STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **14**, September 2025, pp 1139–1152.
Published online in International Academic Press (www.IAPress.org)

McDonald Rayleigh Distribution with Application

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Abstract Statistical modeling of many phenomena is very important topic, especially the phenomena of survival, reliability, economic and financial. Many standard probability distributions lack superiority in modeling data sets of complex phenomenal. In recent years, the design of different forms of probability distributions has received wide attention by using different techniques in statistical theory. In this paper, McDonald family used to extend Rayleigh named as *(McR)* distribution. Some theoretical statistical properties of *McR* distribution are presented and explained. Shape and Scale parameters of *McR* distribution were estimated by maximum likelihood *(ML)* and E-Bayesian *(EB)* methods under square error *(SE)* and linear exponential *(Linex)* loss functions with three different kinds of hyper priors of distributions. The estimation results were applied to a simulation experiment for data generated with different sample sizes. To comparison has been done using *(MSE)* criterion. The experiment showed the superiority of *(EB)* estimators under *(Linex)* loss function with the second joint hyper prior distribution. Two real data sets were fitted by using McR and other models representing especial cases of it. The *(McR)* model demonstrated its flexibility in modeling both type of real data sets.

Keywords Rayleigh distribution, McDonald family, Maximum likelihood method,E-Bayesian method, Square error loss function, linear exponential loss function, McDonald Rayleigh distribution, alternating direction method.

DOI: 10.19139/soic-2310-5070-2531

1. Introduction

Statisticians always resort to creating probability distributions that can accurately phenomena in different fields. These distributions help in understanding their features and making the appropriate decision correctly. There are various methods that have the ability to adapt probability density and hazard rate functions to model different real data sets for complex phenomena.

Many ways proposed to generalize or extend probability distributions based on adding shape parameters to baseline distributions. One of these methods proposed by [17] named as an exponentiated family. Many used the exponentiated family, for example [20, 21], the class of Beta G family proposed by [16]. A McDonald family of distributions introduced by [8] dented with (*Mc*).The *Mc* family has been used to transform the normal distribution into a skewed distribution, offering greater flexibility than the standard normal distribution [8]. The McDonald Exponentiated Gamma distribution was introduced to provide increased flexibility in modeling real data, and the likelihood ratio test was used to compare it with a baseline distribution [2]. The McDonald Quasi-Lindley distribution was proposed as a more flexible model for real data sets [18]. The Mc family was later extended to include the Lindley-Poisson distribution [1]. A new distribution family, the McDonald Generalized Poisson distribution, was proposed [15]. Recently, the *Mc* family was further generalized to the Power Weibull distribution [5]. The probability density function *f(x)* of *Mc* family is defined by:

$$g(x;a,b,c) = \frac{c}{B(a,b)} f(x) F(x)^{(ac-1)} (1 - F(x)^c)^{(b-1)}, \quad x > 0 \tag{1}$$

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

where *f(x)* and *F(x)* are the p.d.f. and c.d.f. in baseline distribution, $a, b, c > 0$ are the additional shape parameters. These parameters have a role in skewness and tail weights. *B(a,b)* is a complete Beta function. Beta generated family is a special case of *Mc* family when $c = 1$. Kumaraswamy family proposed by [9] when $a = 1$ and an eponentiated family when $a = b = 1$. If a random variable X whose density has the form defined in (1) denoted by $X \sim McG(a, b, c)$.Rayleigh distribution plays a major role in modeling and analysis of survival[19], reliability data, communication, physical sciences and clinical research's. Regarding to the importance of this distribution and the reason for the desire to give it greater flexibility, the aim of this paper is to introduce a generalization of Rayleigh distribution by using *(Mc)* family. The motivation behind this generalization is to improve the flexibility of Rayleigh distribution to fit different varieties of real data sets arising from different fields comprising unimootal, to make kurtosis more flexible compared to Rayleigh distribution, to produce skewness for symmetric distribution and to produce increasing, decreasing, bathtub shapes of the hazard function curves. Also this family can be used for modeling positive and negative skewed real data sets.

2. McDonald Rayleigh distribution

The p.d.f. and c.d.f. of baseline Rayleigh distribution [7] are:

$$f(x; \theta) = 2\theta x e^{-\theta x^2}, \quad \theta > 0,\ x > 0. \tag{2}$$

$$F(x) = 1 - e^{-\theta x^2} \tag{3}$$

Substituting (2) and (3) into (1), the p.d.f. of *McR* is:

$$g(x; a, b, c) = \frac{2c\theta x e^{-\theta x^2}}{B(a, b)} (1 - e^{-\theta x^2})^{(ac-1)} \left(1 - (1 - e^{-\theta x^2})^c\right)^{(b-1)}, \quad a, b, c, \theta > 0,\ x > 0. \tag{4}$$

and zero otherwise, where a, b, c are shape parameters and θ is a scale parameter.

From the Figure 1 and for knowing the effect of the scale parameter θ and the shape parameters *(a,b,c)* of the *McR* probability density function curve, the value of one parameter is changed and the values of the other parameters are fixed. Therefore, the following is shown:-

- Increasing of the value of scale parameter θ leads to a decrease in the kurtosis of the function curve, a decrease in the dispersion of the observations and the tail of the curve become less heavier.
- The curve of the function is decreasing when the value of $a < 0.5$. The curve becomes slowly decreasing, has a single inflection point and has a heavy tail when $a = 0.5$ while the curve approaches symmetry, it pulls to the right and the tail becomes less heavy.
- Increasing of the value of b reduces the dispersion of observations, and the curve of the function becomes less kurtotic, more convex, less skewed to the right, and the tail becomes less heavy.
- The function curve is decreasing when $c < 0.5$, and as the value of c increases, the function curve approaches symmetry.

