STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–25.
Published online in International Academic Press (www.IAPress.org)

A comparative study of Hilbert transform and Fourier transform methods to complex-based global minimum variance portfolio

Nurwahidah Nurwahidah [1,2], Mawardi Bahri [1,*], Amran Rahim [1]

[1] *Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia*
[2] *Department of Mathematics, Universitas Islam Negeri Alauddin Makassar, Gowa 92113, Indonesia*

Abstract Quantitative method in portfolio construction is an engaging issue in mathematical finance. A number of studies have shown the role of real numbers in constructing portfolio. However, very little attention has been paid to the role of complex number in finance. The principal objective of this project is to construct complex-based Global Minimum Variance (GMV) portfolio and apply clustering method in asset selection. The findings indicate that the GMV with Hilbert transform method has lower standard deviation in general than the real-based GMV portfolio. On the other hand, GMV portfolio approached by Fourier Transform shows higher standard deviation than complex-based portfolio with Hilbert transform and real-based portfolio. Our findings show how to develop GMV portfolio with Hilbert and Fourier Transform approach for constructing complex-based optimal portfolio.

Keywords Portfolio, Global Minimum Variance, Hilbert Transform, Optimization, Complex Number

AMS 2010 subject classifications: 62P05, 91G10.

DOI: 10.19139/soic-2310-5070-2586

1. Introduction

Risk diversification is a strategy in investment management that aims to reduce risk by spreading investments across different assets or financial instruments. Diversification can be done by investing in various financial assets or real assets. The more capital diversified, the lower risk taken in investing [7]. Thus, it is necessary to diversify the capital into several financial instruments to increase the return and minimize the risk [1].

The construction of investment portfolio is an attempt to diversify risk. A portfolio is a set of assets that can be combined to obtain the lowest risk with a certain return [12]. One method that can minimize risk is constructing a Global Minimum variance (GMV) portfolio [3]. A real-based GMV is designed to minimize risk and maintain expected returns. The real based GMV portfolio can perform risk and return efficiency which means that this portfolio offers the best combination of risk and expected return. By minimizing variance, investors can obtain certain value expected returns with lower risk than a portfolio that is not well-diversified.

Behind the advantages of the real-based GMV portfolio that can minimize risk well, it does not mean that the real-based GMV portfolio does not have disadvantages. The problem with real-based portfolios is that the expected return value in the variance-covariance matrix sometimes can not follow the data pattern and it cause biased. This is affected by high volatility stock price data. High volatility data makes the assumption of constant expected value no longer suitable. One way to handle high volatility data is doing transformation. Transformations supports stabilize the variance and reduce non-normality in the data, which in turn enhance the quality of modeling [15].

*Correspondence to: Mawardi Bahri (Email: mawardibahri@gmail.com). Department of Mathematics, Hasanuddin University. Makassar 90245, Indonesia.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Some researchers have applied Hilbert transformation to solve the problem of data volatility. Drummond, et al. [6] developed a volatility estimation method using Hilbert-Huang transform. Drummond, et al. [6] study is providing a more consistent and accurate solution in analyzing market volatility affected by micro noise. Besides, Kurbatsky, et al. [10] performed the Hilbert transform on highly volatile electricity usage data in Australia to calculate instantaneous amplitude and frequency at any time. This is in line with research conducted by Uchiyama, et al. [16] who performed Hilbert transformation on stock return data to complex numbers to form Maximum Risk Diversification (MRD) portfolio development. The transformation of stock return data to complex numbers is intended to capture dynamic information of time series data. Stock return data transformed to complex numbers can be more detailed and more adaptive in capturing the volatility of stock price movements. The results of Uchiyama's research [16] show that complex-based MRD portfolio is better than real-based MRD portfolio. In additions, [13] also establishes fundamental theory of expected value, variance, and covariance in quaternions that can be applied in portfolio development.

Since the last few decades, Fourier transform has become an important tool in mathematical finance, especially in the portfolio construction. The Fourier transform can be used to decompose the variance, correlation, alpha, and beta of asset returns into separate frequency components as in [4]. This approach allows the construction of frequency-based optimal portfolio that can increase the effectiveness of investment strategies. In addition, the Fourier transform is also combined with the Quasi-Monte Carlo (QMC) method to improve the Value-at-Risk (VaR) efficiency of the portfolio as in [8]. This study shows that the Fourier transform can make the QMC method reach convergence faster thus improving the efficiency of VaR portfolio simulation. Besides, the Fourier transform is applied to calculate the Fourier coefficients of volatility, which allows to obtain an estimate of the volatility spectrum as in [5]. This is useful for analyzing price fluctuations and risk in financial markets. Furthermore, the Fourier transform can be used as a tool to understand the frequency dynamics of market shocks as in [18]. This is particularly useful in the construction of effective portfolio by identifying the response of different assets to market shocks over time. The use of Fourier transform in the study allows investors to optimize portfolio weights for improved diversification strategies among QUAD country markets.

Based on the deep analysis results of the CVRD portfolio, it is found that the portfolio does not consider heterogeneity in choosing portfolio assets. Heterogeneity causes differences in the characteristics of the assets invested. Clustering techniques in portfolio construction reduce the time used in asset selection, because assets with similar characteristics are categorized in one cluster [17]. Park [14] applied K-Means clustering in constructing real based GMV. Gubu, et al. [11] also applied clustering method on the classical portfolio. Thus, this study will apply the clustering method in selecting asset on the portfolio construction. This study aims to construct complex-based GMV portfolio with clustering method in asset selection. Hilbert and Fourier transformation methods will be employed to convert asset return data into complex numbers in this study. The transformed data will be used to construct GMV portfolios, followed by a comparative analysis of the two transformation approaches in terms of expected return, standard deviation, and Sharpe ratio performance of the portfolios.

This research consists into four main sections. Section 1 presents the background of the problem and some review of relevant prior studies. Section 2 outlines the theoretical framework and methodologies employed to address the research questions. Section 3 details the results and provides a comprehensive discussion, while Section 4 concludes the study with a summary of key findings.

2. Materials and Method

2.1. Return

Return is defined as the capital gain or lost over certain investment period. Mathematically, the return of an asset is expressed as:

$$R_{t_i} = \frac{p_{t_i} - p_{t-1_i}}{p_{t-1_i}}, \tag{1}$$

where R_{t_i} denotes the return of asset i at time t, p_{t_i} represents the current closing price asset i at time t, and p_{t-1_i} denotes the previous closing asset price i.

