

McDonald Rayleigh Distribution with application

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Abstract Statistical modeling of many phenomena is very important topic, especially the phenomena of survival, reliability, economic and financial. Many standard probability distributions lack superiority in modeling data sets of complex phenomenal. In recent years, the design of different forms of probability distributions has received wide attention by using different techniques in statistical theory. In this paper, McDonald family used to extend Rayleigh named as (McR) distribution. Some theoretical statistical properties of McR distribution are presented and explained. Shape and Scale parameters of McR distribution were estimated by maximum likelihood (ML) and E-Bayesian (EB) methods under square error (SE) and linear exponential (Linex) loss functions with three different kinds of hyper priors of distributions. The estimation results were applied to a simulation experiment for data generated with different sample sizes. To comparison has been done using (MSE) criterion. The experiment showed the superiority of (EB) estimators under (Linex) loss function with the second joint hyper prior distribution. Two real data sets were fitted by using McR and other models representing especial cases of it. The (McR) model demonstrated its flexibility in modeling both type of real data sets.

Keywords Rayleigh distribution, McDonald family, Maximum likelihood method,E-Bayesian method, Square error loss function, linear exponential loss function, McDonald Rayleigh distribution, alternating direction method.

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1. Introduction

Statisticians always resort to creating probability distributions that can accurately phenomena in different fields. These distributions help in understanding their features and making the appropriate decision correctly. There are various methods that have the ability to adapt probability density and hazard rate functions to model different real data sets for complex phenomena.

Many ways proposed to generalize or extend probability distributions based on adding shape parameters to baseline distributions. One of these methods proposed by [17] named as an exponentiated family. Many used the exponentiated family, for example [20, 21], the class of Beta *G* family proposed by [16]. A McDonald family of distributions introduced by [8] dented with (Mc). The Mc family has been used to transform the normal distribution into a skewed distribution, offering greater flexibility than the standard normal distribution [8]. The McDonald Exponentiated Gamma distribution was introduced to provide increased flexibility in modeling real data, and the likelihood ratio test was used to compare it with a baseline distribution [2]. The McDonald Quasi-Lindley distribution was proposed as a more flexible model for real data sets [18]. The Mc family was later extended to include the Lindley-Poisson distribution [1]. A new distribution family, the McDonald Generalized Poisson distribution, was proposed [15]. Recently, the Mc family was further generalized to the Power Weibull distribution [5]. The probability density function f(x) of Mc family is defined by:

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$$g(x;a,b,c) = \frac{c}{B(a,b)} f(x) F(x)^{(ac-1)} (1 - F(x)^c)^{(b-1)}, \quad x > 0$$
(1)

where f(x) and F(x) are the p.d.f. and c.d.f. in baseline distribution, a, b, c > 0 are the additional shape parameters. These parameters have a role in skewness and tail weights. B(a,b) is a complete Beta function. Beta generated family is a special case of Mc family when c = 1. Kumaraswamy family proposed by [9] when a = 1 and an eponentiated family when a = b = 1. If a random variable X whose density has the form defined in (1) denoted by $X \sim McG(a, b, c)$.Rayleigh distribution plays a major role in modeling and analysis of survival[19], reliability data, communication, physical sciences and clinical research's. Regarding to the importance of this distribution and the reason for the desire to give it greater flexibility, the aim of this paper is to introduce a generalization of Rayleigh distribution to fit different varieties of real data sets arising from different fields comprising unimootal, to make kurtosis more flexible compared to Rayleigh distribution, to produce skewness for symmetric distribution and to produce increasing, decreasing, bathtub shapes of the hazard function curves. Also this family can be used for modeling positive and negative skewed real data sets.

2. McDonald Rayleigh distribution

The p.d.f. and c.d.f. of baseline Rayleigh distribution [7] are:

$$f(x;\theta) = 2\theta x e^{-\theta x^2}, \quad \theta > 0, \ x > 0.$$
⁽²⁾

$$F(x) = 1 - e^{-\theta x^2} \tag{3}$$

Substituting (2) and (3) into (1), the p.d.f. of McR is:

$$g(x;a,b,c) = \frac{2c\theta x e^{-\theta x^2}}{B(a,b)} (1 - e^{-\theta x^2})^{(ac-1)} \left(1 - (1 - e^{-\theta x^2})^c \right)^{(b-1)}, \quad a,b,c,\theta > 0, \ x > 0.$$
(4)

and zero otherwise, where a, b, c are shape parameters and θ is a scale parameter.

From the Figure 1 and for knowing the effect of the scale parameter θ and the shape parameters (a,b,c) of the *McR* probability density function curve, the value of one parameter is changed and the values of the other parameters are fixed. Therefore, the following is shown:-

- Increasing of the value of scale parameter θ leads to a decrease in the kurtosis of the function curve, a decrease in the dispersion of the observations and the tail of the curve become less heavier.
- The curve of the function is decreasing when the value of a < 0.5. The curve becomes slowly decreasing, has a single inflection point and has a heavy tail when a = 0.5 while the curve approaches symmetry, it pulls to the right and the tail becomes less heavy.
- Increasing of the value of *b* reduces the dispersion of observations, and the curve of the function becomes less kurtotic, more convex, less skewed to the right, and the tail becomes less heavy.
- The function curve is decreasing when c < 0.5, and as the value of c increases, the function curve approaches symmetry.



Figure 1. The p.d.f. of McR distribution at different values of shapes and scale parameters.

2.1. Cumulative distribution and hazard rate functions of McR distribution

The cumulative distribution function of the Mc distribution defined by [8] is given by:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{(F(x))^c} u^{a-1} (1-u)^{b-1} du$$
(5)

Putting (3) into (5) and using the Binomial expansion [?] to a term $(1 - u)^{b-1}$, the c.d.f. of *McR* becomes: (i) For (b - 1) is a positive real number:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j}$$
(6)

(ii) For (b-1) is a positive integer:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{b-1} \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j}$$
(7)

From Figure (2), the cumulative distribution function $G_{McR}(x)$ represents the probability that the random variable is less than or equal to x. The accumulation rate increases rapidly with higher θ , reflecting a greater likelihood of smaller values, while lower θ results in slower accumulation, indicating a higher probability of larger values. Therefore, small θ values are used to model wide-spread data, while larger θ values are preferred when focusing on smaller values.