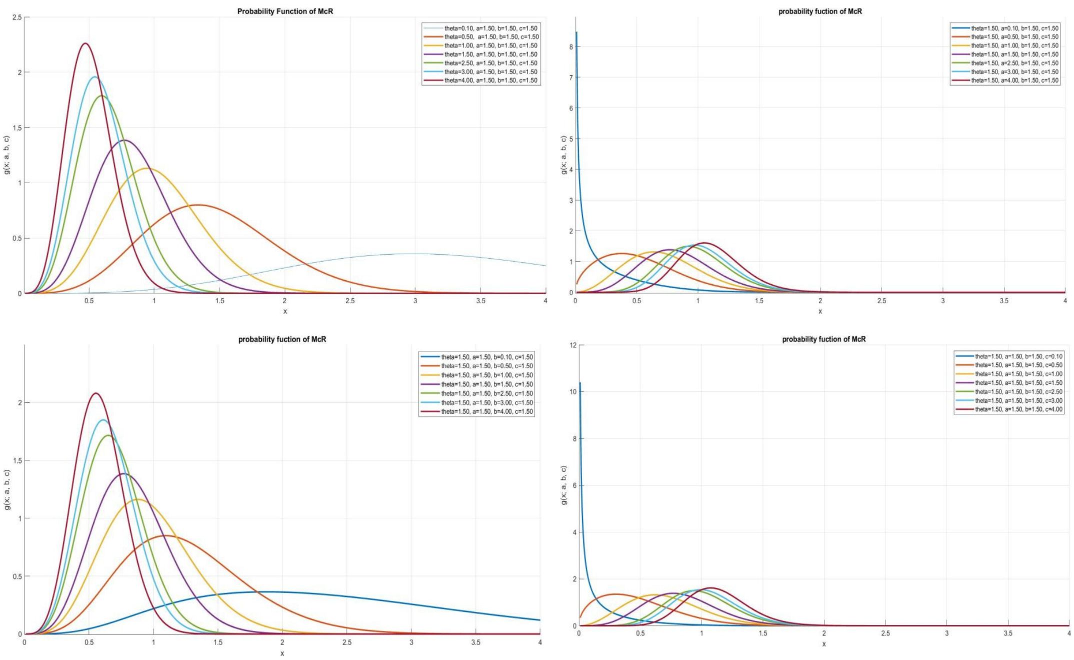

Figure 1. The p.d.f. of *McR* distribution at different values of shapes and scale parameters.

2.1. Cumulative distribution and hazard rate functions of McR distribution

The cumulative distribution function of the Mc distribution defined by [8] is given by:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{(F(x))^c} u^{a-1}(1-u)^{b-1} du \tag{5}$$

Putting (3) into (5) and using the Binomial expansion [6] to a term $(1-u)^{b-1}$, the c.d.f. of *McR* becomes:
(i) For $(b-1)$ is a positive real number:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} \frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j} \tag{6}$$

(ii) For $(b-1)$ is a positive integer:

$$G_{McR}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{b-1} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} \frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j} \tag{7}$$

From Figure (2), the cumulative distribution function $G_{McR}(x)$ represents the probability that the random variable is less than or equal to x. The accumulation rate increases rapidly with higher θ, reflecting a greater likelihood of smaller values, while lower θ results in slower accumulation, indicating a higher probability of larger values. Therefore, small θ values are used to model wide-spread data, while larger θ values are preferred when focusing on smaller values.
The hazard rate function of *McR* is:
(i) When $(b-1)$ is a positive real number:

$$h(x;a,b,c) = \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta x e^{-\theta x^2}(1-e^{-\theta x^2})^{(ac-1)}(1-(1-e^{-\theta x^2})^c)^{(b-1)}}{1-\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{j!\Gamma(b-j)}\frac{(1-e^{-\theta x^2})^{c(a+j)}}{a+j}} \tag{8}$$

(ii) When $(b-1)$ is a positive integer, $h(x;a,b,c)$ has the same formula as in (8) except the upper bound of the sum is $(b-1)$.

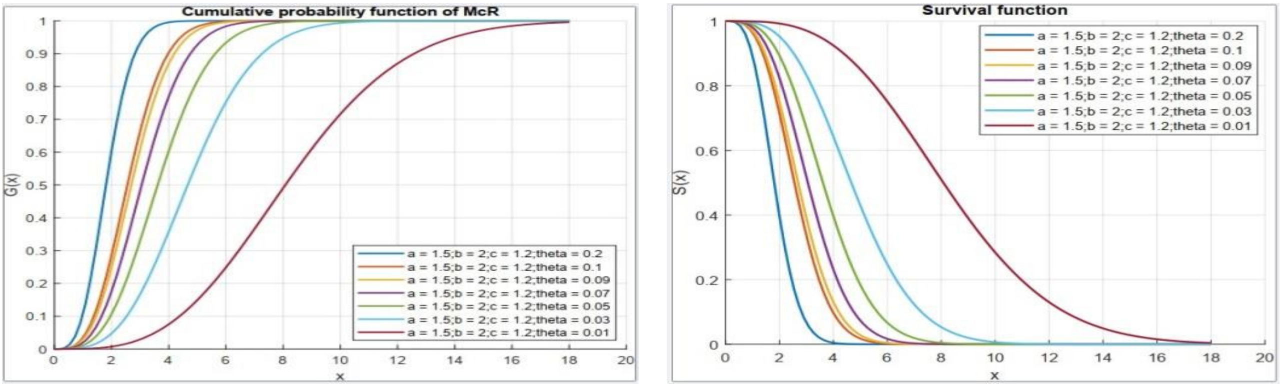

Figure 2. The G(x) and S(x) of *McR* at different values of shape and scale parameters.

3. Statistical Properties of *McR* distribution

This section deals with some statistical properties of *McR* distribution:

3.1. Moments

The r-th moment around zero of *McR* variable is:

$$E(x^r) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta \int_0^\infty x^{r+1} e^{-\theta x^2} (1-e^{-\theta x^2})^{ac-1} (1-(1-e^{-\theta x^2})^c)^{b-1}\, dx \tag{9}$$

By using the Binomial expansion on $(1-(1-e^{-\theta x^2})^c)^{b-1}$ when $(b-1)$ is a positive integer, $E(x^r)$ becomes:

$$E(x^r) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} 2c\theta \int_0^\infty x^{r+1} e^{-\theta x^2} (1-e^{-\theta x^2})^{ac-1} \sum_{j=0}^{b-1} \frac{(-1)^j \Gamma(b)(1-e^{-\theta x^2})^{cj}}{j!\Gamma(b-j)}\, dx \tag{10}$$

Again, the Binomial expansion is used for the term $(1-e^{-\theta x^2})^{(ac+cj-1)}$ when $ac+cj-1$ is a positive integer. After making some simplifications, $E(x^r)$ is:

$$E(x^r) = \frac{c\Gamma(a+b)\Gamma\left(\frac{r}{2}+1\right)}{\Gamma(a)} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k}\Gamma(ac+cj)}{j!k!\Gamma(b-j)\Gamma(ac+cj-k)(k+1)^{\left(\frac{r}{2}+1\right)}} \tag{11}$$

When $(b-1)$ and $ac+cj-1$ are positive real numbers, $E(x^r)$ is the same as in equation (11) except that the upper bounds of the sums tends to infinity.

