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The uniformly more powerful tests than the likelihood ratio test using intersection-union hypotheses for exponential distribution

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Abstract In practice, we may encounter hypotheses that the parameters under test have typical restrictions. These restrictions can be placed in the null or alternative hypotheses. In such a case, the hypothesis is not included in the classical hypothesis testing framework. Therefore, statisticians are looking for the more powerful tests, rather than the most powerful tests. A common method for such tests is to use intersection-union and union-intersection tests. In this paper, we derived the testing procedure of a simple intersection-union and compared it with the likelihood ratio test. We also compare the powers of two exponential sign tests, the rectangle test and smoother test, and the simple intersection-union test with the likelihood ratio test.

Keywords Intersection-union test, Likelihood ratio test, More powerful test, Rectangle test, Smoother test.

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1. Introduction

Suppose that $X_1, \ldots, X_p$ are independent random variables from the exponential distributions $X_i \sim f_i,\ \ x_i \geq 0,\ \theta_i > 0,\ \ i = 1, \ldots, p$ where θ_i is the unknown scale parameter of ith population. We wish to test

$$\begin{aligned} &H_0 : \theta_i \leq \theta_{i0}, \quad \text{for some } i,\ i = 1, \ldots, p \\ &\text{against} \\ &H_1 : \theta_i > \theta_{i0}, \quad \text{for all } i,\ i = 1, \ldots, p \end{aligned} \tag{1}$$

where $\theta_{i0},\ i = 1, \ldots, p$ are positive real value. The test given in (1) can be written as a union of p-subsets of parameter space as the null hypothesis and an intersection of their completeness as the alternative hypothesis:

$$H_0 : \cup_{i=1}^{p} \{\theta_i \leq \theta_{i0}\}, \quad \text{against} \quad H_1 : \cap_{i=1}^{p} \{\theta_i > \theta_{i0}\}. \tag{2}$$

In classical testing, the best tests are the uniformly most powerful (UMP) tests and the uniformly most powerful unbiased (UMPU) tests. These tests are designed for specific hypotheses, such as one-sided and two-sided for the parameters, and can be easily obtained in such for certain families of distributions, such as the exponential family and monotone likelihood ratio family. These tests are well documented and can be found in many textbooks (see for example Davidov and Herman [6], Lehmann and Romano [12]). In many statistical hypotheses, however, the hypotheses on the parameters are complicated, so they do not fall within the framework of classical statistical hypotheses. In such cases, the tests are not the UMP or even the UMPU. For example, to compare several mean

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populations, the null hypothesis may be equality of means against ordered means as the alternative hypothesis. The classical Neyman-Pearson approach can not be used for such hypotheses. Therefore, statisticians are not looking for the UMP tests, but they are looking for the more powerful tests. The likelihood ratio tests (LRTs) are the most commonly used approach to analyzing these hypotheses (see for examples Sasabuchi[21],[22]). The hypotheses in (2) can be replaced with $H_0 : \min_{1\leq i\leq k}\{\eta_i\} \leq 0$ against $H_1 : \min_{1\leq i\leq p}\{\eta_i\} > 0$ with only transfrorming $\eta_i = \ln(\theta_i/\theta_{i0})$, $i = 1, \ldots, p$.

We refer to the problem in (2) as a sign test, because the rejection decision depends solely on whether each parameter θ_i exceeds its benchmark θ_{i0}, which corresponds to the sign of η_i. In our context, since the θ_i are scale parameters of exponential distributions, we call the resulting procedures exponential sign tests. The sign test, which is a nonparametric test, is examined here using the simple intersection-union test (SIUT) framework. Accordingly, the hypotheses of the sign test are formulated to fit within the structure of SIUTs. This leads to sign testing problems that have attracted the attention of many researchers such as Lehmann [11], Sasabuchi [21], [9], Shirley [24], Liu and Berger [15]. Unfortunately, LRTs often lead to weak power tests or even to biased tests ([12]), and sometimes the power of tests is many times smaller than their size. As a result, the tests with the same size of LRTs with uniformly more powerful are needed. Berger [2], Liu and Berger [15], and McDermott and Wang [16] constructed classes of size-α tests that are uniformly more powerful than LRT for the case of covariance matrices, Σ, known. Liu and Berger [15] followed this for p-dimensional normal distribution with unknown mean μ and know nonsingular covariance matrices, $N_p(\mu, \Sigma)$. Berger ([1],[4]) suggested a mixing union-intersection method with the likelihood method, called UIT, and showed that in some cases is more powerful than LRT alone. Gutmann [9] constructs two tests, when $X_1, X_2, \ldots, X_p \sim f(x-\theta)$ are independent and show that they are uniformly more powerful than the uniformly most powerful monotone test in the sign testing problem. Shirley's proposed test [24] is even more powerful than Gutmann's when $p = 3$. Wang and McDermott [16], Berger [2] obtain a size-α test that is uniformly more powerful than LRT when the variance-covariance matrix is diagonal by using the intersection-union test (IUT) for hypotheses inequalities and normal means. They showed that in certain problems the LRT is not very powerful and described a test that has the same size-α and is more powerful than LRT. Also, he showed that the critical region of this test includes the rejection region of the LRT. For the special case of $p = 2$, this provides a test that is uniformly more powerful than a test discussed by Gail and Simon [8]. Berger [2] and Liu and Berger [15] constructed classes of size-α tests that are uniformly more powerful than the LRT for this problem. Their approaches consist of adding sets to the rejection region of the LRT such that tests are larger than the rejection probability of size-α LRT for any points in alternative space. Saikli and Berger [20] considered the sign test problem for a random sample from a normal population with unknown mean μ_i and unknown variance σ_i^2. They first derived the size-α LRT for this problem, and then described an SIUT that is uniformly more powerful than the LRT if the sample sizes are not all equal. Chan et al. [5] constructed two new tests to compare the independent scale parameters of an independent sample of gamma distribution that the rejection region of two news tests is similar to Liu and Berger's [15], Berger and Hsu's [3] and Saikali and Berger's [20]). They constructed a size-α uniformly more powerful test than LRT by adding additional sets to the rejection region of the LRT, named rectangle, and smoother tests. Wu et al.[26] propose a new heuristic testing procedure based on the generalized p-value approach for the sign testing problem of normal variances. Through comprehensive simulation studies, they demonstrate that their method effectively controls the type I error rate and achieves uniformly higher power compared to the likelihood ratio test and several existing methods, especially in small sample scenarios. The authors further illustrate the practical utility of their approach using real data examples. Overall, their work introduces an improved test for comparing normal variances, providing superior error control and statistical power, particularly for small sample sizes.

In this article, we first apply the testing procedure of simple intersection-union and LRT to the exponential distribution, and then we adopt the rectangle test and smoother test to the exponential distribution. We consider the testing problem (2) in the exponential distribution. Two advantages motivated us to do this study. Firstly, there are many applications of the exponential distribution with such hypotheses that can be mentioned. Secondly, the method and the results gained from this study are more analytical rather than just numerical methods gained from other distributions. Therefore, the reader can follow the results easily

$$H_0 : (\theta_1 \leq \theta_{10}) \cup (\theta_2 \leq \theta_{20}), \qquad \text{against} \qquad H_1 : (\theta_1 > \theta_{10}) \cap (\theta_2 > \theta_{20}), \tag{3}$$

where θ_{10}, θ_{20} are fixed constants. However, these tests can be applied to population p and the results are valid. The rest of the paper is as follows:
In the following section, we derive the size-α LRT and SIUT for testing (3) and we show that a SIUT is uniformly more powerful than the LRT if θ_{10} and θ_{20} are different. Section 3, is devoted to the rectangle and smoother tests for (3) in the exponential distribution, uniformly more powerful than the LRT and the SIUT. In Section 4, in a numerical approach, we compare the powers of the rectangle and the smoother test with LRT and SIUT for the sign testing problem (3). In Section 5, integrated size-adjusted and sensitivity analysis, the bootstrap method and empirical estimation of critical values are employed to correct size bias in finite-sample tests, followed by a comprehensive evaluation of the tests performance stability with respect to changes in the initial parameter values. In Section 6, we present a case study that examines the minimum reliability thresholds for a series system whose component lifetimes follow an exponential distribution. In Section 7, some concluding remarks are stated.

2. Likelihood ratio and intersection-union tests

A size-α LRT, for testing (1), rejects H_0 if

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{\sup_{\Theta_0} L(\theta_1,\ldots,\theta_p;\mathbf{x})}{\sup_{\Theta} L(\theta_1,\ldots,\theta_p;\mathbf{x})} \\ &= \max_{1\le i\le p} \frac{\sup_{\Theta_{i0}} L(\theta_1,\ldots,\theta_p;\mathbf{x})}{\sup_{\cup_j^p \Theta_{ij}} L(\theta_1,\ldots,\theta_p;\mathbf{x})} = \max_{1\le i\le p} \lambda_i(\mathbf{x}),\end{aligned} \tag{4}$$

is less than λ_0, where λ_0 is obtained such that

$$\sup_{\Theta_0} P(\lambda(\mathbf{x}) \le \lambda_0) = \alpha, \tag{5}$$

and $\mathbf{x} = (x_1,\ldots,x_p)$. $\lambda_i(\mathbf{x})$ in (4) is the LRT statistic for testing for ith individual test $H_{i0} : \theta_i \le \theta_{i0}$ against $H_{i1} : \theta_i > \theta_{i0}$ which is usaul in one-sided hypotheses testing in LRT. Therefore, the LRT statistics for exponential distribution obtained as follows:

$$\lambda(\mathbf{x}) = \max_{1\le i\le p} \lambda_i(\mathbf{x}) = \max_{1\le i\le p} \frac{x_i}{\theta_i} e^{-\frac{x_i}{\theta_{i0}}+1}.$$

To have a α size test for H_{i0} against H_{i1}, H_{i0} must be rejected if $\frac{x_i}{\theta_{i0}} > -\ln(\alpha)$ or when $x_i > -\theta_{i0}\ln(\alpha) = c_i$. As a result, H_0 is rejected if

$$x_i > \min\{c_1,\ldots,c_p\} = c_0, \quad \text{for any } i = 1,\ldots,p.$$

Berger ([1], [4]) suggested the SIUT for (1) reject H_0 when

$$x_i > -\theta_{i0}\ln(\alpha).$$

2.1. Intersecion-Union test

Consider again the testing problem of (1). The hypotheses can be rewritten as

$$H_0 : \cup_{i=1}^p \{\theta_i \le \theta_{i0}\} \qquad \text{against} \qquad H_1 : \cap_{i=1}^p \{\theta_i > \theta_{i0}\} \tag{6}$$

This is SIUT. Let R_i be the rejection region of an α-level test
$(0 < \alpha < 1)$ for $H_{i0} : \theta_i \le \theta_{i0}$ against $H_{1i} : \theta_i > \theta_{i0}$. It means that

$$P_{\theta_{i0}}(R_i) \le \alpha, \quad \text{for all } \theta_i \le \theta_{i0}.$$

Note that because H_{i0} is one-sided hypothesis testing, there is no difference between the LRT and UMP tests. Take $R = \cap_{i=1}^p R_i$ as the rejection region of H_0 against H_1

$$P_{\theta_i}(R) \le P_{\theta_{i0}}(R_i) \le \alpha, \quad \theta_i \le \theta_{i0}.$$

Therefore, in the test with R set as the rejection region, we have a α-level test for H_0 against H_1. This is the SIUT. A level-α SIUT may be quite a conservative test because its size can be much smaller than the determined value of α.

