STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–14.
Published online in International Academic Press (www.IAPress.org)

A Characterization of a Subclass of Separate Ratio-Type Copulas

Ziad Adwan [1], Nicola Sottocornola [2,*]

[1] *Department of Mathematics, Liwa University, UAE*
[2] *Department of Mathematics, NYUAD, UAE*

Abstract Copulas are essential tools in statistics and probability theory, enabling the study of the dependence structure between random variables independently of their marginal distributions. Among the various types of copulas, Ratio-Type Copulas have gained significant attention due to their flexibility in modeling joint distributions. This paper focuses on Separate Ratio-Type Copulas, where the dependence function is a separate product of univariate functions.
We revisit a theorem characterizing the validity of these copulas under certain assumptions, generalize it to broader settings, and examine the conditions for reversing the theorem in the case of concave generating functions. To address its limitations, we propose new assumptions that ensure the validity of separate copulas under specific conditions. These results refine the theoretical framework for separate copulas, extending their applicability to pure mathematics and applied fields such as finance, risk management, and machine learning.

Keywords Bivariate Copulas, Ratio-Type Copulas

AMS 2010 subject classifications 60E05, 62H05, 62H20.

DOI: 10.19139/soic-2310-5070-2493

1. Introduction

Copulas are powerful tools in statistics and science for modeling and analyzing complex relationships between random variables. Unlike traditional methods that focus on linear correlations, copulas comprehensively capture dependencies, including tail dependence and asymmetries. This makes them particularly valuable in fields where understanding intricate interdependencies is critical [14].

One of the primary advantages of copulas is their ability to separate the marginal distributions of random variables from their dependence structure. This flexibility enables researchers to model the behavior of individual variables using appropriate distributions while independently specifying their dependence [15]. For instance, in finance, copulas are used to analyze the joint risk of assets, allowing for better portfolio optimization and risk management [10].

In environmental science, copulas help model the interplay between variables like rainfall and temperature, enabling more accurate predictions of extreme weather events [11]. Similarly, in medicine, they are used to study the correlation between biomarkers and disease outcomes, advancing personalized treatment plans [12].

Other applications can be found in [1] and [6].

Beyond applied sciences, copulas play a key role in machine learning, reliability analysis, and econometrics. By capturing the essence of dependence structures, copulas provide a versatile framework for addressing real-world problems characterized by uncertainty and interdependence, making them indispensable in modern statistical analysis.

*Correspondence to: Nicola Sottocornola (Email:ns6159@nyu.edu). Department of Mathematics, NYUAD, UAE.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Formally, a bivariate copula $C(u,v)$ is a function that maps the unit square $S=[0,1]^2$ to $I=[0,1]$, satisfying the following conditions:

1. $C(u,0)=C(0,v)=0$ for all $(u,v)\in S$,
2. $C(u,1)=u$ and $C(1,v)=v$ for all $u,v\in I$,
3. For any $(u_1,v_1),(u_2,v_2)\in S$ such that $u_1\le u_2$ and $v_1\le v_2$,

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\ge 0. \tag{1}$$

These properties ensure that a copula captures the dependency structure of a joint distribution while being independent of the marginal distributions of the random variables.

One particularly flexible family of copulas is the *Ratio-Type Copulas*, which are defined as:

$$D_\theta(u,v)=\frac{uv}{1-\theta\phi(u,v)}\,,\qquad 0\le u,\ v\le 1\,,\ \theta\in\mathbb{R}$$

where ϕ is a real function defined on S. In recent years they attracted considerable attention ([2], [3], [4], [5], [8], [13]). To simplify the analysis, researchers have focused on *Separate Ratio-Type Copulas*, a subclass where the dependence function $\phi(u,v)$ is expressed as a separable product of two univariate functions that we assume differentiable a.e.:

$$D_\theta(u,v)=\frac{uv}{1-\theta f(u)g(v)}\,,\qquad 0\le u,\ v\le 1\,,\ \theta\in\mathbb{R}. \tag{2}$$

Now let's consider the function G defined on S in this way:

$$G=(f-uf')(g-vg')-2uvf'g' \tag{3}$$

and define

$$\alpha_1=\min_S(G)\qquad \alpha_2=\max_S(G)\qquad a=-f'(1)\qquad b=-g'(1)\,. \tag{4}$$

In a nice paper published in 2024 [8], El Ktaibi, Bentoumi and Mesfioui examine the conditions under which (2) is a valid copula. Given the assumptions:

A1. $f(1)=g(1)=0$.
A2. f and g are strictly monotonic functions.
A3. $\dfrac{f(u)g(v)}{f(0)g(0)}\le 1-uv$.

they proved that D_θ is a valid copula provided that $1/\alpha_1\le\theta\le 1/\alpha_2$.

