obtain the optimal reinsurance and investment strategies for $t \in [0, \tau \wedge T]$. Using the first order conditions for a regular interior monimizer, Theorem 4.2 describes these optimal strategies and it's associated objective functions.

*Theorem 4.2*
For the dynamics of wealth process of insurer (9), when $t \in [0, \tau \wedge T]$, the optimal investment strategy of corporate bond for $k = 1, 2$, is given by

$$e^*_k(t) = \frac{\hat{e}^*_k(t) + z_k \hat{e}^*_j(t)}{1 - z_k z_j}, \qquad k \neq j \in \{1, 2\}, \tag{20}$$

where $\hat{e}^*_k(t) = \frac{\left(\ln\frac{1}{\gamma}+\gamma-1\right)e^{-\frac{\theta}{\delta}(T-t)}-\gamma+1}{e_k\delta}e^{r(T-t)}$, $k = 1, 2$. The optimal investment strategy of stock asset and reinsurance strategy are given in (16) and (18), respectively.
Moreover, the objective function is given by

$$O_k(t, \hat{d}_k, y, 0) = -\frac{1}{e_k}\exp\big\{ - e_k \hat{d}_k e^{r(T-t)} + l_k(t) + V_k(t) + y g_k(t)\big\} = O_k(t, \hat{d}_k, y, 1)e^{V_k(t)},$$

where

$$V_k(t) = \big(\ln\frac{1}{\gamma} + \gamma - 1\big)e^{-\frac{\theta}{\delta}(T-t)} - \ln\frac{1}{\gamma} - \gamma + 1.$$

*Proof*
When $h = 0$, the HJB equation (15) becomes

$$\begin{aligned} 0 \;=\; & \frac{\partial O_k(t, \hat{d}_k, y, 0)}{\partial t} + \Big\{\big[r\hat{d}_k + (c_k - z_k c_j) - \big(\eta q_k(t) - z_k \eta e^*_j(t)\big) - \big(q_k(t)a_k - z_k e_j(t)a_j\big) \\ & + \big(s_k(t) - z_k s_j(t)\big)\omega y + \big(e_k(t) - z_k e^*_j(t)\big)\theta\big]\frac{\partial O_k(t, \hat{d}_k, y, 0)}{\partial \hat{d}_k} \\ & + \frac{1}{2}\big(s_k(t) - z_k s^*_j(t)\big)^2 y + \zeta_k q^2_k(t) - z^2_k \zeta^2_j e^*_j(t) - 2\rho z_k \zeta_k \zeta_j q_k(t) q^*_j(t)\big]\frac{\partial^2 O_k(t, \hat{d}_k, y, 0)}{\partial \hat{d}^2_k} \\ & + \alpha(\beta - y)\frac{\partial O_k(t, \hat{d}_k, y, 0)}{\partial y} + \frac{1}{2}y\kappa^2\frac{\partial^2 O_k(t, \hat{d}_k, y, 0)}{\partial y^2} \\ & + \hat{\rho}\big(s_k(t) - z_k s^*_j(t)\big)\kappa y\frac{\partial^2 O_k(t, \hat{d}_k, y, 0)}{\partial \hat{d}_k \partial y} \\ & + \big(O_k(t, \hat{d}_k - \big(e_k(t) - z_k e^*_j(t)\big)\delta, y, 1) - O_k(t, \hat{d}_k, y, 0)\big)h^P\Big\}. \end{aligned} \tag{21}$$

To solve this equation, we guess that the objective function has the following form:

$$O_k(t, \hat{d}_k, y, 0) = -\frac{1}{e_k}\exp\big\{ - e_k \hat{d}_k e^{r(T-t)} + l_{0k}(t) + y g_{0k}(t)\big\}, \tag{22}$$

where $l_{0k}(t)$ and $g_{0k}(t)$ are two functions which will be determined later.
Using equation (22), then the equation (21) leads to the differential equation:

$$\begin{aligned} 0 \;=\;& l'_{0k}(t)+g'_{0k}(t)+\alpha(\beta-y)+\frac{1}{2}y\kappa^2 g_{0k}^2 t)+\inf_{q_k(t)}\Big\{\big(-e_k e^{r(T-t)}\big)\big[(c_k-z_kc_j)-\big(\eta q_k(t)-\eta z_k q_j^*(t)\big)\\ &-\big(a_kq_k(t)-z_ka_je_j^*(t)\big)\big]+\frac{1}{2}e_k^2e^{2r(T-t)}\big[\zeta_k^2q_k^2(t)+z_k\zeta_j^2q_j^*(t)-2\rho z_k\zeta_k\zeta_jq_k(t)q_j^*(t)\big]\Big\}\\ &+\inf_{s_k(t)}\Big\{\big(s_k(t)-z_ks_j^*(t)\big)\omega y\big(-e_ke^{r(T-t)}\big)+\frac{1}{2}\Big[\big(s_k(t)-z_ks_k^*(t)\big)^2ye_k^2e^{2r(T-t)}\\ &-2g_{0k}(t)\big(s_k(t)-z_ks_j^*(t)\big)\hat{\rho}\kappa ye_ke^{r(T-t)}\Big]\Big\}+\inf_{e_k(t)}\Big\{\big(-e_ke^{r(T-t)}\big)\big(e_k(t)-z_ke_j^*(t)\big)\theta\\ &+\Big(\exp\Big[e_k\big(e_k(t)-z_ke_j^*(t)\big)\delta e^{r(T-t)}\big(l_k(t)-l_{0k}(t)\big)+\big(g_k(t)-g_{0k}(t)\big)y\Big]-1\Big)h^P\Big\}. \end{aligned} \tag{23}$$

Using the first order conditions for a regular interior minimizer of equation (23), we get

$$\begin{cases} q_1^*(t)=\Big(\frac{2\nu}{e_1e^{r(T-t)}+2\nu}+\frac{\rho z_1\zeta_2e_1e^{r(T-t)}}{e_1\zeta_1e^{r(T-t)}+2\zeta_1\nu}q_2^*(t)\Big)\wedge 1,\\ q_2^*(t)=\Big(\frac{2\nu}{e_2e^{r(T-t)}+2\nu}+\frac{\rho z_2\zeta_1e_2e^{r(T-t)}}{e_2\zeta_2e^{r(T-t)}+2\zeta_2\nu}q_1^*(t)\Big)\wedge 1, \end{cases} \tag{24}$$

$$\begin{cases} s_1^*(t)=e^{r(T-t)}\Big(\frac{\omega}{e_1}+\frac{g_{01}(t)\hat{\rho}\kappa}{e_1}\Big)+z_1s_2^*(t),\\ s_2^*(t)=e^{r(T-t)}\Big(\frac{\omega}{e_2}+\frac{g_{02}(t)\hat{\rho}\kappa}{e_2}\Big)+z_2s_1^*(t), \end{cases}$$

and

$$\begin{cases} e_1^*(t)=e^{r(T-t)}\frac{\ln\frac{1}{\gamma}+\big(l_{01}(t)-l_1(t)\big)+y\big(g_{01}-g_1(t)\big)}{e_1\delta}+z_1e_2^*(t),\\ e_2^*(t)=e^{r(T-t)}\frac{\ln\frac{1}{\gamma}+\big(l_{02}(t)-l_2(t)\big)+y\big(g_{02}-g_2(t)\big)}{e_2\delta}+z_2e_1^*(t). \end{cases}$$

