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Mathematical programming with Semilocally Subconvex functions over cones

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Abstract In this paper, we introduce another generalization of semilocally convex functions over cones, called cone-semilocally subconvex function (C-slsb), and compare it with other generalizations of convex functions through examples. Further, using its properties we establish a theorem of the alternatives for these functions. Then we investigate the optimal solutions of the mathematical programming problem (MP) over cones using these functions, directional derivatives, and the alternative theorem. Investigation of optimal solutions of (MP) is done by deriving optimality and duality results for semilocally subconvex mathematical programming problems over cones (MP).

Keywords Vector Optimization, duality, Alternative Theorem, cones

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1. Introduction

The notion of semilocal convexity reduces the length of the line segment required for the convexity to hold to a locally star-shaped set and expands the definition of convexity. These generalizations are used widely in various fields such as optimization, economics, geometry, etc. In optimization, it allows one to find local and approximate solutions. In the field of economics, the utility and cost functions exhibit semilocal convexity. In geometry, the properties of curves and surfaces that are not convex are studied using locally star-shaped sets and semilocally convex functions.

Ewing [1] introduced semilocally convex functions defined on locally starshaped sets. They are nonconvex functions but satisfy some convex-type properties like non-negative linear combinations of semilocally convex functions are also semilocally convex, and local minima convex functions are also global minima. Several authors introduced various generalizations of semilocally convex functions. Kaur [2], Kaul and Kaur [3, 4] investigated generalizations of these functions and their properties as well. Gupta et al. [5] have studied another generalization of these functions called rho-semilocally preinvex functions over cones and found optimality and duality results for semilocally preinvex mathematical programming problems.

We know that the generalized convex functions are studied in different kinds of manifolds, such as Riemannian manifolds, etc. In 2018 Kılıçman and Saleh [6] defined geodesic semilocal E-preinvex functions on Riemannian manifolds. These functions are the generalization of geodesic semilocal E-convex and geodesic semi-E-preinvex

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functions. Further, for a nonlinear fractional multiobjective programming problem, they obtained sufficient optimality conditions, and finally, they formulated a dual and proved duality results using the above functions for the same problem. Recently, Mayvan and Motallebi [7] studied various optimization problems for locally convex cone-valued functions, and Rimpi and Lalitha [8] established various constraint qualifications by imposing an additional assumption of semilocal convexity at a point on the active constraint.

1.1. Contributions

The theorem of alternatives is one of the fundamental results in mathematical programming problems, from which optimality and duality results may be derived. In this paper, we establish an alternative theorem for semilocally subconvex functions over cones, another generalization of semilocally convex functions is introduced by us to derive optimality and duality results for semilocally subconvex mathematical programming problems over cones.

1.2. Organization

The paper consists of six sections, the first section is introductory. In the second section, we have given some definitions and results, which we have used in the paper, from the literature. Apart from this, we have introduced one generalization of semilocally convex functions known as semilocally subconvex functions over cones. We have depicted its relation with generalizations and subconvex functions with the help of examples. In the third section, we have derived the theorem of alternatives for these functions and proved the necessary Karush Kuhn Tucker (KKT) conditions for the problem (MP). In Section four, sufficient optimality conditions are discussed for a point to be a weak minimizer. Section five focuses on the duality results, and the last section concludes the paper, followed by the references used.

2. Preliminaries and Definitions

Let $E \subseteq \mathbf{R^n}$ be a nonempty convex set and $C \subseteq \mathbf{R^m}$ be a closed, convex, pointed cone with nonempty interior. The positive dual cone C^+ of C is defined as

$$C^+ = \{y^* \in \mathbf{R^m} : y^T y^* \geq 0, \forall\, y \in C\}.$$

The strict positive dual C^{+s} is given by

$$C^{+s} = \{y^* \in \mathbf{R^m} : y^T y^* > 0, \forall\, y \in C\}.$$

Definition 1. *[1] $E \subseteq \mathbf{R^n}$ is said to be a locally starshaped set at $\bar{x}$ if $\forall\, x \in E$ there exists $a(x,\bar{x}) \leq 1$ such that $tx + (1-t)\bar{x} \in E,$ for $0 < t < a(x,\bar{x})$.*

Remark 1
It is clear from the definition that every convex set is locally starshaped; however, a locally starshaped set need not be convex.

Hu and Ling (2004)[9] discussed optimality results for a vector optimization problem using cone subconvex functions.