The hazard rate function of *McR* is:

(i) When (b-1) is a positive real number:

$$h(x;a,b,c) = \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta x e^{-\theta x^2} (1-e^{-\theta x^2})^{(ac-1)} (1-(1-e^{-\theta x^2})^c)^{(b-1)}}{1-\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j}}$$
(8)

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(ii) When (b-1) is a positive integer, h(x; a, b, c) has the same formula as in (8) except the upper bound of the sum is (b-1).



Figure 2. The G(x) and S(x) of *McR* at different values of shape and scale parameters.

3. Statistical Properties of McR distribution

This section deals with some statistical properties of *McR* distribution:

3.1. Moments

The *r*-th moment around zero of *McR* variable is:

$$E(x^{r}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta \int_{0}^{\infty} x^{r+1} e^{-\theta x^{2}} (1-e^{-\theta x^{2}})^{ac-1} (1-(1-e^{-\theta x^{2}})^{c})^{b-1} dx$$
(9)

By using the Binomial expansion on $(1 - (1 - e^{-\theta x^2})^c)^{b-1}$ when (b-1) is a positive integer, $E(x^r)$ becomes:

$$E(x^{r}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta \int_{0}^{\infty} x^{r+1} e^{-\theta x^{2}} (1-e^{-\theta x^{2}})^{ac-1} \sum_{j=0}^{b-1} \frac{(-1)^{j} \Gamma(b) (1-e^{-\theta x^{2}})^{cj}}{j! \Gamma(b-j)} dx$$
(10)

Again, the Binomial expansion is used for the term $(1 - e^{-\theta x^2})^{(ac+cj-1)}$ when ac + cj - 1 is a positive integer. After making some simplifications, $E(x^r)$ is:

$$E(x^{r}) = \frac{c\Gamma(a+b)\Gamma\left(\frac{r}{2}+1\right)}{\Gamma(a)} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k}\Gamma(ac+cj)}{j!k!\Gamma(b-j)\Gamma(ac+cj-k)(k+1)^{\left(\frac{r}{2}+1\right)}}$$
(11)

When (b-1) and ac + cj - 1 are positive real numbers, $E(x^r)$ is the same as in equation (11) except that the upper bounds of the sums tends to infinity.

3.2. Moment generating function (m.g.f.)

The m.g.f of the *McR* variable is:

$$\mu_x^{(t)} = E[e^{tx}] = \frac{2\theta c\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k}\Gamma(b)\Gamma(ac+cj)}{j!k!\Gamma(b-j)\Gamma(ac+cj-k)} \int_0^\infty x^{k+1} e^{-\theta x^2} e^{tx} \, dx \tag{12}$$

By using the Maclaurin series expansion on e^{tx} and making algebraic simplifications, the moment generating function (m.g.f.) becomes:

$$\mu_x^{(t)} = \frac{c\Gamma(a+b)}{\Gamma(a)} \sum_{s=0}^{\infty} \frac{t^s \Gamma(s/2+1)}{s! \theta^{s/2}} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k} \Gamma(ac+cj)}{j! k! \Gamma(b-j) \Gamma(ac+cj-k)(k+1)^{(s/2+1)}}$$
(13)

The m.g.f. defined in (13) holds when (b-1) and ac + cj - 1 are positive integers. However, when they are positive real numbers, the m.g.f. remains the same as in equation (13), except that the upper bounds of the sums tends to infinity.

3.3. Mode and median

The mode of the McR variable is a numerical solution to the following nonlinear equation with respect to x, The R package (uniroot) was used to solve the problem.:

$$\frac{1}{x} - 2\theta x + \frac{2\theta x(ac-1)e^{-\theta x^2}}{(1-e^{-\theta x^2})} - \frac{2c\theta x(b-1)e^{-\theta x^2}(1-e^{-\theta x^2})^c}{(1-e^{-\theta x^2})(1-(1-e^{-\theta x^2})^c)} = 0$$
(14)

The median of the McR variable is a solution to the following equation with respect to x:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{(F(x))^c} u^{a-1} (1-u)^{b-1} \, du = \frac{1}{2}$$
(15)

First, equation (15) must be solved for $(F(x))^c$, and then the inverse function $((F(x))^c)^{-1}$ represents the median of the McR variable.

Data generation from the McR variable can be done using equation (15) by replacing $\frac{1}{2}$ with w, where (w) is a random observation from U(0, 1), and solving the equation with respect to x.

3.4. Skewness, Kurtosis and Bowley Skewness

Skewness measure proposed by [13], that depends on quantiles:

Skewness =
$$\frac{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^3}{\left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2\right)^{3/2}}$$
 (16)

Moors' kurtosis measure, introduced by [14], is defined as:

$$\text{Kurtosis} = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}$$
(17)

The Bowley measure is affected by the shape parameters (a, b, c). Skewness and kurtosis measures can be used when raw moments do not exist or when they are represented as infinite sums.

The Bowley skewness is a measure of the degree of distribution asymmetry, making it more resistant to the effect of outliers, especially when dealing with asymmetric distributions.

Bowley Sk. =
$$\frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$
 (18)

The mode, median, Skewness, and Kurtosis of the McR variable were evaluated at different values of the scale and shape parameters.