Therefore, the optimal investment strategy of stock asset and optimal investment strategy of corporate bond for $k=1,2$, can be computed as given in relations (16) and (20). To obtain the optimal reinsurance strategy $q_k^*(t)$, $k=1,2$, first we consider the equations (17), which are the solution of equations:

$$\hat{q}_1^*(t)=\frac{2\nu}{e_1e^{r(T-t)}+2\nu}+\frac{\rho z_1\zeta_2e_1e^{r(T-t)}}{e_1\zeta_1e^{r(T-t)}+2\zeta_1\nu}\hat{q}_2^*(t),$$

and

$$\hat{q}_2^*(t)=\frac{2\nu}{e_2e^{r(T-t)}+2\nu}+\frac{\rho z_2\zeta_1e_2e^{r(T-t)}}{e_2\zeta_2e^{r(T-t)}+2\zeta_2\nu}\hat{q}_1^*(t).$$

On the other hand, since $0\leq z_k\leq 1$, $\zeta>0$, $\gamma>0$, $e_k>0$ and $-1<\rho<1$, then we have

$$1-x_1(t)x_2(t)=1-\frac{\rho^2z_1z_2\zeta_1\zeta_2e_1e_2e^{r(T-t)}}{\big(e_1\zeta_1e^{r(T-t)}+2\zeta_1\nu\big)\big(e_2\zeta_2e^{r(T-t)}+2\zeta_2\nu\big)}\geq 0,$$

therefore, for $k=1,2$, $\hat{q}_k^*(t)\geq 0$.
Then we consider the following four different cases for $\hat{q}_k^*(t)$, $k=1,2$:
Case 1) If for $k=1,2$, $\hat{q}_k^*(t)\leq 1$, then $q_k^*(t)=\hat{q}_k^*(t)$.
Case 2) If $\hat{q}_1^*(t)\leq 1$ and $\hat{q}_2^*(t)>1$, then $q_k^*(t)=1$, and from (24), we have the following equality

$$\frac{2\nu}{e_1e^{r(T-t)}+2\nu}+\frac{\rho z_1\zeta_2e_1e^{r(T-t)}}{e_1\zeta_1e^{r(T-t)}+2\zeta_1\nu}=\frac{2\nu}{e_1e^{r(T-t)}+2\nu}+\frac{\rho^2\mu_1\mu_2z_1e_1e^{r(T-t)}}{e_1\zeta_1^2e^{r(T-t)}+2\zeta_1^2\nu}.$$

Case 3) If $\hat{q}_1^*(t) > 1$ and $\hat{q}_2^*(t) \le 1$, then it is clear that

$$\big(q_1^*(t), q_2^*(t)\big) = \big(1, \frac{2\nu}{e_2 e^{r(T-t)} + 2\nu} + \frac{\omega\mu_1\mu_2 z_2 e_2 e^{r(T-t)}}{e_2\zeta_2^2 e^{r(T-t)} + 2\nu\zeta_2^2}\big).$$

Case 4) If for $k = 1, 2$, $\hat{q}_k^*(t) > 1$, then $q_1^*(t) = q_2^*(t) = 1$, and we conclude that

$$q_k^*(t) = \hat{q}_k^*(t) \wedge 1, \qquad k = 1, 2.$$

Putting these optimal values in (23), we obtain

$$\begin{aligned} 0 \;=\; & l'_{0k}(t) - \big(e_k e^{r(T-t)}\big)\big[(c_k - z_k c_j) - \big(\eta q_k(t) - \eta z_k q_j^*(t)\big) - \big(a_k q_k(t) - z_k a_j e_j^*(t)\big)\big] \\ & + \frac{1}{2} e_k^2 e^{2r(T-t)}\big[\zeta_k^2 q_k^{2*}(t) + z_k^2\zeta_j^2 q_j^{2*}(t) - 2\rho z_k\zeta_k\zeta_j q_k^*(t) q_j^*(t)\big] + \alpha\beta g_{0k}(t) \\ & - \frac{\theta}{\delta}\ln\frac{1}{\gamma} + h^P\big(\frac{1}{\gamma} - 1\big) + \frac{\theta\big(l_k(t) - l_{0k}(t)\big)}{\delta} + y\big(g'_{0k}(t) + \frac{1}{2}\kappa^2(1-\hat{\rho}^2) g_{0k}^2(t) \\ & - (\hat{\rho}\omega\kappa + \alpha) g_{0k}(t) + \frac{\theta\big(g_k(t) - g_{0k}(t)\big)}{\delta} - \frac{\omega^2}{2}\big) \end{aligned} \tag{25}$$

The equation (25) can divide into two differential equations as follow:

$$\begin{aligned} & l'_{0k}(t) - \frac{\theta l_{0k}(t)}{\delta} - \big(e_k e^{r(T-t)}\big)\big[(c_k - z_k c_j) - \big(\eta q_k(t) - \eta z_k q_j^*(t)\big) - \big(a_k q_k(t) - z_k a_j e_j^*(t)\big)\big] \\ & + \frac{1}{2} e_k^2 e^{2r(T-t)}\big[\zeta_k^2 q_k^{2*}(t) + z_k^2\zeta_j^2 q_j^{2*}(t) - 2\rho z_k\zeta_k\zeta_j q_k^*(t) q_j^*(t)\big] + \alpha\beta g_{0k}(t) \\ & - \frac{\theta}{\delta}\ln\frac{1}{\gamma} + h^P\big(\frac{1}{\gamma} - 1\big) + \frac{\theta l_k(t)}{\delta} = 0, \end{aligned}$$

and

$$g'_{0k}(t) + \frac{1}{2}\kappa^2(1-\hat{\rho}^2) g_{0k}^2(t) - \big(\hat{\rho}\omega\kappa + \alpha + \frac{\theta}{\delta}\big) g_{0k}(t) + \frac{\theta}{\delta} g_k(t) - \frac{\omega^2}{2} = 0, \tag{26}$$

with the condition $l_{0k}(T) = g_{0k}(T) = 0$.
For $k = 1, 2$, define $L_k(t) = l_{0k}(t) - g_{0k}(t)$. It is clear that the function $L_k(t)$ is differentiated with respect to $t$, and we obtain

$$L'_k(t) = l'_{0k}(t) - g'_{0k}(t) = \frac{\theta}{\delta} L_k(t) + \frac{\theta}{\delta}\ln\frac{1}{\gamma} - h^P\big(\frac{1}{\gamma} - 1\big).$$

On the other hand, since $L_k(T) = l_{0k}(T) - g_{0k}(T) = 0$, then

$$L_k(t) = e^{-\frac{\theta}{\delta}(T-t)}\big(\ln\frac{1}{\gamma} + \gamma - 1\big) - \ln\frac{1}{\gamma} - \gamma + 1.$$

Applying Lemma 3.1 given in [31] the solution for equation (26) is $l_{0k}(t) = l_k(t)$, as represented in (19), and this completes the proof. □

**Remark 1.** According to the Theorems 4.1 and 4.2, the optimal investment strategy of stock asset $s_k^*(t)$ and reinsurance strategy $q_k^*(t)$, $k = 1, 2$, for the first and second spaces do not change. This is due to the structure of financial market for surplus process of each insurer that the price process is uncorrelated with corporate coupon bonds.
**Corollary 4. 1.** If $z_1 > 0$ and $z_2 = 0$, i.e., there is no any competition between the two insurance companies, then

the optimal investment and reinsurance strategies are given as follow:

$$\begin{cases} q_1^*(t) = \big(y_1(t) + x_1(t)y_2(t)\big) \wedge 1, & for \quad t \in [0, T], \\ q_2^*(t) = y_2(t), & for \quad t \in [0, T], \end{cases}$$

$$\begin{cases} s_1^*(t) = \hat{s}_1^*(t) + z_1\hat{s}_2^*(t), & for \quad t \in [0, T], \\ s_2^*(t) = \hat{s}_2^*(t), & for \quad t \in [0, T], \end{cases}$$

$$e_1^*(t) = \begin{cases} \hat{e}_1^*(t) + z_1\hat{e}_2^*(t), & for \quad t \in [0, \tau \wedge T], \\ 0, & for \quad t \in [\tau \wedge T, T], \end{cases}$$

and

$$e_2^*(t) = \begin{cases} \hat{e}_2^*(t), & for \quad t \in [0, \tau \wedge T], \\ 0, & for \quad t \in [\tau \wedge T, T]. \end{cases}$$

Corollary 4.1 shows that the optimal investment and reinsurance strategy $\pi_2^*(t) = \big(q_2^*(t), s_2^*(t), e_2^*(t)\big)$ is equal to $\hat{\pi}_2^*(t) = \big(y_2^*(t), \hat{s}_2^*(t), \hat{e}_2^*(t)\big)$, whereas the optimal investment and reinsurance strategy for insurance company 1, i.e., $\pi_1^*(t) = \big(q_1^*(t), s_1^*(t), e_1^*(t)\big)$, can be divided into two parts.