Berger ([1], Theorem 1.1.2) showed that to have a size-α test, we need only one test, say ith which has exactly α size. The important advantage of the SIUT is that in uniformly the most powerful in a class of monotone tests with its size. In a monotone class of tests, the more extreme values of cutoff points belong to the rejection region.

As a result, SIUT is reject H_0 if and only if every H_{i0} rejected;

$$x_i > -\theta_{i0}\ln(\alpha), \quad i = 1, \ldots, p.$$

3. Rectangle test and Smoother test

SIUTs are useful to give more power than LRTs with the same size and also are UMP among size-α monotone tests. However, considering nonmonotone tests, there is no guarantee to have optimal tests. In this section, we obtain the rejection region for two tests, which are not similar and are not unbiased but will be shown that are uniformly more powerful than LRT and SIUT. Having valid controls for the Type I error rate could be the main reason for their increased power on alternative.

3.1. Rectangle test

To have a clear explanation, we restrict the problem to $p = 2$, although the result is valid for any finite integer of p. Let $0 < \alpha < \frac{1}{2}$ and J is given as from the inequality, $J - 1 < \frac{1}{2\alpha} \leq J$. Define $c_1^i, \ldots, c_p^i$, $i = 1, 2$ as $c_j^i = F_i^{-1}(1 - \alpha j) = -\theta_i \ln(j\alpha)$, $j = 1, 2, \ldots, J$ with $c_J^i = F^{-1}(\frac{1}{2}) = m_i$. For $j = 1, 2, ..., J$, define

$$R_j = \Big\{(X_1, X_2) : c_j^1 \leq X_1 < c_{j-1}^1, c_j^2 \leq X_2 < c_{j-1}^2\Big\}. \tag{7}$$

The rejection region for the rectangle test can be expressed as $R = \cup_{i=1}^J R_j$, where R_1 is the rejection region of the SIUT. Now, consider

$$\begin{aligned} R_j &= \Big\{(X_1, X_2); -\theta_1 \ln(j\alpha) \leq X_1 \leq -\theta_1 \ln((j-1)\alpha) \\ &, \quad -\theta_2 \ln(j\alpha) \leq X_2 \leq -\theta_2 \ln((j-1)\alpha)\Big\}, \end{aligned} \tag{8}$$

the test with the rejection region $R = \cup_{j=1}^J R_j$ is a test for H_0 against H_1 with size α (see Theorem 1). It is more powerful than LRT and SIUT because its rejection region includes the rejection region SIUT and has extra sets with positive Lebesgue measures.

In fact

$$\begin{aligned} \beta_R(\theta_1, \theta_2) &= P_{\theta_1,\theta_2}((X_1, X_2) \in R) \\ &= \sum_{j=1}^{J} \alpha^{\frac{\theta_{10}}{\theta_1} + \frac{\theta_{20}}{\theta_2}} [j^{\frac{\theta_{10}}{\theta_1}} - (j-1)^{\frac{\theta_{10}}{\theta_1}}][j^{\frac{\theta_{20}}{\theta_2}} - (j-1)^{\frac{\theta_{20}}{\theta_2}}] \\ &= \beta_{SIUT} + \sum_{j=2}^{J} \alpha^{\frac{\theta_{10}}{\theta_1} + \frac{\theta_{20}}{\theta_2}} \left(\prod_{i=1}^{2} [j^{\frac{\theta_{i0}}{\theta_i}} - (j-1)^{\frac{\theta_{i0}}{\theta_i}}]\right), \end{aligned} \tag{9}$$

where $\beta_{SIUT}(\theta_1, \theta_2)$ is the power of SIUT. Therefore, the power of the rectangle test is larger than the power of SIUT and LRT. It should be mentioned that the rectangle test is not unbiased and not even similar because $\beta_R(\theta_1, \theta_2) = \alpha^2 < \alpha$ as θ_i tends to θ_{i0}, $i = 1, 2$. The rejection region of rectangle test, SIUT, and LRT is illustrated in Figure 1.

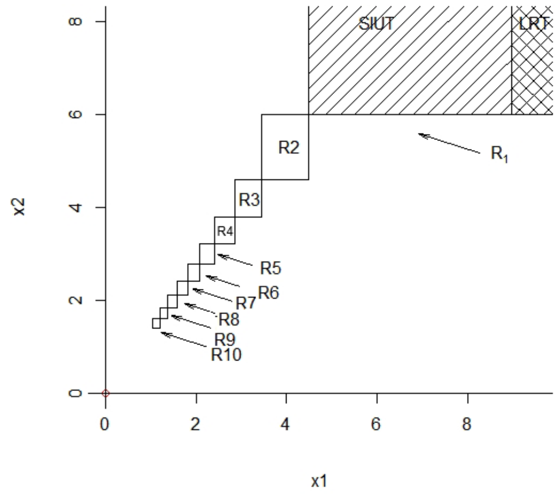

Figure 1. The rejection region of rectangle test, SIUT, and LRT for the case $\theta_{10} = 1, \theta_{20} = 2$ and $\alpha = 0.05$ for the exponential distribution.

Theorem 1
For testing problem (3), the rectangle test is a size-α test, and the rectangle test is uniformly more powerful than the size-α LRT.

Proof
To see that the rectangle test is a size-α test, let there be a θ that maximum of its θ^* and it has at least one equal to 0. Furthermore, the density of the exponential is convex and $\cup_{j=1}^{J} \{-\theta_i \ln(j\alpha) \leqslant X_i \leqslant \theta_i \ln((j-1)\alpha) \quad \text{for all} \quad i = 1, 2\}$ is a convex set. Thus we have

$$\begin{aligned} P_\theta \left(\mathbf{X} \in \cup_{j=1}^{J} R_j\right) &= P\left(\cup_{j=1}^{J} \{-\theta_i \ln(j\alpha) \leqslant X_i \leqslant \theta_i \ln((j-1)\alpha) \quad \text{for all} \quad i = 1, 2\}\right) \\ &\leqslant P_{\theta^*}\left(\cup_{j=1}^{J} \{-\theta_i \ln(j\alpha) \leqslant X_i \leqslant \theta_i \ln((j-1)\alpha) \quad \text{for all} \quad i = 1, 2\}\right) \leqslant \alpha. \end{aligned}$$

3.2. Smoother test

Again consider the test given in (1), Wang and McDermott[25] (see also Saikali and Berger[20]) consider three subsets in $[0,1] \times [0,1]$ plan as follows:

$$\begin{aligned} A_0 &= \left\{(U_1, U_2) : 1-\alpha \le U_1 \le 1, 1-\alpha \le U_2 \le 1\right\} \\ A_1 &= \left\{(U_1, U_2) :\mid U_1 - U_2 \mid \le \frac{\alpha}{2}, \frac{1}{2} \le U_1 \le 1-\alpha, \frac{1}{2} \le U_2 \le 1-\alpha\right\} \\ A_2 &= \left\{(U_1, U_2) : \frac{1}{2} \le U_2 \le U_1 - \frac{1}{2} + \frac{3\alpha}{2}, U_1 < 1-\alpha\right\} \\ &\cup \left\{(U_1, U_2) : \frac{1}{2} \le U_1 \le U_2 - \frac{1}{2} + \frac{3\alpha}{2}, U_2 < 1-\alpha\right\}. \end{aligned} \tag{10}$$

They showed that if $U_1 \sim U(0,1)$, then for $A = A_0 \cup A_1 \cup A_2$

$$P((U_1, U_2) \in A) = \alpha P(\frac{1}{2} \le U_2 \le 1) \le \alpha. \tag{11}$$

Since A is symmetric in U_1 and U_2, it is true for $U_2 \sim U(0,1)$, i. e.

$$P((U_1, U_2) \in A) = \alpha P(\frac{1}{2} \le U_1 \le 1) \le \alpha. \tag{12}$$

3.3. Geometric interpretation of the smoother rejection region:

The smoothed rejection region $A = A_0 \cup A_1 \cup A_2$ is designed based on specific geometric principles to capture both strong marginal evidence (through A_0) and balanced, simultaneous two-sided evidence, even when none of the individual components is very large. This structure is constructed to optimize the power of the test while preserving the size constraint.

Region A_0: This region represents the most powerful part where both test statistics provide strong evidence against their respective null hypotheses, corresponding to the classical intersection–union rule. Geometrically, it is a corner region in the upper-right of the (U_1, U_2) plane where both U_1 and U_2 exceed $1-\alpha$, indicating strong evidence in both variables.

Region A_1: This is a diagonal strip of width α centered around the line $u_1 = u_2$ in the interval $[1/2, 1-\alpha]$. It captures balanced moderate values, where both test statistics are approximately equal and relatively large, a sign of the alternative hypothesis being true even if neither statistic reaches the $1-\alpha$ threshold. The condition$\mid U_1 - U_2 \mid \leq \frac{\alpha}{2}$ in A_1 creates a diagonal band around the line $U_1 = U_2$. This design is motivated both geometrically and statistically:

- *Symmetry exploitation:* when both parameters deviate from their null values by the same magnitude, locations near the diagonal $U_1 \approx U_2$ are particularly informative.
- *Balanced evidence:* points near the diagonal represent cases where both statistics provide similar and moderate evidence against H_0.
- *Power optimization:* instead of requiring both statistics to be large individually (as in A_0), the region accepts moderate values when they are in agreement.

Region A_2: This part consists of asymmetric triangular extensions covering scenarios where one statistic is relatively large while the other provides moderately supportive evidence. Unlike A_0, which requires simultaneous strong evidence from both statistics, A_2 identifies unbalanced but still effective combinations of evidence for rejecting H_0.

