STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **14**, October 2025, pp 2041–2060.
Published online in International Academic Press (www.IAPress.org)

Nonstandard Finite Difference Schemes for Solving Systems of two Linear Fractional Differential Equations: the case of real eigenvalues

Samah Ali [1], Eihab Bashier [2,3,*]

[1] *Department of Modeling and Computational Mathematics - Al Neelain University, Khartoum, Sudan*
[2] *Department of Mathematics, Faculty of Education and Arts, Sohar University, Sohar, Oman*
[3] *Department of Applied Mathematics, Faculty of Mathematical Sciences and Informatics, University of Khartoum, Khartoum, Sudan*

Abstract This paper provides non-standard finite difference methods for solving a Caputo-type fractional linear system with two equations with real eigenvalues. The linear system's real eigenvalues are classified into two types: distinct and repeated eigenvalues. The scenario of repeated eigenvalues is classified into two categories based on whether the dimension of the corresponding eigenspace is one or two. For each of the three scenarios, we obtained exact solution and developed the numerator and denominator functions for the nonstandard finite difference scheme. Each of the three proposed numerical scheme's convergence has been established by proving consistency and stability. We showed that each of the proposed techniques is unconditionally stable when the system's eigenvalues are negative. Moreover, the three developed NSFDMs are explicit, of low computational complexity and are easy to implement. Three examples were used to demonstrate the performance of the proposed methods.

Keywords Denominator function; Mittag-Leffler function; System of linear fractional differential equations; nonstandard finite difference methods.

AMS 2010 subject classifications 26A33, 65L12

DOI: 10.19139/soic-2310-5070-2564

1. Introduction

Fractional differential equations (FDEs) have attracted considerable attention in recent decades due to their ability to accurately describe complex phenomena exhibiting memory and hereditary characteristics, which classical integer-order differential equations often fail to capture. FDEs have found wide applications in various scientific and engineering domains such as physics, biology, control theory, signal processing, and finance—especially in modeling nonlocal dynamics and anomalous diffusion.

Practical applications of FDEs include the following. In viscoelastic materials, FDEs accurately represent the stress-strain relationship in viscoelastic materials, which display both viscous and elastic characteristics [15]. In contrast to traditional models, fractional derivatives encapsulate memory effects and power-law creep behavior seen in polymers, rubbers, and biological tissues. In control systems, FDEs extend traditional PID controllers by including fractional derivatives and integrals, hence enhancing resilience and performance in systems exhibiting long memory effects, such as temperature regulation and robotics [23, 7]. Classical epidemiological models in epidemiology presume exponential, memoryless disease transmission; nevertheless, actual epidemics have memory effects [27, 9]. Fractional-order models more accurately represent outbreaks by including delayed responses and diverse transmission dynamics.

*Correspondence to: Eihab Bashier (Email: ebashier@su.edu.om). Department of Mathematics, Faculty of Education and Arts, Sohar University, Sohar, Oman.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

While many analytical solutions have been developed for certain classes of FDEs, these solutions are generally limited to linear or weakly nonlinear systems with specific structures. For instance, Odibat [21] presented analytical solutions for linear FDEs using Mittag-Leffler functions for both commensurate and incommensurate orders. Techniques like the Adomian Decomposition Method (ADM) [20, 8] and the Homotopy Perturbation Method (HPM) [1] have been employed to construct approximate analytical solutions, often with rapid convergence. Moreover, the Homotopy Analysis Method (HAM) [30] has been used to generalize these methods and address nonlinear systems more flexibly. Despite such advancements, obtaining analytical solutions for general FDEs remains a difficult task, especially for systems involving strong nonlinearities or complex dynamical behavior, thus necessitating the development of robust numerical methods.

Among numerical approaches, the Adams–Bashforth method introduced by Diethelm [10] and the power series method developed by Rida [24] have shown some effectiveness, while Zeid [29] provided a comprehensive review of available techniques. Finite Difference Methods (FDMs) are widely used due to their simplicity and computational efficiency. However, when applied to fractional-order systems—particularly coupled nonlinear systems—traditional FDMs may suffer from significant drawbacks, such as loss of stability, poor convergence, or failure to replicate the qualitative behavior of the original system. To overcome these limitations, Mickens [17, 18] introduced the Nonstandard Finite Difference (NSFD) method, which has proven to be a powerful alternative. NSFD schemes are constructed using modeling principles that aim to preserve key qualitative features of the original continuous system, including positivity, boundedness, equilibrium stability, and dynamical consistency. Unlike standard FDMs, NSFD schemes often utilize nonlocal approximations and custom denominator functions tailored to the problem's structure. Mickens also introduced the notion of exact schemes, further enriching the modeling framework of NSFDs.

Several applications of NSFD methods to chaotic and nonlinear fractional systems have been reported in [26, 25]. Moaddy [19] used NSFD to analyze the chaotic dynamics of the fractional-order Rössler system using the Grünwald–Letnikov approximation. Ongun [22] applied NSFD to the fractional-order Brusselator model, successfully replicating its complex behavior. Hajipour [12] and others extended NSFD methods to chaotic fractional systems such as the Chen system [28] and novel systems involving Caputo derivatives and nonlinearities [5].

Nevertheless, very few investigations have looked at applying NSFD methods to fractional systems with real eigenvalues. This class of systems presents a significant challenge: long-term behavior and asymptotic stability are very sensitive to numerical approximation. In these systems, standard finite difference methods usually produce numerical artifacts or destabilize otherwise stable continuous systems. Therefore, there is an urgent need to formulate NSFD methods that are explicitly tailored to manage real-eigenvalue systems and guarantee accurate representation of their dynamics across time. For example, the homotopy perturbation method and new iterative method [6] used to solve the fractional logistic model were not bounded by the model's carrying capacity. Hence, they lack the dynamical consistency.

This work is to fill this gap by introducing novel exact NSFD methods intended to resolve systems of FDEs with real eigenvalues. This study focuses on maintaining stability and accuracy in systems where real eigenvalues dominate the dynamics, as opposed to earlier research that focuses on chaotic or oscillatory behavior. The proposed methods incorporate adaptive step-size functions and modified discrete approximations that improve the accuracy and qualitative consistency of the numerical solutions. In addition, this work builds on and expands on previous NSFD methods providing a theoretical basis and numerical confirmation for the new method, showing its advantages in maintaining stability, achieving convergence, and long-time behaviour.

A special class of the NSFDm is the class of exact nonstandard finite difference methods, which is characterized by producing zero truncation errors in every mesh of points. Hence, exact NSFDMs are independant of the mesh type, and consequently the step size [13]. Exact NSFDMs were formulated for many types of integer-order differential equations. For instance, in [17], exact finite difference techniques were formulated to deal with the exponential growth model, the logistic model, a system of two linear first-order equations and a higher order fractional differential equations ($1 < \alpha \leq 2$). An exact solution utilizing NSFDMs was formulated for solving a reaction-diffusion problem in [16]. Roeger [26] extended Mickens' work [17] by formulating exact NSFDMs for various cases of linear systems including two first-order differential equations.

For fractional order differential equations, few exact NSFDMs have been constructed in the literature. Recently, exact finite difference techniques were developed to solve the logistic growth model [4] and the fractional decay model [3].

In this paper we consider a linear system of two fractional differential equations of the form

$$
\begin{aligned}
{}_0^C D_t^\alpha x(t) &= ax(t)+by(t),\ t\in[0,T],\ x(0)=x_0 &(1)\\
{}_0^C D_t^\alpha y(t) &= cx(t)+dy(t),\ t\in[0,T],\ y(0)=y_0 &(2)
\end{aligned}
$$

where a,b,c and d are real constants such that $det(A)=ad-bc\neq 0$, $T>0$ is positive real number and ${}_0^C D_t^\alpha$ is the Caputo derivative of order $0<\alpha\leq 1$.

The main objective is to develop exact nonstandard finite difference methods for solving system (1)-(2), relying on the work of Mickens [17] and Roeger [25], particularly in situations where the linear system has real eigenvalues. We investigated two separate cases for the linear system. The first situation involves the system possessing two unique eigenvalues, while the subsequent case entails a repeated eigenvalue. In each of these cases, we computed the numerator and denominator functions and constructed the corresponding exact nonstandard finite difference schemes.

The main contribution of the paper is the introduction of exact nonstandard finite difference methods for solving linear systems of two fractional differential equations with real eigenvalues. If the linear system is stable, the methods exhibit unconditional stability. Each of these NSFDMs have a complexity comparable to that of the improved Euler's method, thereby designating them as low complexity.