### *4.3. More precise analysis of risk model coefficients*

The first part is the strategy $\pi_1^*(t)$, which is likely due to the partial objective of maximizing the terminal wealth; the second part $\big(x_1(t)q_2^*(t), z_1s_2^*(t), z_1e_2^*(t)\big)$ is induced by the relative performance concern. For the insurance company 1, the reinsurance strategy $\hat{q}_1^*(t) = \frac{2\nu}{e_1 e^{r(T-t)}+2\nu}$ decreases in the risk aversion parameter $e_1$, or $1 - q_1^*(t)$ increases in $e_1$. This indicates that a more risk averse insurer will buy more reinsurance contracts. By contract, in a competitive condition, the coefficient $x_1(t)$ increases in $e_1$. Therefore, the optimal reinsurance strategy $q_1^*(t)$ may increase in $e_1$, which depends on the optimal reinsurance strategy $q_2^*(t)$ of insurance company 2. Furthermore, a more risk averse insurer may buy less reinsurance contracts in the presence of competition. In this case, the optimal reinsurance strategy of insurance company 1, is simply to mimic the optimal strategy $q_2^*(t)$ that is followed by the insurance company 1; this reinsurance strategy decreases in the insurance company 2's risk aversion parameter $e_2$. This point is further affirmed after we observe that the optimal strategies are actually riskier than the regular strategies. Next, we will present a details analysis of the optimal reinsurance and investment strategy in a more general competition when both insurance companies have relative concerns.

**Corollary 4. 2.** If $w_z > 0,\ k = 1, 2$, then the optimal strategy $\pi_k^*(t) = \big(q_k^*(t), s_k^*(t), e_k^*(t)\big)$ has the following properties:

(i) If the insurance company 1 increases its optimal reinsurance strategy $q_k^*(t)$ relative to the regular reinsurance strategy $\hat{q}_k^*(t)$, i.e., $z_k = 0, k = 1, 2$, without competition. For this case, the sensitivity of insurance company 1's optimal reinsurance strategy with respect to the parameters are given in Table 1.

Table 1. Sensitivity of insurance company 1's optimal reinsurance strategy $q_k^*(t)$

| $\partial q_k^*(t)/\partial z_k$ | $\partial q_k^*(t)/\partial z_j$ | $\partial q_k^*(t)/\partial \mu_k$ | $\partial q_k^*(t)/\partial \mu_j$ | $\partial q_k^*(t)/\partial \zeta_k$ | $\partial q_k^*(t)/\partial \zeta_j$ |
|---|---|---|---|---|---|
| + | + | + | + | − | − |

(ii) If the insurance company $k$, $k = 1, 2$, holds a positive position of the stock and investment $\big(s_k^*(t)\big)$ will be larger relative to the regular strategy without competition (i.e., $z_k = 0, k = 1, 2$). For this case, the sensitivity of insurance company 1's optimal investment strategy $s_k^*(t)$ with respect to the parameters are given in Table 2.

Table 2. Sensitivity of insurance company 1's optimal investment strategy $s_k^*(t)$

| $\partial s_k^*(t)/\partial z_k$ | $\partial s_k^*(t)/\partial z_j$ | $\partial s_k^*(t)/\partial e_k$ | $\partial s_k^*(t)/\partial e_j$ | $\partial s_k^*(t)/\partial\hat{\rho}$ | $\partial s_k^*(t)/\partial\alpha$ | $\partial s_k^*(t)/\partial\omega$ |
|---|---|---|---|---|---|---|
| $+$ | $+$ | $-$ | $-$ | $-$ | $+(\hat{\rho}>0)$ $-(\hat{\rho}<0)$ | $+$ |

(iii) If each insurance companies choose their investment in the corporate bond relative to the case of no competition (i.e., $z_k = 0, k = 1,2$), and each insurance company will always hold a positive position of the corporate bond with a positive risk premium, whereas $\gamma_k^*(t) = 0$ and $O_k(t,\hat{d}_k,y,1) = O_k(t,\hat{d}_k,y,0)$, if $\gamma = 1$. For this case, the sensitivity of insurance company 1's optimal investment strategy $e_k^*(t)$with respect to the parameters are given in Table 3.

Table 3. Sensitivity of insurance company 1's optimal reinsurance strategy $e_k^*(t)$

| $\partial e_k^*(t)/\partial z_k$ | $\partial e_k^*(t)/\partial z_j$ | $\partial e_k^*(t)/\partial\frac{1}{\gamma}$ | $\partial e_k^*(t)/\partial\zeta$ | $\partial e_k^*(t)/\partial e_k$ | $\partial e_k^*(t)/\partial e_j$ |
|---|---|---|---|---|---|
| $+$ | $+$ | $+$ | $-$ | $-$ | $-$ |

*Proof*

(i) From Theorems 4.1 and 4.2, we know that $q_k^*(t) = \frac{\hat{q}_k^*(t)+x_k(t)\hat{q}_j^*(t)}{1-x_1(t)x_2(t)}$, for $k \neq j = 1,2$. Since $\hat{q}_k^*(t) > 0$, $x_k(t) > 0$ and $1 > 1 - x_1(t)x_2(t)$, therefore, $q_k^*(t) \geq \hat{q}_k^*(t)$. This shows that the insurance company buys fewer reinsurance contracts. We can easily obtain the relationship between the optimal reinsurance strategy and model parameters, and the details of this procedure are omitted here.

(ii) To prove the part (ii), we need to show that $g_k(t) \leq 0$. From Theorems 4.1 and 4.2, we have that $g_k(t) \leq g_k(T) = 0$.