The aim of this paper is:

- To prove that under the Assumptions A1, A2, A3 the theorem cannot be reversed.
- To provide additional Assumptions to make it possible.

2. Bivariate copulas

Before moving forward, we recall the general definition of a bivariate copula C, assuming that all the derivatives involved exist a.e. ([14], [9]):†

Definition 2.1
The function $C\ :\ S\longrightarrow I$ is a bivariate copula if:

†Throughout this paper all the equations and inequalities involving derivatives of f and g are implicitly assumed to hold a.e.

1. $C(u,0) = C(0,v) = 0,\ C(u,1) = u,\ C(1,v) = v,\ \forall\, u,v \in I$
2. $\dfrac{\partial^2 C}{\partial u \partial v} \geq 0,\ \forall\,(u,v) \in S$

Here inequality (1) has been replaced with the more comfortable condition 2. on the second derivative. In the case of copulas of the form (2) the first condition is trivially verified. The second one reduces to (see [8]):

$$\frac{1 - \theta\,[(f - uf')(g - vg') - 2D_\theta f'g']}{(1 - \theta fg)^2} \geq 0$$

or, which is the same, to

$$1 - \theta\,[(f - uf')(g - vg') - 2D_\theta f'g'] \geq 0 \tag{5}$$

3. The Theorem

We start this section with a couple of simple observations:

Remark 3.1
We can assume, without loss of generality, that $f(0) = g(0) = 1$.

Proof
It's enough to remark that, if (2) is a valid copula, so is $\tilde{D} = uv/(1 - \tilde{\theta}\tilde{f}\tilde{g})$ with $\tilde{f} = f/f(0),\ \tilde{g} = g/g(0)$ and $\tilde{\theta} = f(0)g(0)\theta$. □

Remark 3.2
If $M = \max(f - uf')$ and $N = \max(g - vg')$ then $\max\limits_{\partial S}(G) = \max\{M, N\}$.

Proof
Let's find the maximum of G on ∂S:

$$\begin{aligned}
v = 0 \quad &\Longrightarrow \quad G(u,0) = f(u) - uf'(u) \quad \Longrightarrow \quad \max(G(u,0)) = M \\
v = 1 \quad &\Longrightarrow \quad G(u,1) = (f(u) - uf'(u))(-g'(1)) - 2g'(1)uf'(u) = -g'(1)(f(u) + uf'(u)) \\
&\Longrightarrow \quad \max(G(u,1)) = -g'(1)
\end{aligned}$$

because $f + uf' \leq 1$. Analogously

$$\begin{aligned}
u = 0 \quad &\Longrightarrow \quad \max(G(0,v)) = N \\
u = 1 \quad &\Longrightarrow \quad \max(G(1,v)) = -f'(1)
\end{aligned}$$

Remembering that $-f'(1) = (f - uf')\big|_{u=1} \leq M$ and $-g'(1) = (g - vg')\big|_{v=1} \leq N$, we conclude that

$$\max_{\partial S}(G) = \max\{M, N\}\,. \tag{6}$$

□

Conditions A1, A2, A3 can be simplified in light of Remark 3.1:

A1. $f(0) = g(0) = 1\,,\ f(1) = g(1) = 0$.
A2. f and g are strictly decreasing functions.
A3. $fg \leq 1 - uv\,.$

So finally the starting point of our investigation is:

Theorem 3.3
Let f and g verify A1, A2, A3. Then

$$D_\theta \text{ is a valid copula} \quad \Longleftarrow \quad \theta \in \left[\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right]$$

Proof
See Theorem 1 in [8]. □

4. The restriction on the concavity

Assumptions A1, A2, A3 alone cannot guarantee the converse of Theorem 3.3 as shown in the following example:

Example 4.1
Let $f(u) = (1-u)^3$ and $g(v) = (1-v)^3$. The minimum and maximum of G are reached on the diagonal $v = u$ where G has the expression

$$G(u,u) = (f(u) - uf'(u))^2 - 2u^2(f'(u))^2 = -(-1+u)^4\left(14u^2 - 4u - 1\right)$$

so, according to (4), we have

$$\begin{aligned} \alpha_1 &= \min_I(G(u,u)) = G\left(\frac{4}{7}, \frac{4}{7}\right) = -\frac{729}{16807} \Longrightarrow \frac{1}{\alpha_1} \approx -23.0549 \\ \alpha_2 &= G(0,0) = 1. \end{aligned}$$

If Theorem 3.3 were reversible, θ would be constrained to $[-23.0549, 1]$. However, as shown in Figure 1, inequality (5) holds even for $\theta = -30$, indicating that the lower bound is overly restrictive. As a matter of fact, the minimum θ verifying (5) is $\theta_{\min} \approx -36.1903$, significantly smaller than $1/\alpha_1$.