The rest of the paper is organized as follows. In Section 2 we provide some necessary backgrounds. The exact solutions of systems of two linear fractional differential equations are derived in Section 3. The constructions of nonstandard finite difference schemes are discussed in Section 4. The convergences of the proposed NSFDMs are investigated in Section 5. Section 6 illustrates numerical examples for different values of α and compares them to the analytical solution to demonstrate the accuracy of the numerical solution. Section 7 concludes the paper.

2. Preliminaries and definitions

In this section, we introduce the essential mathematical concepts required for developing a nonstandard finite difference method for a system of Caputo type fractional linear equations.

2.1. Caputo derivatives and their Laplace transformations

Definition 2.1 (Caputo Derivatives)
The Caputo fractional derivative of order $\alpha>0$ for a function $f(t)\in C^n([0,T])$, where $n-1<\alpha\leq n, n\in\mathbb{N}$, is defined as:

$$
{}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,\quad t>0
$$

where, Γ denotes the Gamma function.

For $0<\alpha\leq 1$, the Caputo derivative is defined as:

$$
{}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau,
$$

The Laplace transform of the Caputo derivative is given by:

$$
\mathcal{L}\left\{{}_0^C D_t^\alpha f(t)\right\}(s)=s^\alpha F(s)-\sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0^+),
$$

where $F(s) = \mathcal{L}\{f(t)\}(s)$ and $n-1 < \alpha \leq n$.

A special case is $0 < \alpha \leq 1$, where the Laplace transformation of the Caputo derivative is:

$$\mathcal{L}\{{}_0^C D_t^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1} f(0),$$

2.2. The Mittag-Leffler functions and some of their properties

The Mittag-Leffler (ML) function is a generalization of the exponential function that is used extensively in fractional calculus [11].

Definition 2.2 (The Mittag-Leffler functions)
The one-parameter ML function is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}$$

The two-parameter ML function is given by:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha,\beta > 0, \quad z \in \mathbb{C}.$$

We notice that, when $\beta = 1, E_{\alpha,1}(z) = E_\alpha(z)$. Special cases of Mittag-Leffler functions are obtained by setting $\alpha = 1$, where we obtain $E_1(z) = e^z$ and when $\alpha = 2$ where we obtain $E_2(z) = \cosh\sqrt{z}$.

The Mittag-Leffler function $E_\alpha(z)$ is an entire transcendental function. When $\alpha = 1$, $E_1(z) = e^z$. Hence, the function $E_\alpha(z)$ generalizes the exponential function.

When $z = \lambda t^\alpha$, we obtain the Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ which is a basis solution of the fractional growth/decay model

$${}_0^C D_t^\alpha w(t) = \lambda w(t), w(0) = w_0,$$

whose solution is given by $w(t) = w_0 E_\alpha(\lambda t^\alpha)$.

The Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ has the following properties:

(i) For a small value of t:

$$E_\alpha(\lambda t^\alpha) \approx 1 + \frac{\lambda t^\alpha}{\Gamma(\alpha + 1)}$$

(ii) For a large value of t:

$$E_\alpha(\lambda t^\alpha) \sim \frac{1}{\alpha} e^{\lambda^{1/\alpha} t} \quad (\text{ for } \lambda > 0)$$

(iii) For $\lambda < 0$, the function decays as:

$$E_\alpha(\lambda t^\alpha) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

The Laplace transformation of $E_\alpha(\lambda t^\alpha)$ is:

$$\mathcal{L}\{E_\alpha(\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda}$$

Suppose that h is a small positive real number. Then:

$$(t+h)^\alpha = t^\alpha + \alpha h t^{\alpha-1} + \frac{\alpha(\alpha-1)}{2} h^2 t^{\alpha-2} + \mathcal{O}(h^3)$$

Now, $E_\alpha(\lambda(t+h)^\alpha)$ can be approximated as follows:

$$E_\alpha(\lambda(t+h)^\alpha) = E_\alpha\left(\lambda\left(t^\alpha + \alpha h t^{\alpha-1} + O(h^2)\right)\right) \approx E_\alpha\left(\lambda\left(t^\alpha + \alpha h t^{\alpha-1}\right)\right)$$

From Taylor expansion,

$$E_\alpha\left(\lambda(t+h)^\alpha\right) \approx E_\alpha\left(\lambda t^\alpha\right) + \lambda\alpha t^{\alpha-1} h \cdot E'_\alpha\left(\lambda t^\alpha\right)$$

Hence, we find that

$$\frac{E_\alpha\left(\lambda(t+h)^\alpha\right)}{E_\alpha\left(\lambda t^\alpha\right)} \approx 1 + \lambda\alpha t^{\alpha-1} h \cdot \frac{E'_\alpha\left(\lambda t^\alpha\right)}{E_\alpha\left(\lambda t^\alpha\right)}$$

Another important function is $t^\alpha E_\alpha(\lambda t^\alpha)$, which exhibits asymptotic behavior similar to polynomial-exponential functions. The Laplace transformation of $t^\alpha E_\alpha(\lambda t^\alpha)$ is given by:

$$\mathcal{L}\{t^\alpha E_\alpha(\lambda t^\alpha)\} = \frac{\Gamma(\alpha+1)}{(s^\alpha - \lambda)^2}$$

The inverse Laplace transform of $\frac{s^{\alpha-1}}{(s^\alpha-\lambda)^2}$ is given by:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^\alpha-\lambda)^2}\right\} = t^\alpha E_\alpha(\lambda t^\alpha).$$

Hence, the function $t^\alpha E_\alpha(\lambda t^\alpha)$, is a solution of the fractional differential equation:

$${}_0^C D_t^\alpha\left(t^\alpha w(t)\right) = \lambda t^\alpha w(t), \quad w(0) = w_0,$$

whose solution is expressed as:

$$w(t) = t^\alpha E_\alpha(\lambda t^\alpha)$$

3. Deriving Solutions of Systems of linear fractional differential equations

Consider the coupled, linear system of FDEs with constant coefficients (1). It can be written in the matrix form as

$${}_0^C D_t^\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

According to Matignon (1996), the stability of a system of FDEs is assured if the roots of the characteristic polynomial fall outside the closed angular sector. The system of FDEs (3) is asymptotically stable if and only if all roots λ fulfill $|arg(\lambda)| > \alpha\frac{\pi}{2}$ [14]. In accordance with Ahmed [2], fractional order differential equations are equally stable as their integer order counterparts.

Therefore, to ascertain the stability of the linear system (3), we compute the eigenvalues and eigenvectors from the characteristic equation.

$$det(A - \lambda I) = 0 \equiv \lambda^2 - trace(A)\lambda + det(A) = 0 \tag{4}$$

Therefore

$$\lambda_{1,2} = \frac{trace(A) \pm \sqrt{trace(A)^2 - 4det(A)}}{2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2}$$

we have three cases for $\lambda_{1,2}$ which can be distinct, repeated and complex.

3.1. Distinct eigenvalues

If $trace(A)^2 > 4 \cdot det(A)$, the linear system (3) will have two distinct eigenvalues $\lambda_1 and \lambda_2$, where $\lambda_1 \neq \lambda_2$. Then, the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ corresponding to the eigenvalues λ_1 and λ_2 are the basis vectors of $null(A - \lambda I_2)$ given by:

$$\boldsymbol{v}_1 = \begin{pmatrix} \dfrac{b}{\lambda_1 - a} \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} \dfrac{b}{\lambda_2 - a} \\ 1 \end{pmatrix}.$$

where I_2 is the 2×2 identity matrix.

By letting $P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2]$, yields

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = J$$

This implies that the Jordan normal form of matrix A is the diagonal matrix J.