θ	a	b	с	Mean	Median	Mode	Variance	Skewness	Kurtosis	BowleySk.
0.1	0.5	1.01	1.02	1.9984	1.7112	0.36159	2.1661	0.91893	3.6128	0.02951
0.1	0.5	1.01	1.05	2.0312	1.7507	0.56887	2.1712	0.90251	3.5831	0.12474
0.1	0.5	1.1	1.02	1.902	1.6137	0.34326	1.9697	0.9266	3.638	0.29904
0.1	0.5	1.1	1.05	1.9348	1.653	0.54196	1.9761	0.9091	3.6051	0.16393
0.1	1	1.01	1.02	2.8124	2.6401	2.2589	2.1193	0.6535	3.2417	0.07288
0.1	1	1.01	1.05	2.8472	2.6782	2.308	2.1109	0.61718	3.2372	0.07288
0.1	1	1.1	1.02	2.6961	2.4884	2.1663	1.9465	0.6248	3.2407	0.05599
0.1	1	1.1	1.05	2.7312	2.5269	2.2158	1.9397	0.61583	3.2349	0.07288
0.5	0.5	1.01	1.02	0.89369	0.76525	0.1617	0.43321	0.91893	3.6128	0.09245
0.5	0.5	1.01	1.05	0.90838	0.78293	0.25441	0.43425	0.90251	3.5831	0.10609
0.5	0.5	1.1	1.02	0.85059	0.72163	0.1535	0.3934	0.9266	3.638	0.22085
0.5	0.5	1.1	1.05	0.86528	0.73923	0.24238	0.39522	0.9091	3.6051	0.18644
0.5	1	1.01	1.02	1.2577	1.1807	1.0102	0.42385	0.62535	3.2417	0.0664
0.5	1	1.01	1.05	1.2733	1.1977	1.0322	0.42218	0.61718	3.2372	0.11144
0.5	1	1.1	1.02	1.2057	1.1128	0.9688	0.3893	0.6248	3.2107	0.10799
0.5	1	1.1	1.05	1.2214	1.1301	0.99092	0.38793	0.61583	3.234	0.08217
1.5	0.5	1.01	1.02	0.51597	0.44182	0.09335	0.1444	0.91893	3.6128	0.21161
1.5	0.5	1.01	1.05	0.52445	0.45203	0.14689	0.14475	0.90251	3.5831	0.21161
1.5	0.5	1.1	1.02	0.49109	0.41661	0.08861	0.13131	0.9266	3.638	0.17198
1.5	0.5	1.1	1.05	0.49957	0.42684	0.13994	0.13174	0.9091	3.6051	0.20032
1.5	1	1.01	1.02	0.72615	0.68167	0.58325	0.14128	0.62535	3.2417	0.26678
1.5	1	1.01	1.05	0.73515	0.69151	0.59591	0.14073	0.61718	3.2372	0.1994
1.5	1	1.1	1.02	0.69612	0.64248	0.55934	0.12977	0.6248	3.2407	0.19466
1.5	1	1.1	1.05	0.70519	0.65243	0.57211	0.12931	0.61583	3.2349	0.26678
2	0.5	1.01	1.02	0.44685	0.38263	0.08087	0.1083	0.91893	3.6128	0.13868
2	0.5	1.01	1.05	0.45419	0.39147	0.12719	0.10856	0.90251	3.5831	0.1517
2	0.5	1.1	1.02	0.4253	0.36082	0.07674	0.09848	0.9266	3.638	0.2277
2	0.5	1.1	1.05	0.43264	0.36963	0.1212	0.0988	0.9091	3.6051	0.16255
2	1	1.01	1.02	0.62887	0.59034	0.50513	0.10596	0.62535	3.2417	0.24509
2	1	1.01	1.05	0.63666	0.59885	0.51606	0.10554	0.61718	3.2372	0.22808
2	1	1.1	1.02	0.60286	0.55641	0.48442	0.09732	0.6248	3.2407	0.17028
2	1	1.1	1.05	0.61071	0.56504	0.49548	0.09698	0.61583	3.2349	0.22633

Table 1. Some statistical measures of McR variable at different values of parameters.

It is seen from. Table (1) that as the parameter a increases, the mean gradually decreases, reflecting a concentration of values around the mean of the distribution, while the median and mode follow a similar pattern. Variance decreases with increasing a, this indicates a reduction in spread. Skewness, which measures distribution asymmetry, it decreases as a increases, so that the distribution becomes more symmetrical and less right-skewed. Kurtosis remains close to the normal value (around 3), this means, the distribution shape does not deviate significantly from a normal distribution, although there are slight changes depending on parameter values. It can be observed that with smaller parameter values, the distribution is more spread out, whereas with larger values, it becomes narrower and more concentrated. Thus, the table highlights the effect of the parameters a, b, c, e on the distribution's shape and spread, demonstrating how these key statistics shift according to different parameter values, providing deeper insight into the relationship between the parameters and the statistical distribution.

The Figure 3 shows a three- dimensional representation of kurtosis and skewness as a function of the variables b and c with the constant variable a with different values (a = 0.5, 1, 1.5, 2). It is clear that increasing the value of a make the change in kurtosis with b and c more obvious. Also, increasing the value of a reduces the sharp effect of c on the skewness and the shape becomes smoother. We also note that there is a clear inverse relationship between the skewness and the value of c, where the higher value of c leads to reduced skewness. The effect of b does not appear clear at small values of a and its effect increases gradually with increasing value of a.

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Figure 3. Three- dimensional representation of skewness and kurtosis of McR

4. Parameters estimation

Depending on complete data with a random sample of size (n) taken from McR distribution, we consider parameter estimation using maximum likelihood (ML) and Expected Bayesian (EB) estimation methods as follows:

4.1. Maximum Likelihood Method

In this section, we consider the parameters (a, b) are known. Depending on complete sample information, the log-likelihood function is:

$$l(\underline{X};\theta,c) = n \ln\left(\frac{2c}{B(a,b)}\right) + n \ln(\theta) + \sum_{i=1}^{n} \ln(x_i) - \theta \sum_{i=1}^{n} x_i^2 + (ac-1) \sum_{i=1}^{n} \ln\left(1 - e^{-\theta x_i^2}\right) + (b-1) \sum_{i=1}^{n} \ln\left(1 - (1 - e^{-\theta x_i^2})^c\right),$$
(19)

where $\underline{X} = (x_1, x_2, \dots, x_n)'$ and B(a, b) is a complete Beta function. The maximum likelihood estimator (*MLE*) for θ and *c* represents a numerical solution to the following system of non-linear equations with respect to the unknown parameters, The R package (rootSolve) was used to solve the problem.