2.2. Discrete Hilbert Transform

Discrete Hilbert Transform is a transformation that is often used in signal processing, communications, and image processing. This transformation is introduced for performing a phase shift of 90 degrees to original signal. Following [16], the Hilbert transform of a sequence y_m is defined by:

$$\mathcal{H}_D[y_k] = -i \ \text{sgn}\left(k - \frac{M}{2}\right) \sum_{m=0}^{M-1} y_m e^{\frac{2\pi i m}{M}}. \tag{2}$$

In this case, k denotes discrete frequency index, i denotes imaginary unit, $sgn(.)$ denotes sign function, M denotes number of points in the discrete Fourier transform, and y_m denotes the mth element of the sequence.
The analytical signal s_t is obtained by applying the Hilbert transform in (2) to the return of asset in (1)

$$s_t = R_{t_i} + i\mathcal{H}_D[R_{t_i}]. \tag{3}$$

2.3. Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a mathematical technique for converting a sequence of discrete values over a specific time period from the time domain to the frequency domain. DFT transforms continuous (analog) signals into discrete values in the time domain, which are then translated into the frequency domain and processed digitally using microcontrollers or computers. The DFT is defined by:

$$Y_k = \sum_{m=0}^{M-1} y_m e^{-i2\pi \frac{k}{M} m} \tag{4}$$

2.4. Expected Value

The expected value of a random variable is also known as the arithmetic mean of a random variable.

Definition 2.1
If X is a random variable with probability density function $f(x)$, then the expected value of X is defined by:

$$E[X] = \begin{cases} \sum_x xf(x) & \text{if } X \text{is discrete random variable} \\ \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{is continuous random variable.} \end{cases}$$

2.5. Covariance Matrix

The covariance matrix is a basic concept in statistics and data analysis. It is a square matrix that contains all the covariance values of each pair random variables in multivariate data. The covariance matrix of random variable X is defined by

$$\mathbf{\Sigma} = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & Var(X_2) & \cdots & Cov(X_2, X_n) \\ \vdots & \ddots & \vdots & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & \cdots & Var(X_n) \end{bmatrix}. \tag{5}$$

In the covariance matrix $Var(X_a)$ is defined as variance of random variable X_a. The formula of variance is given by

$$Var(X_a) = E[(X_a - E[X_a])(X_a - E[X_a])^T], a = 1, 2, \dots, n. \tag{6}$$

In this case, T is transpose of the matrix.
Furthermore $Cov(X_a, X_b)$ is defined as covariance of random variable X_a and X_b. The formula of covariance is given by

$$Cov(X_a, X_b) = E[(X_a - E[X_a])(X_b - E[X_b])^T], a = b = 1, 2, \dots, n. \tag{7}$$

2.6. K-Means Clustering

Clustering is a methods used to find and categorize high similarity data in a cluster. The study that have been conducted by Wu et al. [17] applied K-Means clustering in portfolio construction. K-Means clustering is a non-hierarchical clustering method that categorize data into one or more cluster. Similar characteristics data are grouped in one cluster, different characteristics data are grouped in other clusters. The initial cluster center is chosen randomly, then the distance from the center to the data can be calculated using the following formula

$$D_{(ac)} = \sqrt{(X_{1a} - X_{1c})^2 + (X_{2a} - X_{2c})^2 + \cdots + (X_{ka} - X_{xc})^2}$$

The distance of data a to cluster center c is denoted by $D_{(ac)}$, data a in data attribute k is denoted by X_{ka}, and center point c in attribute k is denoted by X_{kc}. Recalculation of the distance from the cluster center to the data in each cluster is done until there is no more significant change.

2.7. Sharpe Ratio

Portfolio performance evaluation is one of the most important aspects in the investment decision process. Thus, the evaluation of portfolio performance needs to be done every certain period. Sharpe ratio method is one of the portfolio performance assessment methods. The Sharpe Ratio measures investment returns in excess of the risk-free rate per unit of standard deviation. Mathematically, the the Sharpe ratio is defined by

$$S = \frac{R_p - R_f}{\sigma_p} \tag{8}$$

where R_p represents portfolio return, R_f represents risk-free asset, and σ_p represents standard deviation of portfolio. The greater the Sharpe index value, the better portfolio performance. Conversely, the smaller the Sharpe index value, the worse the portfolio performance [9].

2.8. Quadratic Programming

Quadratic Programming (QP) is an optimization approach to maximize or minimize a quadratic objective function with quadratic or linear inequality constraint function [20]. The general form of QP is given by

$$QP \begin{cases} \text{min} & \frac{1}{2}x^T \boldsymbol{H} x + c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{cases} \tag{9}$$

In the optimization problem (9), x denotes an $n \times 1$ vector of optimization variables, A denotes $n \times m$ matix of constraint coefficients, b represents an $m \times 1$ vector specifying right hand side of the constraints, and c is an $n \times 1$ vector consisting of linear term coefficient in objective function. Additionally, $\boldsymbol{H}$ is an $n \times n$ Hessian matrix which encapsulates the coefficients of the quadratic terms in the objective function.

2.9. Global Minimum Variance Portfolio

Global Minimum Variance (GMV) portfolio is one of the most popular development of the Markowitz portfolio. The GMV portfolio aims to minimize the variance which represents the risk measure of a portfolio. GMV is located at the left end of efficient frontier curve [19]. The efficient frontier is a curve that expresses the relationship between expected return and standard deviation. In other words, the GMV portfolio is a portfolio that allocate principal in several financial assets so that the portfolio risk is minimized. This portfolio is suitable for risk-averse investors and who are interested in reducing investment volatility, even though the return on investment is minimal. The optimization problem of GMV is given by

$$QP_{GMV}\begin{cases} \text{min} & f=\frac{1}{2}w^T\boldsymbol{H}w \\ \text{subject to} & \sum_{i=1}^{N} w_i = 1 \quad i=1,2,\ldots,N \\ & E[R_p]=\sum_{i=1}^{N} w_i E[R_i] \geq \overline{E[R_i]}. \\ & w \geq 0 \end{cases} \tag{10}$$

In the quadratic programming problem (10), $w(w_1, w_2, \ldots, w_N)$ denotes the weight of each asset in the portfolio, $E[R_p]$ denotes the expected value of portfolio, $E[R_i]$ denotes the expected value of each asset in the portfolio, and $\overline{E[R_i]}$ denotes the average of asset return in the portfolio. While $\boldsymbol{H}$ is Hessian matrix assumed to be positive definite for expressing that the variance is a convex function. The weight of each asset can be solved by primal dual interior point method. The first step in determining asset weight in portfolio $(w_1, w_2, \ldots, w_N)$ is making a formula decomposition of (10) to form A, b, c, and $\boldsymbol{H}$ matrices. The formula decomposition of (10) is given by

$$QP_{GMV}\begin{cases} \text{min} & f=w_1^2\sigma_1^2+w_1w_2\sigma_{12}+\cdots+w_1w_N\sigma_{1N} \\ & +w_2w_1\sigma_{21}+w_2^2\sigma_2^2+\cdots+w_2w_N\sigma_{2N}+\cdots+w_Nw_1\sigma_{N1} \\ & +w_Nw_1\sigma_{N2}+\cdots+w_{N^2}\sigma_{N^2} \\ \text{subject to} & w_1+w_2+\cdots+w_N=1 \\ & E[R_1]w_1+E[R_2]w_2+\cdots+E[R_N]w_N-w_{N+1}=\overline{E[R_i]} \\ & w_1\geq 0, w_2\geq 0, \cdots, w_{N+1}\geq 0 \end{cases} \tag{11}$$

where w_{N+1} is an additional slack variable for converting inequality into equality.