Let

$$\boldsymbol{w} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

then the linear system can be written as:

$$\dot{\boldsymbol{w}}(t) = A\boldsymbol{w}, \boldsymbol{w}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{5}$$

Substituting $\boldsymbol{z} = P^{-1}\boldsymbol{w}$ in Equation (5) yields the uncoupled linear system

$${}_0^C D_t^\alpha \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = J\boldsymbol{z} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \boldsymbol{z} = \begin{pmatrix} \lambda_1 z_1(t) \\ \lambda_2 z_2(t) \end{pmatrix} \tag{6}$$

The solution of the fractional equation (6) is given by

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} C_1 E_\alpha(\lambda_1 t^\alpha) \\ C_2 E_\alpha(\lambda_2 t^\alpha) \end{pmatrix} \tag{7}$$

where C_1 and C_2 are constants that shall be evaluated from the initial condition. Finally, by substituting $\boldsymbol{w}(t) = P\boldsymbol{z}(t)$ and solving $\boldsymbol{w}(t_0) = P\boldsymbol{z}(t_0) = \boldsymbol{w}_0$ for the constants C_1 and C_2, we obtain the general solution of the system (1), which can be written as

$$x(t) = -\frac{b}{\lambda_1 - \lambda_2}\left(\left(\frac{\lambda_2 - a}{b}\right)x_0 - y_0\right)\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} + \frac{b}{\lambda_1 - \lambda_2}\left(\left(\frac{\lambda_1 - a}{b}\right)x_0 - y_0\right)\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}$$

and

$$y(t) = -\frac{\lambda_1 - a}{\lambda_1 - \lambda_2}\left(\left(\frac{\lambda_2 - a}{b}\right)x_0 - y_0\right)\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} + \frac{\lambda_2 - a}{\lambda_1 - \lambda_2}\left(\left(\frac{\lambda_1 - a}{b}\right)x_0 - y_0\right)\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}$$

These latest two equations can be written as:

$$x(t) = \left[-\frac{(\lambda_2 - a)}{\lambda_1 - \lambda_2}\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} + \frac{(\lambda_1 - a)}{\lambda_1 - \lambda_2}\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right]x_0 + \frac{b}{(\lambda_1 - \lambda_2)}\left[\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right]y_0 \tag{8}$$

$$y(t) = -\frac{(\lambda_1 - a)(\lambda_2 - a)}{b(\lambda_1 - \lambda_2)}\left[\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right]x_0 + \left[\frac{(\lambda_1 - a)}{(\lambda_1 - \lambda_2)}\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{(\lambda_2 - a)}{(\lambda_1 - \lambda_2)}\frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}\right]y_0 \tag{9}$$

We have $c = -(\lambda_1 - a)(\lambda_2 - a)/b$ and $\lambda_1 + \lambda_2 = a + d$. then $\lambda_1 - a = d - \lambda_2$ and $\lambda_2 - a = d - \lambda_1$. Replacing $\lambda_1 - a$ and $\lambda_2 - a$ by their equivalences in Equation (9) yields:

$$x(t) = \left[a\frac{(\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)})}{\lambda_1 - \lambda_2} + \frac{\frac{\lambda_1 E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}}{\lambda_1 - \lambda_2} \right] x_0 + b\left[\frac{\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}}{\lambda_1 - \lambda_2} \right] y_0 \tag{10}$$

$$y(t) = c\left[\frac{\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)}}{\lambda_1 - \lambda_2} \right] x_0 + \left[d\frac{(\frac{E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)} - \frac{E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)})}{\lambda_1 - \lambda_2} + \frac{\frac{\lambda_1 E_\alpha(\lambda_2 t^\alpha)}{E_\alpha(\lambda_2 t_0^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 t^\alpha)}{E_\alpha(\lambda_1 t_0^\alpha)}}{\lambda_1 - \lambda_2} \right] y_0 \tag{11}$$

3.2. Repeated Eigenvalues

If the linear system (3) has a repeated eigenvalue $\lambda = \frac{a+d}{2}$, then $c = -(a-d)^2/(4b) = -(\lambda - a)^2/b$. There are two different cases for the dimension of the null space of the eigenspace $A - \lambda I$. Accordingly, the analytical solutions for (1) are computed differently.

3.2.1. Dim(null$(A - \lambda I_2)) = 2$ Corresponding to the eigenvalue λ there are two linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. This case occurs when $A - \lambda I$ is the zeros matrix. In this case A is the diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and its eigenvectors $\boldsymbol{v}_1$ and the $\boldsymbol{v}_2$ are the columns of the 2×2 identity matrix. That is:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Jordan form of matrix A is A itself.
Then $E_\alpha(\lambda t^\alpha)\boldsymbol{v}_1$ and $E_\alpha(\lambda t^\alpha)\boldsymbol{v}_2$ are linearly independent. Hence, the general solution of (1) is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2) E_\alpha(\lambda t^\alpha)$$

Hence,

$$x(t) = x(0)E_\alpha(\lambda t^\alpha) = x_0 E_\alpha(\lambda t^\alpha) \tag{12}$$

$$y(t) = y(0)E_\alpha(\lambda t^\alpha) = y_0 E_\alpha(\lambda t^\alpha) \tag{13}$$

3.2.2. Dim(null$(A - \lambda I_2)) = 1$ There is only one eigenvector $\boldsymbol{v}_1$ corresponding to the repeated eigenvalue λ. This case occurs when $A - \lambda I$ is not the zero matrix. The eigenvector $\boldsymbol{v}_1$ is given by

$$\boldsymbol{v}_1 = \begin{pmatrix} \frac{b}{\lambda - a} \\ 1 \end{pmatrix}.$$

Suppose that $\boldsymbol{v}_2$ is the generalized eigenvector that is obtained by solving the linear system $(A - \lambda I)\boldsymbol{v}_2 = \boldsymbol{v}_1$. Then,

$$\boldsymbol{v}_2 = \begin{pmatrix} -\frac{b}{(\lambda - a)^2} \\ 0 \end{pmatrix}$$

Let $P = [\boldsymbol{v}_1 \ \boldsymbol{v}_2]$, then

$$P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where J is the Jordan normal form of matrix A.

Substituting $\boldsymbol{z}(t) = P^{-1}\boldsymbol{w}(t)$, we obtain the linear system:

$$\begin{pmatrix} {}_0^C D_t^\alpha z_1(t) \\ {}_0^C D_t^\alpha z_2(t) \end{pmatrix} = \begin{pmatrix} \lambda z_1(t) + z_2(t) \\ \lambda z_2(t) \end{pmatrix} \tag{14}$$

The solution of the second equation is given by:

$$z_2(t) = C_2 E_\alpha(\lambda t^\alpha)$$

The first equation becomes:

$${}_0^C D_t^\alpha z_1(t) = \lambda z_1(t) + C_2 E_\alpha(\lambda t^\alpha).$$

The Laplace transformation of the equation in z_1 is:

$$s^\alpha Z_1(s) - s^{\alpha-1} z_1(0) = \lambda Z_1(s) + C_2 \frac{s^{\alpha-1}}{s^\alpha - \lambda}$$

which can be simplified to:

$$Z_1(s) = C_1 \frac{s^{\alpha-1}}{s^\alpha - \lambda} + C_2 \frac{s^{\alpha-1}}{(s^\alpha - \lambda)^2}$$

By taking the inverse Laplace transform we obtain:

$$z_1(t) = C_1 E_\alpha(\lambda t^\alpha) + C_2 t^\alpha E_\alpha(\lambda t^\alpha),$$

where C_1 and C_2 are constants that can be obtained by solving the system $P\boldsymbol{z}(t_0) = \boldsymbol{w}_0$ for C_1 and C_2.

Hence, the solution of the linear system (1)-(2) is given by:

$$x(t) = \frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(1 - (\lambda - a)(t^\alpha - t_0^\alpha))x_0 + b\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(t^\alpha - t_0^\alpha)y_0 \tag{15}$$

$$y(t) = -\frac{(\lambda - a)^2}{b}\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(t^\alpha - t_0^\alpha)x_0 + \frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(1 + (\lambda - a)(t^\alpha - t_0^\alpha))y_0 \tag{16}$$

We have $c = -(\lambda - a)^2/b$ and $\lambda = (a+d)/2$, then $\lambda - a = d - \lambda$. Substituting in equations (15) and (16) yields

$$x(t) = \left[a(t^\alpha - t_0^\alpha)\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)} + (1-\lambda)(t^\alpha - t_0^\alpha)\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}\right]x_0 + b\left[\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(t^\alpha - t_0^\alpha)\right]y_0 \tag{17}$$

$$y(t) = c\left[\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}(t^\alpha - t_0^\alpha)\right]x_0 + \left[d(t^\alpha - t_0^\alpha)\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)} + (1-\lambda)(t^\alpha - t_0^\alpha)\frac{E_\alpha(\lambda t^\alpha)}{E_\alpha(\lambda t_0^\alpha)}\right]y_0 \tag{18}$$

4. Nonstandard finite difference schemes for system of linear FDEs

In this section, nonstandard finite difference methods will be developed to solve a system of two fractional differential equations for the cases of two distinct real eigenvalues and a repeated real eigenvalue.

Let N be a positive integer and $h = T/N$. We discretize the interval $[0,T]$ by points $t_k = h \cdot k, k = 0,1,...,N$. Let $x_k \approx x(t_k)$ and $y_k \approx y(t_k)$ for $k = 0,1,...,N$.