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i^2 + (ac-1) \sum_{i=1}^{n} \frac{x_i^2 e^{-\theta x_i^2}}{1 - e^{-\theta x_i^2}} + c(b-1) \sum_{i=1}^{n} \frac{(\theta-1) x_i^2 e^{-\theta x_i^2} (1 - e^{-\theta x_i^2})^c}{(1 - e^{-\theta x_i^2})(1 - (1 - e^{-\theta x_i^2})^c)} = 0,$$
(20)

$$a\sum_{i=1}^{n}\ln(1-e^{-\theta x_{i}^{2}}) + (b-1)\sum_{i=1}^{n}\frac{(1-e^{-\theta x_{i}^{2}})^{c}}{1-(1-e^{-\theta x_{i}^{2}})^{c}}\ln(1-e^{-\theta x_{i}^{2}}) = 0.$$
(21)

The solutions of (20) and (21) satisfy the condition that [23]:

$$J = \begin{bmatrix} \frac{\partial^2 l(\underline{X};\theta,c)}{\partial \theta^2} & \frac{\partial^2 l(\underline{X};\theta,c)}{\partial \theta \partial \phi c} \\ \frac{\partial^2 l(\underline{X};\theta,c)}{\partial c \partial \theta} & \frac{\partial^2 l(\underline{X};\theta,c)}{\partial c^2} \end{bmatrix} \bigg|_{\theta = \hat{\theta}_{ML}, c = \hat{c}_{ML}} \text{is a negative definite matrix.}$$

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4.2. E-Bayesian estimation

E-Bayesian estimators for θ and *c* represent the expectation of Bayesian estimators for each parameter based on all hyperparameter priors [3]. The hyperparameter of prior distributions should be chosen to guarantee a decreasing prior functions [12]. Consider θ and *c* as independent unknown parameters, and let (a, b) be known. Define the informative prior for θ as Gamma (α_0, β_0) and the informative prior for *c* as Gamma (τ_0, κ_0) . We propose the joint prior density function is given by:

$$p(\theta, c) \propto \theta^{\alpha_0 - 1} e^{-\theta \beta_0} c^{\tau_0 - 1} e^{-c\kappa_0}.$$
(22)

Combining the joint prior information for (θ, c) defined in (22) with sample information, the joint posterior distribution of (θ, c) is given by:

$$p(\theta, c|\underline{X}) \propto (c\theta)^{n} e^{-\theta \sum_{i=1}^{n} x_{i}^{2}} \prod_{i=1}^{n} (1 - e^{-\theta x_{i}^{2}})^{ac-1} \prod_{i=1}^{n} (1 - (1 - e^{-\theta x_{i}^{2}})^{c})^{b-1} \\ \times \theta^{\alpha_{0}-1} e^{-\theta \beta_{0}} c^{\tau_{0}-1} e^{-c\kappa_{0}}.$$
(23)

Suppose $l(\theta, c | \underline{X}) = \ln p(\theta, c | \underline{X}).$

$$l(\theta, c|\underline{X}) \propto n \ln(c) + n \ln(\theta) - \theta \sum_{i=1}^{n} x_i^2 + (ac-1) \sum_{i=1}^{n} \ln\left(1 - e^{-\theta x_i^2}\right) + (b-1) \sum_{i=1}^{n} \ln\left(1 - (1 - e^{-\theta x_i^2})^c\right) + (\alpha_0 - 1) \ln(\theta) - \theta \beta_0 + (\tau_0 - 1) \ln(c) - c\kappa_0.$$
(24)

Taking the first partial derivatives of both sides of (24) with respect to θ and c, so that equating the result to zero[4]:

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i^2 + (ac-1) \sum_{i=1}^{n} \frac{x_i^2 e^{-\theta x_i^2}}{1 - e^{-\theta x_i^2}} - c(b-1) \sum_{i=1}^{n} \frac{x_i^2 e^{-\theta x_i^2} (1 - e^{-\theta x_i^2})^{c-1}}{1 - (1 - e^{-\theta x_i^2})^c} + \frac{\alpha_0 - 1}{\theta} - \beta_0 = 0$$
(25)

$$\frac{n}{c} + a \sum_{i=1}^{n} \ln(1 - e^{-\theta x_i^2}) - (b-1) \sum_{i=1}^{n} \frac{(1 - e^{-\theta x_i^2})^c \ln(1 - e^{-\theta x_i^2})}{1 - (1 - e^{-\theta x_i^2})^c} + \frac{\tau_0 - 1}{c} - \kappa_0 = 0.$$
(26)

The numerical solution for the system of non-linear equations with respect to $\underline{\delta}' = (\theta, c)'$ satisfies that $\frac{\partial^2 p(\underline{\delta}|\underline{X})}{\partial \underline{\delta} \partial \underline{\delta}'}$ is a negative definite matrix. This solution, $\underline{\delta}' = (\hat{\theta}, \hat{c})'$, represents the posterior mode. Taking the exponents on both sides of (27), the approximate joint posterior distribution of $(\delta|X)$ is:

 $p(\underline{\delta}|\underline{X}) \propto e^{-\frac{1}{2}(\underline{\delta}-\underline{\hat{\delta}})'V(\underline{\hat{\delta}})^{-1}(\underline{\delta}-\underline{\hat{\delta}})}$ (27)