Based on coefficient of the first and second constrains in (11), the A matrix is constructed as follows

$$A=\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ E[R_1] & E[R_2] & \cdots & E[R_N] & -1 \end{bmatrix}.$$

In this case, b is a vector consist of right hand side of the first and the second constraint coefficient in (11) constructed by

$$b=\begin{bmatrix} 1 \\ \overline{E[R_i]} \end{bmatrix}.$$

The Hessian matrix $\boldsymbol{H}$ is mathematically expressed as the following symmetric matrix

$$\boldsymbol{H}=\begin{bmatrix} 2\sigma_1^2 & 2\sigma_{12} & \cdots & 2\sigma_{1N} \\ 2\sigma_{21} & 2\sigma_2^2 & \cdots & 2\sigma_{2N} \\ \vdots & \ddots & \vdots & \vdots \\ 2\sigma_{N1} & 2\sigma_{N2} & \cdots & 2\sigma_N^2 \end{bmatrix}.$$

Lagrange function of system (11) is expressed by

$$L(w,v,\gamma)=\frac{1}{2}w^T\boldsymbol{H}w+v^T(b-Aw)-\gamma\sum_i ln(w_i) \tag{12}$$

where v represents Lagrange multiplier vector and γ denotes the barrier parameter coefficient. By deriving partially (12) with respect to x and v, then dual and primal feasibility condition are obtained in succession.

$$\boldsymbol{H}w-A^Tv-\gamma W^{-1}e=0 \tag{13}$$

$$b-Aw=0. \tag{14}$$

The complementary slackness conditions of primal dual interior point method is given by the following equation

$$WUe = \gamma e. \tag{15}$$

In (15) W is the diagonal matrix of x_i, U represents diagonal matrix of u_i, and e is matrix of ones.

The Newton method provides an iterative numerical approach for solving the system of nonlinear equations arising in quadratic programming. Given a current iterate consisting of the vectors w, u, and v, then the next iteration will be $w + d_w, u + d_u, v + d_v$. The value of d_w, d_u, d_v are determined by solving the following linear system.

$$\begin{bmatrix} A & 0 & 0 \\ -\boldsymbol{H} & I & A^T \\ U & W & 0 \end{bmatrix} \begin{bmatrix} d_w \\ d_u \\ d_v \end{bmatrix} = - \begin{bmatrix} Aw^k - b \\ -\boldsymbol{H}w + A^T v^k + u^k - c \\ WUe - \gamma^k e \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} d_w &= [X\boldsymbol{H} + U]^{-1}(XA^T d_v - Xr_d + r_c) \\ d_u &= X^{-1}(r_c - Ud_w) \\ d_v &= \big[[X\boldsymbol{H} + U]^{-1} XA^T\big]^{-1} \big[r_p + A[X\boldsymbol{H} + U]^{-1}(Xr_d - r_c)\big], \end{aligned}$$

where r_p, r_d, r_c is given by

$$\begin{aligned} r_p &= -Ax^k + b \\ r_d &= \boldsymbol{H}x^k - A^T v^k - u^k + c \\ rc &= -XUe + u^k e. \end{aligned}$$

Step length is the distance between the initial value w, u, v and the updated value of w, u, v. Formally, the step length is expressed by

$$\begin{aligned} \alpha_p &= \beta Min[1, -\frac{w_i}{d_w i}, d_{wi} < 0], \\ \alpha_d &= \beta Min[1, -\frac{u_i}{d_u i}, d_{ui} < 0]. \end{aligned}$$

The parameter β is a scalar or multiplier that aims to determine the number of portions of the previous direction that will be added to the process of determining the new value. The parameter β value is between 0 and 1. The β value that often used is 0.999. The updated value of w, u, and v is given by

$$\begin{aligned} w^{k+1} &= w^k + \alpha_p d_w \\ u^{k+1} &= u^k + \alpha_d d_u \\ v^{k+1} &= v^k + \alpha_d d_v. \end{aligned}$$

The superscript k represents the k-th iteration.

The optimization algorithm stops when the following three convergence conditions are simultaneously satisfied.

Primal feasibility

$$\sigma_p = \frac{\|Aw^k - b\|}{\|b\| + 1} \leq \epsilon_1$$

Dual feasibility

$$\sigma_p = \frac{\|r_d\|}{\|\boldsymbol{H}x + c\| + 1} \leq \epsilon_2$$

Complementary slackness

$$\gamma^k = \frac{(w^k)^T u}{n}$$

where $r_d = \boldsymbol{H} w^k - A^T v^k - u^k + c$ and $\gamma^k = \frac{(w^k)^T u}{n}$. The notation of ϵ_1, ϵ_2, ϵ_3 are predetermined small positive tolerance values and n denotes the dimension of optimization variable space.

2.10. Mean Variance Portfolio

Mean variance (MV) portfolio is also known as modern model portfolio or Markowitz portfolio. This investment model is developed by Harry Markowitz that explains how investors can maximize returns while minimizing risk by diversifying their portfolio [12]. The optimization problem of MV portfolio with no short-sale is given by

$$QP_{MV} \begin{cases} \min & f = \frac{1}{2} w^T \boldsymbol{H} w \\ \text{subject to} & \sum_{i=1}^{N} w_i = 1 \quad i = 1, 2, \ldots, N \\ & \sum_{i=1}^{N} w_i E[R_i] = R_p \\ & w \geq 0 \end{cases}. \tag{16}$$

In the optimization problem (16), the objective function is minimizing the variance f with 3 constraints. R_p in (16) is defined as target return of portfolio.

2.11. Research Procedure

The stages of this study are as follows:

1. Clustering the data based on standard deviation and transaction volume using K-Means clustering.
2. Determining the return (1) of each asset.
3. Calculating the expected return of each asset (5).
4. Selecting asset in each cluster based on the highest Sharpe ratio value (8).
5. Constructing real based mean variance and global minimum variance portfolio.
6. Transforming return data of the selected asset into complex number by discrete Hilbert and discrete Fourier transform (3).
7. Determining the variance return (6) and covariance return (7) values and forming the covariance matrix (5).
8. Constructing the complex based GMV portfolio (10) using interior point method to solve the quadratic programming problem.
9. Determining expected return, standard deviation and Sharpe ratio of portfolios.
10. Comparing the expected return, standard deviation, and Sharpe ratio of all constructed portfolios.