Let $\kappa_0 = \hat{\rho}\kappa g_k(t)$. Using the equation (26), we can show that the function $\kappa_0$ satisfies the following equation:

$$\kappa_0' + \frac{\kappa}{2\hat{\rho}}(1-\hat{\rho}^2)\kappa_0^2 - (\hat{\rho}\omega\kappa + \alpha)\kappa_0 - \frac{\omega^2\kappa\hat{\rho}}{2} = 0, \tag{27}$$

where $\kappa_0' = \frac{\partial\kappa_0}{\partial t}$. Now, differentiating from the equation (27) with respect to $\hat{\rho}$, we obtain the following equation

$$\frac{\partial\kappa_0'}{\partial\hat{\rho}} - \frac{\kappa}{2}(\frac{1}{\hat{\rho}^2}+1)\kappa_0^2 + \frac{\kappa}{\hat{\rho}}(1-\hat{\rho}^2)\kappa_0\frac{\partial\kappa_0}{\partial\hat{\rho}} - (\hat{\rho}\omega\kappa+\alpha)\frac{\partial\kappa_0}{\partial\hat{\rho}} - \omega\kappa\kappa_0 - \frac{\omega^2\kappa}{2} = 0. \tag{28}$$

Let $\Delta = \frac{\kappa}{\hat{\rho}}(1-\hat{\rho}^2)\kappa_0 - (\hat{\rho}\omega\kappa+\alpha)$. Then the equation (28) becomes as follows:

$$(\frac{\partial\kappa_0}{\partial\hat{\rho}})' + \Delta\frac{\partial\kappa_0}{\partial\hat{\rho}} - \frac{\kappa}{2}\kappa_0^2 - \frac{\kappa}{2}(\kappa_0+\omega)^2 = 0,$$

which the solution to this equation is given by

$$\frac{\partial\kappa_0}{\partial\hat{\rho}} = -\exp\big(\int_t^T \Delta ds\big)\int_t^T \exp\big(-\int_t^T \Delta d\kappa\big)\big[\frac{\kappa}{2}\kappa^2 + \frac{\kappa}{2}(\kappa_0+\omega)^2\big]ds < 0,$$

which shows that $\kappa_0$ decreases as $\hat{\rho}$ increases. On the other hand, the coefficient $\hat{s}_k^*(t)$ is a decreasing function of $\hat{\rho}$, moreover

$$\hat{s}_k^*(t,\hat{\rho}) \geq \hat{s}_k^*(t,1^-) = \lim_{\hat{\rho}\to 1}\frac{\omega+\kappa_0}{e_k}e^{-r(t-T)} = \frac{1}{e_k}\big(\omega + \lim_{\hat{\rho}\to 1}\kappa g_k(t)\big)e^{-r(t-T)}.$$

From (19), we have $\lim_{\hat{\rho}\to 1}\kappa g_k(t) = 0$, and $\hat{s}_k^*(t,\hat{\rho}) \geq \hat{s}_k^*(t,1^-) = \lim_{\hat{\rho}\to 1}\frac{\omega}{e_k}e^{-r(t-T)} > 0$.

Let $\hat{\rho} = 1$, then

$$\hat{s}_k^*(t,\hat{\rho})|_{\hat{\rho}=1} = \frac{1}{e_k}\big(\frac{2\alpha+\omega\kappa}{2(\alpha+\omega\kappa)} + \frac{\omega\kappa}{2(\alpha+\omega\kappa)}e^{-(\alpha+\omega\kappa)(T-t)}\big),$$

which $\hat{s}_k^*(t,\hat{\rho})|_{\hat{\rho}=1} > 0$, since $\alpha > 0$ and $\omega > 0$. Based on these results, we can conclude that $\hat{s}_k^*(t,\hat{\rho}) = \hat{s}_k^*(t) > 0$ for $t \in [0,T]$.
Differentiating equation (27) with respect to $\alpha$, we can obtain the following equation for $t$:

$$\frac{\partial \kappa_0}{\partial \alpha} + \frac{\kappa}{\hat{\rho}}(1-\hat{\rho}^2)\kappa\frac{\partial \kappa}{\partial \alpha} - (\hat{\rho}\omega\kappa + \alpha)\frac{\partial \kappa}{\partial \alpha} - \kappa = 0,$$

which is equavalent to the equation $\big(\frac{\partial \kappa}{\partial \alpha}\big)' + \Delta\kappa_\alpha - \kappa = 0$. Then we have

$$\frac{\partial \kappa}{\partial \alpha} = -\exp\Big(\int_t^T \Delta ds\Big)\int_t^T \exp\Big(-\int_t^T \Delta ds\Big)\kappa ds,$$

for $\hat{\rho} > 0$, $\kappa_\alpha > 0$ and $\kappa_\alpha < 0$ if $\hat{\rho} < 0$. Therefore, the optimal investment strategy $s_k^*(t)$ inceases as $\alpha$ increases if $\hat{\rho} < 0$. Consequently, we have $\frac{\partial}{\partial \alpha}s_k^*(t) > 0$ and $s_t^*(t) > 0$ if $\alpha > 0$.
(iii) To prove the part (iii), consider that if $\frac{1}{\Delta} = \frac{h^Q}{h^P} = 1$, then the optimal result is $\hat{e}_k^*(t) = 0$, $k = 1,2$. thus the optimal investment strategy is given by

$$e_k^*(t) = \frac{\hat{e}_k^*(t) + z_k\hat{e}_j^*(t)}{1 - z_k z_j} = 0. \qquad k \neq j \in \{1,2\},$$

Similarly, if $\frac{1}{\Delta} = \frac{h^Q}{h^P} > 1$, the optimal result is

$$\hat{e}_k^*(t) = \frac{\ln\frac{1}{\gamma}e^{-\frac{\delta}{\zeta}(T-t)} + \big(1 - e^{-\frac{\delta}{\zeta}(T-t)}\big)(1-\gamma)}{\zeta e_k}e^{-r(T-t)},$$

which shows that $\hat{e}_k^*(t) > 0$, and the optimal investment strategy is given by $e_k^*(t) > 0$, and this completes the proof. □

## 5. Verification theorem

In this section, we prove a verification theorem for the optimization problem 3.1. We first present the following Lemma.

*Lemma 3*
Let $V = \mathcal{R} \times \mathcal{R}^+ \times \{0,1\}$ and consider the open sets $V_1, V_2, V_3, \ldots$, with the condition $V_i \subset V_{i+1} \subset V$, $i = 1,2,\ldots$ and $V = \bigcup_{i=1}^{\infty} V_i$. If $\tau_i$ be the the existing time from the open set $V_i$, then for any $\epsilon > 1$ and $i = 1,2,\ldots$, we have

$$\sup_i E\big\{\big|O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k^*}(\tau_i \wedge T), Y(\tau_i \wedge T), H(\tau_i \wedge T)\big)\big|^\epsilon\big\} < \infty.$$

*Proof*
First, consider that

$$\begin{aligned} Q_k\big(t, \hat{D}_k^{\pi_k^*}(t), Y(t), H(t)\big) &= \big(1 - H(t)\big)Q_k\big(t, \hat{D}_k^{\pi_k^*}(t), Y(t), 0\big) \\ &\quad + H(t)Q_k\big(t, \hat{D}_k^{\pi_k^*}(t), Y(t), 1\big). \end{aligned}$$

For $H(t) = 0, 1$, we only need to verify that

$$\sup_i E\big\{\big|O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k^*}(\tau_i \wedge T), Y(\tau_i \wedge T), j\big)\big|^\epsilon\big\} < \infty, \tag{29}$$

for $j = 0, 1$. Therefore, two separate cases are considered as follow.
Case (i) Let $H(t) = 0$. Using the equation (14), we obtain