In this section, we have described a smoother test for exponential distribution for $p = 2$. The rejection region of the smoother test for exponential distribution can be expressed as $A = A_0 \cup A_1 \cup A_2$. By substituting the cumulative distribution function of the exponential distribution in (10), we obtain the rejection region of the smoother test for the sign testing hypothesis in the exponential distribution. Now, we define a smoother test for sign testing problem (3). Let $u_1 = F_1(x_1)$, $u_2 = F_2(x_2)$, smoother test is the test that rejects H_0 if $(X_1, X_2) \in A$, the three sets can be expressed as:

$$
\begin{aligned}
A_0 &= \Big\{(X_1,X_2): X_1 \geq -\theta_1\ln(\alpha), X_2 \geq -\theta_2\ln(\alpha)\Big\}, \qquad (13)\\
A_1 &= \Big\{(X_1,X_2): -\theta_2\ln(e^{\frac{-X_1}{\theta_1}} + \frac{\alpha}{2}) \leq X_2 \leq -\theta_2\ln(e^{\frac{-X_1}{\theta_1}} - \frac{\alpha}{2}),\\
&\qquad \theta_1\ln(2) \leq X_1 \leq -\theta_1\ln(\alpha), \theta_2\ln(2) \leq X_2 \leq -\theta_2\ln(\alpha)\Big\},\\
A_2 &= \Big\{(X_1,X_2): \theta_2\ln(2) \leq X_2 \leq -\theta_2\ln(\frac{1}{2} + e^{\frac{-X_1}{\theta_1}} - \frac{3\alpha}{2}), X_1 < -\theta_1\ln(\alpha)\Big\}\\
&\cup \Big\{(X_1,X_2): \theta_1\ln(2) \leq X_1 \leq -\theta_1\ln(\frac{1}{2} + e^{\frac{-X_2}{\theta_2}} - \frac{3\alpha}{2}), X_2 < -\theta_2\ln(\alpha)\Big\},
\end{aligned}
$$

The power for the smoother test is derived as (see appendix)

$$\begin{aligned}
\beta_S(\theta_1,\theta_2) &= P_{\theta_{10},\theta_{20}}(\text{rejection region}) = P\Big((X_1,X_2)\in A_0\cup A_1\cup A_2\Big)\\
&= P\left((X_1,X_2)\in A_0\right)+P\left((X_1,X_2)\in A_1\right)+P\left((X_1,X_2)\in A_2\right)\\
&= \alpha^{\frac{\theta_{10}}{\theta_1}+\frac{\theta_{20}}{\theta_2}}+2^{-\frac{\theta_{10}}{\theta_1}-\frac{\theta_{20}}{\theta_2}}-2^{-\frac{\theta_{20}}{\theta_2}}(\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\left(2^{-\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}\\
&+ \frac{\left((\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}+\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}+\frac{\left((\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\frac{\left(\alpha^{\frac{\theta_{10}}{\theta_1}}+\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}\\
&- \alpha^{\frac{\theta_{20}}{\theta_2}}(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}+\alpha^{\frac{\theta_{10}}{\theta_1}+\frac{\theta_2}{\theta_{20}}}+2^{\frac{-\theta_{20}}{\theta_2}}(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\alpha^{\frac{\theta_{10}}{\theta_1}}2^{\frac{-\theta_{20}}{\theta_2}}\\
&+ \frac{\left(\frac{1}{2}+\alpha^{\frac{\theta_{10}}{\theta_1}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\frac{\left(\frac{1}{2}+(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}+2^{\frac{-\theta_{10}}{\theta_1}}(\frac{3\alpha}{2})^{\frac{\theta_{20}}{\theta_2}}\\
&- \alpha^{\frac{\theta_{20}}{\theta_2}}2^{\frac{-\theta_{10}}{\theta_1}}+\frac{\left(\frac{1}{2}+\alpha^{\frac{\theta_{20}}{\theta_2}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{10}}{\theta_1}+1}}{\frac{\theta_{10}}{\theta_1}+1}-\frac{\left(\frac{1}{2}+(\frac{3\alpha}{2})^{\frac{\theta_{20}}{\theta_2}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{10}}{\theta_1}+1}}{\frac{\theta_{10}}{\theta_1}+1}. \qquad (14)
\end{aligned}$$

The rejection region of LRT, SIUT (A_0) and smoother test are shown in Figure 2 (the set of $A_0\cup A_1\cup A_2$) for exponential distribution for the case of $\alpha=0.05$ and $\theta_{10}=1,\theta_{20}=2$.
Qualitatively, the test power $\beta_S(\theta_1,\theta_2)$ represents the probability of correctly rejecting H_0 when the true values (θ_1,θ_2) depart from the values assumed under H_0. As θ_1 or θ_2 move farther away from their null values, the ratios $\frac{\theta_1}{\theta_{10}}$ and $\frac{\theta_2}{\theta_{20}}$ in equation (14) change monotonically, causing $\beta_S(\theta_1,\theta_2)$ to increase gradually from the nominal level α (under H_0) toward values close to 1. When these departures are small, $\beta_S(\theta_1,\theta_2)$ increases approximately linearly, and the rate of this increase depends on how sensitive the rejection regions A_0, A_1, A_2 are to the shape of the distribution. When the departures are large, some terms dominate (notably the smaller exponential powers), which makes $\beta_S(\theta_1,\theta_2)$ rise more rapidly toward values near 1 and then remain flat in that region.

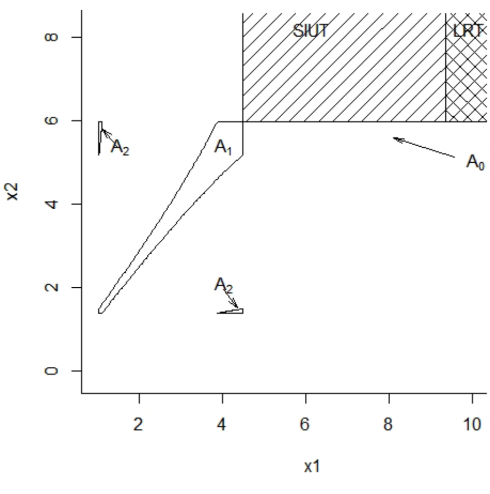

Figure 2. The rejection region of LRT, SIUT (A_0) and smoother test (the set of $A_0\cup A_1\cup A_2$) for exponential distribution

Theorem 2
For testing problem (3), If $0 < \alpha < 0.5$, then smoother test is a size-α test, and smoother test is uniformly more powerful than the size-α LRT.

Proof
Since the rejection region of the size-α, A_0 is a subset of the rejection region smoother test, Hence, the smoother test is uniformly more powerful than the size-α LRT.

$$\text{Size LRT} = \alpha = \sup P_{H_0}((X_1, X_2) \in A_0) \leq \sup P_{H_0}((X_1, X_2) \in A) = \text{Size of smoother test}, \tag{15}$$

Since $\theta_2 \leq \theta_{20}$, we have

$$P_{(\theta_1,\theta_2)}((X_1, X_2) \in A) \leq P_{(\theta_1,\theta_{20})}((X_1, X_2) \in A) \leq \alpha, \tag{16}$$

The size of the smoother test is less than or equal to α. (15) and (16) imply that smoother test has exactly size-α. □

4. Power comparison

In this section, we compare the powers of the four exponential tests, the smoother test and rectangle test, and LRT and SIUT. Figures 3 shows the power of these tests for three popular sizes, 0.01, 0.05 and 0.1. When θ_{10} and θ_{20} are fixed and θ_i is changed from $\theta_{i0}(i = 1, 2)$. As is expected, the power for four tests are increasing with incresing the parameter. However, for all values of the parameter, smoother test and rectangle are more power than two others. SIUT has slightly more power than LRT. To compare the power of these tests when two parameters change simultanseouly, the surface plot of the power against θ_1 and θ_2 has been shown in Figure 4. The power of four tests has increases as $(\frac{\theta_1}{\theta_{10}}, \frac{\theta_2}{\theta_{20}})$ gets large and more increases for the smoother and rectangle tests.

Some numerical results of these four functions for certain values of $\theta_1, \theta_2, \theta_{10}, \theta_{20}$ and $\alpha = 0.05$ are given in Table 1 when $\theta_{10} = 1.5, \theta_{20} = 2$ and $\alpha = 0.01, 0.05$ and 0.1. Different values of $\theta_2(0.09\theta_{20}, 0.4\theta_{20}, 0.6\theta_{20}, 0.8\theta_{20}, \theta_{20}, 2\theta_{20}, 5\theta_{20}, 10\theta_{20}, 50\theta_{20})$ are considered. As can be seen from the Table, for the many of four tests the poer is less than size. It means that four test may be biased. However, the powers increase when θ_2 tends away θ_{20}. In all cases, The power of the smoother test and rectangle test, is larger than the power of SIUT and LRT.

5. Integrated size-adjusted and sensitivity analysis

In this section, we integrate two complementary strands of numerical investigation to strengthen the validity of our conclusions. First, we conduct a thorough sensitivity analysis to assess the robustness of test performance across a broad spectrum of baseline parameter configurations. Second, we apply a size-adjustment procedure to address finite-sample bias arising from reliance on asymptotic critical values, ensuring fairer statistical comparisons. Together, these analyses provide a more reliable and nuanced perspective on the comparative strengths of the considered tests.

5.1. Sensitivity analysis and parameter justification

In the numerical studies of this research, the baseline null parameter values $(\theta_{10}, \theta_{20})$ were set to commonly used reference configurations in the statistical literature, such as $(1, 1)$, $(1.2, 1.5)$, $(1.5, 2)$, and $(2, 2.5)$. To examine the robustness of the findings with respect to these choices, we carried out a comprehensive sensitivity analysis using the power estimates reported in Tables 2, 3.