3.2. Moment generating function (m.g.f.)

The m.g.f of the *McR* variable is:

$$\mu_x^{(t)} = E[e^{tx}] = \frac{2\theta c\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k}\Gamma(b)\Gamma(ac+cj)}{j!k!\Gamma(b-j)\Gamma(ac+cj-k)} \int_0^\infty x^{k+1} e^{-\theta x^2} e^{tx}\, dx \tag{12}$$

By using the Maclaurin series expansion on e^{tx} and making algebraic simplifications, the moment generating function (m.g.f.) becomes:

$$\mu_x^{(t)} = \frac{c\Gamma(a+b)}{\Gamma(a)} \sum_{s=0}^{\infty} \frac{t^s \Gamma(s/2+1)}{s!\theta^{s/2}} \sum_{j=0}^{b-1} \sum_{k=0}^{ac+cj-1} \frac{(-1)^{j+k}\Gamma(ac+cj)}{j!k!\Gamma(b-j)\Gamma(ac+cj-k)(k+1)^{(s/2+1)}} \tag{13}$$

The m.g.f. defined in (13) holds when $(b-1)$ and $ac+cj-1$ are positive integers. However, when they are positive real numbers, the m.g.f. remains the same as in equation (13), except that the upper bounds of the sums tends to infinity.

3.3. Mode and median

The mode of the McR variable is a numerical solution to the following nonlinear equation with respect to x,The R package (uniroot) was used to solve the problem.:

$$\frac{1}{x} - 2\theta x + \frac{2\theta x(ac-1)e^{-\theta x^2}}{(1-e^{-\theta x^2})} - \frac{2c\theta x(b-1)e^{-\theta x^2}(1-e^{-\theta x^2})^c}{(1-e^{-\theta x^2})(1-(1-e^{-\theta x^2})^c)} = 0 \tag{14}$$

The median of the McR variable is a solution to the following equation with respect to x:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{(F(x))^c} u^{a-1}(1-u)^{b-1}\, du = \frac{1}{2} \tag{15}$$

First, equation (15) must be solved for $(F(x))^c$, and then the inverse function $((F(x))^c)^{-1}$ represents the median of the McR variable.

Data generation from the McR variable can be done using equation (15) by replacing $\frac{1}{2}$ with w, where (w) is a random observation from $U(0,1)$, and solving the equation with respect to x.

3.4. Skewness, Kurtosis and Bowley Skewness

Skewness measure proposed by [13], that depends on quantiles:

$$\text{Skewness} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{3/2}} \tag{16}$$

Moors' kurtosis measure, introduced by [14], is defined as:

$$\text{Kurtosis} = \frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)} \tag{17}$$

The Bowley measure is affected by the shape parameters (a,b,c). Skewness and kurtosis measures can be used when raw moments do not exist or when they are represented as infinite sums.
The Bowley skewness is a measure of the degree of distribution asymmetry, making it more resistant to the effect of outliers, especially when dealing with asymmetric distributions.

$$\text{Bowley Sk.} = \frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)} \tag{18}$$

The mode, median, Skewness, and Kurtosis of the McR variable were evaluated at different values of the scale and shape parameters.

Table 1. Some statistical measures of McR variable at different values of parameters.

θ	a	b	c	$Mean$	$Median$	$Mode$	$Variance$	$Skewness$	$Kurtosis$	$BowleySk.$
0.1	0.5	1.01	1.02	1.9984	1.7112	0.36159	2.1661	0.91893	3.6128	0.02951
0.1	0.5	1.01	1.05	2.0312	1.7507	0.56887	2.1712	0.90251	3.5831	0.12474
0.1	0.5	1.1	1.02	1.902	1.6137	0.34326	1.9697	0.9266	3.638	0.29904
0.1	0.5	1.1	1.05	1.9348	1.653	0.54196	1.9761	0.9091	3.6051	0.16393
0.1	1	1.01	1.02	2.8124	2.6401	2.2589	2.1193	0.6535	3.2417	0.07288
0.1	1	1.01	1.05	2.8472	2.6782	2.308	2.1109	0.61718	3.2372	0.07288
0.1	1	1.1	1.02	2.6961	2.4884	2.1663	1.9465	0.6248	3.2407	0.05599
0.1	1	1.1	1.05	2.7312	2.5269	2.2158	1.9397	0.61583	3.2349	0.07288
0.5	0.5	1.01	1.02	0.89369	0.76525	0.1617	0.43321	0.91893	3.6128	0.09245
0.5	0.5	1.01	1.05	0.90838	0.78293	0.25441	0.43425	0.90251	3.5831	0.10609
0.5	0.5	1.1	1.02	0.85059	0.72163	0.1535	0.3934	0.9266	3.638	0.22085
0.5	0.5	1.1	1.05	0.86528	0.73923	0.24238	0.39522	0.9091	3.6051	0.18644
0.5	1	1.01	1.02	1.2577	1.1807	1.0102	0.42385	0.62535	3.2417	0.0664
0.5	1	1.01	1.05	1.2733	1.1977	1.0322	0.42218	0.61718	3.2372	0.11144
0.5	1	1.1	1.02	1.2057	1.1128	0.9688	0.3893	0.6248	3.2107	0.10799
0.5	1	1.1	1.05	1.2214	1.1301	0.99092	0.38793	0.61583	3.234	0.08217
1.5	0.5	1.01	1.02	0.51597	0.44182	0.09335	0.1444	0.91893	3.6128	0.21161
1.5	0.5	1.01	1.05	0.52445	0.45203	0.14689	0.14475	0.90251	3.5831	0.21161
1.5	0.5	1.1	1.02	0.49109	0.41661	0.08861	0.13131	0.9266	3.638	0.17198
1.5	0.5	1.1	1.05	0.49957	0.42684	0.13994	0.13174	0.9091	3.6051	0.20032
1.5	1	1.01	1.02	0.72615	0.68167	0.58325	0.14128	0.62535	3.2417	0.26678
1.5	1	1.01	1.05	0.73515	0.69151	0.59591	0.14073	0.61718	3.2372	0.1994
1.5	1	1.1	1.02	0.69612	0.64248	0.55934	0.12977	0.6248	3.2407	0.19466
1.5	1	1.1	1.05	0.70519	0.65243	0.57211	0.12931	0.61583	3.2349	0.26678
2	0.5	1.01	1.02	0.44685	0.38263	0.08087	0.1083	0.91893	3.6128	0.13868
2	0.5	1.01	1.05	0.45419	0.39147	0.12719	0.10856	0.90251	3.5831	0.1517
2	0.5	1.1	1.02	0.4253	0.36082	0.07674	0.09848	0.9266	3.638	0.2277
2	0.5	1.1	1.05	0.43264	0.36963	0.1212	0.0988	0.9091	3.6051	0.16255
2	1	1.01	1.02	0.62887	0.59034	0.50513	0.10596	0.62535	3.2417	0.24509
2	1	1.01	1.05	0.63666	0.59885	0.51606	0.10554	0.61718	3.2372	0.22808
2	1	1.1	1.02	0.60286	0.55641	0.48442	0.09732	0.6248	3.2407	0.17028
2	1	1.1	1.05	0.61071	0.56504	0.49548	0.09698	0.61583	3.2349	0.22633