3. Main results

3.1. Data Descriptions

The portfolio selection is based on Indonesia Stock Exchange Industrial Classification data, especially Jakarta Islamic Index (JII) data. This study applied monthly closing price data of 30 issuers from January, 1, 2013 to December, 1, 2024. Besides that, this study also applied monthly closing price data of the 30 most traded commodities with the same time frame as the stock data. JII stocks were selected because they fulfill sharia principles and provide consistent market characteristics, while liquid commodities were applied because they provide stable and reliable data for technical analysis using Hilbert and Fourier transform methods. Furthermore, this study incorporates Indonesia interest rate decision as risk-free asset, with time frame corresponding to the assets data utilized in the analysis.

The monthly return data movement of the selected asset (Orange Juice Futures, US Sugar #11 Futures, ADRO, MDKA and INKP) during the period displays in Figure 1.

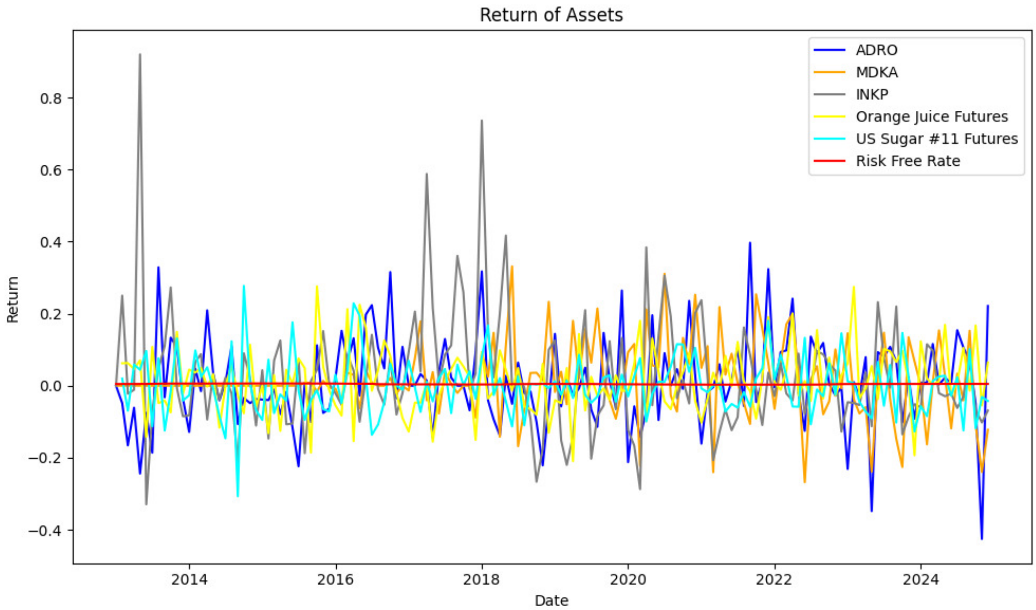

Figure 1. Monthly historical return

Based on the Figure 1, it can be seen that the returns of the six assets are very volatile. INKP has the highest data spike among all assets and ADRO has the lowest data spike. This result aligns with the characteristic analysis, which represents that stocks in Cluster 1 expose low price volatility with fluctuating standard deviation (ADRO), whereas stocks in Cluster 3 (INKP) show high price volatility. The result of expected return and standard deviation computation of each asset is contained in the Table 1.

Table 1. Expected return and standard deviation of each asset

Assets	$E[R_i]$	σ_i
ADRO	0.011029976	0.126261671
MDKA	0.015560837	0.1034473
INKP	0.027857707	0.168507397
Orange Juice Futures	0.014438854	0.089897864
US Sugar #11 Futures	0.003490062	0.080295915

Table 1 contains the expected return and standard deviation of five selected assets. INKP has the highest expected return (0.0279), but it also has the highest standard deviation (0.1685) among these assets. The expected return and standard deviation value of INKP indicate that its profit potential comes with a greater degree of risk. On the other hand, US Sugar #11 Futures showed the lowest expected return (0.0035), but it was accompanied by relatively less volatility (0.0803). The expected return and standard deviation value of US Sugar #11 Futures reflect the more stable yet less aggressive return-generating characteristics of defensive assets 1. Generally, the commodity assets such as Orange Juice Futures and US Sugar #11 Futures exhibit a lower risk profile than equity assets, even though with more moderate expected returns. Meanwhile, stocks such as MDKA shows a relatively optimal balance between expected return (0.0156) and risk (0.1034). The combination value of expected return and standard deviation making MDKA as attractive candidates in a portfolio that emphasizes Sharpe ratio efficiency. This data supports the significance of a quantitative approach in asset selection. An optimally diversified portfolio constructed by the trade-off between risk and return.

The heatmap in Figure 2 represents the correlation matrix between five assets; ADRO, MDKA, INKP, Orange

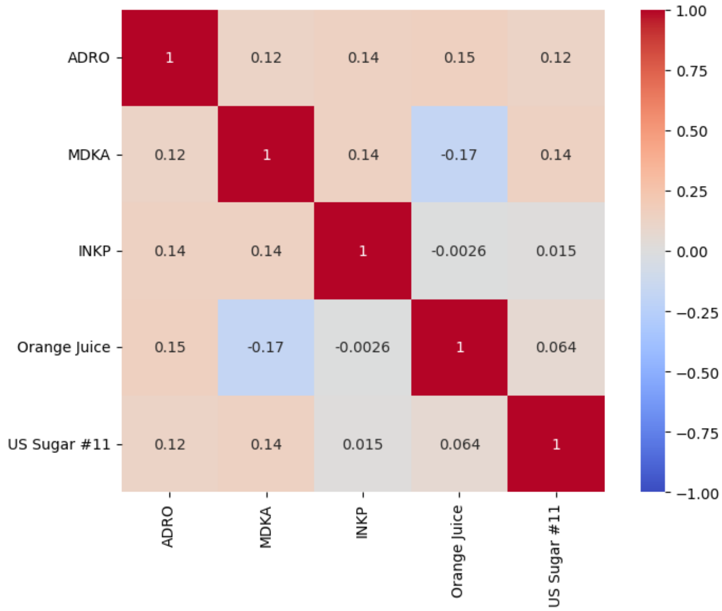

Figure 2. Heatmap of Asset Correlation

Juice, and US Sugar #11, which reflects the linear relationship between asset pairs. In general, the correlations between the assets are at low to moderate levels, with most values hovering around zero to 0.15. A notable negative correlation is found between MDKA and Orange Juice at -0.17, indicating a potential risk diversification contribution if both assets are included in one portfolio. INKP assets appear to have a very weak correlation relationship with Orange Juice and US Sugar #11, which could theoretically strengthen the minimum variance portfolio structure. Such correlation patterns are important as inputs in complex Hilbert and Fourier transform-based approaches for constructing global minimum variance portfolios.