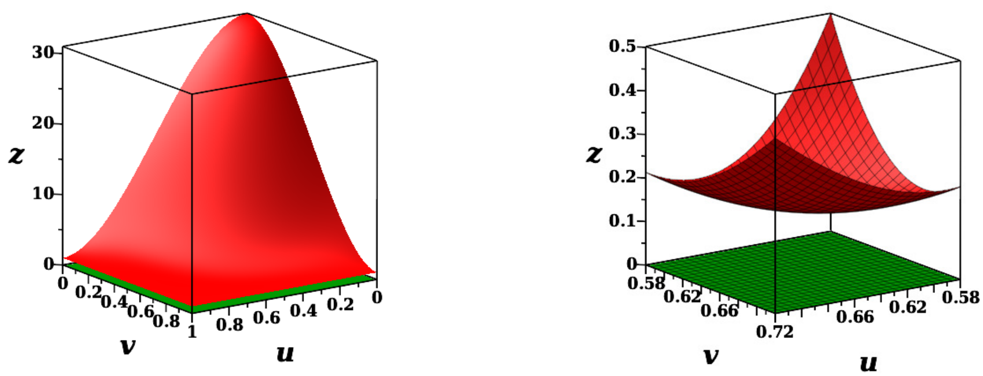

Figure 1. Inequality (5) with $\theta = -30$ on S (left) and zoomed around the minimum point (right).

In order to avoid these situations we restrict our analysis to the case where f and g are globally concave down.[‡] Therefore we will use the following assumptions:

[‡]What we mean is that, if A and B are two points of the graph, the segment $[AB]$ is below the graph itself. This is obviously stronger than the condition that the second derivative is less or equal to zero a.e. For instance our condition, together with B2, implies that the function is continuous.

B1. $f(0) = g(0) = 1\,,\ f(1) = g(1) = 0$.
B2. f and g are strictly decreasing functions.
B3. f and g are globally concave functions.

We introduce now a family of continuous functions $h_M,\ M \geq 1$:

$$h_M(u) = \begin{cases} 1 & 0 \leq u \leq 1 - \dfrac{1}{M}\,, \\ M \cdot (1-u) & 1 - \dfrac{1}{M} < u \leq 1\,. \end{cases} \tag{7}$$

Remark 4.2
If f and g verify B1, B2, B3 we have $fg \leq \alpha_2(1-uv)$.

Proof
Because $f - uf'$ and $g - vg'$ are increasing functions, we assume $M = \max(f-uf') = a,\ N = \max(g - vg') = b$ and, without loss of generality, $\alpha_2 = \max\{M, N\} = M$. Because

$$f \leq h_M \quad \text{and} \quad g \leq h_N$$

it is enough to prove the result with $f = h_M$ and $g = h_N$.

- $f = g = 1\,.$

$$\begin{aligned} 1 &\leq -\frac{1}{M} + 2 \\ 1 &\leq M\left[1 - \left(1 - \frac{1}{M}\right)^2\right] \\ 1 &\leq M\left[1 - \left(1 - \frac{1}{M}\right)\left(1 - \frac{1}{N}\right)\right] \\ fg &\leq M(1-uv) \end{aligned}$$

- $f = 1,\ g = N \cdot (1-v)\,.$

$$\begin{aligned} N(1-v) &\leq M(1-v) \\ N(1-v) &\leq M\left[1 - \left(1 - \frac{1}{M}\right)v\right] \\ N(1-v) &\leq M(1-uv) \\ fg &\leq M(1-uv) \end{aligned}$$

- $f = M \cdot (1-u),\ g = N \cdot (1-v)\,.$

$$\begin{aligned} 1 &\leq \frac{1-uv}{1-u} \\ \frac{1}{1-v} &\leq \frac{1-uv}{(1-u)(1-v)} \\ N &\leq \frac{1-uv}{(1-u)(1-v)} \\ MN(1-u)(1-v) &\leq M(1-uv) \\ fg &\leq M(1-uv) \end{aligned}$$

where the third inequality follows from $N \leq \dfrac{1}{1-v}$ (see the definition of h_N).

□

5. A preliminary result

The possibility to reverse Theorem 3.3 is related to the position of the maximum of G in S.

Theorem 5.1
Let D_θ be the copula (2). Assume that f and g verify B1, B2, B3 and $\alpha_2 = \max_{\partial S}(G)$. Then

$$D_\theta \text{ is a valid copula} \quad \Longrightarrow \quad \theta \in \left[\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right].$$

Proof
First note B3 implies that $f - uf'$ and $g - vg'$ are increasing functions so $M = a$ and $N = b$. The maximum of G on ∂S is now $\max\{a, b\}$ because of (6). But the maximum of G is reached on the boundary of S by hypothesis, so

$$\alpha_2 = \max\{a, b\}. \tag{8}$$

D_θ being a valid copula, θ has to verify (5) for all (u, v) in S. In particular:

$$v = 0 \quad \Longrightarrow \quad 1 - \theta(f - uf') \geq 0 \quad \Longrightarrow \quad \theta \leq \frac{1}{a}$$
$$u = 0 \quad \Longrightarrow \quad 1 - \theta(g - vg') \geq 0 \quad \Longrightarrow \quad \theta \leq \frac{1}{b}$$

so finally $\theta \leq 1/\max\{a, b\} = 1/\alpha_2$ because of (8).