It is seen from.Table (1) that as the parameter a increases, the mean gradually decreases, reflecting a concentration of values around the mean of the distribution, while the median and mode follow a similar pattern. Variance decreases with increasing a, this indicates a reduction in spread. Skewness, which measures distribution asymmetry, it decreases as a increases, so that the distribution becomes more symmetrical and less right-skewed. Kurtosis remains close to the normal value (around 3), this means, the distribution shape does not deviate significantly from a normal distribution, although there are slight changes depending on parameter values. It can be observed that with smaller parameter values, the distribution is more spread out, whereas with larger values, it becomes narrower and more concentrated. Thus, the table highlights the effect of the parameters a, b, c, e on the distribution's shape and spread, demonstrating how these key statistics shift according to different parameter values, providing deeper insight into the relationship between the parameters and the statistical distribution.

The Figure 3 shows a three- dimensional representation of kurtosis and skewness as a function of the variables b and c with the constant variable a with different values $(a = 0.5, 1, 1.5, 2)$. It is clear that increasing the value of a make the change in kurtosis with b and c more obvious. Also, increasing the value of a reduces the sharp effect of c on the skewness and the shape becomes smoother. We also note that there is a clear inverse relationship between the skewness and the value of c, where the higher value of c leads to reduced skewness. The effect of b does not appear clear at small values of a and its effect increases gradually with increasing value of a.

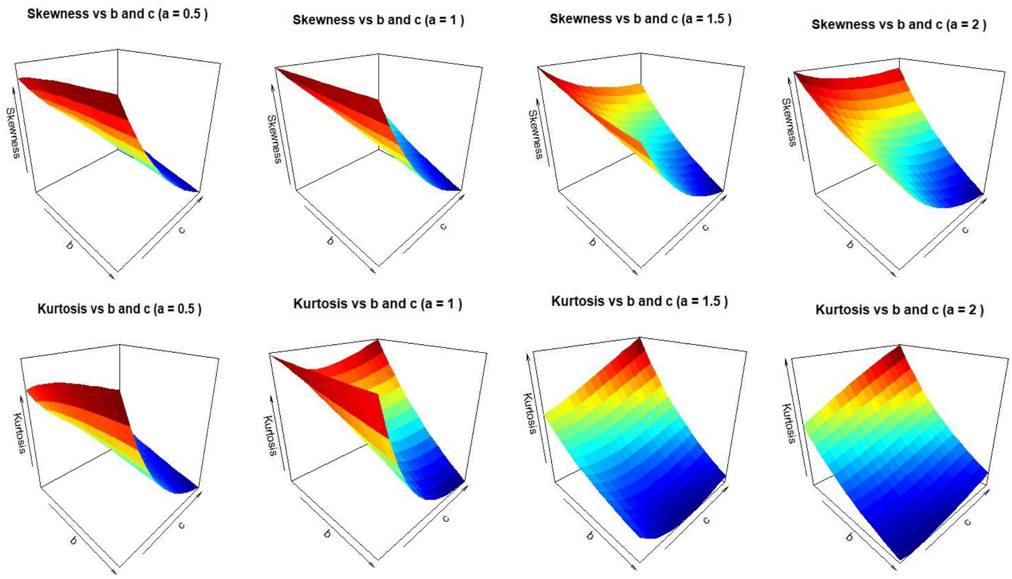

Figure 3. Tthree- dimensional representation of skewness and kurtosis of McR

4. Parameters estimation

Depending on complete data with a random sample of size (n) taken from McR distribution, we consider parameter estimation using maximum likelihood (ML) and Expected Bayesian (EB) estimation methods as follows:

4.1. Maximum Likelihood Method

In this section, we consider the parameters (a, b) are known. Depending on complete sample information, the log-likelihood function is:

$$l(\underline{X}; \theta, c) = n \ln\left(\frac{2c}{B(a,b)}\right) + n\ln(\theta) + \sum_{i=1}^{n} \ln(x_i) - \theta \sum_{i=1}^{n} x_i^2 + (ac-1)\sum_{i=1}^{n} \ln\left(1 - e^{-\theta x_i^2}\right) + (b-1)\sum_{i=1}^{n} \ln\left(1 - (1 - e^{-\theta x_i^2})^c\right), \tag{19}$$

where $\underline{X} = (x_1, x_2, \ldots, x_n)'$ and $B(a, b)$ is a complete Beta function. The maximum likelihood estimator (MLE) for θ and c represents a numerical solution to the following system of non-linear equations with respect to the unknown parameters,The R package (rootSolve) was used to solve the problem.