$$\begin{aligned} Q_k\big(t, \hat{D}_k^{\pi_k^*}(t), Y(t), 0\big) &= -\frac{1}{e_k}\exp\big\{-e_k\hat{D}_k^{\pi_k^*}(t)e^{r(T-t)} + l_{0k}(t) + g_{0k}(t)Y(t)\big\} \\ &= -\frac{1}{e_k}\exp\big\{-e_k\hat{d}_k e^{r(T-t)} - e_k\int_0^t e^{r(T-s)}\big[(c_k - z_k c_j) - \eta q_k^*(s) - z_k\eta q_j^*(s) \\ &\quad -\big(a_k q_k^*(s) - z_k a_j q_j^*(s)\big) + \big(s_k^*(s) - z_k s_j^*(s)\big)\omega Y(s) + \big(e_k^*(s) - z_k e_j^*(s)\big)\delta(1-\gamma)\big] \\ &\quad -e_k\hat{\rho}\int_0^t e^{r(T-s)}\big(s_k^*(s) - z_k s_j^*(s)\big)\big(Y(s)\big)^{\frac{1}{2}}dW_2(s) \\ &\quad -e_k\int_0^t e^{r(T-s)}\sqrt{1-\hat{\rho}^2}\big(e_k^*(s) - z_k e_j^*(s)\big)\big(Y(s)\big)^{\frac{1}{2}}d\bar{W}(s) \\ &\quad +e_k\int_0^t e^{r(T-s)}\big(e_k^*(s) - z_k e_j^*(s)\big)\theta dC_s^P - e_k\int_0^t e^{r(T-s)}\zeta_k q_k^*(s)dB_k(s) \\ &\quad +e_k\int_0^t e^{r(T-s)}z_k\zeta_j q_j^*(s)dB_j(s) + l_{0k}(t) + f_{0k}(t)Y(t)\big\}, \end{aligned}$$

where we assume that $W_1(t) = \hat{\rho} + \sqrt{1-\hat{\rho}^2}\hat{W}(t)$, and $\hat{W}(t)$ is a standard Brownian motion independent of $W_2(t)$. Define

$$P_{1k}(t) = -e_k\omega\big(s_k^*(t) - z_k s_j^*(t)\big)e^{r(T-t)} + \frac{e_k^2(1-\hat{\rho}^2)}{2}\big(s_k^*(t) - z_k s_j^*(t)\big)^2 e^{2r(T-t)},$$

$$P_{2k}(t) = -e_k\hat{\rho}\big(s_k^*(t) - z_k s_j^*(t)\big)e^{r(T-t)},$$

and $P_{3k}(t) = g_{0k}(t)$. From Corollary 4.1, the function $g_{0k}(t)$ satisfies the following differential equation:

$$g_{0k}'(t) + \frac{1}{2}\kappa^2(1-\hat{\rho}^2)g_{0k}^2(t) - (\hat{\rho}\omega\kappa + \alpha)g_{0k}(t) - \frac{\omega^2}{2} = 0.$$

With some calculations, we have

$$P_{1k}(t) + P_{3k}'(t) - \alpha P_{3k}(t) + \frac{1}{2}\big(\kappa P_{3k}(t) + P_{2k}(t)\big)^2 = 0.$$

On the other hand, since $\kappa$, $\omega$, $g_{0k}(t)$ and $\pi_k^*(t)$, $k = 1, 2$, are bounded, therefore there exist positive constants $\Omega$, $\Theta$, $\Delta$, and $0 \le b \le 1$, such that

$$\frac{\Delta^2 - \Delta}{2}\big(e^{r(T-t)}e_k^2 b^2(1-\hat{\rho}^2)s_k^{*2} + \big(bP_{3k}(t) + P_{2k}(t)\big)^2 < \Omega,$$

and

$$\omega b\hat{\rho}\Delta + b^2\big(1+\hat{\rho}^2\big)g_{0k}(t) < \Theta.$$

Let

$$A_k^+ = \frac{-\Theta + \sqrt{\Theta^2 + 2(\Omega+1)\Delta}}{2}, \qquad A_k^- = \frac{-\Theta - \sqrt{\Theta^2 + 2(\Omega+1)\Delta}}{2},$$

and $A(t) = \frac{A_k^+\exp\big(tA_k^+\big) - A_k^-\exp\big(tA_k^-\big)}{\exp\big(tA_k^+\big) - \exp\big(tA_k^-\big)}$. Then $A(t) > 0$ and $A'(t) + \Theta A(t) + \frac{A^2(t)}{2} = -(\Omega+1)$. Now, define

$$Q_{1k}(t) = \Delta P_{1k}(t) + e^{2r(T-t)}\frac{\Delta^2 - \Delta}{2}e_k^2 b^2(1-\hat{\rho}^2)s_k^{*2}, \qquad Q_{2k}(t) = \Delta P_{2k}(t),$$

and

$$Q_{3k}(t) = \Delta\big(P_{3k}(t) + A(t)\big).$$

Then the inequalities

$$\begin{aligned}
&E\big|O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k^*}(\tau_i \wedge T), Y(\tau_i \wedge T), 0\big)\big|^{\epsilon} \\
&\leq e_k^{-\Delta} \exp\big\{e_k \Delta \hat{d}_k e^{rT} - e_k \Delta \int_0^t e^{r(T-s)} \big[(c_k - z_k c_j) - \eta q_k^*(s) - z_k \eta q_j^*(s) \\
&-\big(a_k q_k^*(s) - z_k a_j q_j^*(s)\big) + \big(e_k^*(s) - z_k e_j^*(s)\big)\theta(1-\gamma)\big] ds - e_k \int_0^t e^{r(T-s)} \Delta \zeta_k q_k^*(s) dB_k(s) \\
&+e_k \int_0^t e^{r(T-s)} \Delta \zeta_k q_j^*(s) dB_j(s) + \Delta f_{0k}(t) + e_k \int_0^t e^{r(T-s)} \Delta\big(e_k^*(s) - z_k e_j^*(s)\big)\theta dC^P(t)\big\} \\
&\times \exp\big\{e_k \int_0^t e^{r(T-s)} \Delta \sqrt{1-\hat{\rho}^2}\big(e_k^*(s) - z_k e_j^*(s)\big)\big(Y(s)\big)^{\frac{1}{2}} d\bar{W}(s) \\
&-\frac{e_k^2(1-\hat{\rho}^2)}{2}\Delta^2 \int_0^t e^{2r(T-s)}\big(s_k^*(s) - z_k s_j^*(s)\big)^2 Y(s) ds\big\} \\
&\times \exp\big\{\int_0^t Q_{1k}(s) Y(s) ds + \int_0^t Q_{2k}(s)\big(Y(s)\big)^{\frac{1}{2}} dW_2(s) + Q_{3k}(t) Y(t)\big\},
\end{aligned}$$

and

$$\begin{aligned}
&Q_{1k}(t) + Q_{3k}'(t) - \alpha Q_{3k}(t) + \frac{\big(bQ_{3k}(t) + Q_{2k}(t)\big)^2}{2} \\
&= \Delta\big(A'(t) - \alpha A(t) + \omega b \hat{\rho} \Delta + \Delta A(t)\big(b\hat{\rho}\Delta + b^2\big) g_{0k}(t) + \frac{b\Delta A^2(t)}{2}\big) \\
&+\frac{\Delta^2 - \Delta}{2}\big(e^{2r(T-s)} b^2 e_k^2 (1-\hat{\rho}^2) s_k^{*2} + \big(bQ_{3k}(t) + Q_{2k}(t)\big)^2\big) \\
&\leq \Delta\big(A'(t) + \Theta A(t) + \frac{\Delta b^2 A^2(t)}{2} + \Theta\big) = \Delta\big(-\Theta - 1 + \Theta\big) = -\Delta < 0,
\end{aligned}$$

hold. Moreover, we have the inequality

$$E\big[\exp\big(\int_0^t Q_{1k}(s) Y(s) ds + \int_0^t Q_{2k}(s)\big(Y(s)\big)^{\frac{1}{2}} dW_2(s) + Q_{3k}(t) Y(t)\big] \leq \exp\big(\alpha\beta Q_{3k}(t)\big),$$

and

$$\begin{aligned}
&E\big(\exp\big\{-e_k \Delta \int_0^t e^{r(T-s)}\big(e_k^*(s) - z_k e_j^*(s)\big)\theta(1-\gamma) ds \\
&+\Delta e_k \int_0^t e^{r(T-s)}\big(e_k^*(s) - z_k e_j^*(s)\big)\theta dC^P(t)\big\}\big) < \infty.
\end{aligned}$$

Therefore, for $i = 1, 2, \ldots$, we have

$$E\big\{\big|O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k^*}(\tau_i \wedge T), Y(\tau_i \wedge T), 0\big)\big|^{\epsilon}\big\} < \infty. \tag{30}$$

Case (ii) Let $H(t) = 1$. Using the same method in case (i), for $i = 1, 2, \ldots$, we have

$$E\big\{\big|O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k^*}(\tau_i \wedge T), Y(\tau_i \wedge T), 1\big)\big|^{\epsilon}\big\} < \infty. \tag{31}$$

Therefore, the relations (30) and (31) complete the proof. □

*Theorem 5.1*
Let the integrable function $Q_k, k = 1, 2$, on the time $[0, T]$ is a solution to the equation (15). Then the investment and reinsurance strategies $\pi^*(t)$ given in Theorems 4.1 and 4.2 are the optimal strategies and $Q_k$ is the corresponding objective function.