The proof for $1/\alpha_1$ is similar:
$G = fg - vfg' - uf'g - uvf'g'$ where the first three terms are non-negative. So

$$G \geq -uvf'g' \geq -f'g' \geq -ab \quad \Longrightarrow \quad \alpha_1 \geq -ab. \tag{9}$$

But $G(1, 1) = -ab$ so $\alpha_1 = -ab$. Replacing now $u = v = 1$ in (5) we get

$$1 - \theta(-ab) \geq 0 \quad \Longrightarrow \quad \theta \geq \frac{1}{-ab} = \frac{1}{\alpha_1}.$$

□

So the problem is now to identify those functions f and g that generate the function G in (3) with the property that the maximum is reached on the boundary of S.

6. The restriction on the derivatives

In this section, we will impose another constraint on f' and g' to ensure that the maximum of G is on the boundary of S as it is in Theorem 5.1. If $1 \leq \max\{a, b\} < 2$, then we can construct functions f and g for which the maximum of G is not on the boundary of S and the following is one such example. To start, consider the continuous functions (7) with $M \geq 1$.

For such functions we can observe that we have, in the right neighborhood of $u = 1 - 1/M$:

$$u \approx 1 - \frac{1}{M} \qquad h(u) \approx 1 \qquad -h'(u) = M. \tag{10}$$

Restricting G to the diagonal of S, and replacing these values of h for both f and g, we find:

$$G(u, u) = (h(u) - uh'(u))^2 - 2u^2(h'(u))^2 \approx -M^2 - 2 + 4M > M = \max_{\partial S}(G)$$

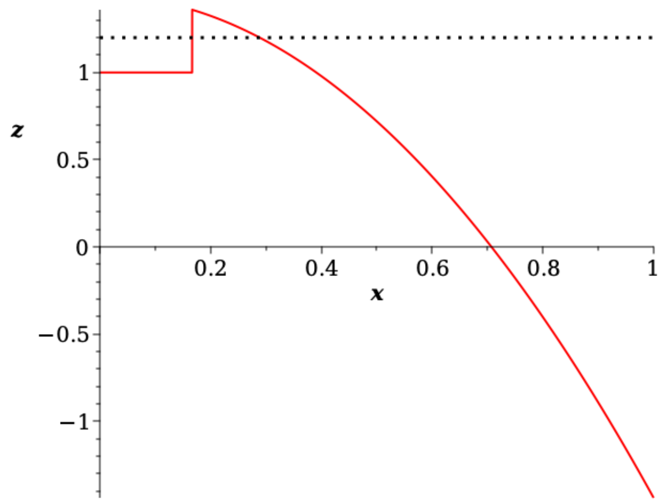

Figure 2. The function $z = G(u,u)$ with $f = h_{1.2}$ and $z = 1.2 = \max_{\partial S}(G)$ (dotted).

if $1 < M < 2$ (see Figure 2).

The problem with this (symmetric) example is that f and $-f'$ can simultaneously have values that are very close to their maximum possible values, that are $f \approx 1,\ -f' \approx a$. If we want to rule out such cases we have to introduce an additional bound on the values of the function and its derivative:

B4. $H = f \cdot (1 - vg') + g \cdot (1 - uf') \leq \max\{a, b\} + 1.$

Before moving further it could be useful to check if this new condition is reasonable, testing some simple examples where the maximum of G is clearly reached on the boundary of S. We use as a benchmark the examples provided in Table 1 in [8].

The results in Table **??** (see Appendix 1) should convince the reader that B4 is a reasonable choice if we want to get rid of functions like (7), while preserving a great variety of well known functions.

Example 6.1
We examine condition B4 on one example:

$$f(x) = 1 - u^n \qquad g(v) = 1 - v^m \qquad 1 \leq m \leq n$$

we have $a = -f'(1) = n,\ b = -g'(1) = m$ and so $\max\{a, b\} = n$. It can be easily checked that there is only one critical point (u_0, v_0) of H in the interior of $S:\ (u_0, v_0) = \left(\left(\frac{m-1}{n+m}\right)^{1/n}, \left(\frac{n-1}{n+m}\right)^{1/m}\right)$. A simple calculation gives

$$H(u_0, v_0) = 2 + \frac{(n-1)(m-1)}{n+m} \leq 2 + (n-1) = n+1\,.$$

On the boundary of S the maximum of H is $H(1,0) = n+1$. So finally condition B5 reduces to

$$\max_S(H) = n + 1 \leq \max\{a, b\} + 1 = n + 1.$$

Now, we state the main result of this paper.