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i^2 + (ac-1)\sum_{i=1}^{n} \frac{x_i^2 e^{-\theta x_i^2}}{1 - e^{-\theta x_i^2}} + c(b-1)\sum_{i=1}^{n} \frac{(\theta-1)x_i^2 e^{-\theta x_i^2}(1 - e^{-\theta x_i^2})^c}{(1 - e^{-\theta x_i^2})(1 - (1 - e^{-\theta x_i^2})^c)} = 0, \tag{20}$$

$$a\sum_{i=1}^{n} \ln(1 - e^{-\theta x_i^2}) + (b-1)\sum_{i=1}^{n} \frac{(1 - e^{-\theta x_i^2})^c}{1 - (1 - e^{-\theta x_i^2})^c} \ln(1 - e^{-\theta x_i^2}) = 0. \tag{21}$$

The solutions of (20) and (21) satisfy the condition that[23]:

$$J = \begin{bmatrix} \frac{\partial^2 l(\underline{X};\theta,c)}{\partial\theta^2} & \frac{\partial^2 l(\underline{X};\theta,c)}{\partial\theta\partial c} \\ \frac{\partial^2 l(\underline{X};\theta,c)}{\partial c\partial\theta} & \frac{\partial^2 l(\underline{X};\theta,c)}{\partial c^2} \end{bmatrix}\Bigg|_{\theta=\hat{\theta}_{ML}, c=\hat{c}_{ML}} \quad \text{is a negative definite matrix.}$$

4.2. E-Bayesian estimation

E-Bayesian estimators for θ and c represent the expectation of Bayesian estimators for each parameter based on all hyperparameter priors [3]. The hyperparameter of prior distributions should be chosen to guarantee a decreasing prior functions [12]. Consider θ and c as independent unknown parameters, and let (a,b) be known. Define the informative prior for θ as Gamma(α_0,β_0) and the informative prior for c as Gamma(τ_0,κ_0). We propose the joint prior density function is given by:

$$p(\theta,c)\propto \theta^{\alpha_0-1}e^{-\theta\beta_0}c^{\tau_0-1}e^{-c\kappa_0}. \tag{22}$$

Combining the joint prior information for (θ,c) defined in (22) with sample information, the joint posterior distribution of (θ,c) is given by:

$$\begin{aligned} p(\theta,c|\underline{X}) \propto\, & (c\theta)^n e^{-\theta\sum_{i=1}^n x_i^2}\prod_{i=1}^n(1-e^{-\theta x_i^2})^{ac-1}\prod_{i=1}^n(1-(1-e^{-\theta x_i^2})^c)^{b-1} \\ & \times \theta^{\alpha_0-1}e^{-\theta\beta_0}c^{\tau_0-1}e^{-c\kappa_0}. \end{aligned} \tag{23}$$

Suppose $l(\theta,c|\underline{X})=\ln p(\theta,c|\underline{X})$.

$$\begin{aligned} l(\theta,c|\underline{X}) \propto\, & n\ln(c)+n\ln(\theta)-\theta\sum_{i=1}^n x_i^2+(ac-1)\sum_{i=1}^n\ln\left(1-e^{-\theta x_i^2}\right)+(b-1)\sum_{i=1}^n\ln\left(1-(1-e^{-\theta x_i^2})^c\right) \\ & +(\alpha_0-1)\ln(\theta)-\theta\beta_0+(\tau_0-1)\ln(c)-c\kappa_0. \end{aligned} \tag{24}$$

Taking the first partial derivatives of both sides of (24) with respect to θ and c, so that equating the result to zero[4]:

$$\frac{n}{\theta}-\sum_{i=1}^n x_i^2+(ac-1)\sum_{i=1}^n\frac{x_i^2e^{-\theta x_i^2}}{1-e^{-\theta x_i^2}}-c(b-1)\sum_{i=1}^n\frac{x_i^2e^{-\theta x_i^2}(1-e^{-\theta x_i^2})^{c-1}}{1-(1-e^{-\theta x_i^2})^c}+\frac{\alpha_0-1}{\theta}-\beta_0=0 \tag{25}$$

$$\frac{n}{c}+a\sum_{i=1}^n\ln(1-e^{-\theta x_i^2})-(b-1)\sum_{i=1}^n\frac{(1-e^{-\theta x_i^2})^c\ln(1-e^{-\theta x_i^2})}{1-(1-e^{-\theta x_i^2})^c}+\frac{\tau_0-1}{c}-\kappa_0=0. \tag{26}$$

The numerical solution for the system of non-linear equations with respect to $\underline{\delta}'=(\theta,c)'$ satisfies that $\frac{\partial^2 p(\underline{\delta}|\underline{X})}{\partial\underline{\delta}\,\partial\underline{\delta}'}$ is a negative definite matrix.This solution, $\hat{\underline{\delta}}'=(\hat{\theta},\hat{c})'$, represents the posterior mode.
Taking the exponents on both sides of (27), the approximate joint posterior distribution of $(\underline{\delta}|\underline{X})$ is:

$$p(\underline{\delta}|\underline{X})\propto e^{-\frac{1}{2}(\underline{\delta}-\hat{\underline{\delta}})'V(\hat{\underline{\delta}})^{-1}(\underline{\delta}-\hat{\underline{\delta}})} \tag{27}$$

So that $\underline{\delta}|\underline{X}\sim N_2(\hat{\underline{\delta}},V(\hat{\underline{\delta}})^{-1})$, where $V(\hat{\underline{\delta}})=\left.\frac{\partial^2 p(\underline{\delta}|\underline{X})}{\partial\underline{\delta}\,\partial\underline{\delta}'}\right|_{\underline{\delta}=\hat{\underline{\delta}}}$. Under square error (SE) and linear exponential $(Linex)$ loss functions, Bayesian and Expected Bayesian (EB) estimators, suppose the joint hyperprior for τ_0 and α_0 has three forms:

$$\begin{aligned} & p_1(\alpha_0,\tau_0)=1, && 0<\alpha_0,\tau_0<1 \\ & p_2(\alpha_0,\tau_0)=6\alpha_0(1-\tau_0), && 0<\alpha_0,\tau_0<1 \\ & p_3(\alpha_0,\tau_0)=\frac{3}{4}\alpha_0^{-1/2}\tau_0^{-1/2}, && 0<\alpha_0,\tau_0<1 \end{aligned} \tag{28}$$

(EB) estimators for θ and c under (SE) Loss function are the solutions of the following system of non-linear equations with respect to θ and c:

$$\int_0^1\int_0^1\frac{\partial l(\underline{\delta},\underline{X})}{\partial\theta}p_i(\alpha_0,\tau_0)\,d\alpha_0 d\tau_0=0 \quad ; \quad \int_0^1\int_0^1\frac{\partial l(\underline{\delta},\underline{X})}{\partial c}p_i(\alpha_0,\tau_0)\,d\alpha_0 d\tau_0=0 \tag{29}$$

where $p_i(\alpha_0, \tau_0)$ has three forms defined in (28) for $i = 1, 2, 3$.
E-Bayesian estimators for θ and c under $(Linex)$ Loss function, firstly the marginal posterior density for each parameter of $\underline{\delta}$ is given by:

$$\theta|\underline{X} \sim N(\hat{\theta}, V_{11}^*) \quad ; \quad c|\underline{X} \sim N(\hat{c}, V_{22}^*)$$