4.1. Linear system of FDEs with distinct eigenvalues

To obtain a nonstandard finite difference scheme for (1), following Mickens's method and making following substitutions in (10) and (11)

$$\begin{cases} t_0 \to t_k = hk, & t \to t_{k+1} = h(k+1) \\ x_0 \to x_k, & x(t) \to x_{k+1} \\ y_0 \to y_k, & y(t) \to y_{k+1} \end{cases} \tag{19}$$

then

$$\begin{aligned} x_{k+1} &= \left[a\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)}{\lambda_1 - \lambda_2} + \frac{\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} \right] x_k + \\ & \qquad b\left[\frac{\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} \right] y_k \end{aligned} \tag{20}$$

$$\begin{aligned} y_{k+1} &= c\left[\frac{\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} \right] x_0 + \\ & \qquad \left[d\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)}{\lambda_1 - \lambda_2} + \frac{\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} \right] y_0 \end{aligned} \tag{21}$$

Equations (20) and (21) can be written as:

$$x_{k+1} - \frac{\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} x_k = \left[\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)}{\lambda_1 - \lambda_2} \right] (ax_k + by_k) \tag{22}$$

$$y_{k+1} - \frac{\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2} y_k = \left[\frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)}{\lambda_1 - \lambda_2} \right] (cx_k + dy_k) \tag{23}$$

Let

$$\phi_k = \frac{\left(\frac{E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)} - \frac{E_\alpha(\lambda_2 h^\alpha k^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)}\right)}{\lambda_1 - \lambda_2},$$

and

$$\psi_k = \frac{\frac{\lambda_1 E_\alpha(\lambda_2 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_2 h^\alpha k^\alpha)} - \frac{\lambda_2 E_\alpha(\lambda_1 h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda_1 h^\alpha k^\alpha)}}{\lambda_1 - \lambda_2}$$

The NSFD scheme for the linear system (1) with distinct eigenvalues as

$$\frac{x_{k+1} - \psi_k x_k}{\phi_k} = ax_k + by_k, \tag{24}$$

$$\frac{y_{k+1} - \psi_k y_k}{\phi_k} = cx_k + dy_k, \tag{25}$$

4.2. System of Linear FDEs with Repeated Eigenvalues

4.2.1. Case 1: Dim$(A - \lambda I) = 2$ A nonstandard finite difference scheme for (5), when matrix A has a repeated eigenvalue λ and corresponding two linearly independent eigenvectors is obtained by making substituting (19) in

(12) and (13), then

$$x_{k+1} = x_k \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(a h^\alpha (k)^\alpha)} \tag{26}$$

$$y_{k+1} = y_k \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(a h^\alpha (k)^\alpha)} \tag{27}$$

Subtracting x_k and y_k from both sides of (26) and (27), gives

$$\frac{x_{k+1} - x_k}{\frac{\left(\frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha (k)^\alpha)} - 1\right)}{\lambda}} = \lambda x_k, \tag{28}$$

$$\frac{y_{k+1} - y_k}{\frac{\left(\frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha (k)^\alpha)} - 1\right)}{\lambda}} = \lambda y_k. \tag{29}$$

Hence, the denominator function is given by

$$\phi_k = \frac{\left(\frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha (k)^\alpha)} - 1\right)}{\lambda},$$

while the numerator function $\psi_k = 1$ for $k = 0, 1, ..., N-1$.

It was proven in [3] that the nonstandard schemes (28) and (29) are exact nonstandard finite difference forms for solving the fractional decay model.

4.2.2. Case 2: Dim$(A - \lambda I) = 1$ The nonstandard finite difference scheme for (1) with a repeated eigenvalue and one corresponding eigenvector is obtained by making substitutions (19) in (17) and (18). Then,

$$\begin{aligned} x_{k+1} &= \left[a h^\alpha (k+1^\alpha - k^\alpha) \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)} + (1-\lambda)(h^\alpha((k+1)^\alpha - k^\alpha)) \frac{E_\alpha(\lambda h^\alpha (k+1))}{E_\alpha(\lambda h^\alpha k^\alpha)} \right] x_k + \\ &\quad b \left[\frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)} (t^\alpha - h^\alpha k^\alpha) \right] y_k \end{aligned} \tag{30}$$

$$\begin{aligned} y_{k+1} &= c \left[\frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)} (h^\alpha((k+1)^\alpha - k^\alpha)) \right] x_k + \\ &\quad \left[d\, h^\alpha((k+1)^\alpha - k^\alpha) \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)} + (1-\lambda) h^\alpha((k+1)^\alpha - k^\alpha) \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)} \right] y_k \end{aligned} \tag{31}$$

By taking

$$\psi_k = (1 - \lambda h^\alpha((k+1)^\alpha - (k)^\alpha)) \frac{E_\alpha(\lambda (hk+h)^\alpha)}{E_\alpha(\lambda (hk)^\alpha)}$$

and

$$\phi_k = h^\alpha((k+1)^\alpha - (k)^\alpha)) \frac{E_\alpha(\lambda h^\alpha (k+1)^\alpha)}{E_\alpha(\lambda h^\alpha k^\alpha)}$$

The NSFD scheme for (1) repeated eigenvalues as

$$\frac{x_{k+1}-\psi_k x_k}{\phi_k} = ax_k + by_k \tag{32}$$

$$\frac{y_{k+1}-\psi_k y_k}{\phi_k} = cx_k + dy_k \tag{33}$$

5. Convergence and Stability for Nonstandard finite difference scheme for system of linear FDEs

This section presents proof for the convergence of each constructed nonstandard finite difference scheme by demonstrating its consistency and numerical stability.

5.1. Consistency of the numerical schemes

We prove the convergence of the NSFD scheme for (24) and (25), from fractional Taylor series expansion

$$x(t_{k+1}) = x(t_k) + {}_0^C D_t^\alpha x(t_k)\frac{h^\alpha}{\Gamma(\alpha+1)} + O(h^{2\alpha})$$

similarly

$$y(t_{k+1}) = y(t_k) + {}_0^C D_t^\alpha y(t_k)\frac{h^\alpha}{\Gamma(\alpha+1)} + O(h^{2\alpha})$$

Therefore

$${}_0^C D_t^\alpha x(t_k) = \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + O(h^\alpha)$$

and

$${}_0^C D_t^\alpha y(t_k) = \frac{y(t_{k+1})-y(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + O(h^\alpha)$$

The local truncation error (LTE) of (24) is given by

$$LTE_x = \frac{x(t_{k+1})-\varphi_k(h,\lambda,\phi)x(t_k)}{\phi_k(h,\lambda,\phi)} - ax(t_k) - by(t_k)$$

It can be expressed as

$$LTE_x = \frac{x(t_{k+1})-\varphi_k(h,\lambda,\phi)x(t_k)}{\phi_k(h,\lambda,\phi)} - \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k)$$

$$LTE_x = \frac{x(t_{k+1})}{\phi_k} - \frac{x(t_k)}{\frac{\phi_k}{\varphi_k}} - \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} + \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k)$$

$$|LTE_x| = \left|\left(\frac{1}{\phi_k} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_{k+1}) - \left(\frac{1}{\frac{\phi_k}{\varphi_k}} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_k) + \frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k) + O(h)\right|$$

$$|LTE_x| \le \left|\left(\frac{1}{\phi_k} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_{k+1}) - \left(\frac{1}{\frac{\phi_k}{\varphi_k}} - \frac{1}{\frac{h^\alpha}{\Gamma(\alpha+1)}}\right)x(t_k)\right| + \left|\frac{x(t_{k+1})-x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k) + O(h^\alpha)\right|$$

$$|LTE_x| \leq \left| \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \phi_k}{\phi_k \frac{h^\alpha}{\Gamma(\alpha+1)}} \right) x(t_{k+1}) - \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \frac{\phi_k}{\varphi_k}}{\frac{\phi_k}{\varphi_k} \frac{h^\alpha}{\Gamma(\alpha+1)}} \right) x(t_k) \right| + \left| \frac{x(t_{k+1}) - x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - ax(t_k) - by(t_k) \right| + |O(h^\alpha)|$$

$$|LTE_x| \leq \left| \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \phi_k}{\phi_k} \right) \frac{x(t_{k+1})}{\frac{h^\alpha}{\Gamma(\alpha+1)}} - \left(\frac{\frac{h^\alpha}{\Gamma(\alpha+1)} - \frac{\phi_k}{\varphi_k}}{\frac{\phi_k}{\varphi_k}} \right) \frac{x(t_k)}{\frac{h^\alpha}{\Gamma(\alpha+1)}} \right| + |O(h^\alpha)| + |O(h^\alpha)| \tag{34}$$

from (34) we realize that $|LTE_x| \longrightarrow 0$ as $h \longrightarrow 0$, and similarly $|LTE_y| \longrightarrow 0$ as $h \longrightarrow 0$.