7. The inverse of the Theorem

We are finally ready to prove the following

Theorem 7.1
Let

$$D_\theta(u,v) = \frac{uv}{1-\theta f(u)g(v)}\,, \qquad 0 \le u,\, v \le 1\,,\, \theta \in \mathbb{R}$$

where f and g verify the following conditions:

B1. $f(0) = g(0) = 1\,,\; f(1) = g(1) = 0.$
B2. f and g are strictly decreasing functions.
B3. f and g are globally concave down.
B4. $f \cdot (1 - vg') + g \cdot (1 - uf') \le \max\{a, b\} + 1.$

Then:

$$D_\theta \text{ is a valid copula} \qquad \Longleftrightarrow \qquad \theta \in \left[\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right]$$

Proof

$\Longleftarrow$) We have $\alpha_2 \ge 1$ and so $\theta \le \dfrac{1}{\alpha_2} \le 1$. We start our proof checking that $0 \le D_\theta \le 1$.

$$\theta fg \le fg \quad \Longrightarrow \quad 1 - \theta fg \ge 1 - fg \ge 0 \quad \Longrightarrow \quad D_\theta \ge 0\,.$$

$$\text{Remark 4.2} \quad \Longrightarrow \quad fg \le \alpha_2(1-uv) \quad \Longrightarrow \quad \frac{fg}{\alpha_2} \le 1 - uv \quad \Longrightarrow \quad \theta fg \le 1 - uv$$
$$\Longrightarrow \quad 1 - \theta fg \ge uv \quad \Longrightarrow \quad D_\theta \le 1\,.$$

The last thing we have to prove is (5). If $\theta > 0$ we have

$$1 - \theta\,[(f - uf')(g - vg') - 2D_\theta f'g'] \ge 1 - \theta G \ge 1 - \theta\alpha_2 \ge 0$$

and similarly, if $\theta < 0$,

$$1 - \theta\,[(f - uf')(g - vg') - 2D_\theta f'g'] \ge 1 - \theta G \ge 1 - \theta\alpha_1 \ge 0\,.$$

$\Longrightarrow$) According to Theorem 5.1 we only have to prove that $\alpha_2 = \max\limits_{\partial S}(G) = \max\{a, b\}$. If we remember that $f - uf'$ and $g - vg'$ are increasing functions we immediately realize that their minimum is 1 and so

$$uf' \le f - 1 \le 0 \qquad vg' \le g - 1 \le 0\,.$$

Multiplying the previous inequalities, we get $uvf'g' \ge (f-1)(g-1)$ that implies

$$-uvf'g' \le -fg + f + g - 1\,. \tag{11}$$

Now

$$\begin{aligned} G &= (f - uf')(g - vg') - 2uvf'g' \\ &= fg - vfg' - ugf' - uvf'g' \\ &\overset{(11)}{\le} fg - vfg' - ugf' - fg + f + g - 1 \\ &= [f(1 - vg') + g(1 - uf')] - 1 \\ &\overset{(B4)}{\le} [\max\{a,b\} + 1] - 1 = \max\{a, b\}\,. \end{aligned}$$

So $\alpha_2 \leq \max\{a,b\}$. But $\max\{a,b\} = \max_{\partial S}(G)$ because of (6) so $\alpha_2 \geq \max\{a,b\}$.
We finally obtain that $\alpha_2 = \max\{a,b\}$.

□

8. Final remarks and open problems

1. **Inequality in B4.** In Example 6.1 we showed that condition B4 reduced to $\max(H) = \max\{a,b\}+1$. This is not a specific feature of this particular example. It is always the case. A simple calculation shows that, if B4 is verified, then:

$$H(1,0) = a+1\,, \quad H(0,1) = b+1 \quad \Longrightarrow \quad \max(H) = \max\{a,b\}+1.$$

2. **Numeric check of condition B4.** In Appendix 2 we added a Python code verifying condition B4. It also produces a contour plot of H with the localization of the point of maximum.
3. **Condition A3** This condition is quite restrictive. Note that, at least in the case where f and g are concave functions, it can be replaced by B4 (Theorem 7.1). This is a significant generalization. It is quite easy to find example of copulas verifying B4 and not A3. Take for instance the symmetric copula generated by

$$f = -0.6u^5 + 2.4u^4 - 2.8u^3 + 1 \tag{12}$$

In the symmetric case, A3 becomes $f(u)f(v) \leq 1 - uv$ and, on the diagonal, it reduces to $f \leq \sqrt{1-u^2}$. This inequality is clearly not verified by the function in (12) (Figure 3, left) so A3 is not verified. On the other hand the function H has its maximum at $(0,1)$ and $H(0,1) = a+1$ so B4 is verified (Figure 3, right).