Across all null parameter settings, the smoother test demonstrated consistently strong and stable performance. Its power increased gradually and smoothly as both θ_1 and θ_2 deviated from their null values, maintaining appreciable levels even for small or moderate deviations. In large deviations, the Smoother approached maximum power

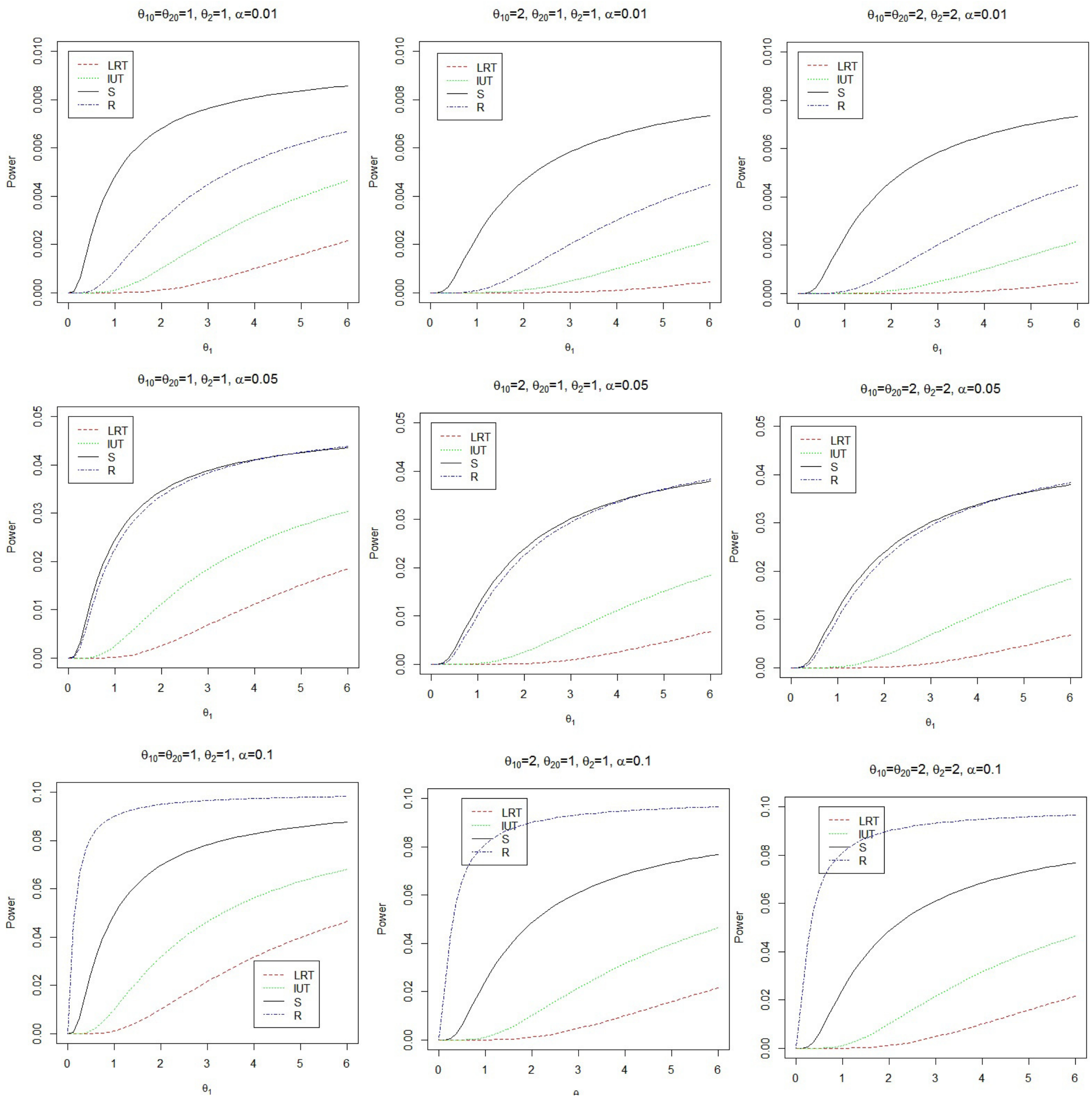

Figure 3. The power of four tests, the LRT, SIUT, smoother test, and rectangle test for different values of the parameter.

quickly, without requiring abrupt changes in parameter values. This stability across the entire parameter space makes it suitable for reliably detecting both subtle and pronounced effects. The rectangle test also exhibited relatively stable performance, with a similar gradual gain in power; however, in certain parameter regions, its power trajectories were slightly less smooth compared to the Smoother. While it maintained moderate-to-high power for small to moderate deviations and eventually achieved high power for large departures, the increases sometimes occurred less uniformly and, in rare cases, with minor fluctuations. Nevertheless, the Rectangle remains a solid general-choice test, particularly when scenarios are expected to involve steady parameter shifts. By contrast, the LRT and SIUT tests showed marked power improvements mainly for large departures from the null (e.g., $\theta_1 \geq 1.5\,\theta_{10}$ and $\theta_2 \geq 1.5\,\theta_{20}$), while for small or moderate deviations, their rejection rates were often below the nominal level α, reflecting conservative behavior. Overall, this sensitivity analysis confirms that the main qualitative

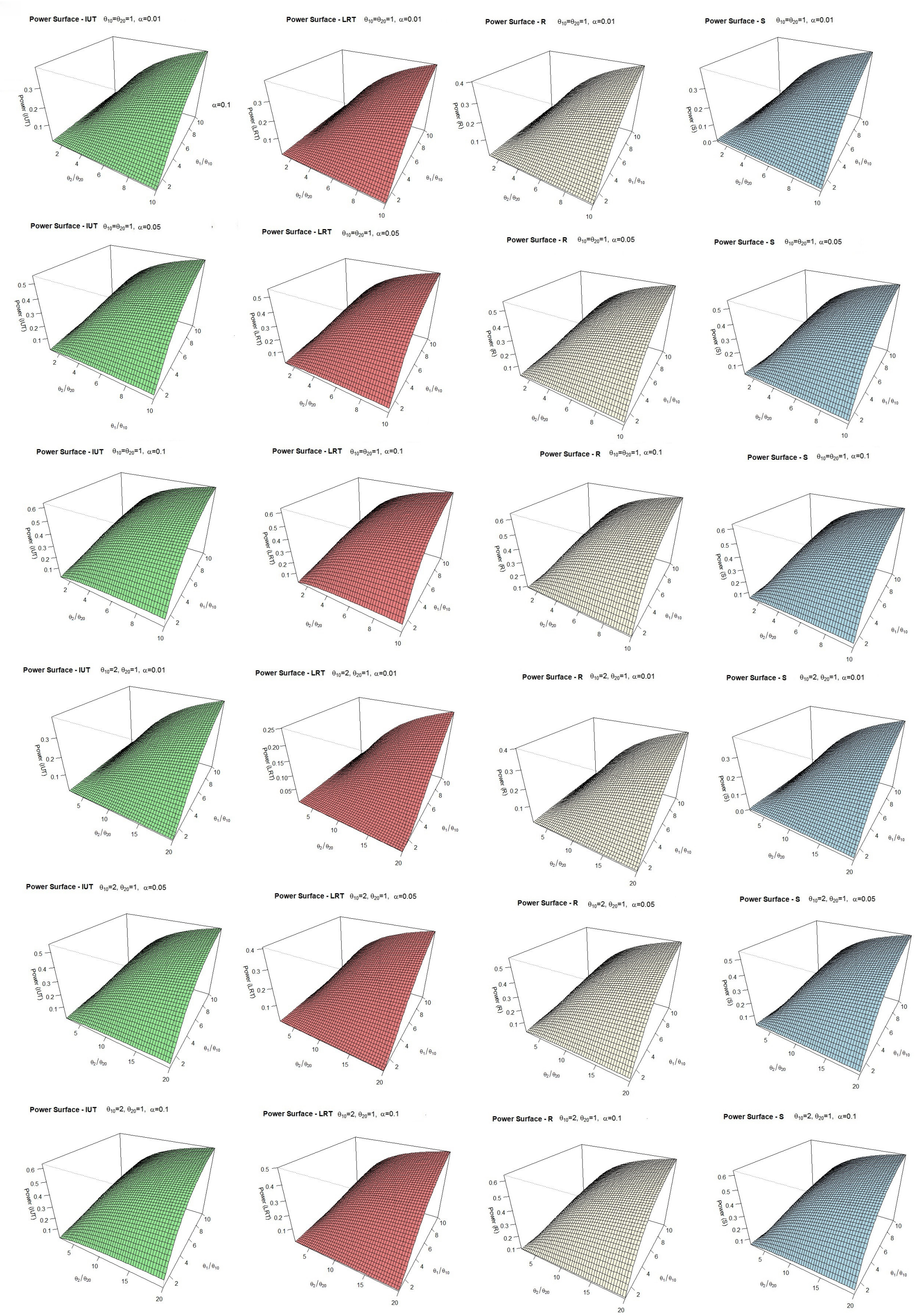

Figure 4. The surface power plot of four tests for different values of the parameter $(\frac{\theta_1}{\theta_{10}}, \frac{\theta_2}{\theta_{20}})$. Not that there are SIUT, LRT, rectangle test, and smoother test in columns one to four, respectively.

Table 1. The power of LRT (first row), SIUT (second row), Rectangle (third row) and Smoother (forth row) tests for different θ_1 and θ_2 when $\theta_{10} = 1.5$, $\theta_{20} = 2$.The expression outside the parentheses, in parentheses and inside the brackets indicates the test power value for $\alpha = 0.01, 0.05$ and 0.1, respectively.