5.2. *Stability of the numerical schemes*

To prove the stability of the NSFD scheme for (24) and (25)
Let $e_k^x = x(t_k) - x_k$ and $e_k^y = y(t_k) - y_k$, the NSFD scheme (24) and (25) can be written as

$$\begin{aligned} x_{k+1} &= (\varphi + a\phi)x_k + b\phi y_k \\ y_{k+1} &= c\phi x_k + (\varphi + d\phi)y_k \end{aligned} \tag{35}$$

substituting $x(t_k)$ and $y(t_k)$ instead of x_k and y_k in (35), we obtain

$$\begin{aligned} x(t_{k+1}) &= (\varphi + a\phi)x(t_k) + b\phi y(t_k) \\ y(t_{k+1}) &= c\phi x(t_k) + (\varphi + d\phi)y(t_k) \end{aligned} \tag{36}$$

subtracting (36) from (35), substituting $e_k^x = x(t_k) - x_k$ and $e_k^y = y(t_k) - y_k$

$$\begin{aligned} e_{k+1}^x &= (\psi_k + a\phi_k)e_k^x + b\phi_k e_k^y \\ e_{k+1}^y &= c\phi_k e_k^x + (\psi_k + d\phi_k)e_k^y \end{aligned} \tag{37}$$

we can write (37) as

$$\begin{pmatrix} e_{k+1}^x \\ e_{k+1}^y \end{pmatrix} = \begin{pmatrix} \varphi + a\phi_k & b\phi_k \\ c\phi_k & \varphi + d\phi_k \end{pmatrix} \begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix}$$

Generally

$$\begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix} = \begin{pmatrix} (\psi_k + a\phi_k)^k & (b\phi_k)^k \\ (c\phi_k)^k & (\psi_k + d\phi_k)^k \end{pmatrix} \begin{pmatrix} e_0^x \\ e_0^y \end{pmatrix}$$

or

$$\begin{pmatrix} e_k^x \\ e_k^y \end{pmatrix} = \begin{pmatrix} \psi_k + a\phi_k & b\phi_k \\ c\phi_k & \psi_k + d\phi_k \end{pmatrix}^k \begin{pmatrix} e_0^x \\ e_0^y \end{pmatrix}$$

Let

$$B^{(k)} = \begin{pmatrix} \psi_k + a\phi_k & b\phi_k \\ c\phi_k & \psi_k + d\phi_k \end{pmatrix} = \psi_k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of $B^{(k)}$ are $\psi_k + \lambda_1\phi_k$ and $\psi_k + \lambda_2\phi_k$. Hence, $e_k^x \longrightarrow 0$ and $e_k^y \longrightarrow 0$ as $k \longrightarrow \infty$ if and only if $P(B^{(k)}) < 1$.

we have

$$\sigma_1^k = \psi_k + \lambda_1\phi_k = \frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)}$$

where

$$\psi_k = \frac{\left[\lambda_1 \frac{E_\alpha(\lambda_2(hk+h)^\alpha)}{E_\alpha(\lambda_2(hk)^\alpha)} - \lambda_2 \frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)} \right]}{\lambda_1 - \lambda_2} \quad \text{and} \quad \phi_k = \frac{\left[\frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)} - \frac{E_\alpha(\lambda_2(hk+h)^\alpha)}{E_\alpha(\lambda_2(hk)^\alpha)} \right]}{\lambda_1 - \lambda_2}$$

then

$$|\sigma_1^k| < 1 \quad \Rightarrow \quad \frac{E_\alpha(\lambda_1(hk+h)^\alpha)}{E_\alpha(\lambda_1(hk)^\alpha)} < 1$$

that is

$$E_\alpha(\lambda_1(hk+h)^\alpha) < E_\alpha(\lambda_1(hk)^\alpha) \quad \Rightarrow \quad \lambda_1 < 0$$

similarly

$$\sigma_2^k = \psi_k + \lambda_2\phi_k \qquad \text{and} \qquad |\sigma_2^k| < 1 \quad \Rightarrow \quad \lambda_2 < 0$$

Similarly with repeated eigenvalues where

$$\psi_k = (1 - \lambda h^\alpha((k+1)^\alpha - (k)^\alpha))\frac{E_\alpha(\lambda(hk+h)^\alpha)}{E_\alpha(\lambda(hk)^\alpha)}$$

and

$$\phi_k = h^\alpha((k+1)^\alpha - (k)^\alpha))\frac{E_\alpha(\lambda(hk+h)^\alpha)}{E_\alpha(\lambda(hk)^\alpha)},$$

then

$$\sigma_{1,2}^k = \psi_k + \lambda\phi_k \qquad \text{and} \qquad |\sigma_{1,2}^k| < 1 \quad \Rightarrow \quad \lambda < 0$$

We end up with the following theorem.

Theorem 5.1
If the eigenvalues of the fractional linear system (1) are negative, either of the nonstandard finite difference scheme described by equations (24) and (25) or the nonstandard finite difference scheme described by equations (32) and (33) is unconditionally stable.

Theorem 5.1 asserts that if the two eigenvalues of the linear system are negative, the NSFDM will exhibit unconditional stability for any step size and fractional order. A modification to the coefficients a, b, c, or d of the linear system may impact the stability of the numerical methods if it alters the sign of one or both eigenvalues of the system. Any perturbation of the linear system parameters that does not change an eigenvalue from negative to positive will guarantee the unconditional stability of the NSFDM.

6. Numerical results

This section will present three examples to demonstrate the performance of the nonstandard finite difference methods established in this research.

We ran the numerical simulations in Python 3.12 on a computer with an Intel core i7 CPU. We used Khensin's implementation of the Mittag-Leffler function which can be found on GitHub at https://github.com/khinsen/mittag-leffler.

Example 6.1
Consider the linear system of FDEs [21]

$$\begin{aligned} {}_0^C D_t^\alpha x(t) &= 2x(t) - y(t), x(0) = 1.2 \\ {}_0^C D_t^\alpha y(t) &= 4x(t) - 3y(t), y(0) = 4.2 \end{aligned}$$

Let

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$$

The eigenvalues of A are

$$\lambda_1 = 1 \quad \textit{and} \quad \lambda_2 = -2$$

The analytical solution is given by

$$\begin{aligned} x(t) &= 0.2E_\alpha(t^\alpha) + E_\alpha(-2t^\alpha) \\ y(t) &= 0.2E_\alpha(t^\alpha) + 4E_\alpha(-2t^\alpha) \end{aligned}$$

The numerator and denominator functions are given by

$$\psi_k = \frac{\left[\dfrac{E_\alpha(-2h^\alpha(k+1)^\alpha)}{E_\alpha(-2h^\alpha k^\alpha)} + \dfrac{2E_\alpha(h^\alpha(k+1)^\alpha)}{E_\alpha(h^\alpha k^\alpha)}\right]}{3}$$

and

$$\phi_k = \frac{\left[\dfrac{E_\alpha(h^\alpha(k+1)^\alpha)}{E_\alpha(h^\alpha(k)^\alpha)} - \dfrac{E_\alpha(-2h^\alpha(k+1)^\alpha)}{E_\alpha(-2h^\alpha k^\alpha)}\right]}{3}.$$

The NSFD scheme from ((24)) and (25) as

$$\frac{x_{k+1} - \varphi x_k}{\phi} = 2x_k - y_k$$

$$\frac{y_{k+1} - \varphi y_k}{\phi} = 4x_k - 3y_k$$

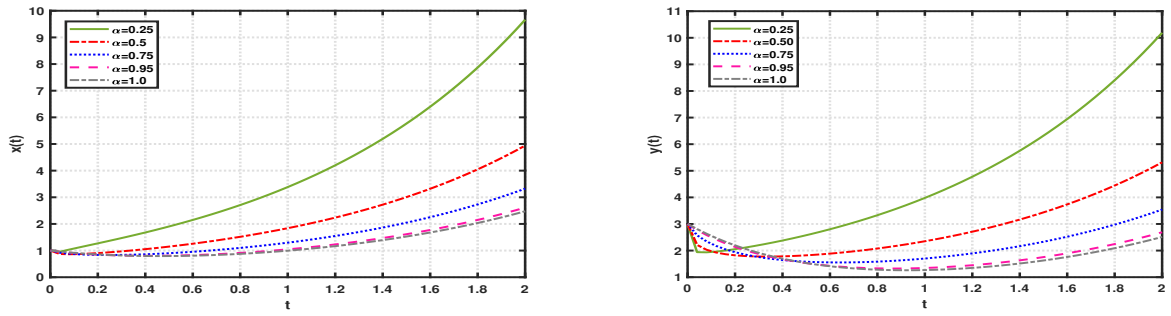
We solved the problem for $\alpha = 0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0 with step sizes $h = 0.0625, 0.1, 0.125, 0.2, 0.25, 0.4, 0.5, 1$ and 2.0. The infinity norm error corresponding to each couple h and α is illustrated in Table 1.