3.2. Clustering Process

Asset selection process with K-Means clustering method is based on return standard deviation and average transaction volume. Standard deviation is used to measure volatility or changes in asset price movements while average transaction volume is used to describe the liquidity of asset or how easily the asset is sold and bought without affecting the asset price.

There are 9 outliers out of 30 commodity during the clustering process. These commodities are crude oil WTI futures (CL), fresh hen egg futures (DJDc1), gold futures (GC), lean hogs futures (LHc1), lumber futures (LXRc1), natural gas futures (NG), nickel futures (NICKEL), US cocoa futures (CC), and US corn futures (ZC). Furthermore, the number of clusters in this study is determined by Within Sum of Square (WSS) method, which optimizes the distance between cluster center data. The number of optimal clusters can be seen in the Figure 3.

Figure 3 represents Elbow Method in determining the optimal number of commodity clusters based on Total Within-Cluster Sum of Squares (WSS). The plot indicates a noticeable inflection point at $k = 2$. The elbow elbow proposes that partitioning the commodity set into two different clusters captures the essential underlying structure

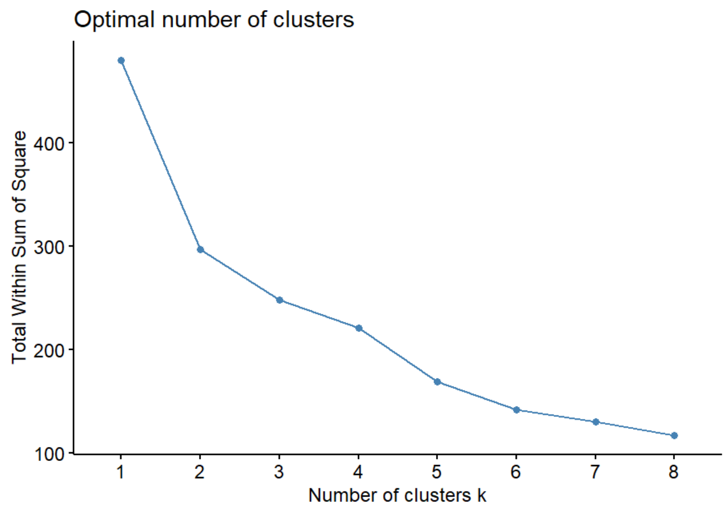

Figure 3. Optimal cluster of commodity

while avoiding over fitting and redundancy. From the value of k it can be conclude that 2 is chosen as the optimum number of stock cluster. The results of stocks clustering can be seen in Figure 4 below:

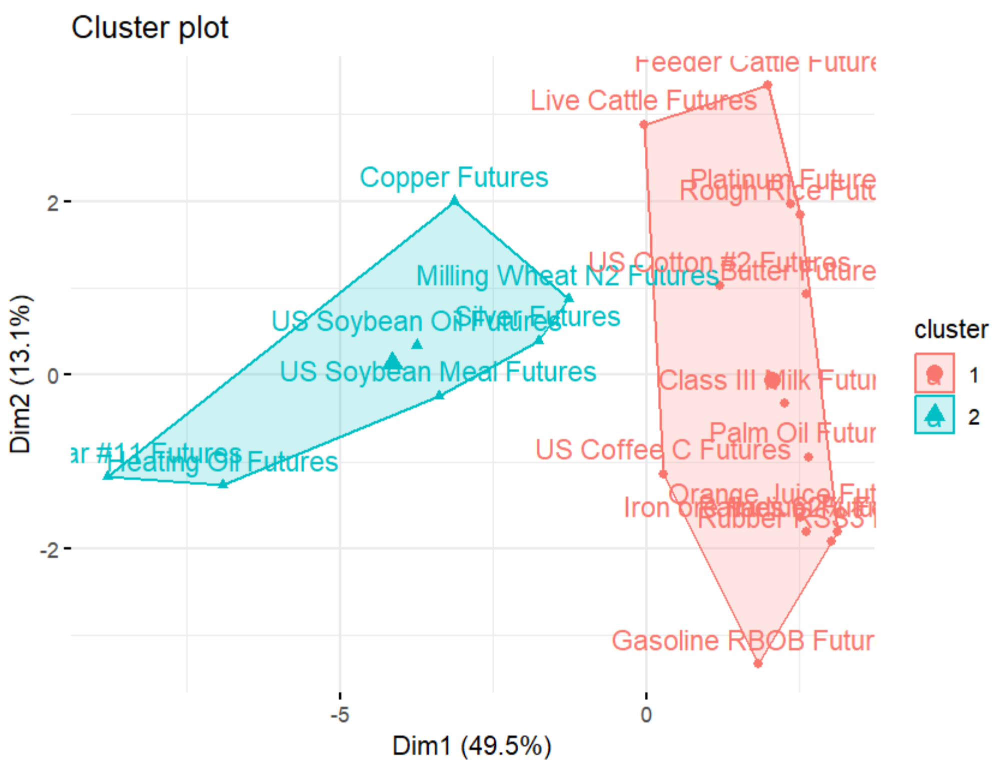

Figure 4. Result of commodity clustering

Based on Figure 4, it can be seen that the results of commodity clustering with the K-Means method are obtained 2 optimal clusters with each cluster members presented in Table 2 below. The first cluster is consist of 14 commodities, while the second cluster is consist of 7 commodities. Meanwhile, There are 8 out of 30 stocks listed in

Table 2. Member of each commodity cluster

Cluster	Number of members	Group of Stocks
1	14	Butter Futures (CBc1), Gasoline RBOB Futures (GPR) Iron ore fines 62% Fe CFR Futures (TIOc1), Palladium Futures (PA) Platinum Futures (PL), Rubber RSS3 Futures (SRUc1), US Coffee C Futures (KC) Class III Milk Futures (DCSc1), Feeder Cattle Futures (FC), Live Cattle Futures (LCc1), Orange Juice Futures (OJ), Palm Oil Futures (FCPOc1), Rough Rice Futures (RR), US Cotton #2 Futures (CT)
2	7	Copper Futures (HG), Milling Wheat N2 Futures (BL2c1), US Soybean Meal Futures (ZM), US Sugar #11 Futures (SB), Heating Oil Futures (NYF), Silver Futures (SI) US Soybean Oil Futures (ZL)

JII not included in the analysis because these stocks indicated outliers during the clustering process. These stocks are BRIS, BRMS, BRPT, ESSA, HRUM, SMGR, TLKM, and TPIA. Meanwhile Figure 5 displays the Elbow Method used to identify the optimal number of commodities clusters based on the Total Within-Cluster Sum of Squares (WSS). The graph demonstrate a visible bending at $k = 3$. Further increment in the clusters number only result minor reductions in WSS. This indicates that three clusters are adequate to interpret the primary structure of the data without irrelevant complexity. The curve's slower decrease long way off this point confirms that adding more clusters offers limited improvement in clustering quality. Therefore $k = 3$ is a statistically and practically efficient choice of cluster number.