So that $\underline{\delta}|\underline{X} \sim N_2(\hat{\underline{\delta}}, V(\hat{\underline{\delta}})^{-1})$, where $V(\hat{\underline{\delta}}) = \frac{\partial^2 p(\underline{\delta}|\underline{X})}{\partial \underline{\delta} \partial \underline{\delta}'} \Big|_{\underline{\delta} = \underline{\delta}}$. Under square error (SE) and linear exponential (Linex) loss functions, Bayesian and Expected Bayesian (EB) estimators, suppose the joint hyperprior for τ_0 and α_0 has three forms:

$$p_{1}(\alpha_{0},\tau_{0}) = 1, \qquad 0 < \alpha_{0}, \tau_{0} < 1$$

$$p_{2}(\alpha_{0},\tau_{0}) = 6\alpha_{0}(1-\tau_{0}), \qquad 0 < \alpha_{0}, \tau_{0} < 1$$

$$p_{3}(\alpha_{0},\tau_{0}) = \frac{3}{4}\alpha_{0}^{-1/2}\tau_{0}^{-1/2}, \qquad 0 < \alpha_{0}, \tau_{0} < 1$$
(28)

(EB) estimators for θ and c under (SE) Loss function are the solutions of the following system of non-linear equations with respect to θ and c:

$$\int_{0}^{1} \int_{0}^{1} \frac{\partial l(\underline{\delta}, \underline{X})}{\partial \theta} p_{i}(\alpha_{0}, \tau_{0}) \, d\alpha_{0} d\tau_{0} = 0 \quad ; \quad \int_{0}^{1} \int_{0}^{1} \frac{\partial l(\underline{\delta}, \underline{X})}{\partial c} p_{i}(\alpha_{0}, \tau_{0}) \, d\alpha_{0} d\tau_{0} = 0 \tag{29}$$

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where $p_i(\alpha_0, \tau_0)$ has three forms defined in (28) for i = 1, 2, 3. E-Bayesian estimators for θ and c under (*Linex*) Loss function, firstly the marginal posterior density for each parameter of $\underline{\delta}$ is given by:

$$\theta | \underline{X} \sim N(\hat{\theta}, V_{11}^*) \quad ; \quad c | \underline{X} \sim N(\hat{c}, V_{22}^*)$$

Where V_{11}^* is an element in the first row and first column of $(-V^{-1})$

$$\theta | X \sim \mathcal{N}(\hat{\theta}, V_{11}^*) \quad ; \quad c | X \sim \mathcal{N}(\hat{c}, V_{22}^*)$$

Where V_{11}^* is an element in the first row and first column of $(-\mathbf{V}^{-1})$, where $\mathbf{V}^{-1} = \frac{\partial^2 L(\mathbf{X};\theta,c)}{\partial \eta \partial \eta'}\Big|_{\eta=\hat{\eta}_{\text{ML}}}$, where $\eta = (\theta, c)'$ and $\hat{\eta}_{\text{ML}} = (\hat{\theta}_{\text{ML}}, \hat{c}_{\text{ML}})'$, and V_{22}^* is an element in the second row and second column of $(-\mathbf{V}^{-1})$, we need to find $E(e^{-\theta}|\underline{X})$ and $E(e^{-c}|\underline{X})$ where:

$$E(e^{-\theta}|\underline{X}) = e^{-\hat{\theta} + \frac{1}{2}V_{11}^*} \quad ; \quad E(e^{-c}|\underline{X}) = e^{-\hat{c} + \frac{1}{2}V_{22}^*} \tag{30}$$

So that the (EB) estimators for θ and c are:

$$\hat{\theta}_{EBMcR_lines} = \hat{\theta} + \frac{1}{2} \int_{0}^{1} \int_{0}^{1} V_{11}^{*} p_{i}(\alpha_{0}, \tau_{0}) \, d\alpha_{0} \, d\tau_{0}$$

$$\hat{c}_{EBMcR_lines} = \hat{c} + \frac{1}{2} \int_{0}^{1} \int_{0}^{1} V_{22}^{*} p_{i}(\alpha_{0}, \tau_{0}) \, d\alpha_{0} \, d\tau_{0}$$
(31)

for i = 1,2,3. Where $\hat{\theta}_{EBMcR_lines}$ and \hat{c}_{EBMcR_lines} are the solutions of (31), $\hat{\theta}_{BR_lines}$ and \hat{c}_{ER_lines} are:

$$\hat{\theta}_{BR,lines} = -\hat{\theta} + \frac{1}{2}V_{11}^*, \quad ; \quad \hat{c}_{ER,lines} = -\hat{c} + \frac{1}{2}V_{22}^*.$$
 (32)

5. Simulation Study

In this section, different random sample sizes (25, 50, 100) taken from McR with different values (a = 0.5, 1), (b = 1.01, 1.1), (c = 1.02, 1.05), ($\theta = 0.1, 0.5, 1.5, 2$). The experiment replicated 1000 times. The parameters estimated using the methods described in the previous section, and the efficiency of the estimators evaluated based on the Mean Squared Error (MSE) criterion. The results presented in Tables (2, 3, 4) respectively.