Where V_{11}^* is an element in the first row and first column of $(-V^{-1})$

$$\theta|X \sim \mathcal{N}(\hat{\theta}, V_{11}^*) \quad ; \quad c|X \sim \mathcal{N}(\hat{c}, V_{22}^*)$$

Where V_{11}^* is an element in the first row and first column of $(-\mathbf{V}^{-1})$, where $\mathbf{V}^{-1} = \left.\frac{\partial^2 L(\mathbf{X};\theta,c)}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}'}\right|_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}_{\mathrm{ML}}}$, where $\boldsymbol{\eta} = (\theta, c)'$ and $\hat{\boldsymbol{\eta}}_{\mathrm{ML}} = (\hat{\theta}_{\mathrm{ML}}, \hat{c}_{\mathrm{ML}})'$, and V_{22}^* is an element in the second row and second column of $(-\mathbf{V}^{-1})$, we need to find $E(e^{-\theta}|\underline{X})$ and $E(e^{-c}|\underline{X})$ where:

$$\begin{aligned} E(e^{-\theta}|\underline{X}) &= e^{-\hat{\theta}+\frac{1}{2}V_{11}^*} \\ E(e^{-c}|\underline{X}) &= e^{-\hat{c}+\frac{1}{2}V_{22}^*} \end{aligned} \tag{30}$$

So that the (EB) estimators for θ and c are:

$$\begin{aligned} \hat{\theta}_{EBMcR\_lines} &= \hat{\theta} + \frac{1}{2}\int_0^1 \int_0^1 V_{11}^*\, p_i(\alpha_0, \tau_0)\, d\alpha_0\, d\tau_0 \\ \hat{c}_{EBMcR\_lines} &= \hat{c} + \frac{1}{2}\int_0^1 \int_0^1 V_{22}^*\, p_i(\alpha_0, \tau_0)\, d\alpha_0\, d\tau_0 \end{aligned} \tag{31}$$

for i = 1,2,3. Where $\hat{\theta}_{EBMcR\_lines}$ and $\hat{c}_{EBMcR\_lines}$ are the solutions of (31), $\hat{\theta}_{BR\_lines}$ and $\hat{c}_{ER\_lines}$ are:

$$\begin{aligned} \hat{\theta}_{BR\_lines} &= -\hat{\theta} + \frac{1}{2}V_{11}^* \\ \hat{c}_{ER\_lines} &= -\hat{c} + \frac{1}{2}V_{22}^*. \end{aligned} \tag{32}$$

5. Simulation Study

In this section, different random sample sizes (25, 50, 100) taken from McR with different values $(a = 0.5, 1)$,$(b = 1.01, 1.1)$, $(c = 1.02, 1.05)$, $(\theta = 0.1, 0.5, 1.5, 2)$. The experiment replicated 1000 times. The parameters estimated using the methods described in the previous section, and the efficiency of the estimators evaluated based on the Mean Squared Error (MSE) criterion. The results presented in Tables (2 ,3 ,4) respectively.

Table 2. The value of MSE of estimated parameters θ, c of McR under SE and $Linex$ Loss functions for $n = 25$

Estimation	$b = 1.01, c = 1.02, c_0 = \kappa_0 = \tau_0 = 0.7$				$b = 1.01, c = 1.05, c_0 = \kappa_0 = \tau_0 = 0.01$			
	$a = 1.5, \alpha_0 = 0.9, \beta_0 = 0.5$				$a = 1.5, \alpha_0 = 0.5, \beta_0 = 0.9$			
	$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$	
	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$
ML	0.801 68	3.2776	1.4473	3.0688	0.562 54	3.4469	1.5758	2.8238
MSE	0.346 55	1.5488	0.3869	0.672 74	0.167 78	0.752 67	0.679 86	1.455 15
$Bayes_{SE}$	0.592 14	3.1875	1.5083	2.9628	0.523 25	3.4011	1.6135	2.6813
MSE	0.234 51	1.0668	0.262 29	0.772 73	0.069 941	0.6565	0.6638	0.478 33
$Bayes_{linex}$	0.508 94	3.0666	1.4573	3.0558	0.2913	3.2123	1.5934	2.8567
MSE	0.132 41	1.0555	0.164 29	0.472 74	0.068 81	0.577 18	0.551 21	0.459 46
$EBayes_{SE1}$	0.517 96	3.3414	1.5175	3.1311	0.533 15	3.1087	1.6078	2.8812
MSE	0.146 82	1.0532	0.2602	0.676 62	0.067 901	0.7565	0.550 88	0.458 43
$EBayes_{SE2}$	0.512 69	5.0162	2.2763	3.6967	0.799 72	2.6631	1.4116	2.3217
MSE	0.1274	1.0316	0.986 73	0.5227	0.064 462	0.636 48	1.2485	0.369 89
$EBayes_{SE3}$	0.562 69	5.0162	2.0963	3.7397	0.819 22	2.4655	1.4056	2.1255
MSE	0.432 44	1.0786	0.162 38	1.2227	0.087 62	0.669 84	0.6435	1.8973
$EBayes_{linex1}$	0.510 71	3.4285	1.5028	3.135	0.528 93	3.2641	1.5932	2.8798
MSE	0.136 63	1.0216	0.164 53	0.377 11	0.067 83	0.499 88	0.550 16	0.358 28
$EBayes_{linex2}$	0.515 22	3.5039	1.5106	3.168	0.532 13	3.3728	1.6019	2.9078
MSE	0.124 74	1.0012	0.151 83	0.282 58	0.062 381	0.308 89	0.450 54	0.3221
$EBayes_{linex3}$	0.516 12	3.6688	1.4606	3.3548	0.539 54	3.5128	1.811 34	2.9438
MSE	0.156 84	1.1672	0.188 83	0.722 28	0.072 871	0.689 23	0.592 51	0.4644

Table 3. The value of MSE of estimated parameters θ, c of McR under SE and $Linex$ Loss functions for $n = 50$