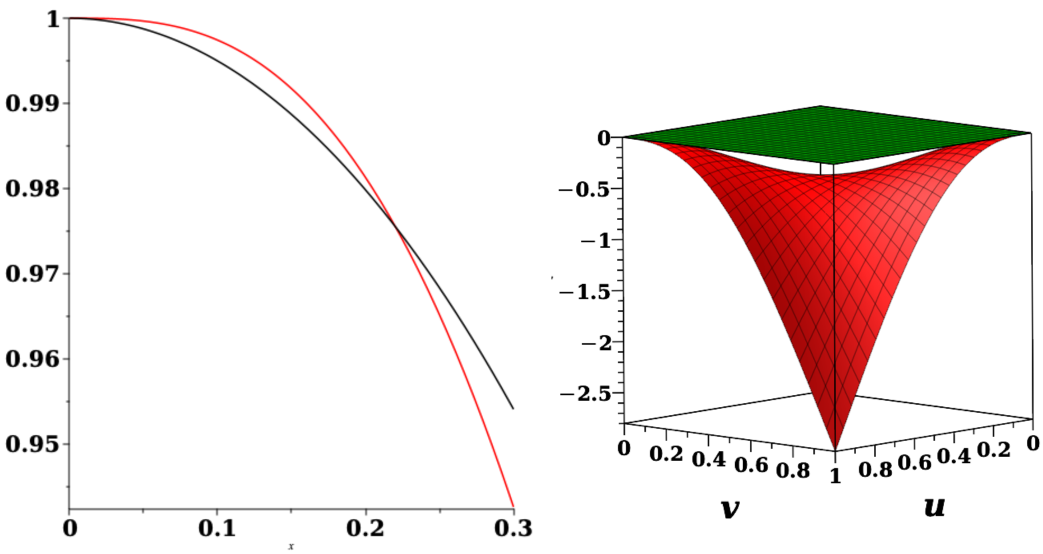

Figure 3. The graph of f in red and the curve $y = \sqrt{1-x^2}$ in black (left). The surface $z = H - a - 1$ (right).

Other examples are given in Table 1. Graphical evidence collected on a large number of examples suggests that $A3 \implies B4$ and so B4 always represents an extension of the previous condition (at least in the case of concave functions). Nevertheless this result needs a theoretical proof.

4. **Geometric interpretation of B4.** The question of what exactly condition B4 implies for the geometry of the graphs of f and g remains an open problem. Numerical analysis conducted on the symmetric cases in Table 1

gives an upper bound for the curvature k of 3.4. We are, at present, unable to provide any theoretical support for this estimation.

5. **Extension to piecewise functions.**
Extension to (some) piecewise functions can be immediately obtained replacing B3 with the more general condition:
$\tilde{\text{B}}3$: f' and g' are decreasing functions.
This condition allows for jump discontinuities in the graphs of f and g. All the proofs remain the same. As a matter of fact the condition B3 on the concavity was only used, in this paper, to infer $\tilde{\text{B}}3$.
6. **Extension to concave up functions.**
First of all B4 cannot be stated in the form
B4. $H = f \cdot (1 - vg') + g \cdot (1 - uf') \leq \max\{a, b\} + 1$.
This would rule out all f and g with derivative zero at 1 ($a = b = 0$). Even simple copulas, verifying Theorem 7.1, like:
$$f(u) = \frac{1-u}{1+u} \qquad g(v) = \frac{1-v}{1+v} \tag{13}$$
would be rejected. In this case indeed $\max\{a, b\} = 1/2$ but $\max H = 2$. We can observe that $\max\{a, b\}$ was the maximum of G on the boundary of S in the case of concave functions. This suggests the following modified version of B4:
$\tilde{\text{B}}4$: $H = f \cdot (1 - vg') + g \cdot (1 - uf') \leq \max_{\partial S}(G) + 1$.
This new condition works fine with (13) but unfortunately it is also verified by the copula in Example 4.1, where Theorem 7.1 does not hold. So finally it looks like B4 is excessively restrictive while $\tilde{\text{B}}4$ is not restrictive enough. This is definitely a tricky problem. It will be the subject of another paper.