	b = 1.01	l, c = 1.02	$, c_0 = \kappa_0 = \kappa_0$	$ au_0 = 0.7$	$\frac{b = 1.01, c = 1.05, c_0 = \kappa_0 = \tau_0 = 0.01}{a = 1.5, \alpha_0 = 0.5, \beta_0 = 0.9}$				
Estimation	<i>a</i> =	$= 1.5, \alpha_0 =$	$= 0.9, \beta_0 = 0$).5					
Estimation	$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		
	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	
ML	0.80168	3.2776	1.4473	3.0688	0.56254	3.4469	1.5758	2.8238	
MSE	0.34655	1.5488	0.3869	0.67274	0.16778	0.75267	0.67986	1.45515	
$Bayes_{SE}$	0.59214	3.1875	1.5083	2.9628	0.52325	3.4011	1.6135	2.6813	
MSE	0.23451	1.0668	0.26229	0.77273	0.069941	0.6565	0.6638	0.47833	
$Bayes_{linex}$	0.50894	3.0666	1.4573	3.0558	0.2913	3.2123	1.5934	2.8567	
MSE	0.13241	1.0555	0.16429	0.47274	0.06881	0.57718	0.55121	0.45946	
$EBayes_{SE1}$	0.51796	3.3414	1.5175	3.1311	0.53315	3.1087	1.6078	2.8812	
MSE	0.14682	1.0532	0.2602	0.67662	0.067901	0.7565	0.55088	0.45843	
$EBayes_{SE2}$	0.51269	5.0162	2.2763	3.6967	0.79972	2.6631	1.4116	2.3217	
MSE	0.1274	1.0316	0.98673	0.5227	0.064462	0.63648	1.2485	0.36989	
$EBayes_{SE3}$	0.56269	5.0162	2.0963	3.7397	0.81922	2.4655	1.4056	2.1255	
MSE	0.43244	1.0786	0.16238	1.2227	0.08762	0.66984	0.6435	1.8973	
$EBayes_{linex1}$	0.51071	3.4285	1.5028	3.135	0.52893	3.2641	1.5932	2.8798	
MSE	0.13663	1.0216	0.16453	0.37711	0.06783	0.49988	0.55016	0.35828	
$EBayes_{linex2}$	0.51522	3.5039	1.5106	3.168	0.53213	3.3728	1.6019	2.9078	
MSE	0.12474	1.0012	0.15183	0.28258	0.062381	0.30889	0.45054	0.3221	
$EBayes_{linex3}$	0.51612	3.6688	1.4606	3.3548	0.53954	3.5128	1.81134	2.9438	
MSE	0.15684	1.1672	0.18883	0.72228	0.072871	0.68923	0.59251	0.4644	

Table 2. The value of MSE of estimated parameters θ , c of McR under SE and Linex Loss functions for n = 25

	b = 1.01	, c = 1.02,	$c_0 = \kappa_0 = c_0$	$ au_0 = 0.7$	$\frac{b = 1.01, c = 1.05, c_0 = \kappa_0 = \tau_0 = 0.01}{a = 1.5, \alpha_0 = 0.5, \beta_0 = 0.9}$				
Estimation	<i>a</i> =	$= 1.5, \alpha_0 =$	$0.9, \beta_0 = 0$).5					
Estimation	$\theta = 0.5$, c = 3	$\theta = 1.5, c = 3$		$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		
	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	
ML	0.44594	2.8935	2.1819	2.7521	0.4976	2.8669	1.5758	2.7669	
MSE	0.14325	0.46169	0.49144	0.20095	0.171761	0.3256	0.29196	0.32098	
$Bayes_{SE}$	0.47896	2.8956	1.4619	2.9012	0.49877	2.72051	1.5316	2.8132	
MSE	0.04155	0.02517	0.09144	0.08775	0.054863	0.029938	0.02697	0.02277	
$Bayes_{linex}$	0.49424	2.9005	1.4819	3.0921	0.50911	3.1156	1.5923	2.8256	
MSE	0.04125	0.02411	0.08144	0.08695	0.052722	0.026911	0.019215	0.02267	
$EBayes_{SE1}$	0.49169	2.523	1.4262	2.8079	0.50771	2.9251	1.6106	2.8231	
MSE	0.04351	0.02415	0.05341	0.020408	0.051863	0.025939	0.019299	0.021292	
$EBayes_{SE2}$	0.51875	2.4284	1.4339	2.4119	0.76156	0.3877	2.4159	2.2346	
MSE	0.04127	0.06176	0.05311	0.03321	0.051433	0.025688	0.08931	0.021364	
$EBayes_{SE3}$	0.56875	3.4284	2.4188	2.1123	0.77155	0.39711	2.5159	2.2139	
MSE	0.14127	0.16176	0.08232	0.0833	0.101413	0.07621	0.040151	0.15614	
$EBayes_{linex1}$	0.58262	2.96864	1.50995	2.7801	0.50218	3.0189	1.5523	2.8393	
MSE	0.02329	0.02032	0.02918	0.020174	0.051782	0.027908	0.019204	0.021157	
$EBayes_{linex2}$	0.501096	3.5039	1.52083	2.7941	0.50447	3.0948	1.5593	2.8006	
MSE	0.01335	0.01811	0.02901	0.02027	0.041208	0.020794	0.081024	0.021082	
$EBayes_{linex3}$	0.59086	3.5328	1.5893	2.6982	0.51344	3.5948	1.6012	2.7826	
MSE	0.04335	0.08142	0.02991	0.08022	0.061007	0.13084	0.10221	0.029812	

Table 3. The value of MSE of estimated parameters θ , c of McR under SE and Linex Loss functions for n = 50

	b = 1.0	01, c = 1.02	$c_0 = \kappa_0 = \tau_0$	$_0 = 0.7$	$\frac{b = 1.01, c = 1.05, c_0 = \kappa_0 = \tau_0 = 0.01}{a = 1.5, \alpha_0 = 0.5, \beta_0 = 0.9}$				
Estimation	a	$x = 1.5, \alpha_0 =$	$= 0.9, \beta_0 = 0.0$.5					
Estimation	$\theta = 0.5$	5, c = 3	$\theta = 1.5, c = 3$		$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		
	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	$\hat{ heta}$	\hat{c}	
ML	0.57688	2.7463	2.2018	2.6841	0.47929	2.7809	1.5614	2.7352	
MSE	0.11641	0.1967	0.13009	0.18635	0.17539	0.11342	0.176181	0.199008	
$Bayes_{SE}$	0.53618	2.8463	1.9018	2.8841	0.4812	2.4384	1.5946	2.7608	
$M \tilde{S} E$	0.01589	0.0967	0.01231	0.0560	0.004819	0.01686	0.076814	0.098493	
$Bayes_{linex}$	0.51612	2.8477	2.018	2.8533	0.4819	2.62125	1.55115	2.7713	
$M\tilde{S}E$	0.01381	0.0867	0.011192	0.04465	0.004622	0.02169	0.075124	0.068233	
$EBayes_{SE1}$	0.54225	2.802	2.2465	2.7386	0.48302	2.8374	1.5931	2.7908	
MSE	0.014644	0.0978	0.01321	0.089316	0.004634	0.01066	0.077186	0.098093	
$EBayes_{SE2}$	0.54584	2.2030	1.6698	2.1079	0.53354	2.2561	1.3897	2.1862	
MSE	0.01276	1.2189	0.00494	0.011360	0.04218	0.01195	0.07622	0.020021	
$EBayes_{SE3}$	0.51258	2.8213	1.49179	2.708	0.4831	2.9092	1.5435	2.7725	
MSE	0.02776	1.00889	0.00994	0.01936	0.027228	0.03715	0.06564	0.060342	
$EBayes_{linex1}$	0.51258	2.8213	1.49179	2.708	0.4831	2.9092	1.5435	2.7725	
MSE	0.011344	0.01093	0.001303	0.00886	0.004554	0.001199	0.005763	0.009813	
$EBayes_{linex2}$	0.48543	2.8588	1.42259	2.72	0.48501	2.9733	1.5396	2.7911	
MSE	0.01248	0.01024	0.001407	0.008763	0.004472	0.001004	0.006512	0.009634	
$EBayes_{linex3}$	0.51543	2.8177	1.32254	2.7721	0.42301	2.9531	1.59645	2.735511	
MSE	0.01341	0.01294	0.01107	0.009333	0.07532	0.018492	0.009611	0.0124634	