	θ_1				
$\theta_2 \downarrow$	$0.09\theta_{10}$	$0.4\theta_{10}$	$0.6\theta_{10}$	$0.8\theta_{10}$	θ_{10}
$0.09\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
$0.4\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.001]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.004) [0.009]	0.002 (0.009) [0.013]	0.001 (0.009) [0.017]	0.002 (0.009) [0.019]
	0.000 (0.000) [0.000]	0.001 (0.005) [0.009]	0.001 (0.007) [0.014]	0.002 (0.008) [0.017]	0.002 (0.009) [0.018]
$0.6\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.004]	0.000 (0.001) [0.004]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.002]
	0.000 (0.000) [0.000]	0.002 (0.006) [0.014]	0.002 (0.012) [0.022]	0.002 (0.016) [0.030]	0.003 (0.016) [0.031]
	0.000 (0.000) [0.000]	0.001 (0.007) [0.015]	0.002 (0.012) [0.023]	0.003 (0.015) [0.028]	0.003 (0.016) [0.031]
$0.8\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.003]	0.000 (0.001) [0.006]	0.000 (0.004) [0.011]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.004]	0.000 (0.001) [0.005]
	0.000 (0.000) [0.000]	0.002 (0.009) [0.016]	0.003 (0.014) [0.030]	0.003 (0.020) [0.036]	0.004 (0.021) [0.039]
	0.000 (0.000) [0.000]	0.002 (0.008) [0.016]	0.003 (0.014) [0.029]	0.004 (0.019) [0.036]	0.004 (0.021) [0.042]
θ_{20}	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.001) [0.004]	0.000 (0.002) [0.012]	0.000 (0.005) [0.018]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.003]	0.000 (0.002) [0.006]	0.000 (0.002) [0.012]
	0.000 (0.000) [0.000]	0.002 (0.008) [0.019]	0.003 (0.014) [0.031]	0.004 (0.020) [0.045]	0.005 (0.024) [0.053]
	0.000 (0.000) [0.000]	0.002 (0.009) [0.018]	0.003 (0.016) [0.032]	0.004 (0.021) [0.043]	0.005 (0.026) [0.050]
	θ_{10}	$2\theta_{10}$	$5\theta_{10}$	$10\theta_{10}$	$50\theta_{10}$
θ_{20}	0.000 (0.005) [0.018]	0.004 (0.022) [0.059]	0.011 (0.060) [0.113]	0.023 (0.078) [0.138]	0.027 (0.104) [0.170]
	0.000 (0.002) [0.012]	0.001 (0.011) [0.032]	0.004 (0.029) [0.064]	0.007 (0.040) [0.082]	0.009 (0.045) [0.097]
	0.005 (0.024) [0.053]	0.006 (0.034) [0.074]	0.010 (0.044) [0.085]	0.010 (0.045) [0.091]	0.011 (0.053) [0.101]
	0.005 (0.026) [0.050]	0.007 (0.035) [0.070]	0.008 (0.043) [0.086]	0.010 (0.047) [0.094]	0.010 (0.050) [0.098]
$2\theta_{20}$	0.002 (0.016) [0.041]	0.018 (0.071) [0.136]	0.070 (0.178) [0.265]	0.115 (0.239) [0.332]	0.162 (0.309) [0.407]
	0.001 (0.011) [0.032]	0.009 (0.051) [0.100]	0.039 (0.123) [0.198]	0.066 (0.164) [0.256]	0.091 (0.218) [0.305]
	0.007 (0.037) [0.073]	0.020 (0.080) [0.147]	0.047 (0.141) [0.223]	0.068 (0.179) [0.266]	0.091 (0.211) [0.305]
	0.007 (0.035) [0.071]	0.019 (0.079) [0.141]	0.048 (0.142) [0.225]	0.070 (0.177) [0.266]	0.092 (0.212) [0.304]
$5\theta_{20}$	0.004 (0.031) [0.069]	0.050 (0.143) [0.226]	0.203 (0.350) [0.447]	0.310 (0.471) [0.573]	0.457 (0.604) [0.667]
	0.004 (0.027) [0.067]	0.040 (0.123) [0.196]	0.153 (0.297) [0.390]	0.251 (0.408) [0.502]	0.364 (0.518) [0.609]
	0.009 (0.043) [0.095]	0.050 (0.137) [0.217]	0.162 (0.314) [0.419]	0.270 (0.414) [0.518]	0.361 (0.524) [0.605]
	0.008 (0.044) [0.088]	0.049 (0.142) [0.226]	0.169 (0.316) [0.415]	0.255 (0.416) [0.509]	0.365 (0.520) [0.605]
$10\theta_{20}$	0.006 (0.042) [0.081]	0.073 (0.182) [0.267]	0.280 (0.442) [0.532]	0.445 (0.600) [0.675]	0.647 (0.754) [0.808]
	0.006 (0.040) [0.078]	0.063 (0.172) [0.255]	0.253 (0.412) [0.501]	0.390 (0.546) [0.636]	0.580 (0.702) [0.758]
	0.009 (0.047) [0.099]	0.071 (0.177) [0.262]	0.256 (0.416) [0.509]	0.403 (0.564) [0.634]	0.581 (0.698) [0.759]
	0.009 (0.047) [0.093]	0.068 (0.177) [0.264]	0.259 (0.416) [0.511]	0.402 (0.558) [0.635]	0.577 (0.699) [0.760]
$50\theta_{20}$	0.009 (0.050) [0.099]	0.091 (0.216) [0.303]	0.377 (0.524) [0.601]	0.581 (0.699) [0.764]	0.851 (0.901) [0.924]
	0.008 (0.051) [0.093]	0.092 (0.213) [0.300]	0.362 (0.517) [0.607]	0.580 (0.699) [0.754]	0.826 (0.891) [0.909]
	0.009 (0.055) [0.103]	0.095 (0.220) [0.309]	0.350 (0.522) [0.602]	0.578 (0.705) [0.760]	0.833 (0.894) [0.914]
	0.010 (0.050) [0.099]	0.091 (0.216) [0.306]	0.366 (0.518) [0.602]	0.576 (0.698) [0.759]	0.832 (0.888) [0.912]

conclusions are robust for a broad range of null parameter values: the Rectangle and Smoother tests consistently achieve stable gains across scenarios, whereas the LRT and SIUT tests are most effective in the presence of strong joint deviations from the null hypothesis.

Table 2. The power of LRT, SIUT, Rectangle and Smoother tests (row1–row4) for different θ_1 and θ_2 for different θ_{10}, θ_{20} and the expression outside the parentheses, in parentheses, and inside the brackets indicates the test power value for $\alpha = 0.01$, 0.05, and 0.1, respectively.

	$\theta_{10} = 1, \theta_{20} = 1$				
	θ_1				
$\theta_2 \downarrow$	θ_{10}	$2\theta_{10}$	$5\theta_{10}$	$10\theta_{10}$	$50\theta_{10}$
θ_{20}	0.000 (0.003) [0.009]	0.001 (0.011) [0.032]	0.002 (0.030) [0.063]	0.007 (0.036) [0.078]	0.008 (0.048) [0.096]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
$2\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.001]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]
	0.000 (0.000) [0.000]	0.000 (0.004) [0.009]	0.002 (0.009) [0.013]	0.001 (0.009) [0.017]	0.002 (0.009) [0.019]
	0.000 (0.000) [0.000]	0.001 (0.005) [0.009]	0.001 (0.007) [0.014]	0.002 (0.008) [0.017]	0.002 (0.009) [0.018]
$5\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.004]	0.000 (0.001) [0.004]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.002]
	0.000 (0.000) [0.000]	0.002 (0.006) [0.014]	0.002 (0.012) [0.022]	0.002 (0.016) [0.030]	0.003 (0.016) [0.031]
	0.000 (0.000) [0.000]	0.001 (0.007) [0.015]	0.002 (0.012) [0.023]	0.003 (0.015) [0.028]	0.003 (0.016) [0.031]
$10\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.003]	0.000 (0.001) [0.006]	0.000 (0.004) [0.011]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.001]	0.000 (0.000) [0.004]	0.000 (0.001) [0.005]
	0.000 (0.000) [0.000]	0.002 (0.009) [0.016]	0.003 (0.014) [0.030]	0.003 (0.020) [0.036]	0.004 (0.021) [0.039]
	0.000 (0.000) [0.000]	0.002 (0.008) [0.016]	0.003 (0.014) [0.029]	0.004 (0.019) [0.036]	0.004 (0.021) [0.042]
$50\theta_{20}$	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.001) [0.004]	0.000 (0.002) [0.012]	0.000 (0.005) [0.018]
	0.000 (0.000) [0.000]	0.000 (0.000) [0.000]	0.000 (0.000) [0.003]	0.000 (0.002) [0.006]	0.000 (0.002) [0.012]
	0.000 (0.000) [0.000]	0.002 (0.008) [0.019]	0.003 (0.014) [0.031]	0.004 (0.020) [0.045]	0.005 (0.024) [0.053]
	0.000 (0.000) [0.000]	0.002 (0.009) [0.018]	0.003 (0.016) [0.032]	0.004 (0.021) [0.043]	0.005 (0.026) [0.050]
	$\theta_{10} = 1.2, \theta_{20} = 1.5$				
	θ_{10}	$2\theta_{10}$	$5\theta_{10}$	$10\theta_{10}$	$50\theta_{10}$
θ_{20}	0.000 (0.005) [0.018]	0.004 (0.022) [0.059]	0.011 (0.060) [0.113]	0.023 (0.078) [0.138]	0.027 (0.104) [0.170]
	0.000 (0.002) [0.012]	0.001 (0.011) [0.032]	0.004 (0.029) [0.064]	0.007 (0.040) [0.082]	0.009 (0.045) [0.097]
	0.005 (0.024) [0.053]	0.006 (0.034) [0.074]	0.010 (0.044) [0.085]	0.010 (0.045) [0.091]	0.011 (0.053) [0.101]
	0.005 (0.026) [0.050]	0.007 (0.035) [0.070]	0.008 (0.043) [0.086]	0.010 (0.047) [0.094]	0.010 (0.050) [0.098]
$2\theta_{20}$	0.002 (0.016) [0.041]	0.018 (0.071) [0.136]	0.070 (0.178) [0.265]	0.115 (0.239) [0.332]	0.162 (0.309) [0.407]
	0.001 (0.011) [0.032]	0.009 (0.051) [0.100]	0.039 (0.123) [0.198]	0.066 (0.164) [0.256]	0.091 (0.218) [0.305]
	0.007 (0.037) [0.073]	0.020 (0.080) [0.147]	0.047 (0.141) [0.223]	0.068 (0.179) [0.266]	0.091 (0.211) [0.305]
	0.007 (0.035) [0.071]	0.019 (0.079) [0.141]	0.048 (0.142) [0.225]	0.070 (0.177) [0.266]	0.092 (0.212) [0.304]
$5\theta_{20}$	0.004 (0.031) [0.069]	0.050 (0.143) [0.226]	0.203 (0.350) [0.447]	0.310 (0.471) [0.573]	0.457 (0.604) [0.667]
	0.004 (0.027) [0.067]	0.040 (0.123) [0.196]	0.153 (0.297) [0.390]	0.251 (0.408) [0.502]	0.364 (0.518) [0.609]
	0.009 (0.043) [0.095]	0.050 (0.137) [0.217]	0.162 (0.314) [0.419]	0.270 (0.414) [0.518]	0.361 (0.524) [0.605]
	0.008 (0.044) [0.088]	0.049 (0.142) [0.226]	0.169 (0.316) [0.415]	0.255 (0.416) [0.509]	0.365 (0.520) [0.605]
$10\theta_{20}$	0.006 (0.042) [0.081]	0.073 (0.182) [0.267]	0.280 (0.442) [0.532]	0.445 (0.600) [0.675]	0.647 (0.754) [0.808]
	0.006 (0.040) [0.078]	0.063 (0.172) [0.255]	0.253 (0.412) [0.501]	0.390 (0.546) [0.636]	0.580 (0.702) [0.758]
	0.009 (0.047) [0.099]	0.071 (0.177) [0.262]	0.256 (0.416) [0.509]	0.403 (0.564) [0.634]	0.581 (0.698) [0.759]
	0.009 (0.047) [0.093]	0.068 (0.177) [0.264]	0.259 (0.416) [0.511]	0.402 (0.558) [0.635]	0.577 (0.699) [0.760]
$50\theta_{20}$	0.009 (0.050) [0.099]	0.091 (0.216) [0.303]	0.377 (0.524) [0.601]	0.581 (0.699) [0.764]	0.851 (0.901) [0.924]
	0.008 (0.051) [0.093]	0.092 (0.213) [0.300]	0.362 (0.517) [0.607]	0.580 (0.699) [0.754]	0.826 (0.891) [0.909]
	0.009 (0.055) [0.103]	0.095 (0.220) [0.309]	0.350 (0.522) [0.602]	0.578 (0.705) [0.760]	0.833 (0.894) [0.914]
	0.010 (0.050) [0.099]	0.091 (0.216) [0.306]	0.366 (0.518) [0.602]	0.576 (0.698) [0.759]	0.832 (0.888) [0.912]

Table 3. The power of LRT, SIUT, Rectangle and Smoother tests (row1-row4) (second row), test (third row), and Smoother test (fourth row) for different θ_1 and θ_2 when $\theta_{10} = 1.5$, $\theta_{20} = 2$ and the expression outside the parentheses, in parantheses and inside the brackets indicates the test power value for $\alpha = 0.01, 0.05$ and 0.1, respectively.