Appendix 1. Synoptic comparative table

$f(u)$	$g(v)$	Conditions	B1	B2	A3	B3	B4
$1-u^n$	$1-v^n$	$1 \leq n \leq 2$	T	T	T	T	T
		$2 < n$	T	T	**F**	T	T
$\log_b(u + b(1-u))$	$\log_b(v + b(1-v))$	$1 < b$	T	T	T	T	T
$\cos(\pi u/2)$	$\cos(\pi v/2)$		T	T	T	T	T
$1-u$	$\log_b(v + b(1-v))$	$1 < b \leq 41$	T	T	T	T	T
		$41 < b$	T	T	**F**	T	T
$\cos(\pi u/2)$	$1-v$		T	T	T	T	T
$\log_b(v + b(1-v))$	$\cos(\pi v/2)$	$1 < b$	T	T	T	T	T
$(1-u)e^{cu}$	$(1-v)e^{cv}$	$0 \leq c \leq 1$	T	T	T	T	T
$\dfrac{e^{au} - e^a}{1 - e^a}$	$\dfrac{e^{av} - e^a}{1 - e^a}$	$0 < a \leq 3.7$	T	T	T	T	T
		$3.7 < a$	T	T	**F**	T	T
$h_M(u)$	$h_M(v)$	$1 < M < 2$	T	T	**F**	T	**F**

Table 1. Testing conditions B1,. . . ,B4 (T=True, F=False). We added A3 to show how restrictive this condition is.

Appendix 2. Python code to check condition B4

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from sympy.utilities.lambdify import lambdify
import plotly.graph_objects as go
import os
import webbrowser
import sys

def run_analysis():
    u, v = sp.symbols('u v')

    # User inputs the functions
    f_expr = sp.sympify(input("\nEnter the function f(u): "))
    g_expr = sp.sympify(input("Enter the function g(v): "))

    # Derivatives
    f_prime = sp.diff(f_expr, u)
    g_prime = sp.diff(g_expr, v)

    # Define H(u, v)
    H_expr = f_expr * (1 - v * g_prime) + g_expr * (1 - u * f_prime)
    H_simplified = sp.simplify(H_expr)

    print(f"\nH(u, v) = f(u) * [1 - v * g'(v)] + g(v) * [1 - u * f'(u)]")
    print(f"Simplified H(u, v) = {H_simplified}")

    # Compute symbolic a, b, max(a,b), and max(a,b)+1
    a_symbolic = -f_prime.subs(u, 1)
    b_symbolic = -g_prime.subs(v, 1)
    max_ab_symbolic = sp.Max(a_symbolic, b_symbolic)
    max_ab_plus_1 = max_ab_symbolic + 1

    print(f"\na = -f'(1) = {a_symbolic}")
    print(f"b = -g'(1) = {b_symbolic}")
    print(f"max(a, b) = {max_ab_symbolic}")
    print(f"max(a, b) + 1 = {max_ab_plus_1}")

    # H_max symbolic over corners and value 2
    H_max_symbolic = sp.Max(
        sp.simplify(H_simplified.subs(u, 0).subs(v, 0)),
        sp.simplify(H_simplified.subs(u, 0).subs(v, 1)),
```

```
    sp.simplify(H_simplified.subs(u, 1).subs(v, 0)),
    sp.simplify(H_simplified.subs(u, 1).subs(v, 1)),
    sp.simplify(2)
)
print(f"\nSymbolic H_max = {H_max_symbolic}")

# Numerical evaluation
H_func = lambdify((u, v), H_simplified, modules=["numpy"])
U = np.linspace(0, 1, 100)
V = np.linspace(0, 1, 100)
U_mesh, V_mesh = np.meshgrid(U, V)

try:
    H_values = H_func(U_mesh, V_mesh)
    H_values = np.nan_to_num(H_values)
except Exception as e:
    print(f"Error in numerical evaluation: {e}")
    H_values = np.zeros_like(U_mesh)

# Find maximum value
H_max_numeric = np.max(H_values)
max_indices = np.unravel_index(np.argmax(H_values, axis=None), H_values.shape)
max_u, max_v = U_mesh[max_indices], V_mesh[max_indices]
max_ab_plus_1_numeric = float(max_ab_plus_1.evalf())

# Show numeric results
print(f"\nNumerical H_max = {H_max_numeric} at (u, v) = ({max_u}, {max_v})")
print(f"Numerical max(a, b) + 1 = {max_ab_plus_1_numeric}")

if np.isclose(H_max_numeric, max_ab_plus_1_numeric, atol=1e-14):
    comparison = "H_max is equal to max(a, b) + 1"
elif H_max_numeric > max_ab_plus_1_numeric:
    comparison = "H_max is greater than max(a, b) + 1"
else:
    comparison = "max(a, b) + 1 is greater than H_max"
print(f"Comparison: {comparison}")

# Contour Plot
plt.figure(figsize=(8, 6))
contour = plt.contourf(U_mesh, V_mesh, H_values, cmap="viridis", levels=50)
plt.colorbar()
plt.scatter(max_u, max_v, color="red", label=f"Max H = {H_max_numeric:.4f} at ({max_u:.4f}, {max_v:.4f})")
plt.xlabel("u")
plt.ylabel("v")
plt.title("Contour Plot of H(u, v)")
plt.legend()
```