Table 4. The value of MSE of estimated parameters θ , c of McR under SE and Linex Loss functions for n = 100

From tables(2,3,4) above, it is seen that the estimates for these two parameters are better with increasing sample size according to the mean square error (MSE) criterion, which indicates the accuracy of the estimations. We also note the superiority of Bayesian method under the quadratic loss function (SE) and the linear exponential loss function (Linex) over the maximum likelihood method (ML) in estimation. Moreover, we note that the estimators using E - Bayes method under the loss function SE and the loss function Linex are better than the Bayesian estimators under the same loss functions, the results showed that $E - Bayes_{SE2}$ was better in estimation than the Bayesian method under the loss function SE, and that $E - Bayes_{SE2}$ was better the loss function Linex. In general, the best method for estimation was E - Bayes under Linex loss function with second hyper prior at $(\alpha_0 = 0.5, \tau_0 = 0.01)$ with all sample sizes. because it obtained the lowest mean square error for all cases.

6. Real data application

In this section, we study the flexibility of McR distribution by fitting complete real data sets in different domains. Each real data set was fitted by using the McR. Beta-Rayleigh(BR), Kumaraswamy Rayleigh(KR), and Rayleigh(R) distributions. The comparison were made using the Kolmogorov-Smirnov statistic, its p-value, Akaikes information criterion(AIC), and Bayesian information criterion(BIC). We used two different types of real data sets to illustrate the usefulness and effectiveness of this distribution. The first data represents Machine downtime at the Kut factory of the Iraqi textile Industries Company[11] The sample size is 100., while the second data represent the 40 leukemia patient treated at KSA health ministry hospital[22]. Note that the observations of two real data sets are show in table5.

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Table 5. Raw values of the observations in Data set I and Data set II.

Data setI	10, 9.5, 8, 6.5, 10, 12, 11, 18.5, 8, 12, 9, 17, 22, 26.5, 14, 4.5, 8, 9, 5, 12, 1.5, 20, 7, 12, 18, 12, 15, 5.5, 2, 3, 9, 2, 7, 14, 21.5, 18, 14, 8, 1, 9.5, 11.5, 15, 8, 9.5, 13, 8.5, 20, 21.5, 19, 18.5, 18, 14.5, 3, 14, 3, 5, 12, 7, 16.5, 8.5, 21, 7.5, 5, 9.5, 12, 22.5, 20.5, 17.5, 13, 13.5, 4, 7, 15.5, 8, 3.5, 1, 4, 13, 7.5, 15.5, 2, 14.5, 17.5, 4.5, 3, 27, 10, 16, 4, 16, 4, 18.5, 6.5, 20, 6.5, 8.5, 15, 14
Data setII	0.315, 0.496, 0.616, 1.145, 1.208, 2.211, 2.370, 2.532, 2.693, 2.805, 3.348, 3.427, 3.499, 3.534, 3.767, 4.323, 4.381, 4.392, 4.397, 4.647, 1.263, 2.910, 3.751, 4.753, 1.414, 2.912, 3.858, 4.929, 2.025, 3.192, 3.986, 4.973, 2.036, 3.263, 4.049, 5.074, 2.162, 3.348, 4.244, 4.381

Table 6 shows the goodness of the two data sets based on McR, ER, KR, BR, R models. According to (K - S) statistic, p - value, AIC, BIC, it is seen that the (McR) is the best fit to two real data sets with smallest of AIC and BIC. The p-values Corresponding to smallest AIC and BIC for two data sets are greater than 0.05.

Data set	Model	$\hat{ heta}$	\hat{c}	AIC	BIC	K-S	p-value
	McR	2.01381	1.49934	76.04669	78.03815	0.091863	0.6799
	ER	1.99926	1.49931	80.65457	82.64604	0.07665	0.6124
Data1	KR	2.45734	1.49944	79.74704	81.7385	0.11496	0.1499
	BR	1.93734	—	88.32726	90.31872	0.25547	5.567e-06
	R	2.45749	—	85.34032	87.33179	0.063515	0.2241
	McR	0.00497	9.99937	135.0787	141.8342	0.18701	0.4219
	ER	0.00457	9.99983	143.6080	146.9858	0.14987	0.3301
Data2	KR	0.00354	9.92537	193.9642	199.0308	0.21672	0.04669
	BR	0.00421		208.642	220.308	0.3067	0.001079
	R	0.00449	—	144.3313	146.0202	0.16523	0.2249

Table 6. Estimated parameters with goodness of fit criteria for two real data sets.



Figure 4. The estimated p.d.f and c.d.f for two data sets based on four models.