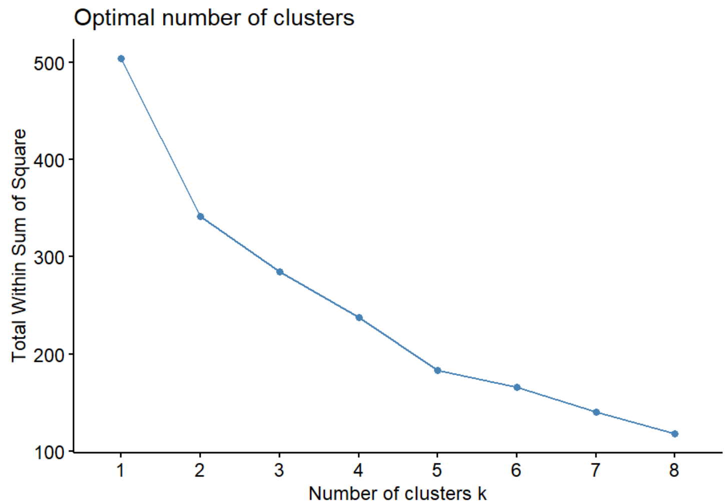

Figure 5. Optimal cluster of stock

Figure 6 represents the stocks clustering outcomes. The analysis shows a clear partition into three different clusters. Each cluster is characterized by unique spatial and structural properties. Cluster 1 (red) consists of stocks from the energy and mining sectors such as ANTM, PGAS, and ADRO. Cluster 2 (green) is the largest cluster that contain consumer goods, telecommunications, and health care sectors stocks. Cluster 3 (blue) includes industrial and basic material stocks such as INKP and INDY.

Referring to Figure 6, it can be seen that the results of stocks clustering with the K-Means method obtained 3 optimal clusters with each cluster members presented in Table 3 below.

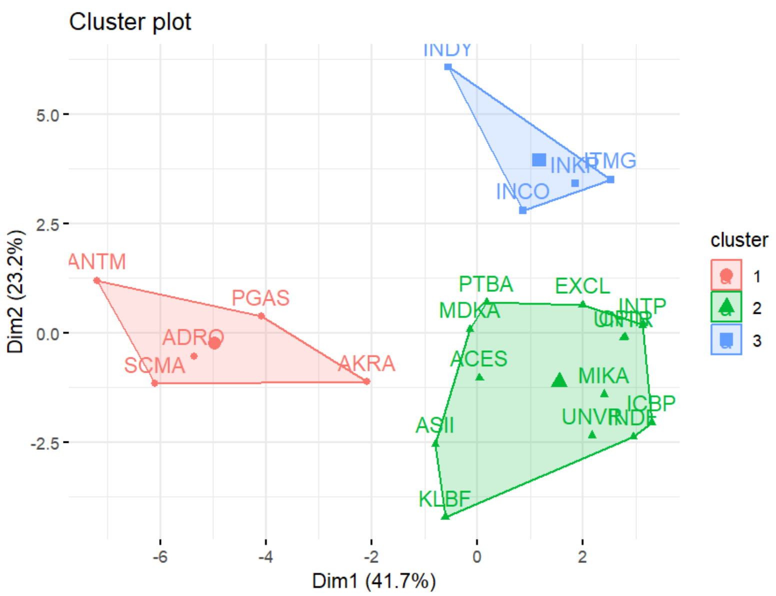

Figure 6. Result of stocks clustering

Table 3. Member of each stock cluster

Cluster	Number of members	Group of Commodity
1	5	ADRO, AKRA, ANTM, PGAS, SCMA
2	13	ACES, ASII, CPIN, EXCL, ICBP INDF, INTP, KLBF, MDKA, MIKA PTBA, UNTR, UNVR
3	4	INCO, INDY, INKP, ITMG

Data in this study is clustered based on return standard deviation and average transaction volume of the assets. The characteristics analysis result of each commodity cluster presented in Table 4 below:

Table 4. Result of commodity characteristics analysis

Cluster	Sd 2013	Sd 2014	Sd 2015	Sd 2016	Sd 2017	Sd 2018
1	0.1060766	-0.0543264	0.04778555	0.09256864	0.2550069	-0.0697972
2	-0.2121531	0.1086528	-0.09557110	-0.18513728	-0.5100139	0.1395944
Cluster	Sd 2019	Sd 2020	Sd 2021	Sd 2022	Sd 2023	Sd 2024
1	0.2958186	0.1481993	-0.03326674	0.0004548896	-0.1614830	0.02308501
2	-0.5916373	-0.2963986	0,67244	-0.0009097793	0.3229661	-0.04617003
Cluster	mean 2013	mean 2014	mean 2015	mean 2016	mean 2017	mean 2018
1	-0.5541464	-0.5735949	-0.5699479	-0.6026366	-0.6157273	-0.6126008
2	1.1082928	1.1471898	1.1398958	1.2052733	1.2314546	1.2252016
Cluster	mean 2019	mean 2020	mean 2021	mean 2022	mean 2023	mean 2024
1	-0.5956619	-0.6122596	-0.5725497	-0.5321325	-0.5688545	-0.6099631
2	1.1913238	1.2245191	1.1450994	1.0642650	1.1377090	1.2199262

Table 4 shows the characteristics of commodities data based on cluster analysis using the standard deviation of closing prices (Sd) and average transaction volume (mean). Cluster 1 consistently shows lower price volatility with standard deviations that close to zero during the analysis period. It is indicating stable price fluctuations. In contrast, cluster 2 expose more significant price varieties, with wider standard deviations and higher fluctuations than cluster 1. In terms of average transaction volume, cluster 1 exhibits constantly negative values. It is reflecting less trading activity or transaction volumes below the market average. On the other hand, cluster 2 reveals consistently positive and higher average transaction volume values. It is describes more active trading activity and transaction dominance in commodities within this group. This combination is explained significant differences in volatility characteristics and trading activity between the two clusters.