Definition 2. *$\phi : E \subseteq \mathbf{R^n} \longrightarrow \mathbf{R^m}$ is said to be C-subconvex at $\bar{x} \in E$ if there exits $v \in int\ C$ such that for any $t \in (0,1),\ \epsilon > 0,\ \epsilon v + t\,\phi(x) + (1-t)\,\phi(\bar{x}) - \phi(tx + (1-t)\,\bar{x}) \in C$.*

We now introduce semilocally subconvex functions over cones defined on locally starshaped sets.

Definition 3. *$\phi : E \subset \mathbf{R^n} \longrightarrow \mathbf{R^m}$ is said to be C-semilocally subconvex (C-slsb) at $\bar{x} \in E$ if corresponding to each $x \in E$ there exists a positive number $d(x,\bar{x}) \leq a(x,\bar{x}), v \in int\,C$, such that $\forall \epsilon > 0$,*

$$\epsilon v + t\,\phi(x) + (1-t)\,\phi(\bar{x}) - \phi(tx + (1-t)\,\bar{x}) \in C,\ 0 < t < d(x,\bar{x})$$

C-Semilocally Convex	$\Rightarrow$ $\not\Leftarrow$	**C-Semilocally preinvex**	$\Rightarrow$ $\not\Leftarrow$	**C ρ-Semilocally preinvex**	$\Rightarrow$ $\not\Leftarrow$	**C-Semilocally Subconvex**

Figure 1. Relation between generalizations of Cone-semilocally convex functions

ϕ is said to be C-slsb on E if it is C-slsb at each $x \in E$.

Remark 2
The following remarks and Figure1 relate the C-slb functions to other such functions available in the literature.

1. If $\epsilon = 0$, then the C-slb function reduces to the C-semilocally convex functions defined by Weir [10].
2. If $d(x, \bar{x}) = a(x, \bar{x}) = 1$ or $t \in [0, 1]$ then the above definition reduces to C-subconvex functions [9] over cones, as discussed in Definition 2.
3. If $\epsilon = 0$ and $x - \bar{x}$ is replaced by $\eta(x, \bar{x})$ where $\eta : E \times E \longrightarrow \mathbf{R^m}$ is any vector-valued function, then semilocally subconvex function over cones reduces to semilocally preinvex function over cones, introduced by Suneja et.al.[11]

This example shows that there exist functions that are C-slsb but not C-semilocally convex.

Example 1
Let $E = \mathbf{R} \backslash S$ where $S = [\frac{-1}{2}, \frac{1}{2}] \cup \{2\}$. Then E becomes a locally starshaped set where,

$$a(x, \bar{x}) = \begin{cases} \frac{\bar{x}-2}{\bar{x}-x} & 2 < \bar{x}, \frac{1}{2} < x < 2, \\ |\frac{2-\bar{x}}{x-\bar{x}}| & \frac{1}{2} < \bar{x} < 2, 2 < x, \text{ or } \frac{1}{2} < \bar{x} < 2, x < -\frac{1}{2}, \\ 1 + x & \text{elsewhere.} \end{cases}$$

Let $C = \{(x, y) : y \leq -x,\ y \leq 0\}$ be the convex, closed and pointed cone. Define the function $\phi : E \to \mathbf{R}^2$ by
$$\phi(x) = \begin{cases} (0, -x), & x < -\frac{1}{2}, \\ (x, 0) & \frac{1}{2} < x, x \neq 2. \end{cases}$$
Then ϕ is C-slsb at $\bar{x} = -1$ as
$\epsilon v + t\phi(x) + (1-t)\phi(\bar{x}) - \phi(tx + (1-t)\bar{x}) \in C$ for $0 < t < d(x, \bar{x})$ where $d(x, \bar{x}) < a(x, \bar{x})$. However ϕ fails to be C-semilocally convex at $\bar{x} = -1$ since corresponding to $x = \frac{7}{3}$ there exists no positive number $d(x, \bar{x}) < a(x, \bar{x})$ such that $t\phi(x) + (1-t)\phi(\bar{x}) - \phi(tx + (1-t)\bar{x}) \in C$.

Note that the above function ϕ as in the example 1 which is C-slsb is also not semilocally preinvex over cones.