Table 1. The infinity norm errors obtained by the NSFD scheme (24)-(25) for Example (6.1), for different step sizes h and orders α.

$h \backslash \alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.0625	4.97E-14	5.33E-15	1.78E-15	7.11E-15	2.22E-15	7.99E-15	8.88E-16	6.22E-15
0.1000	1.42E-14	8.88E-15	2.66E-15	7.99E-15	4.88E-15	1.33E-15	3.11E-15	1.78E-15
0.1250	4.26E-14	3.55E-15	4.44E-15	6.22E-15	2.22E-15	8.88E-16	1.33E-15	3.11E-15
0.2000	1.42E-14	1.78E-15	5.33E-15	2.66E-15	1.33E-15	1.78E-15	6.66E-16	6.66E-16
0.2500	2.84E-14	7.11E-15	5.33E-15	2.66E-15	1.78E-15	6.66E-16	6.66E-16	1.78E-15
0.4000	2.13E-14	3.55E-15	1.78E-15	8.88E-16	2.66E-15	4.44E-16	6.66E-16	4.44E-16
0.5000	2.84E-14	3.55E-15	3.55E-15	8.88E-16	1.33E-15	6.66E-16	8.88E-16	4.44E-16
1.0000	1.42E-14	3.55E-15	1.78E-15	2.66E-15	2.22E-16	1.55E-15	1.33E-15	4.44E-16
2.0000	2.13E-14	1.07E-14	1.78E-15	8.88E-16	1.78E-15	4.44E-16	4.44E-16	3.11E-15

Although the system is unstable, Table 1 demonstrates a steady performance of the numerical scheme, where the infinity norm errors computed by the numerical scheme (24)-(25) are of order 10^{-14} or less, including large step sizes. This indicates that the nonstandard finite difference scheme is almost exact and is highly stable.

The solution of Example 6.1, obtained from the NSFDM (24)-(25) for $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0,2]$, is shown in Figure 1. The values of the variables x_k and y_k for $k = 0, ..., N$ are illustrated in Figure 1.



$x(t)$ $y(t)$

Figure 1. The solution of Example 6.2 for $\alpha = 0.25, 0.50, 0.75, 0.95$ and 1.00

From Figure 1, we notice that the presence of the positive eigenvalue $\lambda_1 = 1$ caused the solution components $x(t)$ and $y(t)$ to grow towards $+\infty$ as t increases, therefore the system is unstable.

Example 6.2
Consider the linear system of FDEs

$$
\begin{aligned}
{}_0^C D_t^\alpha x(t) &= x(t) + 5y(t), \; x(0) = 1, t \in [0,8] \\
{}_0^C D_t^\alpha y(t) &= -5x(t) - 9y(t), \; y(0) = 3, t \in [0,8]
\end{aligned}
$$

In this example, A is the matrix

$$
A = \begin{pmatrix} 1 & 5 \\ -5 & -9 \end{pmatrix},
$$

whose eigenvalues are:

$$
\lambda_1 = \lambda_2 = -4
$$

The eigenvector $\boldsymbol{v}_1$ and generalized eigenvector $\boldsymbol{v}_2$ (obtained by solving $(A - \lambda I)\, \boldsymbol{v}_2 = \boldsymbol{v}_1$) corresponding to $\lambda = -4$ are

$$
\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}
$$

The exact solution of Example 6.2 is given by:

$$
\begin{aligned}
x(t) &= (1 + 20t^\alpha) E_\alpha(-4t^\alpha) \\
y(t) &= (3 - 20t^\alpha) E_\alpha(-4t^\alpha)
\end{aligned}
$$

The NSFD scheme from (32) and (33) as

$$
x_{k+1} = \psi_k x_k - \phi_k (x_k + 5y_k)
$$

$$
y_{k+1} = \psi_k y_k + \phi_k (-5x_k - 9y_k)
$$

where

$$
\psi_k = (1 + 4h^\alpha ((k+1)^\alpha - (k)^\alpha)) \frac{E_\alpha(-4h^\alpha (k+1)^\alpha)}{E_\alpha(-4h^\alpha k^\alpha)}
$$

and

$$
\phi_k = h^\alpha ((k+1)^\alpha - (k)^\alpha) \frac{E_\alpha(-4h^\alpha (k+1)^\alpha)}{E_\alpha(-4h^\alpha k^\alpha)}
$$

We let α takes the values $0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0, and h takes the values $0.1, 0.125, 0.2, 0.25, 0.4, 0.5, 0.8, 1.0, 2.0, 4.0$ and 8.0. By applying the proposed nonstandard finite difference scheme, we computed the infinity norm errors, defined as

$$\|Error\|_{\infty}^{\alpha,h} = \max_{k=0,\ldots,N} \max \{|x_k - x(t_k)|, |y_k - y(t_k)|\}.$$

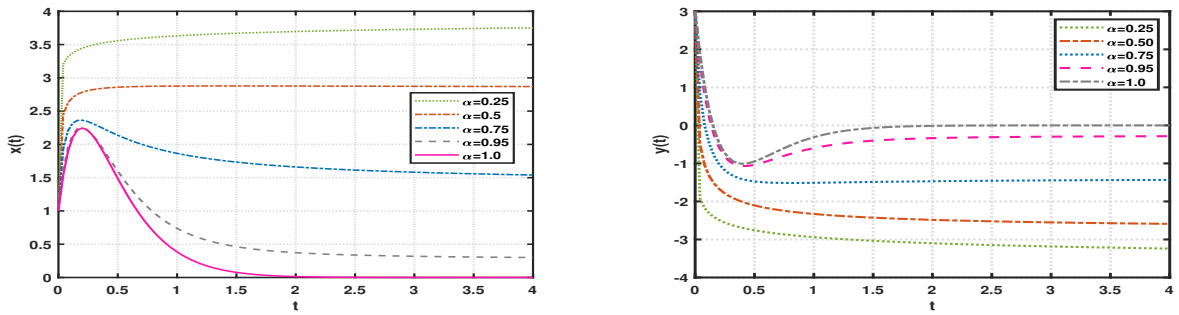
These errors are illustrated in Table 2.

Table 2. The infinity norm error obtained by the NSFDM (32) and (33) for Example 6.2, for different step sizes h and orders α

$h\backslash\alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.25	3.55E-15	4.88E-15	3.55E-15	4.00E-15	3.11E-15	1.20E-14	1.17E-15	2.22E-16
0.40	2.66E-15	3.11E-15	5.77E-15	5.33E-15	1.11E-14	1.04E-14	1.05E-15	1.10E-15
0.50	5.33E-15	4.00E-15	3.11E-15	2.66E-15	2.22E-15	2.00E-15	1.44E-15	2.22E-16
0.80	3.55E-15	1.33E-15	3.11E-15	4.44E-15	3.55E-15	4.00E-15	6.66E-16	7.62E-16
1.00	8.88E-16	1.78E-15	1.02E-14	7.55E-15	2.22E-15	4.00E-15	6.66E-16	1.08E-19
2.00	8.88E-16	1.78E-15	8.88E-16	3.55E-15	2.66E-15	3.11E-15	4.44E-16	6.78E-21
4.00	0.00E+00	8.88E-16	8.88E-16	2.66E-15	4.44E-15	0.00E+00	2.22E-16	0.00E+00
8.00	8.88E-16	4.44E-16	8.88E-16	4.44E-16	0.00E+00	2.22E-16	0.00E+00	0.00E+00

Many of the infinity norm errors obtained by (32)-(33) that are shown in Table 2 lie within the machine precision. The other infinity norm errors computed by the numerical scheme are of order 10^{-16} or less, including large step sizes such as $h = 3$ and $h = 1.5$. This indicates that the nonstandard finite difference scheme is almost exact and is highly stable.