Estimation	$b = 1.01, c = 1.02, c_0 = \kappa_0 = \tau_0 = 0.7$				$b = 1.01, c = 1.05, c_0 = \kappa_0 = \tau_0 = 0.01$			
	$a = 1.5, \alpha_0 = 0.9, \beta_0 = 0.5$				$a = 1.5, \alpha_0 = 0.5, \beta_0 = 0.9$			
	$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$		$\theta = 0.5, c = 3$		$\theta = 1.5, c = 3$	
	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$
ML	0.445 94	2.8935	2.1819	2.7521	0.4976	2.8669	1.5758	2.7669
MSE	0.143 25	0.461 69	0.491 44	0.200 95	0.171 761	0.3256	0.291 96	0.320 98
$Bayes_{SE}$	0.478 96	2.8956	1.4619	2.9012	0.498 77	2.720 51	1.5316	2.8132
MSE	0.041 55	0.025 17	0.091 44	0.087 75	0.054 863	0.029 938	0.026 97	0.022 77
$Bayes_{linex}$	0.494 24	2.9005	1.4819	3.0921	0.509 11	3.1156	1.5923	2.8256
MSE	0.041 25	0.024 11	0.081 44	0.086 95	0.052 722	0.026 911	0.019 215	0.022 67
$EBayes_{SE1}$	0.491 69	2.523	1.4262	2.8079	0.507 71	2.9251	1.6106	2.8231
MSE	0.043 51	0.024 15	0.053 41	0.020 408	0.051 863	0.025 939	0.019 299	0.021 292
$EBayes_{SE2}$	0.518 75	2.4284	1.4339	2.4119	0.761 56	0.3877	2.4159	2.2346
MSE	0.041 27	0.061 76	0.053 11	0.033 21	0.051 433	0.025 688	0.089 31	0.021 364
$EBayes_{SE3}$	0.568 75	3.4284	2.4188	2.1123	0.771 55	0.397 11	2.5159	2.2139
MSE	0.141 27	0.161 76	0.082 32	0.0833	0.101 413	0.076 21	0.040 151	0.156 14
$EBayes_{linex1}$	0.582 62	2.968 64	1.509 95	2.7801	0.502 18	3.0189	1.5523	2.8393
MSE	0.023 29	0.020 32	0.029 18	0.020 174	0.051 782	0.027 908	0.019 204	0.021 157
$EBayes_{linex2}$	0.501 096	3.5039	1.520 83	2.7941	0.504 47	3.0948	1.5593	2.8006
MSE	0.013 35	0.018 11	0.029 01	0.020 27	0.041 208	0.020 794	0.081 024	0.021 082
$EBayes_{linex3}$	0.590 86	3.5328	1.5893	2.6982	0.513 44	3.5948	1.6012	2.7826
MSE	0.043 35	0.081 42	0.029 91	0.080 22	0.061 007	0.130 84	0.102 21	0.029 812

Table 4. The value of MSE of estimated parameters θ, c of McR under SE and $Linex$ Loss functions for $n = 100$

Estimation	$b=1.01, c=1.02, c_0=\kappa_0=\tau_0=0.7$				$b=1.01, c=1.05, c_0=\kappa_0=\tau_0=0.01$			
	$a=1.5, \alpha_0=0.9, \beta_0=0.5$				$a=1.5, \alpha_0=0.5, \beta_0=0.9$			
	$\theta=0.5, c=3$		$\theta=1.5, c=3$		$\theta=0.5, c=3$		$\theta=1.5, c=3$	
	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$	$\hat{\theta}$	$\hat{c}$
ML	0.576 88	2.7463	2.2018	2.6841	0.479 29	2.7809	1.5614	2.7352
MSE	0.116 41	0.1967	0.130 09	0.186 35	0.175 39	0.113 42	0.176 181	0.199 008
$Bayes_{SE}$	0.536 18	2.8463	1.9018	2.8841	0.4812	2.4384	1.5946	2.7608
MSE	0.015 89	0.0967	0.012 31	0.0560	0.004 819	0.016 86	0.076 814	0.098 493
$Bayes_{linex}$	0.516 12	2.8477	2.018	2.8533	0.4819	2.621 25	1.551 15	2.7713
MSE	0.013 81	0.0867	0.011 192	0.044 65	0.004 622	0.021 69	0.075 124	0.068 233
$EBayes_{SE1}$	0.542 25	2.802	2.2465	2.7386	0.483 02	2.8374	1.5931	2.7908
MSE	0.014 644	0.0978	0.013 21	0.089 316	0.004 634	0.010 66	0.077 186	0.098 093
$EBayes_{SE2}$	0.545 84	2.2030	1.6698	2.1079	0.533 54	2.2561	1.3897	2.1862
MSE	0.012 76	1.2189	0.004 94	0.011 360	0.042 18	0.011 95	0.076 22	0.020 021
$EBayes_{SE3}$	0.512 58	2.8213	1.491 79	2.708	0.4831	2.9092	1.5435	2.7725
MSE	0.027 76	1.008 89	0.009 94	0.019 36	0.027 228	0.037 15	0.065 64	0.060 342
$EBayes_{linex1}$	0.512 58	2.8213	1.491 79	2.708	0.4831	2.9092	1.5435	2.7725
MSE	0.011 344	0.010 93	0.001 303	0.008 86	0.004 554	0.001 199	0.005 763	0.009 813
$EBayes_{linex2}$	0.485 43	2.8588	1.422 59	2.72	0.485 01	2.9733	1.5396	2.7911
MSE	0.012 48	0.010 24	0.001 407	0.008 763	0.004 472	0.001 004	0.006 512	0.009 634
$EBayes_{linex3}$	0.515 43	2.8177	1.322 54	2.7721	0.423 01	2.9531	1.596 45	2.735 511
MSE	0.013 41	0.012 94	0.011 07	0.009 333	0.075 32	0.018 492	0.009 611	0.012 463 4

From tables(2,3,4) above, it is seen that the estimates for these two parameters are better with increasing sample size according to the mean square error (MSE) criterion, which indicates the accuracy of the estimations. We also note the superiority of Bayesian method under the quadratic loss function (SE) and the linear exponential loss function ($Linex$) over the maximum likelihood method (ML) in estimation. Moreover, we note that the estimators using $E - Bayes$ method under the loss function SE and the loss function $Linex$ are better than the Bayesian estimators under the same loss functions, the results showed that $E - Bayes_{SE2}$ was better in estimation than the Bayesian method under the loss function SE, and that $E - Bayes_{linex2}$ was better the loss function $Linex$. In general, the best method for estimation was $E - Bayes$ under $Linex$ loss function with second hyper prior at $(\alpha_0 = 0.5, \tau_0 = 0.01)$ with all sample sizes. because it obtained the lowest mean square error for all cases.