Example 2
Let $\eta(x, \bar{x}) = \frac{1}{2}(x - \bar{x})^2$, the function ϕ considered in example 1 fails to be C-semilocally preinvex at $\bar{x} = -1$ because for $x = \frac{11}{3}, t\phi(x) + (1-t)\phi(\bar{x}) - \phi(\bar{x} + t\eta(x, \bar{x})) \notin C$ for any t lying between 0 and $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$, where $a_\eta(x, \bar{x}) = a(x, \bar{x})$ as considered in example 1.

The following function ξ fails to be cone- ρ semilocally preinvex and is C-slsb.

Example 3
Consider the function $\xi : E \to \mathbf{R}$ given by $\xi(x) = \begin{cases} (-x - \frac{1}{2}, 0), & x < -\frac{1}{2}, \\ (0, x - \frac{1}{2}) & \frac{1}{2} < x, x \neq 2. \end{cases}$ Let $D = \{(x, y) : x \geq 0, y \geq 0\}$, then ξ is D-slsb at $\bar{x} = -1$ but fails to be $\rho-$semilocally preinvex over cone D for $\eta(x, \bar{x}) = \frac{(x-\bar{x})^2}{2}$, $\rho = 1$, $\theta(x, \bar{x}) = x - \bar{x}$ with $t = \frac{1}{2}$ and $x = 3$.

Definition 4. *The function $\phi : E \longrightarrow \mathbf{R^m}$ is said to be directionally differentiable at $\bar{x} \in E$ in the direction $d \in \mathbf{R^n}$ if*

$$\phi^{'}(\bar{x}, d) = \lim_{t \to 0^+} \frac{\phi(\bar{x} + td) - \phi(\bar{x})}{t}, \textit{exists.}$$

The following theorem reduces the definition of C-slsb in terms of directional derivative.

Theorem 1. *Let ϕ be C-slsb on E then $\epsilon v + \phi(x) - \phi(\bar{x}) - \phi^{'}(\bar{x}, x - \bar{x}) \in C$, for all $x \in E$.*

Proof
Let ϕ be C-slsb on E then, there exist $v \in$ int C, $d(x, \bar{x}) < a(x, \bar{x})$ such that for all $\epsilon > 0$, $\epsilon t v + t\phi(x) + (1-t)\phi(\bar{x}) - \phi(tx + (1-t)\,\bar{x}) \in C$, $0 < t < d(x, \bar{x})$, which can be rewritten as $\epsilon v + \phi(x) - \phi(\bar{x}) - \frac{\phi(tx+(1-t)\bar{x}) - \phi(\bar{x})}{t} \in C$ or, $\epsilon v + \phi(x) - \phi(\bar{x}) - \frac{\phi(\bar{x}+t(x-\bar{x})) - \phi(\bar{x})}{t} \in C$. Since C is a closed cone, therefore, taking the limit as $t \longrightarrow 0+$ we get $\epsilon v + \phi(x) - \phi(\bar{x}) - \phi^{'}(\bar{x}, x - \bar{x}) \in C$, for all x in E. □

Theorem 2. *If ϕ is C-slsb on E then for every $x_1, x_2 \in E$ there exist $d(x_1, x_2) < a(x_1, x_2)$ such that for $0 < t < d(x_1, x_2)$, there exists $v \in int\, C$ such that $\forall\, \epsilon > 0, \epsilon\, v + t\, \phi(x_1) + (1-t)\, \phi(x_2) \in \phi(E) + C$.*

The proof of the above result holds trivially.

Theorem 3. *ϕ is C-slsb on E if and only if $\forall\, v^{'} \in int\, C$, $x_1, x_2 \in E$ there exists $d(x_1, x_2) < a(x_1, x_2)$ such that for $0 < t < d(x_1, x_2)$, $v^{'} + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C$.*

Proof
Let ϕ be the C-slsb on E and let $v^{'} \in intC, x_1, x_2 \in E$, there exist $0 < t < d(x_1, x_2)$ such that $\forall\, \epsilon > 0$, using Theorem 2 $\epsilon v + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C$. As $v^{'} \in int\, C$, we can choose $\epsilon_0 > 0, v_0 \in int\, C$ such that $v^{'} - \epsilon_0 v = v_0$, which gives $v^{'} + t\phi(x_1) + (1-t)\phi(x_2) = \epsilon_0 v + v_0 + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C + v_0 \in \phi(E) + C + int\, C \subset \phi(E) + int\, C$.
Conversely let $v^{'} + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C, 0 < t < d(x_1, x_2)$. As $int\, C$ is nonempty, there exists $v \in int\, C$. Let $\bar{v} = \epsilon v, \epsilon > 0$, then $\bar{v} \in int\, C$ and $\bar{v} + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C$. In other words, $\epsilon v + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + C$. Therefore, ϕ is a C-slsb on a locally starshaped set E. □

The next result establishes a characterization for semilocally subconvex functions over cones.