The solution of Example 6.2, obtained from the NSFDM (32)-(33) for $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0,4]$, is shown in Figure 2. The values of the variables x_k and y_k for $k = 0, \ldots, N$ are illustrated in Figure 2.



$x(t)$ $y(t)$

Figure 2. The solution of Example 6.2 for $\alpha = 0.25, 0.50, 0.75, 0.95$ and 1.00

Figure 2 illustrates that the solution components $x(t)$ and $y(t)$ of the linear system in Example 6.2 exhibit stability, as the recurrent eigenvalue $\lambda = -4$ is negative.

Example 6.3
Consider the linear system of FDEs

$$\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= -x(t), x(0) = 1, \\ {}_0^C D_t^{\alpha} y(t) &= -y(t), y(0) = 3. \end{aligned}$$

The eigenvalues of the system are given by

$$\lambda_1 = \lambda_2 = -1$$

Since

$$A - \lambda I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then the eigenvectors corresponding to $\lambda = -1$ are

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, the solution of Example 6.3 is given by:

$$x(t) = E_\alpha(-t^\alpha)$$
$$y(t) = 3E_\alpha(-t^\alpha)$$

The NSFD scheme from (28) and (29) as

$$\frac{x_{k+1} - x_k}{1 - \left(\dfrac{E_\alpha(-(hk+h)^\alpha)}{E_\alpha(-(hk)^\alpha)}\right)} = -x_k$$

$$\frac{y_{k+1} - y_k}{1 - \left(\dfrac{E_\alpha(-(hk+h)^\alpha)}{E_\alpha(-(hk)^\alpha)}\right)} = -y_k$$

We let α takes the values $0.05, 0.2, 0.35, 0.5, 0.65, 0.8, 0.95$ and 1.0, and h takes the values $0.05, 0.1, 0.125, 0.25, 0.5, 1.0, 1.25, 2.5$ and 5.0. By applying the proposed nonstandard finite difference scheme (28)-(29), we computed the infinity norm errors, defined as

$$\|Error\|_\infty^{\alpha,h} = \max_{k=0,\ldots,N} \max \{|x_k - x(t_k)|, |y_k - y(t_k)|\}.$$

These errors are illustrated in Table 3.

Table 3. The infinity norm errors obtained by the NSFD (28)-(29) for different step sizes h and values of α.

$h\backslash\alpha$	0.05	0.20	0.35	0.50	0.65	0.80	0.95	1.00
0.50	0.00E+00	0.00E+00	0.00E+00	2.22E-16	1.11E-16	2.22E-16	5.55E-17	2.22E-16
1.00	0.00E+00	0.00E+00	0.00E+00	1.11E-16	0.00E+00	2.22E-16	5.55E-17	1.11E-16
1.25	0.00E+00	0.00E+00	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16
1.50	2.22E-16	2.22E-16	1.11E-16	2.22E-16	1.11E-16	1.11E-16	2.22E-16	2.22E-16
2.50	2.22E-16	0.00E+00	2.22E-16	2.22E-16	1.11E-16	2.22E-16	2.78E-16	2.78E-16
3.00	0.00E+00	2.22E-16	2.22E-16	1.11E-16	1.11E-16	1.67E-16	1.67E-16	1.11E-16
5.00	0.00E+00	2.22E-16	4.44E-16	5.55E-17	2.22E-16	1.11E-16	9.02E-17	2.01E-16
15.00	2.22E-16	2.22E-16	1.11E-16	1.67E-16	1.11E-16	2.50E-16	1.46E-16	8.56E-17

We notice in Table 3 that all the infinity norm errors obtained by the NSFDM (28)-(29) for Example 6.3 lie within the machine precision, for all values of h and α. This shows that the proposed NSFDM solved Example 6.3 exactly.

The solution of Example 6.3, obtained from the NSFDM (28)-(29) for $\alpha = 0.25, 0.50, 0.75, 0.95$, and 1.00, with $t \in [0,5]$, is shown in Figure 5. The values of the variables x_k and y_k for $k = 0, ..., N$ are illustrated in figures 3 and 4 respectively.

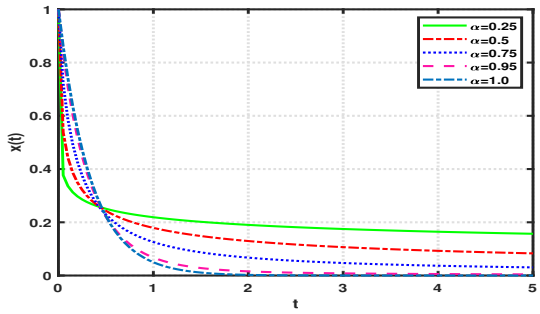


Figure 3. $x(t)$

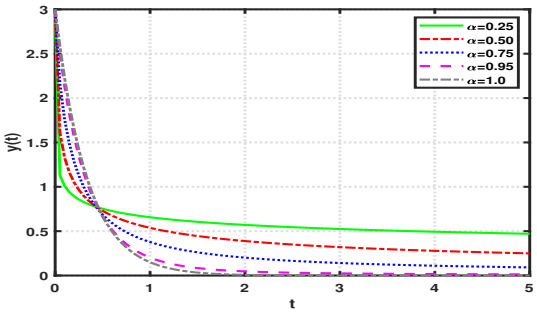


Figure 4. $y(t)$

Figure 5. The solution of Example 6.3 for $\alpha = 0.25, 0.50, 0.75, 0.95$ and 1.00

The solution components $x(t)$ and $y(t)$ from Example 6.3 demonstrate stability, as illustrated in Figure 5, partly due to the negative recurring eigenvalue $\lambda = -1$.

As demonstrated by the numerical examples, the proposed method exhibits a good degree of numerical stability, maintaining consistent performance across a wide range of fractional orders α and time step sizes h, even though robustness to noise was not a primary focus of this study. Future work will involve simulations with perturbed initial conditions to evaluate this aspect more comprehensively.

It is worth noting that the fractional order α plays a crucial role in determining the system's behavior, smaller values lead to more diffusive dynamics, while values approaching one recover the behavior of classical integer-order systems. From an implementation perspective, the nonstandard finite difference methods presented in this research need suitable choices of time step sizes (neither too small nor very large). Choosing an exceedingly small step size may greatly influence the method's accuracy, however it does not affect its stability, owing to the accumulation of round-off errors. Conversely, using an excessively high step size obscures the solution behavior between the mesh points.

The most apparent challenges to the proposed exact nonstandard finite difference methods arise when at least one eigenvalue of the linear fractional system is positive. The approach exhibits instability if any eigenvalue is positive, as shown in the numerical stability section. Despite the eigenvalues of the linear fractional system in Example 1 being positive, the method's performance has been quite satisfactory, with the highest infinity norm error on the order of 10^{-14}. Typically, methods do not ensure the attainment of minimal errors for linear fractional systems with positive eigenvalues.

7. Conclusions

This paper has introduced non-standard finite difference methods for solving a Caputo-type fractional linear system, characterized by equations with real eigenvalues. These eigenvalues are distinguished into unique and repeated types, with repeated eigenvalues further classified based on the dimension of their corresponding eigenspace. For each of these scenarios, exact solutions were derived, and the numerator and denominator functions for the non-standard finite difference scheme were developed.

The convergence of each proposed numerical method was proven by proving consistency and stability. We asserted that the methods remain robust against changes in the step size h and fractional order α. They also maintain resilience to changes in the system coefficients a, b, c, and d, provided that none of the two eigenvalues changes its

sign from negative to positive. This assures that the proposed methods are unconditionally stable when the system's eigenvalues are negative.

Letting N represent the total number of mesh points. At iteration k, each of the three generated NSFDMs necessitates 3 multiplications to compute either the numerator or denominator functions ψ_k and ϕ_k, along with 2 multiplication operations for scheme application at each step. This results in 8 multiplications every iteration throughout N iterations, yielding a cumulative total of $8N$ multiplications. The temporal complexity is minimal and similar to the Improved Euler's approach, which requires $8N$ multiplication operations.

The performances of the proposed nonstandard finite difference methods are illustrated using three numerical examples.

Table 1 shows the infinity norm error of the NSFDM (24)-(25), which is designed for linear systems of two linear FDEs. Several values of the step sizes h and orders α were used to demonstrate these errors. We see that the largest resulting error is of order 10^{-14} and the least is of order 10^{-16}, which is close to the machine precision.