It was observed that E - Bayesian estimation outperformed the Linex loss function based on th second type of joint hyper prior distribution, so that the parameters of (McR) were estimated based on two real data sets. Furthermore, Figure 4 shows the estimated p.d.f and c.d.f plots based on four models for the two data sets. This Figure shows that the (McR) distribution is a best fit to two real data sets.

7. Conclusion

In this article, the McDonald family is used to extend Rayleigh distribution. This new distribution named McDonald Rayleigh (McR) distribution. Some statistical properties of McR have been studied. Some parameters of McR were estimated by ML, Bayesian and E-Bayesian methods under square error and linex exponential loss functions with three different forms of hyperparameter priors of distributions. The simulation study and real data sets used to estimate the parameters of McR. In simulation experiment. It is seen that E - Bayesian estimators based on type(2) hyperprior is the best. To demonstrate the flexibility of McR, two real data sets were fitted using the McR and its submodels. The McR proved the best fit among other submodels used. For future work, we will consider ML and Bayesian estimation methods based on progressive type two censored data.

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REFERENCES

- 1. A. Percontini, F. V. Gomes-Silva, R. da Silva, L. Handique, and P. R. Diniz Marinho, The McDonald Lindley-Poisson Distribution, Pakistan Journal of Statistics and Operation Research, vol. 17, no. 4, pp. 1095–1112,2021.
- 2. A.A. Al-Babtain, F.Merovci, I. Elbatal, The McDonald exponentiated gamma distribution and its statistical properties, Springerplus, vol. 4, no. 1, pp. 1-22,2015.
- 3. A. Alhamaidah, M. N. Qomi and A. Kiapour, E-Bayesian and Robust Bayesian Estimation and Prediction for the Exponential Distribution based on Record Values, Journal of the Iranian Statistical Society, vol. 21, no. 2, pp. 133-147,2022.
- 4. B., G. AL-Ani, R., S.AL-Rassam, S., N. Rashed, Bayesian estimation for two parameter exponential distribution using linear transformation of reliability function, Periodicals of Engineering and Natural Sciences, vol. 8, no. 1, pp. 2303-4521,2020.
- 5. A. Sayibu, Shei, A. Luguterah, A. Luguterah, and S. Nasiru, McDonald Generalized Power Weibull Distribution: Properties, and Applications, Journal of Statistics Applications and Probability, vol. 13, no. 1, pp. 297–322, 2024.
 6. F. Chipepa , B. Oluyede and B. Makubate , *The Topp-Leone Marshall-Olkin-G Family of Distributions With Applications*,
- International Journal of Statistics and Probability, vol. 9, no. 4, pp. 15–32, 2020.
- 7. F. Olayode, The Topp-Leone Rayleigh Distribution with Application, American Journal of Mathematics and Statistics, vol. 6, no. 6, pp. 215-220, 2019.
- 8. G. M. Cordeiro, R. J. Cintra, L. C. Rêgo and, E. M. M. Ortiga, The McDonald normal distribution, Pakistan journal of statistics and operation research, vol. 3, no. 8, pp. 301-329, 2012.
- 9. G. M. Cordeiro, and M., Castro, Anew family of generalized distributions, Journal od statistical computation and simulation, vol.7, no. 81, pp. 883-898, 2010.
- 10. G. M. Cordeiro, R. J. Cintra, L. C. Rêgo, and E. M. M. Ortega, The McDonald Normal Distribution, Pakistan Journal of Statistics and Operation Research, vol.8, no. 3, pp. 301-329, 2012.
- 11. H. R. Talib, Using some methods to estimate the parameters and dependencies of the composite probit model (Exponential-Weibull) with a practical application, Iraqi Journal for Administrative Sciences, vol.13, no. 52, pp. 246–268, 2017.
- 12. Han, K. T., WinGen: Windows software that generates IRT parameters and item responses, Applied Psychological Measurement, vol. 31, no. 5, pp. 457--459, 2007.
- 13. J. F. Kenny and E. S. Keeping, Mathematics of statistics, Past 1, 3rd edition, Prinstion NJ., 1962.
- 14. J. J. A. Moors, A quartile alternative for kurtosis, Journal of the royal statistical society D, vol. 1, no. 37, pp. 25–32, 1988.
- 15. K. Pokhrel, G. R. Åryal, R. C. Kafle, B. Tharu, and N. Khanal, McDonald-G Poisson Family of Distributions, Statistica, vol. 82, no. 2, pp. 119-144, 2022.
- 16. N. Eugene, C. Lee and F. Famoye, eta normal distribution and its applications, Communications in statistics theory and methods, vol. 4, no. 31, pp. 497-512, 2002.
- 17. R. C. Gupta, Modelling failure time data by by Lehman alternative, Communications in statistics theory and methods, vol. 7, no. 2, pp. 887–904, 1998.
- 18. R. Roozegar, and F. Esfandiyari, The McDonald quasi lindley distribution and its statistical properties with applications, Journal of Statistics Applications and Probability, vol. 4, no. 3, pp. 375-385, 2015.

- 19. R. S. Al-Rassam, K. A. Mohammed and S. N. Rashed, Different Methods to Estimate Stress-Strength Reliability Function for Modified Exponentiated Lomax Distribution, Iraqi Journal for Computer Science and Mathematics, vol. 6, pp. 99-106, 2025.
- 20. S. Nadirjah, The exponentiated Gumbel distribution with climate application, iEnviron metrics, vol. 1, no. 17, pp. 13–31, 2006.
- 21. S. Nadirjah and S. Kotz, The exponentiated Frechet distribution, Interstate electronic journal, pp. 1–7, 2003.
- 22. I. Alkhairy, A new approach of generalized Rayleigh distribution with analysis of asymmetric data sets, Alexandria Engineering Journal, vol 100, pp. 1–14, 2024.
 A. A. Ahmed, Z. Y. Algamal and O. Albalawi, *Bias reduction of maximum likelihood estimation in exponentiated Teissier*
- distribution, Applied Mathematics and Statistics, pp. 1-7, 2024.