Theorem 4. *If ϕ is C-slsb on E, then $\phi(E) + int\, C$ is locally starshaped.*

Proof
Let ϕ be C-slsb on E and $y_1, y_2 \in \phi(E) + int\, C$. Then there exist $x_1, x_2 \in E, c_1, c_2 \in int\, C, d(x_1, x_2) < a(x_1, x_2)$, such that $y_1 = \phi(x_1) + c_1, y_2 = \phi(x_2) + c_2$. As $int\, C$ is convex, $c^{'} = tc_1 + (1-t)c_2 \in int\, C$ for $0 < t < d(x_1, x_2) < 1$. Now using theorem 3 for $c^{'} \in int\, C, c^{'} + t\phi(x_1) + (1-t)\phi(x_2) \in \phi(E) + int\, C$.
Now consider, $ty_1 + (1-t)y_2 = t(\phi(x_1) + c_1) + (1-t)(\phi(x_2) + c_2) = t\phi(x_1) + (1-t)\phi(x_2) + c^{'} \in \phi(E) + intC$, for $0 < t < d(x_1, x_2)$. Therefore, $\phi(E) + int\, C$ is a locally starshaped set. □

3. Necessary Optimilaty Conditions

This section discusses the theorem of Alternatives and the KKT necessary optimality conditions for the vector optimization problem.

(MP) C-min $\phi(x)$

subject to $-\xi(x) \in D$,

where $\phi = (\phi_1, \phi_2, ..., \phi_m)^T : \mathbb{R}^n \to \mathbb{R}^m, \xi = (\xi_1, \xi_2, ..., \xi_p)^T : \mathbb{R}^n \to \mathbb{R}^p$ are directionally differentiable functions on $\mathbb{R}^n$. C and D are closed, convex, and pointed cones with nonempty interiors in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. Let $F_p = \{x \in \mathbb{R}^n : -\xi(x) \in D\}$ be the set of all feasible solutions of (VP).

Definition 5. *Let $\bar{x} \in F_p$, then*

1. *$\bar{x}$ is called a weak minimizer of (MP) if for all $x \in F_p, \phi(\bar{x}) - \phi(x) \notin int\, C$.*
2. *$\bar{x}$ is called a minimizer of (MP) if for all $x \in F_p, \phi(\bar{x}) - \phi(x) \notin C \backslash \{0\}$.*

Kaur [2] has established that every closed locally starshaped set is convex.
Let $F = (\phi,\ \xi) : E \subseteq \mathbf{R^n} \longrightarrow \mathbf{R^m} \times \mathbf{R^p}$ and $K = C \times D$. If F is K-slsb on E, that is ϕ is C-slsb on E and ξ is D-slsb on E, then by theorem 4, $F(E) + int\, K$ is locally starshaped.

Remark 3
Suppose that we assume $F(E) + int\, K$ is closed then using C-semilocally subconvex functions, $F(E) + int\, K$ becomes convex and the following theorem of alternatives is proved on the lines of Illes and Kassay [12].

Theorem 5. *Let F be K-slsb on E, such that $F(E) + int\, K$ is closed with a nonempty interior, then exactly one of the following holds:*

(i) there exists $x \in E$ such that $-\phi(x) \in int\, C$ and $-\xi(x) \in int\, D$,
(ii) there exist $\alpha \in C^+, \beta \in D^+$ such that
$\alpha^T\phi(x) + \beta^T\xi(x) \geq 0$,
$(\alpha, \beta) \neq (0, 0)$, for all $x \in E$.