Figure 1 illustrates the NSFD solutions with a step size $h = 0.04$ and orders $\alpha = 0.25, 0.5, 0.75, 0.95$ and 1.0. The existence of a positive eigenvalue of the system, caused its components to grow unbounded, indicating that the system of linear FDE in Example 6.1 is unstable. These results agree with those found in [21].

Table 2 illustrates the infinity norm error derived from solving Example 6.2 using the NSFDM (32)-(33) across various step sizes h and orders α. The maximum error is on the order of 10^{-15}, while the minimum is 0.00, indicating that the suggested NSFDM resolved the issue precisely, particularly for high step sizes. We see that at substantial step sizes, namely $h = 2, 4$, and 8, the NSFDM yielded few mistakes, demonstrating the method's stability.

In Figure 2, we set a step size of $h = 0.04$ and then show the solution components $x(t)$ and $y(t)$ derived from the NSFDM (32)-(33) for $\alpha = 0.25, 0.5, 0.75, 0.95$ and 1.0. It is evident that the solution remains stable due to the negative value of the repeated eigenvalue λ. The solution for $\alpha = 1$ confirms that the NSFDM encompasses the first-order ODE scenario.

Example 6.3 is an uncoupled system of two linear FDEs. We solved the example using the NSFDM (28)-(29). Table 3 shows the infinity norm errors obtained by the scheme for several step sizes h and orders α. All errors included in the table are within machine precision $\varepsilon = 2.2204 \times 10^{-16}$, indicating that the proposed method is exact. This result agrees with those in [3].

Figure 5 illustrates the components of the solution $x(t)$ and $y(t)$ obtained from NSFDM (32)-(33) for values of $\alpha = 0.25, 0.5, 0.75, 0.95$ and 1.0. The solutions converge to a stable level for each value of α, due to the negative value of the eigenvalue $\lambda = -1$. For $\alpha = 1$, the solutions converge to $x^* = y^* = 0$, which indicates that the proposed numerical methods extend the classical systems of two first-order ODEs.

In our implementation of the exact finite difference schemes, tables 1-3 show that the numerical methods work consistently, no matter what step size or fractional order is utilized in the simulations. The methods can obtain accurate solutions (within the level of machine precision) for large step sizes such as $h = 2, 4$ and 8. A standard finite-difference method or any other standard numerical method can never have such a feature.

The most apparent challenges to the proposed exact nonstandard finite difference methods arise when at least one eigenvalue of the linear fractional system is positive. The approach exhibits instability if any eigenvalue is positive, as shown in the numerical stability section. Despite the eigenvalues of the linear fractional system in Example 1 being positive, the performance of the method has been quite satisfactory, with the highest infinity norm error on the order of 10^{-14}. Typically, methods do not ensure the attainment of minimal errors for linear fractional systems with positive eigenvalues.

In future research, we plan to develop exact non-standard finite difference methods to solve linear systems of FDEs with complex eigenvalues, nonlinear FDEs, FDEs of order $1 < \alpha \leq 2$, and classes of fractional partial differential equations (PDEs).

REFERENCES

1. O. Abdulaziz, I. Hashim, and S. Momani. Solving systems of fractional differential equations by homotopy-perturbation method. *Physics Letters A*, 372(4):451–459, jan 2008.

2. E. Ahmed, A.M.A. El-Sayed, and H.A.A. El-Saka. Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models. *Journal of Mathematical Analysis and Applications*, 325(1):542–553, jan 2007.
3. SAID AL KATHIRI, EIHAB BASHIER, NUR NADIAH ABD HAMID, and NORSHAFIRA RAMLI. Development of a non-standard finite difference method for solving a fractional decay model. *Journal of applied mathematics & informatics*, 42(3):695–708, 2024.
4. Samah Ali and Eihab Bashier. A nonstandard finite difference method for solving the fractional logistic model. *Journal of Applied Mathematics and Computational Mechanics*, 23(4):18–29, December 2024.
5. Dumitru Baleanu, Sadegh Zibaei, Mehran Namjoo, and Amin Jajarmi. A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system. *Advances in Difference Equations*, 2021(1), jun 2021.
6. Sachin Bhalekar and Varsha Daftardar-Gejji. Solving fractional-order logistic equation using a new iterative method. *International Journal of Differential Equations*, 2012:1–12, 2012.
7. Sebastián Buedo-Fernández and Juan J. Nieto. Basic control theory for linear fractional differential equations with constant coefficients. *Frontiers in Physics*, 8, September 2020.
8. Varsha Daftardar-Gejji and Hossein Jafari. Adomian decomposition: a tool for solving a system of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 301(2):508–518, jan 2005.
9. K. Diethelm. Efficient solution of multi-term fractional differential equations using p(ec)m e methods. *Computing*, 71(4):305–319, November 2003.
10. Kai Diethelm and Walz Guido. Numerical solution of fractional order differential equations by extrapolation. *Numerical algorithms, Vol. 16, No. 3, p. 231-253*, 1997.
11. Rudolf Gorenflo, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications: Theory and Applications*. Springer Berlin Heidelberg, 2014.
12. Mojtaba Hajipour, Amin Jajarmi, and Dumitru Baleanu. An efficient nonstandard finite difference scheme for a class of fractional chaotic systems. *Journal of Computational and Nonlinear Dynamics*, 13(2), dec 2017.
13. Sergey Lemeshevsky. *Exact Finite-Difference Schemes.* De Gruyter, Inc., Berlin/Boston, 2016. Description based on publisher supplied metadata and other sources.
14. Denis Matignon. Stability results for fractional differential equations with applications to control processing. *In Computational Engineering in Systems Applications*, 1996.
15. Mohammad Amirian Matlob and Yousef Jamali. The concepts and applications of fractional order differential calculus in modeling of viscoelastic systems: A primer. *Critical Reviews in Biomedical Engineering*, 47(4):249–276, 2019.
16. Ronald E. Mickens. Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis. *Numerical Methods for Partial Differential Equations*, 5(4):313–325, December 1989.
17. Ronald E. Mickens. *Nonstandard finite difference models of differential equations*. World Scientific, 1994.
18. Ronald E. Mickens. *Applications of Nonstandard Finite Difference Schemes*. World Scientific Publishing Company, 2000.
19. K. Moaddy, I. Hashim, and S. Momani. Non-standard finite difference schemes for solving fractional-order rössler chaotic and hyperchaotic systems. *Computers and amp mathsemicolon Mathematics with Applications*, 62(3):1068–1074, aug 2011.
20. Shaher Momani and Kamel Al-Khaled. Numerical solutions for systems of fractional differential equations by the decomposition method. *Applied Mathematics and Computation*, 162(3):1351–1365, mar 2005.
21. Zaid M. Odibat. Analytic study on linear systems of fractional differential equations. *Computers and amp mathsemicolon Mathematics with Applications*, 59(3):1171–1183, feb 2010.
22. Mevlüde Yakıt Ongun, Damla Arslan, and Roberto Garrappa. Nonstandard finite difference schemes for a fractional-order brusselator system. *Advances in Difference Equations*, 2013(1), apr 2013.
23. Sakthivel Rathinasamy and Ren Yong. Approximate controllability of fractional differential equations with state-dependent delay. *Results in Mathematics*, 63(3–4):949–963, April 2012.
24. S. Z. Rida and A. A. M. Arafa. New method for solving linear fractional differential equations. *International Journal of Differential Equations*, 2011:1–8, 2011.
25. Lih-Ing W. Roeger. Exact finite-difference schemes for two-dimensional linear systems with constant coefficients. *Journal of Computational and Applied Mathematics*, 219(1):102–109, sep 2008.
26. Lih-Ing W. Roeger. Exact nonstandard finite-difference methods for a linear system – the case of centers. *Journal of Difference Equations and Applications*, 14(4):381–389, apr 2008.
27. Ndolane Sene. Sir epidemic model with mittag–leffler fractional derivative. *Chaos, Solitons & Fractals*, 137:109833, August 2020.
28. Beijia Wang, Liang Li, and Yaowu Wang. An efficient nonstandard finite difference scheme for chaotic fractional-order chen system. *IEEE Access*, 8:98410–98421, 2020.
29. Samaneh Soradi Zeid. Approximation methods for solving fractional equations. *Chaos, Solitons and amp mathsemicolon Fractals*, 125:171–193, aug 2019.
30. Mohammad Zurigat, Shaher Momani, Zaid Odibat, and Ahmad Alawneh. The homotopy analysis method for handling systems of fractional differential equations. *Applied Mathematical Modelling*, 34(1):24–35, jan 2010.