	$\theta_{10} = 1.5, \theta_{20} = 2$				
	θ_1				
$\theta_2 \downarrow$	θ_{10}	$2\theta_{10}$	θ_{10}	$0.8\theta_{10}$	θ_{10}
θ_{20}	0.000 (0.003) [0.009]	0.001 (0.011) [0.032]	0.002 (0.030) [0.063]	0.007 (0.036) [0.078]	0.008 (0.048) [0.096]
	0.000 (0.002) [0.010]	0.001 (0.011) [0.032]	0.004 (0.029) [0.064]	0.007 (0.040) [0.082]	0.009 (0.045) [0.097]
	0.004 (0.026) [0.052]	0.006 (0.034) [0.074]	0.010 (0.044) [0.085]	0.010 (0.045) [0.091]	0.011 (0.053) [0.101]
	0.005 (0.026) [0.049]	0.007 (0.035) [0.070]	0.008 (0.043) [0.086]	0.010 (0.047) [0.094]	0.010 (0.050) [0.098]
$2\theta_{20}$	0.001 (0.011) [0.031]	0.010 (0.049) [0.102]	0.040 (0.124) [0.194]	0.063 (0.167) [0.245]	0.090 (0.209) [0.304]
	0.001 (0.011) [0.032]	0.009 (0.051) [0.100]	0.039 (0.123) [0.198]	0.066 (0.164) [0.256]	0.091 (0.218) [0.305]
	0.007 (0.037) [0.073]	0.020 (0.080) [0.147]	0.047 (0.141) [0.223]	0.068 (0.179) [0.266]	0.091 (0.211) [0.305]
	0.007 (0.035) [0.071]	0.019 (0.079) [0.141]	0.048 (0.142) [0.225]	0.070 (0.177) [0.266]	0.092 (0.212) [0.304]
$5\theta_{20}$	0.003 (0.026) [0.060]	0.040 (0.122) [0.201]	0.161 (0.298) [0.400]	0.244 (0.406) [0.509]	0.358 (0.520) [0.594]
	0.004 (0.027) [0.067]	0.040 (0.123) [0.196]	0.153 (0.297) [0.390]	0.251 (0.408) [0.502]	0.364 (0.518) [0.609]
	0.009 (0.043) [0.095]	0.050 (0.137) [0.217]	0.162 (0.314) [0.419]	0.270 (0.414) [0.518]	0.361 (0.524) [0.605]
	0.008 (0.044) [0.088]	0.049 (0.142) [0.226]	0.169 (0.316) [0.415]	0.255 (0.416) [0.509]	0.365 (0.520) [0.605]
$10\theta_{20}$	0.005 (0.038) [0.077]	0.064 (0.171) [0.253]	0.250 (0.411) [0.503]	0.399 (0.561) [0.638]	0.576 (0.702) [0.765]
	0.006 (0.040) [0.078]	0.063 (0.172) [0.255]	0.253 (0.412) [0.501]	0.390 (0.546) [0.636]	0.580 (0.702) [0.758]
	0.009 (0.047) [0.099]	0.071 (0.177) [0.262]	0.256 (0.416) [0.509]	0.403 (0.564) [0.634]	0.581 (0.698) [0.759]
	0.009 (0.047) [0.093]	0.068 (0.177) [0.264]	0.259 (0.416) [0.511]	0.402 (0.558) [0.635]	0.577 (0.699) [0.760]
$50\theta_{20}$	0.009 (0.050) [0.098]	0.089 (0.213) [0.299]	0.369 (0.516) [0.594]	0.567 (0.689) [0.756]	0.832 (0.886) [0.911]
	0.008 (0.051) [0.093]	0.092 (0.213) [0.300]	0.362 (0.517) [0.607]	0.580 (0.699) [0.754]	0.826 (0.891) [0.909]
	0.009 (0.055) [0.103]	0.095 (0.220) [0.309]	0.350 (0.522) [0.602]	0.578 (0.705) [0.760]	0.833 (0.894) [0.914]
	0.010 (0.050) [0.099]	0.091 (0.216) [0.306]	0.366 (0.518) [0.602]	0.576 (0.698) [0.759]	0.832 (0.888) [0.912]
	$\theta_{10} = 2, \theta_{20} = 2.5$				
	θ_{10}	$2\theta_{10}$	$5\theta_{10}$	$10\theta_{10}$	$50\theta_{10}$
θ_{20}	0.000 (0.005) [0.014]	0.003 (0.019) [0.054]	0.008 (0.052) [0.099]	0.018 (0.066) [0.123]	0.020 (0.090) [0.151]
	0.000 (0.002) [0.010]	0.001 (0.011) [0.032]	0.004 (0.029) [0.064]	0.007 (0.040) [0.082]	0.009 (0.045) [0.097]
	0.004 (0.026) [0.052]	0.006 (0.034) [0.074]	0.010 (0.044) [0.085]	0.010 (0.045) [0.091]	0.011 (0.053) [0.101]
	0.005 (0.026) [0.049]	0.007 (0.035) [0.070]	0.008 (0.043) [0.086]	0.010 (0.047) [0.094]	0.010 (0.050) [0.098]
$2\theta_{20}$	0.002 (0.015) [0.039]	0.016 (0.066) [0.130]	0.062 (0.167) [0.252]	0.103 (0.224) [0.312]	0.146 (0.285) [0.383]
	0.001 (0.011) [0.032]	0.009 (0.051) [0.100]	0.039 (0.123) [0.198]	0.066 (0.164) [0.256]	0.091 (0.218) [0.305]
	0.007 (0.037) [0.073]	0.020 (0.080) [0.147]	0.047 (0.141) [0.223]	0.068 (0.179) [0.266]	0.091 (0.211) [0.305]
	0.007 (0.035) [0.071]	0.019 (0.079) [0.141]	0.048 (0.142) [0.225]	0.070 (0.177) [0.266]	0.092 (0.212) [0.304]
$5\theta_{20}$	0.004 (0.030) [0.067]	0.048 (0.137) [0.220]	0.193 (0.339) [0.435]	0.294 (0.457) [0.559]	0.435 (0.585) [0.651]
	0.004 (0.027) [0.067]	0.040 (0.123) [0.196]	0.153 (0.297) [0.390]	0.251 (0.408) [0.502]	0.364 (0.518) [0.609]
	0.009 (0.043) [0.095]	0.050 (0.137) [0.217]	0.162 (0.314) [0.419]	0.270 (0.414) [0.518]	0.361 (0.524) [0.605]
	0.008 (0.044) [0.088]	0.049 (0.142) [0.226]	0.169 (0.316) [0.415]	0.255 (0.416) [0.509]	0.365 (0.520) [0.605]
$10\theta_{20}$	0.006 (0.041) [0.080]	0.071 (0.179) [0.264]	0.275 (0.434) [0.526]	0.437 (0.590) [0.668]	0.630 (0.741) [0.800]
	0.006 (0.040) [0.078]	0.063 (0.172) [0.255]	0.253 (0.412) [0.501]	0.390 (0.546) [0.636]	0.580 (0.702) [0.758]
	0.009 (0.047) [0.099]	0.071 (0.177) [0.262]	0.256 (0.416) [0.509]	0.403 (0.564) [0.634]	0.581 (0.698) [0.759]
	0.009 (0.047) [0.093]	0.068 (0.177) [0.264]	0.259 (0.416) [0.511]	0.402 (0.558) [0.635]	0.577 (0.699) [0.760]
$50\theta_{20}$	0.009 (0.050) [0.099]	0.091 (0.215) [0.302]	0.376 (0.523) [0.600]	0.578 (0.697) [0.762]	0.846 (0.897) [0.922]
	0.008 (0.051) [0.093]	0.092 (0.213) [0.300]	0.362 (0.517) [0.607]	0.580 (0.699) [0.754]	0.826 (0.891) [0.909]
	0.009 (0.055) [0.103]	0.095 (0.220) [0.309]	0.350 (0.522) [0.602]	0.578 (0.705) [0.760]	0.833 (0.894) [0.914]
	0.010 (0.050) [0.099]	0.091 (0.216) [0.306]	0.366 (0.518) [0.602]	0.576 (0.698) [0.759]	0.832 (0.888) [0.912]

5.2. *Size-adjusted power*

To address the well-known issue of bias (i.e., deflated power, $\text{Power} < \alpha$) under the classic implementation of finite-sample tests which arises from reliance on asymptotic critical values, we incorporated a size-adjustment strategy, as suggested by Davison and Hinkley[7]. This method involves empirically estimating the critical values for each test via repeated simulations under the null, ensuring the actual Type I error rate matches the nominal significance level. We examined the behavior of the parameter pairs $\theta_{10} = 1.5$ and $\theta_{20} = 2$ with $\theta_1 = 1.5$ and $\theta_2 = 2$, and $\theta_{10} = 1$ and $\theta_{20} = 1$ with $\theta_1 = 3$ and $\theta_2 = 4$ and $\alpha = 0.05$. In the first case, where the true values equal the null hypothesis, the classical tests behaved very conservatively: rejection rates were far below the nominal significance level α and the empirical powers were near zero. After applying size adjustment using empirical critical values, rejection rates moved closer to α, indicating that size adjustment substantially improves Type I error control and reduces the conservative bias. In the second case, where the null ($\theta_{10} = 1$, $\theta_{20} = 1$) is substantially different from the true values $\theta_1 = 3$ and $\theta_2 = 4$, the classical tests showed a tendency for excessive rejection (inflated rejection rates relative to α). After size adjustment, the test's powers increased markedly while maintaining better control of Type I error. Therefore, size adjustment makes comparisons among methods fairer and the results more reliable.

6. A numerical study for sign testing in exponential distribution

In this section, we study the testing minimum reliability thresholds for p components in a series system lifetimes each exponential with the parameters, θ_i, $i = 1, \ldots, p$.
It is desired test that no component performs better than the specified minimum acceptable threshold, i.e $H_0 : \min\{\theta_1, \ldots, \theta_p\} \leq \theta_0$ against $H_1 : \min\{\theta_1, \ldots, \theta_p\} > \theta_0$. Or, equivalently in terms of failure rates: $\max\{\lambda_1, \ldots, \lambda_p\} \geq \lambda_0$ against $H_1 : \max\{\lambda_1, \ldots, \lambda_p\} < \lambda_0$.