Definition 6. *The constraint function ξ is said to satisfy the generalized Slater-type constraint qualification at $\bar{x}$ if there exists $x^* \in E$ such that $-\xi(x^*) \in int\, D$*

Theorem 6. *Let $F(x) = (\phi(x) - \phi(\bar{x}),\ \xi(x)), \forall\, x \in E$ and $F(E) + int\,(C \times D)$ be closed with nonempty interior. Let $\bar{x} \in F_p$ be a weak minimizer of (MP), ϕ be C-slsb and ξ be D-slsb on E. Suppose that ξ satisfies the generalized Slater-type constraint qualification, then there exist $0 \neq \bar{\alpha} \in C^+, \bar{\beta} \in D^+$ such that*

$$\bar{\alpha}^T\phi^{'}(\bar{x}, x - \bar{x}) + \bar{\beta}^T\xi^{'}(\bar{x}, x - \bar{x}) \geq 0, \forall\, x \in E, \tag{1}$$

and,

$$\bar{\beta}^T\xi(\bar{x}) = 0. \tag{2}$$

Proof
Let $\bar{x}$ be a weak minimizer of (MP), then there does not exist any $x \in E$ such that $F(x) \in (C \times D)$, or

$$-[\phi(x) - \phi(\bar{x}), \xi(x)] \in int\,(C \times D).$$

Using Theorem of Alternative (Theorem 5), there exist $\bar{\alpha} \in C^+,\ \bar{\beta} \in D^+, (\bar{\alpha}, \bar{\beta}) \neq (0, 0)$ such that $\bar{\alpha}^T(\phi(x) - \phi(\bar{x})) + \bar{\beta}^T\xi(x) \geq 0, \forall\, x \in E$ which implies,

$$\bar{\alpha}^T\phi(x) + \bar{\beta}^T\xi(x) \geq \bar{\alpha}^T\phi(\bar{x}), \forall\, x \in E. \tag{3}$$

Since $\bar{\beta} \in D^+, -\xi(\bar{x}) \in D$ therefore, $\bar{\beta}^T\xi(\bar{x}) \leq 0$. Consider $x = \bar{x}$ in (3) we have $\bar{\beta}^T\xi(x) \geq\ 0\, \forall\, x \in E$. Therefore,

$$\bar{\beta}^T\xi(\bar{x}) = 0. \tag{4}$$

Using (2) and (3) we get,

$$(\bar{\alpha}^T\phi + \bar{\beta}^T\xi)(x) - (\bar{\alpha}^T\phi + \bar{\beta}^T\xi)(\bar{x}) \geq 0, \forall\, x \in E.$$

As E is a locally starshaped set, $t\bar{x} + (1-t)x \in E, 0 < t < a(x, \bar{x})$, which gives, $(\bar{\alpha}^T\phi + \bar{\beta}^T\xi)(t\bar{x} + (1-t)x) - (\bar{\alpha}^T\phi + \bar{\beta}^T\xi)(\bar{x}) \geq 0$, implies, $\bar{\alpha}^T(\phi(t\bar{x} + (1-t)x) - \phi(\bar{x})) + \bar{\beta}^T(\xi(t\bar{x} + (1-t)x) - \xi(\bar{x}) \geq 0$. Dividing the above by $t > 0$ and taking lim $t \longrightarrow 0^+$, we obtain

$$\bar{\alpha}^T\phi^{'}(\bar{x}, x - \bar{x}) + \bar{\beta}^T\xi^{'}(\bar{x}, x - \bar{x}) \geq 0, \forall\, x \in E. \tag{5}$$

Next, let if possible $\bar{\alpha} = 0$, then $\bar{\beta} \neq 0$ and (5) reduces to,

$$\bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0, \forall\, x \in E. \tag{6}$$

Since ξ is D-slsb at $\bar{x}$, for every $x \in S, \epsilon v + \xi(x) - \xi(\bar{x}) - \xi^{'}(\bar{x}, x - \bar{x}) \in D$, as $\bar{\beta}^T \in D^+$,

$$\bar{\beta}^T \epsilon v + \bar{\beta}^T \xi(x) - \bar{\beta}^T \xi(\bar{x}) - \bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0. \tag{7}$$

Adding (6) and (7) and using $\bar{\beta}^T \epsilon v > 0$ we have, $\bar{\beta}^T \xi(x) - \bar{\beta}^T \xi(\bar{x}) \geq 0, \forall\, x \in E$. Using (4) we get,

$$\bar{\beta}^T \xi(x) \geq 0, \forall\, x \in E. \tag{8}$$

In view of the generalized Slater-type constraint qualification, there exists $x^* \in E$ such that $-\xi(x^*) \in int\, D$ resulting in, $\bar{\beta}^T \xi(x^*) < 0$, which contradicts (8), hence $\bar{\alpha} \neq 0$. □