*Proof*
Consider the conditions given in Lemma 5.1. For $(\hat{d}_k, y, h) \in V$, let $\tau_i$ be the exit time of $\big(\hat{D}_k^{\pi_k}(t), Y(t), H(t)\big)$ from $V_i$. Then $\tau_i \wedge T \to T$, a.s., as $i \to \infty$. Applying Itô's formula to the function $Q_k$, we have

$$\begin{aligned}
&Q_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k}(\tau_i \wedge T), Y(\tau_i \wedge T), H(\tau_i \wedge T)\big)\\
&= Q_k(t, \hat{d}_k, y, h) + \int_0^{\tau_i \wedge T} \mathcal{B}_k^{\pi_k} Q_k\big(u, \hat{D}_k^{\pi_k}(u), Y(u), H(u)\big) du\\
&+ \int_t^{\tau_i \wedge T} \zeta_k q_k^*(u) \frac{\partial Q_k}{\partial \hat{d}_k} dB_k(u) - \int_t^{\tau_i \wedge T} z_k \zeta_j q_j^*(u) \frac{\partial Q_k}{\partial \hat{d}_k} dB_j(u)\\
&+ \int_t^{\tau_i \wedge T} \big(s_k^*(u) - z_k s_j^*(u)\big) Q_y^k \big(Y(u)\big)^{\frac{1}{2}} dW_1(u)\\
&+ \int_t^{\tau_i \wedge T} \big[Q_k\big(u, \hat{D}_k^{\pi_k}(u) - \big(e_k^*(u) - z_k e_j^*(u)\big)\theta \hat{D}_k^{\pi_k}(u), Y(u), 1\big)\\
&- Q_k\big(u, \hat{D}_k^{\pi_k}(u), Y(u), 0\big)\big] dC_u^P.
\end{aligned}$$

Taking conditional expectation given $(t, \hat{d}, y, h)$ on both sides of the equation (27) and using (15), we get

$$\begin{aligned}
&Q_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k}(\tau_i \wedge T), Y(\tau_i \wedge T), H(\tau_i \wedge T)\big) \big| \hat{D}_k^{\pi_k}(t) = \hat{d}_k, Y(t) = y, H(t) = h\big)\\
&= E\Big[\int_0^{\tau_i \wedge T} \mathcal{B}_k^{\pi_k} Q_k\big(u, \hat{D}_k^{\pi_k}(u), Y(u), H(u)\big) du \big| \hat{D}_k^{\pi_k}(t) = \hat{d}_k, Y(t) = y, H(t) = h\Big]\\
&+ Q_k(t, \hat{d}_k, y, h)\\
&\le Q_k(t, \hat{d}_k, y, h).
\end{aligned}$$

From Lemma 5.1, since for any $i = 1, 2, \ldots,$ $Q_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k}(\tau_i \wedge T), Y(\tau_i \wedge T), H(\tau_i \wedge T)\big)$, is uniformly integrable, we have

$$\begin{aligned}
O_k(t, \hat{d}_k, y, h) &= \sup_{\pi_k \in \Pi_k} E\big[U_k\big(\hat{D}_k^{\pi_k}(t)\big) \big| \hat{D}_k^{\pi_k}(t) = \hat{d}_k, Y(t) = y, H(t) = h\big]\\
&= \lim_{i \to \infty} E\big[O_k\big(\tau_i \wedge T, \hat{D}_k^{\pi_k}(\tau_i \wedge T), Y(\tau_i \wedge T), H(\tau_i \wedge T)\big) \big| \hat{D}_k^{\pi_k}(t) = \hat{d}_k, Y(t) = y, H(t) = h\big)\\
&\quad + Q_k(t, \hat{d}_k, y, h),
\end{aligned}$$

when $\pi_k(t) = \pi_k^*(t)$, then the above inequality becomes an equality, and $O_k(t, \hat{d}_k, y, h) = Q_k(t, \hat{d}_k, y, h)$. Therefore, the proof is complete. □

## 6. Numerical experiments: effects of model parameters

In this section, we conduct the numerical examples to investigate the effects of risk model parameters on the optimal strategies. We set the default risk model parameters of each insurance companies as given in Table 1. In addition, the base parameters are presented in Table 2.

### *6.1. Optimal investment strategy for corporate bond*

In this subsection, we analyze the effect of parameters $\gamma$ and $\delta$ on the optimal investment strategy for corporate bond. These parameters change within the reasonable intervals. Figures 1 and 2 show the effect of parameter $\gamma$

Table 4. Default risk model parameters

| Insurance company 1 | | Insurance company 2 | |
|---|---|---|---|
| Notation | Value | Notation | Value |
| $\mu_1$ | 0.8 | $\mu_2$ | 1 |
| $\zeta_1$ | $(0, 5)$ | $\zeta_2$ | $(0, 5)$ |
| $e_1$ | 2 | $e_2$ | 3 |
| $z_1$ | $(0, 1)$ | $z_2$ | $(0, 1)$ |

Table 5. Base parameters

| $r$ | $\nu$ | $T$ | $\gamma$ | $\delta$ | $\kappa$ | $\omega$ | $\alpha$ | $\hat{\rho}$ | $\lambda$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.08 | 2 | 10 | $(0, 1)$ | $(0, 1)$ | 2.5 | 0.5 | $(1, 4)$ | $(-1, 1)$ | 1 |

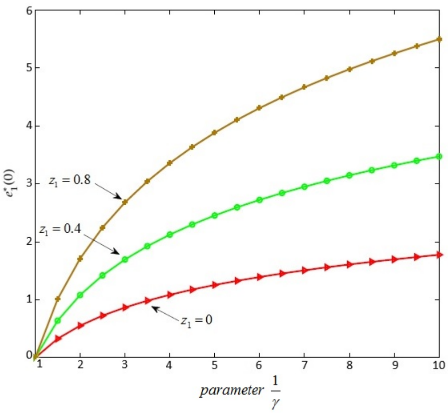


Figure 1. The effect of parameter $\frac{1}{\gamma}$ on the optimal investment strategy $e_1^*(0)$

on the optimal investment strategies $e_1^*(0)$ and $e_2^*(0)$. We can see that there is a positive correlation between the optimal strategies $e_1^*(0)$ and $e_2^*(0)$ and parameter $\gamma$. The slope of both graphs decreases with increasing the value of parameter. Figures 3 and 4 show the effect of loss rate of the corporate coupon bond parameter $\delta$ on the optimal investment strategies $e_1^*(0)$ and $e_2^*(0)$. We can see that there is a negative correlation between the optimal strategies $e_1^*(0)$ and $e_2^*(0)$ and parameter $\delta$. According to the Figures 3 and 4, the insurance company $k$, $k = 1, 2$, reduces their value of investment in the corporate coupon bond as the loss rate increases. Figures 5 and 6 show the effect of parameters $z_1$ and $z_2$ on the optimal investment strategies $e_1^*(0)$ and $e_2^*(0)$, respectively. It is clear that the optimal investment strategy $e_k^*(0)$, increases as the parameter $z_k$, $k = 1, 2$, increases, respectively. In this case, the insurance company can maximize the probability of generating greater terminal wealth against its competitor at the finite time $T$.