```
plt.tight_layout()
plt.show()

# 3D Plot
fig = go.Figure()
fig.add_trace(
    go.Surface(
        z=H_values,
        x=U,
        y=V,
        colorscale="Viridis",
        name="H(u, v)"
    )
)

# Add red plane: z = max(a, b) + 1
red_plane = np.full_like(H_values, max_ab_plus_1_numeric)
fig.add_trace(
    go.Surface(
        z=red_plane,
        x=U,
        y=V,
        colorscale=[[0, "rgba(255,0,0,0.4)"], [1, "rgba(255,0,0,0.4)"]],
        showscale=False,
        name=f"z = max(a, b) + 1 = {max_ab_plus_1_numeric:.4f}"
    )
)

# Highlight max point
fig.add_trace(
    go.Scatter3d(
        x=[max_u],
        y=[max_v],
        z=[H_max_numeric],
        mode="markers",
        marker=dict(size=8, color="red"),
        name=f"Max H = {H_max_numeric:.4f}"
    )
)

fig.update_layout(
    title="3D Plot of H(u, v) with Max Point and Plane z = max(a, b) + 1",
    scene=dict(
        xaxis_title="u",
        yaxis_title="v",
        zaxis_title="H(u, v)"
```

```
        )
    )

    output_file = "3D_Plot_H_function.html"
    fig.write_html(output_file)
    print(f"\n3D plot saved to {output_file}. Opening in web browser...")

    try:
        webbrowser.open('file://' + os.path.realpath(output_file))
    except Exception as e:
        print(f"Could not open in browser: {e}")

# ==============================
# Loop to allow multiple runs
# ==============================
print("=== Welcome to the H(u,v) Analyzer ===")

while True:
    run_analysis()
    again = input("\n Would you like to enter new functions f(u) and g(v)? (yes/no): ").strip().lower()
    if again not in ['yes', 'y']:
        print("Goodbye!")
        break
```

REFERENCES

1. E. M. Almetwally, A. Fayomi, M. E. Qura, *Advanced Copula-Based Models for Type II Censored Data: Applications in Industrial and Medical Settings*, Mathematics, MDPI, vol. 12 (12), pages 1-35, 2024.
2. C. Chesneau, *Some new ratio-type copulas: Theory and properties*, Appl. Math. 49, 79–101, 2022.
3. C. Chesneau, *A Collection of Two-Dimensional Copulas Based on an Original Parametric Ratio Scheme*, Symmetry, 15, 977, 2023. https://doi.org/10.3390/sym15050977
4. C. Chesneau, *Theoretical Advancements on a Few New Dependence Models Based on Copulas with an Original Ratio Form*, Modelling 4(2), 102-132, 2023. https://doi.org/10.3390/modelling4020008
5. C. Chesneau, *Exploring Two Modified Ali-Mikhail-Haq Copulas and New Bivariate Logistic Distributions*, Pan-Am. J. Math. 3, 4, 2024. https://doi.org/10.28919/cpr-pajm/3-4
6. E. Dehghan et al. *Introducing Copula as a Novel Statistical Method in Psychological Analysis*, Int. J. Environ. Res. Public Health, 18(15), 7972, 2021. https://doi.org/10.3390/ijerph18157972
7. F. Durante, J. Fernández-Sánchez, W. Trutschnig, *Solution to an open problem about a transformation on the space of copulas*, Depend. Model. 2, 65–72, 2014.
8. F. El Ktaibi, R. Bentoumi and M. Mesfioui, *On the Ratio-Type Family of Copulas*, Mathematics, 12, 1743, 2024. https://doi.org/10.3390/math12111743
9. F. El Ktaibi, R. Bentoumi, N. Sottocornola, M. Mesfioui, *Bivariate Copulas Based on Counter-Monotonic Shock Method*, Risks, 10, 202. 2022. https://doi.org/10.3390/risks10110202
10. P. Embrechts, A. J. McNeil & D. Straumann, *Correlation and Dependence in Risk Management: Properties and Pitfalls*, In M. A. H. Dempster (Ed.), Risk Management: Value at Risk and Beyond, 176-223, Cambridge University Press, 2022. http://dx.doi.org/10.1017/CBO9780511615337.008
11. C. Genest, A.-C. Favre, *Everything you always wanted to know about copula modeling but were afraid to ask*, Journal of Hydrologic Engineering, 12(4), 347–368, 2007.
12. H. Joe, *Dependence Modeling with Copulas*, CRC Press, 2014.
13. R. Mesiar, A. Stupňanová, *Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications*, Fuzzy Sets Syst. 261, 112–123, 2015.
14. R. B. Nelsen *An Introduction to Copulas*, Springer, 2006.
15. A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publications de l'Institut de Statistique de l'Université de Paris, 8, 229–231. 1959.