4. Sufficient Optimality Conditions

Theorem 7. *Let $\bar{x}$ be a feasible point of (MP) and ϕ be C-slsb and ξ be D-slsb at $\bar{x}$. If there exist $0 \neq \bar{\alpha} \in C^+$ and $\bar{\beta} \in D^+$ such that*

$$\bar{\alpha}^T \xi^{'}(\bar{x}, x - \bar{x}) + \bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0, \forall\, x \in E, \tag{9}$$

and

$$\bar{\beta}^T \xi(\bar{x}) = 0, \tag{10}$$

hold for all $x \in F_p$, then $\bar{x}$ is a weak minimizer of (MP).

Proof
Let if possible $\bar{x}$ is not a weak minimizer of (MP). Then there exists $x \in F_p$ such that

$$\phi(\bar{x}) - \phi(x) \in int\, C. \tag{11}$$

Since ϕ is C-slsb, there exist $v \in int\, C, d_1(x, \bar{x}) < a(x, \bar{x}), 0 < t < d_1(x, \bar{x}), \epsilon > 0$ such that $\epsilon v + \phi(x) - \phi(\bar{x}) - \phi^{'}(\bar{x}, x - \bar{x}) \in C$, as $\bar{\alpha} \in C^+ \backslash \{0\}$,

$$\bar{\alpha}^T \epsilon v + \bar{\alpha}^T \phi(x) - \bar{\alpha}^T \phi(\bar{x}) - \bar{\alpha}^T \phi^{'}(\bar{x}, x - \bar{x}) \geq 0. \tag{12}$$

Again as ξ is D-slsb there exist $w \in int\, D, d_2(x, \bar{x}) < a(x, \bar{x}), 0 < t < d_2(x, \bar{x}), \delta > 0$ such that

$$\bar{\beta}^T \delta w + \bar{\beta}^T \xi(x) - \bar{\beta}^T \xi(\bar{x}) - \bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0. \tag{13}$$

Adding (12) and (13) we have,

$$\bar{\alpha}^T \epsilon v + \bar{\beta}^T \delta w + \bar{\alpha}^T (\phi(x) - \phi(\bar{x})) + \bar{\beta}^T (\xi(x) - \xi(\bar{x})) - \bar{\alpha}^T \phi^{'}(\bar{x}, x - \bar{x}) - \bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0. \tag{14}$$

Using (9) and (10) and as $\phi^{'}$ and $\xi^{'}$ are linear we get,

$$\bar{\alpha}^T \epsilon v + \bar{\beta}^T \delta w + \bar{\alpha}^T (\phi(x) - \phi(\bar{x})) + \bar{\beta}^T \xi(x) \geq 0. \tag{15}$$

Since $x \in F_p, -\xi(x) \in int\, D$ therefore, $\bar{\beta}^T \xi(x) \leq 0$. As ϵ and δ are arbitrarily chosen using (15) we have, $\bar{\alpha}^T (\phi(x) - \phi(\bar{x})) \geq 0$. Now using, $\bar{\alpha} \neq 0$ and above we have $\phi(\bar{x}) - \phi(x) \notin int\, C$ which contradicts (11) thus $\bar{x}$ is a weak minimizer of (MP). □

5. Duality

(MD)
$$C\text{-max } \phi(u) + \beta^T \xi(u)c$$
$$\text{subject to} \quad \alpha^T \phi^{'}(\bar{x}, x - \bar{x}) + \beta^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0 \tag{16}$$

where $c \in \text{int}\, C$ is a fixed element, $\alpha \in C^+ \backslash \{0\}, \alpha^T c = 1,\ \beta \in D^+$ and $u \in E$.
The feasible set of (MD) is denoted by F_d.