This test specifically focuses on the worst-performing component Suppose that we have a computer system consisting of three main components: Processor (CPU), Random Access Memory (RAM), Hard Disk Drive (HDD). These three components operate in series and the entire system will stop working if any of these components fail. Assuming the failures are independent, the critical component in a series system in terms of failure rate is the component that has the maximum failure rate ($\lambda_{\max}$).

Failure rate of RAM, HDD and CPU

Schroeder et al. [23] is comprehensive field on RAM module failure rates in more than two million memory modules over 2.5 years, reported to be about 8 percent of modules experience at least one bit error annually. Due to the lack of access to the raw data of this study, in the performed simulations, the hourly failure rate ($\lambda = 0.00046$), equivalent to $MTBF = 2174$ hours.

Pinheiro et al. [18] in a comprehensive analysis of the failure rates of over one hundred thousand hard drives in Google datacenters over a five-year period, reported that annual failure ratevaries between 1.7 and 8.6 percent. Based on the reported values an hourly failure rate of $\lambda = 0.000002$ ($MTBF = 500,000$ hours) was used for simulating the related data. According to HP Reliability Data, 2011, and IBM/Google documents, the annual failure rate of server processors is usually between 0.01 and 0.5 percent and their mean time between failures (MTBF) is between one and one and a half million hours.

Four tests LRT, SIUT, smoother, rectangular at significance level $\alpha = 0.05$ were applied to the sample data. The observed values and the critical values of the LRT and $SIUT$ are calculated and given in Table 4. The LRT does

Table 4. Comparison of statistical test results for the RAM, HDD, and CPU system

c_i	$\lambda_{\text{HDD}}^{(LRT)}$	$\lambda_{\text{RAM}}^{(LRT)}$	$\lambda_{\text{CPU}}^{(LRT)}$	$\lambda_0^{(LRT)}$	$\bar{x}_{\text{HDD}}$	$\bar{x}_{\text{RAM}}$	$\bar{x}_{\text{CPU}}$
7489.331	0.9907	0.000	0.000	0.6268	1204819.28	2173.9	500000

not reject H_0 since calculated statistic $\lambda_{\text{stat}} = \max\{\lambda_{RAM}^{(LRT)}, \lambda_{HDD}^{(LRT)}, \lambda_{CPU}^{(LRT)}\} = 0.9907$ is greater than the critical value $\lambda_0 = 0.6269$, which was obtained via simulations with $n = 100000$ repetitions.
For SIUT, the null hypothesis is rejected if, for all i, $\bar{x}_i > c = c_i = -\theta_0 \ln(\alpha) = 7489.33$. As this not satisfied for all $\bar{x}_i$, the conclusion for SIUT is not rejecting of H_0.

Three component pairs (RAM-CPU, RAM-HDD and CPU-HDD) are evaluated for the Smoother test rejection regions A_0, A_1, and A_2. For each pair, the mean times to failure $(\bar{x}_i, \bar{x}_j)$, the normalized values (u_i, u_j), and the specific conditions required for each rejection region are computed and presented.

RAM-CPU: This pair does not satisfy the conditions for rejection regions A_0, A_1, or A_2. $\bar{x}_{RAM} = 2173.91$ is below the A_0 threshold of 7489.25 although $\bar{x}_{CPU} = 1204819.28$ exceeds this threshold A_1 fails because $|u_{RAM} - u_{CPU}| = |0.5804 - 1| = 0.4196 > 0.025$ and A_2 is not satisfied since $u_{CPU} = 1.0000 > 0.95$ and the allowed range condition for the second variable is not met. Therefore H_0 is not rejected for this pair.

RAM-HDD: Similar to RAM-CPU, $\bar{x}_{RAM} = 2173.91$ is below the threshold of 7489.25 and A_1 fails because the u difference is $0.4196 > 0.025$. A_2 also fails since $u_{HDD} = 1.0000 > 0.95$ and the allowed range condition for the second variable is not satisfied. Therefore H_0 is nor rejected.

CPU-HDD: Both $\bar{x}_{CPU}$ and $\bar{x}_{HDD}$ exceed the A_0 threshold, so A_0 is satisfied. However A_1 is not satisfied since u_{CPU} and u_{HDD} are both approximately 1 and the condition $u \leq 0.95$ is not met.A_2 is also not satisfied because both u values are greater than 0.95 Nevertheless, due to A_0 being satisfied, this pair is rejected.
Since not all component pairs are rejected simultaneously the null hypothesis H_0 in the Smoother test is not rejected overall.

In the Rectangle test, the three component pairs (RAM–CPU), (RAM–HDD) and (CPU–HDD) were examined using the parameters $\alpha = 0.05$, $\theta_0 = 2500$, and $J = 10$. The threshold values c_j were computed according to $c_j = -\theta_0 \ln(j\alpha)$ for $j = 1$ to 10 (with $c_0 = \infty$).

Based on these thresholds, each pair was evaluated as follows:

Pair RAM-CPU: $\bar{x}_{RAM}$ lies between $L_1 = 2161.97$ and $U_1 = 2448.54$ $(j = 8)$. Also, $\bar{x}_{CPU}$ is greater than $L_2 = 7498.71$ $(j = 1)$. Thus, this pair lies within one of the rectangular rejection regions and is therefore rejected.

Pair RAM-HDD: $\bar{x}_{RAM}$ lies between $L_1 = 2161.97$ and $U_1 = 2448.54$ $(j = 8)$. Also, $\bar{x}_{HDD}$ is greater than $L_2 = 7498.71$ $(j = 1)$. Thus, this pair lies within one of the rectangular rejection regions and is therefore rejected.

Pair CPU-HDD: For $j = 1$, $L_1 = 7498.71$ and $U_1 = \infty$, likewise $L_2 = 7498.71$ and $U_2 = \infty$. Both $\bar{x}_{CPU}$ and $\bar{x}_{HDD}$ are greater than these limits. Thus, this pair lies within one of the rectangular rejection regions and is therefore rejected. Since all three component pairs were simultaneously located in the rejection regions of the Rectangle test, the null hypothesis H_0 is rejected.

7. Conclusion

In this paper, we are looking for an approach to construct size-α tests that are more powerful than LRT for the special sign testing problem (3). For exponential distribution, the SIUT is a uniformly most powerful monotone test with higher power than the LRT. Although the SIUT is more powerful, both tests are not unbiased. two rectangular and smoothed tests have been examined for a more powerful test. Numerical results show that two rectangular and smoothed tests have much more power than the SIUT and the LRT. Rectangular and smoothing tests have rejection regions that encompass not only the likelihood ratio test's rejection region and SIUT, but also other areas. This broader scope can make them more powerful. Essentially, statisticians have expanded the likelihood ratio test's rejection region in these tests. The key is that under the null hypothesis (H_0), the test's error rate (α) remains the same, while the test's power increases because of these added regions, so, a strategy for developing more powerful tests involves adding rejection regions without increasing the test's size (α). Note that the results are similar for the Weibull and Gamma distributions, but further research is required to determine whether this property holds for the other distributions. In the Rectangle/Smoother tests, if the number of dimensions (p) is large and α is small, computing the power may become computationally time-consuming.

Appendix . Proof of Formulas (13) and (14)

Proof of Formulas (13): Consider

$$u_1 = F_1(x_1) \Rightarrow x_1 = F_1^{-1}(u_1) = -\theta_1 \ln(1-u_1),$$
$$u_2 = F_2(x_2) \Rightarrow x_2 = F_2^{-1}(u_2) = -\theta_2 \ln(1-u_2),$$

so

$$1-\alpha \le u_1 < 1 \Rightarrow 1-\alpha \le 1-e^{-\frac{x_1}{\theta_1}} < 1 \Rightarrow x_1 \ge -\theta_1 \ln\alpha,$$
$$1-\alpha \le u_2 < 1 \Rightarrow 1-\alpha \le 1-e^{-\frac{x_1}{\theta_1}} < 1 \Rightarrow x_2 \ge -\theta_2 \ln\alpha,$$

so, we have

$$A_0 = \Big\{(x_1,x_2) : x_1 \ge -\theta_1 \ln\alpha, x_2 \ge -\theta_2 \ln\alpha\Big\}.$$

For A_1 we have

$$\begin{aligned}
\mid u_1 - u_2 \mid \le \frac{\alpha}{2} \quad &\Rightarrow \quad \frac{-\alpha}{2} \le u_1 - u_2 \le \frac{\alpha}{2} \Rightarrow \frac{-\alpha}{2} \le 1 - e^{\frac{-x_1}{\theta_1}} - (1-e^{\frac{-x_2}{\theta_2}}) \le \frac{\alpha}{2}\\
&\Rightarrow \quad \frac{-\alpha}{2} \le e^{\frac{-x_2}{\theta_2}} - e^{\frac{-x_1}{\theta_1}} \le \frac{\alpha}{2} e^{\frac{-x_1}{\theta_1}} \Rightarrow \frac{\alpha}{2} \le e^{\frac{-x_2}{\theta_2}} \le e^{\frac{-x_1}{\theta_1}} + \frac{\alpha}{2}\\
&\Rightarrow \quad \ln(e^{\frac{-x_1}{\theta_1}} - \frac{\alpha}{2}) \le \frac{-x_2}{\theta_2} \le \ln(e^{\frac{-x_1}{\theta_1}} + \frac{\alpha}{2})\\
&\Rightarrow \quad -\theta_2 \ln(e^{\frac{-x_1}{\theta_1}} + \frac{\alpha}{2}) \le x_2 \le -\theta_2 \ln(e^{\frac{-x_1}{\theta_1}} - \frac{\alpha}{2}),
\end{aligned}$$