### 6.2. *Optimal reinsurance strategy*

Figures 7 and 8 show the effect of parameters $\zeta_1$ and $\zeta_2$ on the optimal reinsurance strategies $q_1^*(0)$ and $q_2^*(0)$ when $z_1 > 0$ and $z_2 > 0$, respectively. According to the figures, the optimal reinsurance strategies $q_k^*(0)$ of insurance company $k$ decreases as $\zeta_k$, $k = 1, 2$, increases. In these figures, the optimal reinsurance strategies $q_k^*(0)$ is constant when $z_k = 0$, $k = 1, 2$. These results are completely consistent with Theorems 4.1 and 4.2. Figures 9 and 10, show the effect of parameters $z_1$ and $z_2$ on the optimal reinsurance strategies $q_1^*(0)$ and $q_2^*(0)$, respectively. According to these figures, the optimal reinsurance strategies $q_k^*(0)$ of insurance company $k$ increases as $z_k$, $k = 1, 2$, increases.

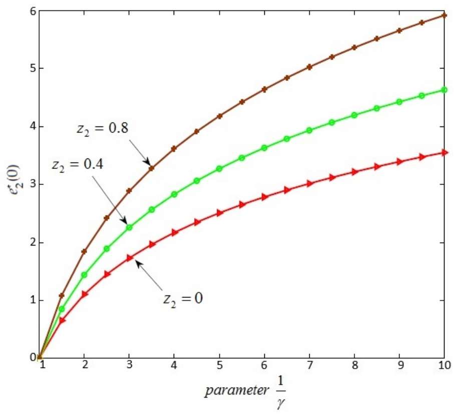


Figure 2. The effect of parameter $\frac{1}{\gamma}$ on the optimal investment strategy $e_2^*(0)$

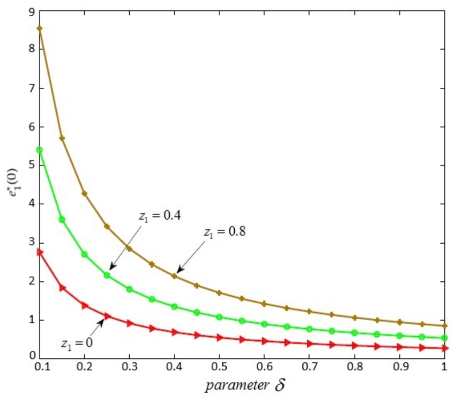


Figure 3. The effect of parameter $\delta$ on the optimal investment strategy $e_1^*(0)$

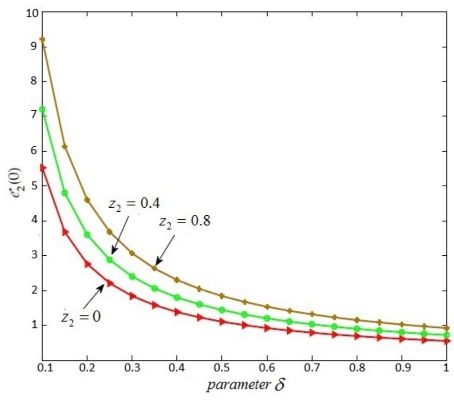


Figure 4. The effect of parameter $\delta$ on the optimal investment strategy $e_2^*(0)$

### *6.3. Optimal investment strategy*

Figures 11 and 12 show the effect of correlation coefficient $\hat{\rho}$ on the optimal reinsurance strategies $s_1^*(0)$ and $s_2^*(0)$ for different values of $z_1$ and $z_2$, respectively. It can be seen that in each figure with increasing the correlation coefficient, the optimal reinsurance strategy decreases and this result is completely consistent with economic

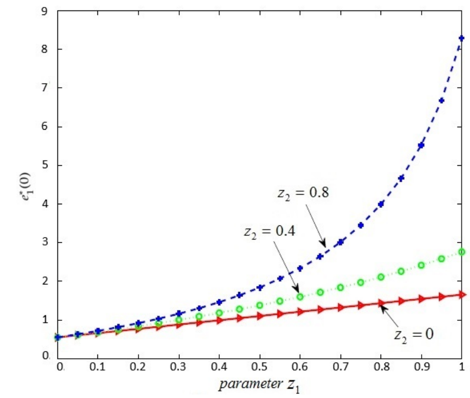


Figure 5. The effect of parameter $z_1$ on the optimal investment strategy $e_1^*(0)$

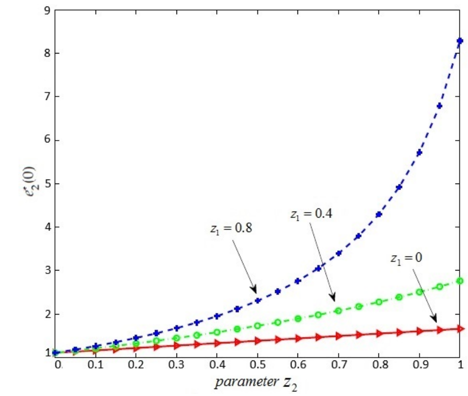


Figure 6. The effect of parameter $z_2$ on the optimal investment strategy $e_2^*(0)$

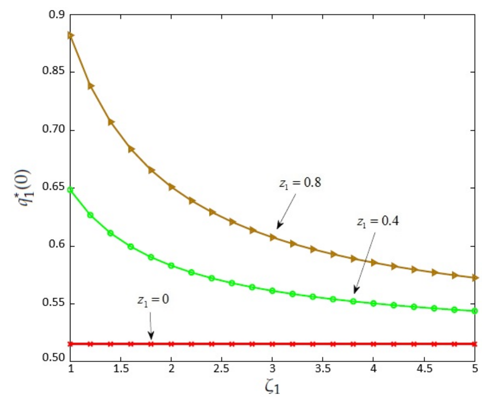


Figure 7. The effect of parameter $\zeta_1$ on the optimal reinsurance strategy $q_1^*(0)$

conditions. Moreover, the figures 13-16 show the effect of parameter $\alpha$ on the optimal reinsurance strategies $s_1^*(0)$ and $s_2^*(0)$ for different values of $z_1$ and $z_2$. According to these figures, the optimal reinsurance strategy $s_k^*(0)$, $k = 1, 2$, increases as the parameter $\alpha$ increase for positive correlation coefficient and the strategy decreases as the parameter $\alpha$ increase for negative correlation coefficient.