Definition 7. *A point $(u, \alpha, \beta) \in F_d$ is called a weak maximizer of (MD) if $\phi(z) + \beta^T \xi(z)c - \phi(u) - \beta^T \xi(u)c \notin int\, C, \forall\, (z, \alpha, \beta) \in F_d$.*

Theorem 8. *Let $x \in F_p$ and $(u, \alpha, \beta) \in F_d$. If ϕ is C -slsb and ξ is D-slsb at u then*
$$\phi(u) + \beta^T \xi(u)c - \phi(x) \notin int\, C.$$

Proof
Suppose if possible
$$\phi(u) + \beta^T \xi(u)c - \phi(x) \in int\, C. \tag{17}$$
Since ϕ is C-slsb at u there exists $v \in int\, C$ such that for all $\epsilon > 0$,
$$\epsilon v + \phi(x) - \phi(u) - \phi^{'}(u, x - u) \in C. \tag{18}$$
Adding (17) and (18) we get, $\epsilon v + \beta^T \xi(u)c - \phi^{'}(u, x - u) \in int\, C$. Using $\alpha \in C^+ \backslash \{0\}$ and $\alpha^T c = 1$, we have,
$$\alpha^T \epsilon v + \beta^T \xi(u) - \alpha^T \phi^{'}(u, x - u) > 0. \tag{19}$$
As $v \in int\, C, \epsilon > 0, \alpha \in C^+ \backslash \{0\},\ \alpha^T \epsilon v > 0$ therefore,
$$\beta^T \xi(u) - \alpha^T \phi^{'}(u, x - u) > 0. \tag{20}$$
Since (u, α, β) is a feasible solution of (MD) so (16) holds, adding (16) and (20) we get
$$\beta^T \xi(u) + \beta^T \xi^{'}(u, x - u) > 0. \tag{21}$$
Now as ξ is D-slsb at u there exists $w \in int\, D$ such that for all $\delta > 0,\ \ \delta w + \xi(x) - \xi(u) - \xi^{'}(u, x - u) \in D$, using $\beta \in D^+$,
$$\beta^T \delta w + \beta^T \xi(x) - \beta^T \xi(u) - \beta^T \xi^{'}(u, x - u) \geq 0. \tag{22}$$
Again adding (21) and (22) and using $w \in int\, D, \delta > 0,\ \beta^T \delta w > 0$ we have
$$\beta^T \xi(x) > 0 \tag{23}$$
Now, as $x \in F_p, -\xi(x) \in D$ consequently $\beta^T \xi(x) \leq 0$, for $\beta \in D^+$ which contradicts (23).

□

Theorem 9. *Let ϕ be C-slsb and ξ be D-slsb on E. Suppose $F(E) + int\, (C \times D)$ be closed with nonempty interior and ξ satisfies the Slater-type constraint qualification. If $\bar{x} \in F_p$ is a weak minimizer of (MP) then there exists $0 \neq \bar{\alpha} \in C^+, \bar{\beta} \in D^+$ such that $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is a feasible solution of (MD). Moreover, if the conditions of the Weak Duality theorem 8 are satisfied for all feasible solutions of (MP) and (MD) then, $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is a weak maximizer of (MD).*

Proof
Since $\bar{x}$ is a weak minimizer of (MP), by theorem 6 there exist multipliers $\bar{\alpha} \in C^+ \backslash \{0\}, \bar{\beta} \in D^+$ such that
$$\bar{\alpha}^T \phi^{'}(\bar{x}, x - \bar{x}) + \bar{\beta}^T \xi^{'}(\bar{x}, x - \bar{x}) \geq 0, \forall\, x \in E, \tag{24}$$
$$\bar{\beta}^T \xi(\bar{x}) = 0, \tag{25}$$
so $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is a feasible solution of (MD). Let if possible $(\bar{x}, \bar{\alpha}, \bar{\beta})$ be not a weak maximizer of (MD), then there exists a feasible solution $(u, \bar{\alpha}, \bar{\beta})$ of (MD) such that $\phi(u) + \bar{\beta}^T \xi(u)c - \phi(\bar{x}) - \bar{\beta}^T \xi(\bar{x})c \in int\, C$. In view of (25) we get, $\phi(u) + \bar{\beta}^T \xi(u)c - \phi(\bar{x}) \in int\, C$. The above is a contradiction to the Weak Duality theorem 8, for the feasible solutions $\bar{x}$ of (MP) and $(u, \bar{\alpha}, \bar{\beta})$ of (MD). Therefore $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is a weak maximizer of (MD). □

6. Conclusion

The article introduces a new generalization of semilocally convex functions over cones, known as cone semilocally subconvex functions. We establish the theorem of alternatives for a mathematical programming problem (MP) involving cones and investigate the necessary and sufficient optimality conditions for the problem using the defined generalizations. We examine a Wolfe-type dual corresponding to the programming problem (MP) and demonstrate the weak duality and strong duality theorems for weak minimum between the primal problem (MP) and the associated dual (MD).

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