From sided:

$$\begin{aligned}
\frac{1}{2} \le u_1 \le 1-\alpha \quad &\Rightarrow \quad \frac{1}{2} \le 1 - e^{\frac{-x_1}{\theta_1}} \le 1-\alpha \Rightarrow \frac{-1}{2} \le -e^{\frac{-x_1}{\theta_1}} \le -\alpha\\
&\Rightarrow \quad \alpha \le e^{\frac{-x_1}{\theta_1}} \le \frac{1}{2} \ln\alpha \le \frac{-x_1}{\theta_1} \le -\ln 2 \Rightarrow \theta_1 \ln 2 \le x_1 \le -\theta_1 \ln\alpha,\\
\frac{1}{2} \le u_2 \le 1-\alpha \quad &\Rightarrow \quad \frac{1}{2} \le 1 - e^{\frac{-x_2}{\theta_2}} \le 1-\alpha \Rightarrow \frac{-1}{2} \le -e^{\frac{-x_2}{\theta_2}} \le -\alpha \Rightarrow \alpha \le e^{\frac{-x_2}{\theta_2}} \le \frac{1}{2}\\
&\Rightarrow \quad \ln\alpha \le \frac{-x_2}{\theta_2} \le -\ln 2 \Rightarrow \theta_2 \ln 2 \le x_2 \le -\theta_2 \ln\alpha,
\end{aligned}$$

so, we have

$$\begin{aligned}
A_1 \quad = \quad &\Big\{(X_1,X_2) : -\theta_2 \ln(e^{\frac{-X_1}{\theta_1}} + \frac{\alpha}{2}) \le X_2 \le -\theta_2 \ln(e^{\frac{-X_1}{\theta_1}} - \frac{\alpha}{2}),\\
&\theta_1 \ln 2 \le X_1 \le -\theta_1 \ln\alpha, \theta_2 \ln 2 \le X_2 \le -\theta_2 \ln\alpha\Big\}.
\end{aligned}$$

For A_2 we have:

$$\frac{1}{2} \le u_2 \le u_1 - \frac{1}{2} + \frac{3\alpha}{2} \Rightarrow \frac{1}{2} \le 1 - e^{\frac{-x_2}{\theta_2}} \le 1 - e^{\frac{-x_1}{\theta_1}} - \frac{1}{2} + \frac{3\alpha}{2} \Rightarrow \frac{1}{2} \le 1 - e^{\frac{-x_2}{\theta_2}} \le \frac{1}{2} - e^{\frac{-x_1}{\theta_1}} + \frac{3\alpha}{2}$$

$$\Rightarrow \frac{-1}{2} \le -e^{\frac{-x_2}{\theta_2}} \le \frac{-1}{2} - e^{\frac{-x_1}{\theta_1}} + \frac{3\alpha}{2} \Rightarrow \frac{1}{2} + e^{\frac{-x_1}{\theta_1}} - \frac{3\alpha}{2} \le e^{\frac{-x_2}{\theta_2}} \le \frac{1}{2}$$

$$\Rightarrow \ln(\frac{1}{2}+e^{\frac{-x_1}{\theta_1}}-\frac{3\alpha}{2}) \leq \frac{-x_2}{\theta_2} \leq -\ln 2 \Rightarrow \theta_2 \ln 2 \leq x_2 \leq -\theta_2 \ln(\frac{1}{2}+e^{\frac{-x_1}{\theta_1}}-\frac{3\alpha}{2}),$$

$$u_2 < 1-\alpha \Rightarrow 1-e^{\frac{-x_2}{\theta_2}} < 1-\alpha \Rightarrow -e^{\frac{-x_2}{\theta_2}} < -\alpha \Rightarrow e^{\frac{-x_2}{\theta_2}} > \alpha \Rightarrow \frac{-x_2}{\theta_2} > \ln\alpha \Rightarrow x_2 < -\theta_2 \ln\alpha,$$

Similarly for u_1:

$$u_1 < 1-\alpha \Rightarrow 1-e^{\frac{-x_1}{\theta_1}} < 1-\alpha \Rightarrow -e^{\frac{-x_1}{\theta_1}} < -\alpha \Rightarrow e^{\frac{-x_1}{\theta_1}} > \alpha \Rightarrow \frac{-x_1}{\theta_1} > \ln\alpha \Rightarrow x_1 < -\theta_1 \ln\alpha,$$

$$\frac{1}{2} \leq u_1 \leq u_2 - \frac{1}{2} + \frac{3\alpha}{2} \Rightarrow \frac{1}{2} \leq 1-e^{\frac{-x_1}{\theta_1}} \leq 1-e^{\frac{-x_2}{\theta_2}} - \frac{1}{2} + \frac{3\alpha}{2} \Rightarrow \frac{-1}{2} \leq -e^{\frac{-x_1}{\theta_1}} \leq \frac{-1}{2} - e^{\frac{-x_2}{\theta_2}} + \frac{3\alpha}{2},$$
$$\Rightarrow \frac{1}{2}+e^{\frac{-x_2}{\theta_2}}-\frac{3\alpha}{2} \leq e^{\frac{-x_1}{\theta_1}} \leq \frac{1}{2} \Rightarrow \ln(\frac{1}{2}+e^{\frac{-x_2}{\theta_2}}-\frac{3\alpha}{2}) \leq \frac{-x_1}{\theta_1} \leq -\ln 2,$$
$$\Rightarrow \theta_1 \ln 2 \leq x_1 \leq -\theta_1 \ln(\frac{1}{2}+e^{\frac{-x_2}{\theta_2}}-\frac{3\alpha}{2}),$$

so, we have

$$A_2 = \left\{(x_1,x_2): \theta_2 \ln 2 \leq x_2 \leq -\theta_2 \ln(\frac{1}{2}+e^{\frac{-x_1}{\theta_1}}-\frac{3\alpha}{2}), x_1 < -\theta_1 \ln\alpha\right\}$$
$$\cup \left\{(x_1,x_2): \theta_1 \ln 2 \leq x_1 \leq -\theta_1 \ln(\frac{1}{2}+e^{\frac{-x_2}{\theta_2}}-\frac{3\alpha}{2}), x_2 < -\theta_2 \ln\alpha\right\},$$

Proof of formula14: The power of the S test consists of three parts

$$\begin{aligned} P((X_1,X_2)\in A_0) &= \int_{-\theta_{10}\ln\alpha}^{+\infty}\int_{-\theta_{20}\ln\alpha}^{+\infty} \frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\,dx_1dx_2 \\ &= \left(\int_{-\theta_{10}\ln\alpha}^{+\infty}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\,dx_1\right)\left(\int_{-\theta_{20}\ln\alpha}^{+\infty}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\,dx_2\right) \\ &= \left(e^{\frac{\theta_{10}}{\theta_1}\ln\alpha}-e^{+\infty}\right)\times\left(e^{\frac{\theta_{20}}{\theta_2}\ln\alpha}-e^{+\infty}\right) = \alpha^{\frac{\theta_{10}}{\theta_1}}\times\alpha^{\frac{\theta_{20}}{\theta_2}} = \alpha^{\frac{\theta_{10}}{\theta_1}+\frac{\theta_{20}}{\theta_2}}, \end{aligned}$$

and

$$\begin{aligned} P((X_1,X_2)\in A_1) &= \int_{\theta_{10}\ln 2}^{-\theta_{10}\ln(\frac{1}{2}-\frac{\alpha}{2})}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\left(\int_{\theta_{20}\ln 2}^{-\theta_{20}\ln(e^{-\frac{x_1}{\theta_1}}-\frac{\alpha}{2})}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\,dx_2\right)dx_1 \\ &+ \int_{-\theta_{10}\ln(\frac{1}{2}-\frac{\alpha}{2})}^{-\theta_{10}\ln(\frac{3\alpha}{2})}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\left(\int_{-\theta_{20}\ln(e^{\frac{-x_1}{\theta_1}}+\frac{\alpha}{2})}^{-\theta_{20}\ln(e^{-\frac{x_1}{\theta_1}}-\frac{\alpha}{2})}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\,dx_2\right)dx_1 \\ &+ \int_{-\theta_{10}\ln(\frac{3\alpha}{2})}^{-\theta_{10}\ln\alpha}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\left(\int_{-\theta_{20}\ln(e^{\frac{-x_1}{\theta_1}}+\frac{\alpha}{2})}^{-\theta_{20}\ln\alpha}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\,dx_2\right)dx_1 \end{aligned} \tag{17}$$

$$= 2^{-\frac{\theta_{20}}{\theta_2}}\times 2^{-\frac{\theta_{10}}{\theta_1}} + \frac{\left((\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1} - 2^{-\frac{\theta_{20}}{\theta_2}}(\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}$$
$$-\frac{\left(2^{-\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1} - \frac{\left((\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}+\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1} + \frac{\left((\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}+\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}$$

$$+\frac{\left((\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\frac{\left((\frac{1}{2}-\frac{\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}$$

$$+\frac{\left((\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\frac{\left(\alpha^{\frac{\theta_{10}}{\theta_1}}+\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\alpha^{\frac{\theta_{20}}{\theta_2}}(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}+\alpha^{\frac{\theta_{10}}{\theta_1}+\frac{\theta_2}{\theta_{20}}}$$

and

$$\begin{aligned}P\left((X_1,X_2)\in A_2\right) &= \int_{-\theta_{20}\ln(\frac{3\alpha}{2})}^{-\theta_{20}\ln\alpha}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}\left(\int_{\theta_{10}\ln 2}^{-\theta_{10}\ln(\frac{1}{2}+e^{-\frac{x_2}{\theta_2}}-\frac{3\alpha}{2})}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}dx_1\right)dx_2 \\ &+ \int_{-\theta_{10}\ln(\frac{3\alpha}{2})}^{-\theta_{10}\ln\alpha}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}\left(\int_{\theta_{20}\ln 2}^{-\theta_{20}\ln(\frac{1}{2}+e^{-\frac{x_1}{\theta_1}}-\frac{3\alpha}{2})}\frac{1}{\theta_2}e^{-\frac{x_2}{\theta_2}}dx_2\right)dx_1\end{aligned} \tag{18}$$

$$=\frac{\left(\frac{1}{2}+\alpha^{\frac{\theta_{10}}{\theta_1}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}-\frac{\left(\frac{1}{2}+(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{20}}{\theta_2}+1}}{\frac{\theta_{20}}{\theta_2}+1}+2^{\frac{-\theta_{10}}{\theta_1}}(\frac{3\alpha}{2})^{\frac{\theta_{20}}{\theta_2}}$$

$$-\alpha^{\frac{\theta_{20}}{\theta_2}}2^{\frac{-\theta_{10}}{\theta_1}}+\frac{\left(\frac{1}{2}+\alpha^{\frac{\theta_{20}}{\theta_2}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{10}}{\theta_1}+1}}{\frac{\theta_{10}}{\theta_1}+1}-\frac{\left(\frac{1}{2}+(\frac{3\alpha}{2})^{\frac{\theta_{20}}{\theta_2}}-3\frac{\alpha}{2}\right)^{\frac{\theta_{10}}{\theta_1}+1}}{\frac{\theta_{10}}{\theta_1}+1}$$

$$+2^{\frac{-\theta_{20}}{\theta_2}}(\frac{3\alpha}{2})^{\frac{\theta_{10}}{\theta_1}}-\alpha^{\frac{\theta_{10}}{\theta_1}}2^{\frac{-\theta_{20}}{\theta_2}}$$

By changing the variable in integrals of the above formulas (17) and (18), we obtained (14).

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