Table 5. Result of stock characteristics analysis

Cluster	Sd 2013	Sd 2014	Sd 2015	Sd 2016	Sd 2017	Sd 2018
1	-0.5050284	0.2398691	-0.2157952	0.04654022	-0.2302533	0.3113692
2	0.1386838	-0.3909179	-0.2174859	-0.38116124	-0.4302715	-0.5394380
3	0.1805632	0.9706469	0.9765732	1.18059876	1.6861991	1.3639620
Cluster	Sd 2019	Sd 2020	Sd 2021	Sd 2022	Sd 2023	Sd 2024
1	0.5309501	0.7294213	0.6998530	0.5989415	0.1780466	0.7421625
2	-0.6363920	-0.6101381	-0.5109491	-0.4958673	-0.3122322	-0.1874151
3	1.4045862	1.0711721	0.7857685	0.8628919	0.7921964	-0.3186041
Cluster	mean 2013	mean 2014	mean 2015	mean 2016	mean 2017	mean 2018
1	0.9140179	1.1960187	1.4396967	1.3214061	1.3534019	1.6332001
2	-0.1873229	-0.26265818	-0.3057304	-0.3809739	-0.4836286	-0.4657925
3	-0.5337228	-0.6413846	-0.8059972	-0.4135925	-0.1199595	-0.5276747
Cluster	mean 2019	mean 2020	mean 2021	mean 2022	mean 2023	mean 2024
1	1.5196026	1.5271477	1.4145319	1.5159600	1.0424193	1.2616570
2	-0.4050061	-0.3818686	-0.3816007	-0.3949131	-0.1588017	-0.2542029
3	-0.5832336	-0.6678617	-0.5279625	-0.6114824	-0.7869185	-0.7509120

Table 5 shows the standard deviation of commodity closing price and average transaction volume for each cluster. Cluster 1 have relatively low price volatility with a fluctuating standard deviation. However, cluster 1 remains smaller standard deviation than the other clusters in most periods, such as standard deviation value in 2013 (-0.5050284) and standard deviation value in 2023 (0.1780466). Based on the average transaction volume, cluster 1 expose relatively high and stable positive values, such as average transaction volume in 2013(0.9140179) and average transaction volume in 2024 (1.2616570). The high and stable positive average transaction volume is reflecting significant trading activity. Cluster 2 shows more negative price volatility pattern or close to zero, such as standard deviation value in 2014 (-0.3909179) and standard deviation value in 2023 (-0.3122322). Meanwhile, cluster 2 mostly have negative average transaction volume, such as average transaction volume in 2013 (-0.1873229) and average transaction volume in 2024 (-0.2542209), this indicates weaker commodity market activity. In contrast, cluster 3 exhibits higher price volatility, such as standard deviation in 2017 (1.6861991). Besides, the average transaction volume in cluster 3 have lower and negative pattern, such as average transaction volume in 2013 (-0.5337228) and average transaction volume in 2024 (-0.7509120). The average transaction volume in cluster 3 reflecting more volatile market activity than the other clusters. This combination highlight the significant differences in price volatility patterns and trading intensity between the three clusters.

After clustering the commodity, the next step is determining Sharpe ratio of each commodity in each cluster. In calculating the Sharpe ratio, the risk-free rate used is the average of Indonesia interest rate decision appropriate with the assets data timeline. Based on the Sharpe ratio calculation, the stocks that represent each cluster to construct the portfolio is presented in Table 6.

Table 6. Sharpe ratio of each stocks in each clusters

Cluster	Commodity	Sharpe ratio
1	Butter Futures (CBc1)	0.012893323
	Gasoline RBOB Futures (GPR)	-0.00147879
	Iron ore fines 62% Fe CFR Futures (TIOc1)	-0.024989274
	Palladium Futures (PA)	0.005596242
	Platinum Futures (PL)	-0.114207506
	Rubber RSS3 Futures (SRUc1)	-0.020480822
	US Coffee C Futures (KC)	0.054670495
	Class III Milk Futures (DCSc1)	0.010781388
	Feeder Cattle Futures (FC)	0.012696919
	Live Cattle Futures (LCc1)	-0.003466598
	Orange Juice Futures (OJ)	0.110488999
	Palm Oil Futures (FCPOc1)	0.045769033
	Rough Rice Futures (RR)	-0.043180107
	US Cotton #2 Futures (CT)	-0.047866682
2	Copper Futures (HG)	-0.038840889
	Milling Wheat N2 Futures (BL2c1)	-0.036876257
	US Soybean Meal Futures(ZM)	-0.05192541
	US Sugar #11 Futures (SB)	-0.012654035
	Heating Oil Futures (NYF)	-0.027474581
	Silver Futures (SI)	-0.022717076
	US Soybean Oil Futures (ZL)	-0.046376469

Table 6 presents the Sharpe ratios of each commodities in cluster 1 and 2. The Sharpe ratio provides insights into their risk-adjusted performance. Cluster 1 contains a diverse range of commodities, including agricultural products (e.g., US Coffee C Futures, Palm Oil Futures), energy derivatives (e.g., Gasoline RBOB Futures), and metals (e.g., Palladium, Platinum). Within cluster 1, several assets shows positive Sharpe ratios. The positive Sharpe ratio implies favorable returns relative to their risk, with Orange Juice Futures (0.1105) and US Coffee C Futures (0.0547) standing out as particularly strong performers. Meanwhile, the negative Sharpe ratios in assets such as Platinum Futures (-0.1142) and Rough Rice Futures (-0.0432) implies that not all assets in this cluster offer attractive risk-adjusted returns. This is highlighting the heterogeneity of performance within the cluster.

On the other hand, cluster 2 consist of energy and agricultural commodities such as Heating Oil Futures, US Soybean derivatives, and various grains and metals. If cluster 2 is compared to cluster 1, this cluster demonstrates a consistently weaker risk-adjusted performance, with all assets showing negative Sharpe ratios. The lowest Sharpe ratio is noticed in US Soybean Meal Futures (-0.0519), meanwhile even traditionally stable assets such as Silver Futures and Heating Oil Futures fail to show positive excess returns. In this case, excess return is the difference between return of an investment compared to a benchmark return, such as market return, risk-free rate or a certain index.

Table 5 not only provides an evaluation of performance based on risk and return or Sharpe ratio but also provides an overview of the stability and flexibility of each cluster to volatile market. Cluster 1 exhibits various risk-return profile, which indicates the possibility of more diversification opportunities for investors. On the contrary, Cluster 2 presents less performance consistency, which reflects high certain risk factors sensitivity such as energy price fluctuations or agricultural supply chain disruptions.

After commodity clustering process in Table 6, the commodity with maximum Sharpe ratio value is selected for representing each cluster to construct the optimum portfolio. Orange Juice Futures presents the maximum value of Sharpe ratio in the first cluster (0.110488999). Meanwhile US Sugar #11 Futures in the second cluster shows the highest Sharpe ratio value (-0.12654035).Thereby, Orange Juice Futures is chosen as the representative asset of cluster 1 to be included in the constructed portfolio and US Sugar #11 Futures is selected to represent the second cluster in the portfolio.