6. Real data application

In this section, we study the flexibility of McR distribution by fitting complete real data sets in different domains. Each real data set was fitted by using the McR. Beta-Rayleigh(BR), Kumaraswamy Rayleigh(KR), and Rayleigh(R) distributions. The comparison were made using the Kolmogorov-Smirnov statistic, its p-value, Akaikes information criterion(AIC), and Bayesian information criterion(BIC). We used two different types of real data sets to illustrate the usefulness and effectiveness of this distribution. The first data represents Machine downtime at the Kut factory of the Iraqi textile Industries Company[11] The sample size is 100., while the second data represent the 40 leukemia patient treated at KSA health ministry hospital[22]. Note that the observations of two real data sets are show in table5.

Table 5. Raw values of the observations in Data set I and Data set II.

Data setI	10, 9.5, 8, 6.5, 10, 12, 11, 18.5, 8, 12, 9, 17, 22, 26.5, 14, 4.5, 8, 9, 5, 12, 1.5, 20, 7, 12, 18, 12, 15, 5.5, 2, 3, 9, 2, 7, 14, 21.5, 18, 14, 8, 1, 9.5, 11.5, 15, 8, 9.5, 13, 8.5, 20, 21.5, 19, 18.5, 18, 14.5, 3, 14, 3, 5, 12, 7, 16.5, 8.5, 21, 7.5, 5, 9.5, 12, 22.5, 20.5, 17.5, 13, 13.5, 4, 7, 15.5, 8, 3.5, 1, 4, 13, 7.5, 15.5, 2, 14.5, 17.5, 4.5, 3, 27, 10, 16, 4, 16, 4, 18.5, 6.5, 20, 6.5, 8.5, 15, 14
Data setII	0.315, 0.496, 0.616, 1.145, 1.208, 2.211, 2.370, 2.532, 2.693, 2.805, 3.348, 3.427, 3.499, 3.534, 3.767, 4.323, 4.381, 4.392, 4.397, 4.647, 1.263, 2.910, 3.751, 4.753, 1.414, 2.912, 3.858, 4.929, 2.025, 3.192, 3.986, 4.973, 2.036, 3.263, 4.049, 5.074, 2.162, 3.348, 4.244, 4.381

Table 6 shows the goodness of the two data sets based on McR, ER, KR, BR, R models. According to $(K-S)$ statistic, $p-value$, AIC, BIC, it is seen that the (McR) is the best fit to two real data sets with smallest of AIC and BIC. The p-values Corresponding to smallest AIC and BIC for two data sets are greater than 0.05.

Table 6. Estimated parameters with goodness of fit criteria for two real data sets.

Data set	**Model**	$\hat{\theta}$	$\hat{c}$	AIC	BIC	$K-S$	$p-value$
Data1	McR	2.01381	1.49934	76.04669	78.03815	0.091863	0.6799
	ER	1.99926	1.49931	80.65457	82.64604	0.07665	0.6124
	KR	2.45734	1.49944	79.74704	81.7385	0.11496	0.1499
	BR	1.93734	—	88.32726	90.31872	0.25547	5.567e-06
	R	2.45749	—	85.34032	87.33179	0.063515	0.2241
Data2	McR	0.00497	9.99937	135.0787	141.8342	0.18701	0.4219
	ER	0.00457	9.99983	143.6080	146.9858	0.14987	0.3301
	KR	0.00354	9.92537	193.9642	199.0308	0.21672	0.04669
	BR	0.00421	—	208.642	220.308	0.3067	0.001079
	R	0.00449	—	144.3313	146.0202	0.16523	0.2249

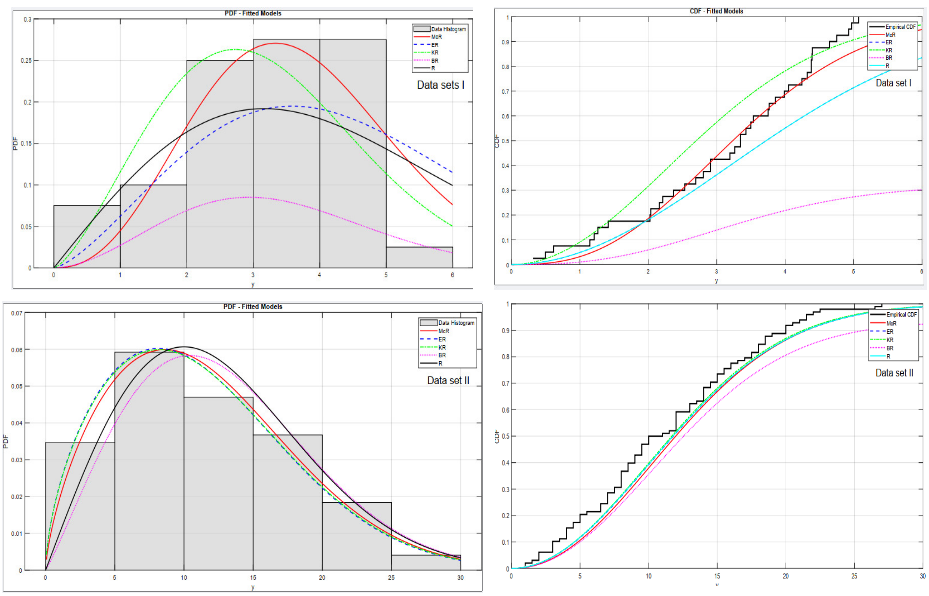

Figure 4. The estimated p.d.f and c.d.f for two data sets based on four models.

It was observed that $E - Bayesian$ estimation outperformed the $Linex$ loss function based on th second type of joint hyper prior distribution, so that the parameters of (McR) were estimated based on two real data sets. Furthermore, Figure 4 shows the estimated p.d.f and c.d.f plots based on four models for the two data sets. This Figure shows that the (McR) distribution is a best fit to two real data sets.

7. Conclusion

In this article, the McDonald family is used to extend Rayleigh distribution. This new distribution named McDonald Rayleigh (McR) distribution. Some statistical properties of McR have been studied. Some parameters of McR were estimated by ML, $Bayesian$ and $E - Bayesian$ methods under square error and $linex$ exponential loss functions with three different forms of hyperparameter priors of distributions. The simulation study and real data sets used to estimate the parameters of McR. In simulation experiment. It is seen that $E - Bayesian$ estimators based on type(2) hyperprior is the best. To demonstrate the flexibility of McR, two real data sets were fitted using the McR and its submodels. The McR proved the best fit among other submodels used. For future work, we will consider ML and Bayesian estimation methods based on progressive type two censored data.

Acknowledgement

We are very grateful to experts for their appropriate and constructive suggestions to improve this template.

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