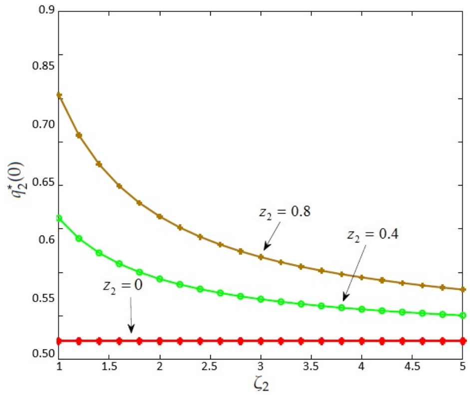


Figure 8. The effect of parameter $\zeta_2$ on the optimal reinsurance strategy $q_2^*(0)$

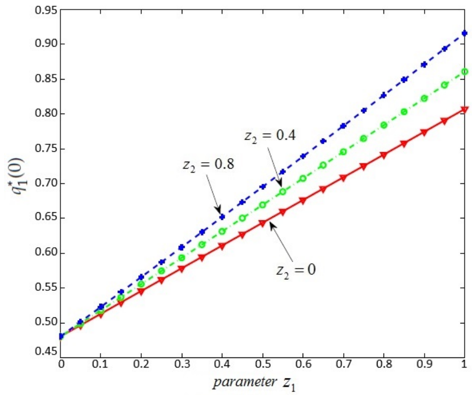


Figure 9. The effect of parameter $z_1$ on the optimal reinsurance strategy $q_1^*(0)$

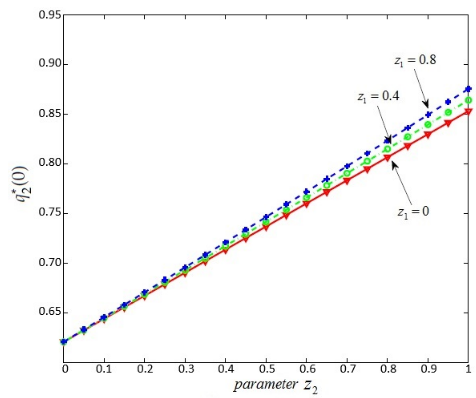


Figure 10. The effect of parameter $z_2$ on the optimal reinsurance strategy $q_2^*(0)$

## 7. Conclusions

In this paper, we considered two insurance companies under the purchasing proportional reinsurance, a risk-free asset, and defaultable grade bond in a financial market and studied the objective function to maximize his expected utility of wealth process the difference between his terminal surplus and that of his competitor. We supposed that

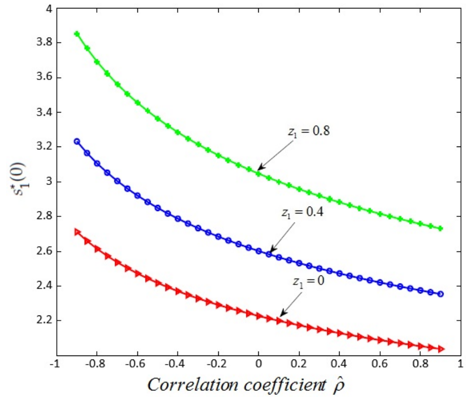


Figure 11. The effect of correlation coefficient $\hat{\rho}$ on the optimal reinsurance strategy $s_1^*(0)$

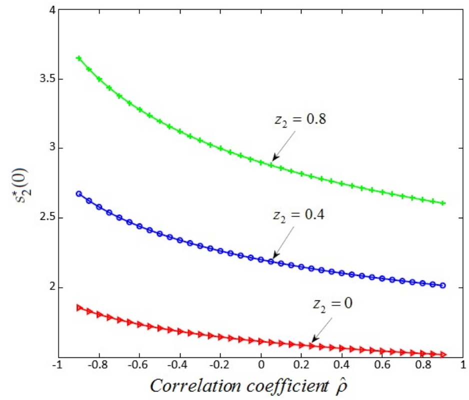


Figure 12. The effect of correlation coefficient $\hat{\rho}$ on the optimal reinsurance strategy $s_2^*(0)$

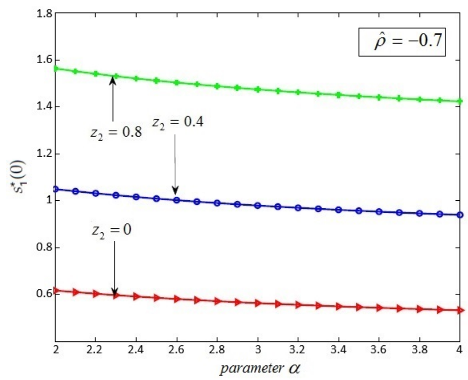


Figure 13. The effect of parameter $\alpha$ on the optimal reinsurance strategy $s_1^*(0)$ when $\hat{\rho} = -0.7$

both insurers having the compound Poisson process and they are allowed to purchase proportional reinsurance and invest in a financial market which consists of a risk-free asset, a defaultable grade bond whose the price process of each insurer is governed by a standard Brownian motion and dynamics of defaultable bond price is represented as an exponential form of stochastic differential equation. The optimization problems are presented and we solved the nonlinear HJB equations related to our optimal control problems, the optimal investment and reinsurance strategies

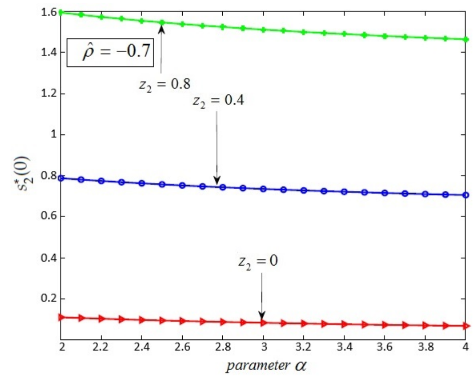


Figure 14. The effect of parameter $\alpha$ on the optimal reinsurance strategy $s_2^*(0)$ when $\hat{\rho} = -0.7$

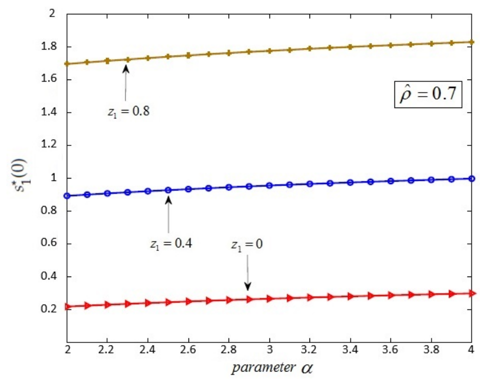


Figure 15. The effect of parameter $\alpha$ on the optimal reinsurance strategy $s_1^*(0)$ when $\hat{\rho} = 0.7$

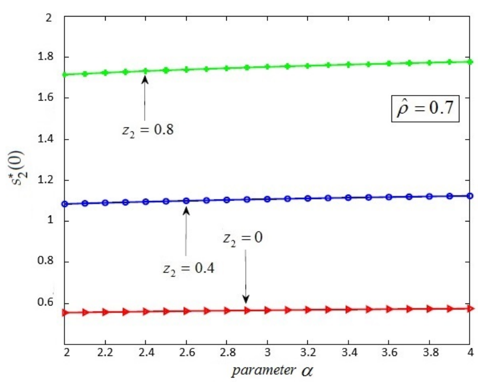


Figure 16. The effect of parameter $\alpha$ on the optimal reinsurance strategy $s_2^*(0)$ when $\hat{\rho} = 0.7$

are derived for both insurers. Finally, numerical examples proposed to illustrate the impacts of model parameters on the optimal strategies. In these examples, we set different values for risk model parameters. In Subsection 6.1, we investigated the effect of parameters $\gamma$, loss rate of the corporate coupon bond parameter and $z_1$ and $z_2$ on the optimal investment strategies $e_1^*(0)$ and $e_2^*(0)$. In Subsection 6.2, we investigated the effect of parameters $\zeta_1$, $\zeta_2$, $z_1$ and $z_2$ on the optimal reinsurance strategies $q_1^*(0)$ and $q_2^*(0)$, separately. In addition, in Subsection 6.3, the effects of correlation coefficient $\hat{\rho}$ and parameter $\alpha$ on the optimal reinsurance strategies are studied.

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