Table 7. Sharpe ratio of each stocks in each clusters

Cluster	Stocks	Sharpe ratio
1	ADRO	0.051669262
	AKRA	0.027946388
	ANTM	0.048963066
	PGAS	-0.037161036
	SCMA	-0.059865927
2	ACES	0.007503785
	ASII	-0.059350099
	CPIN	0.014456849
	EXCL	-0.052226123
	ICBP	0.073571527
	INDF	-0.012984951
	INTP	-0.084216699
	KLBF	-0.020493209
	MDKA	0.10686318
	MIKA	-0.026511972
	PTBA	0.007612597
	UNTR	0.012950795
	UNVR	-0.126749226
3	INCO	0.04787413
	INDY	0.060123553
	INKP	0.138578947
	ITMG	0.015432683

Table 7 displays the Sharpe ratio values for individual stocks in three different clusters. Cluster 1 contains of stocks that show a positive Sharpe ratio such as ADRO (0.051669262), AKRA (0.027946388), and ANTM (0.048963066). The stocks indicating that the assets in this group are able to provide good returns relative to the risk. However, there are also stocks that present negative values of Sharpe ratio, such as PGAS (-0.037161036) and SCMA (-0.059865927). Both of the stocks with negative value of Sharpe ratio reflecting that not all members of this cluster perform well.

Cluster 2 presents a more heterogeneous Sharpe ratio. Although there are stocks with significant positive Sharpe ratios such as MDKA (0.10686318) and ICBP (0.073571527), cluster 2 also includes stocks with the poorest Sharpe ratio performance such as INTP (-0.084216699) and UNVR (-0.126749226). This inequality illustrates the non-uniformity in sensitivity to common market factors faced by the stocks in this cluster. Therefore, cluster 2 can be categorized as the group with the highest level of risk diversification, but also demands more attention in portfolio management to minimize the negative contribution of underperforming members.

Cluster 3 contains stocks that show positive Sharpe ratio values such as INCO (0.04787413), INDY (0.060123553), INKP (0.138578947), and ITMG (0.015432683). The positive consistency in this cluster indicates its solid risk-return performance. The positive values of Sharpe ratio can be considered as a relatively more stable and potential cluster for strategic allocation in the portfolio. Stocks in this cluster can be prime candidates for investment strategies based on risk-return ratio optimization, because the Sharpe ratio value in this cluster indicates good efficiency in risk-to-return conversion.

Refers to Table 7, ADRO shows the maximum value of Sharpe ratio in the first cluster (-0.51669262). MDKA has the highest value of Sharpe ratio in the second cluster (0.10686318). The best value of Sharpe ratio in the third cluster is showed by INKP (0.138578947) (INKP). Hence, there are 3 stocks selected to construct the portfolio. Those are ADRO, MDKA, and INKP.

3.3. Real-based Mean Variance Portfolio Construction

Figure 6 presents the asset allocation movement of the real-based mean variance portfolio without short selling from 2013 to 2024. The mean variance portfolio is constructed using a target return constraint 0.75. In the early years

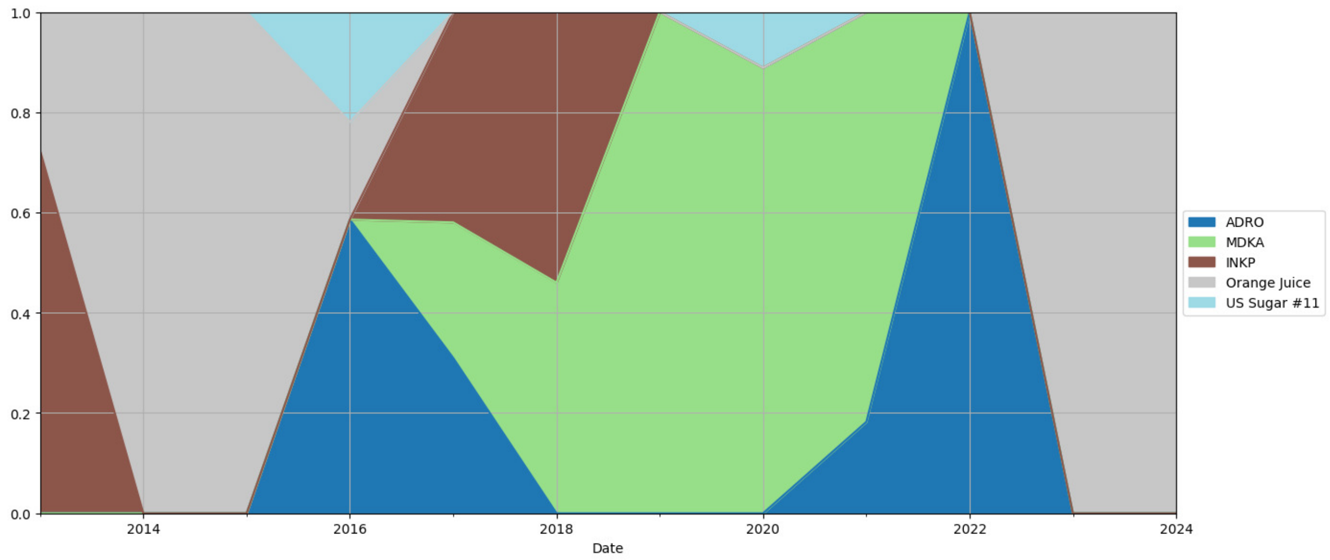

Figure 7. Asset allocation of real based mean variance portfolio with no short sale

2013-2015 the asset allocation is dominated by INKP and US Sugar #11. This is indicates limited diversification. A shift towards greater diversification is seen in 2016-2018. In the 2016-2018 period, the increase in allocation occurred in ADRO and MDKA. From 2019 to 2021, MDKA became the dominant asset. This indicates an increase in risk-adjusted returns over the 2019-2021 period. In 2022-2023, the allocation returned to ADRO and INKP that reflects the changing market dynamics.

Table 8. Result of real-based MV Portfolio

$Period$	w_{ADRO}	w_{MDKA}	w_{INKP}	$w_{OrangeJuice}$	$w_{USSugar\#11}$
2013	0	0	0.724	0.276	0
2014	0	0	0	0	1
2015	0	0	0	0	1
2016	0.5859	0	0	0.1948	0.2193
2017	0.3099	0.2704	0.4197	0	0
2018	0	0.4601	0.5399	0	0
2019	0	1	0	0	0
2020	0	0.8888	0	0	0.1112
2021	0.1824	0.8176	0	0	0
2022	1	0	0	0	0
2023	0	0	0	1	0
2024	0	0	0	1	0

Table 8 exhibits the asset allocation of a no-short-sale mean-variance portfolio targeting a 0.75 return from 2013–2024. The frequent 0 or 1 asset weights arise because the optimizer, constrained from short-selling. This concentrates capital in the few assets that best meet the return target while minimizing risk. For example, US Sugar #11 dominates in 2014–2015, while MDKA and ADRO take full allocation in 2019 and 2022, respectively that reflects their temporary efficiency. Zero asset weights indicate assets excluded for insufficient returns or excessive risk. These extreme allocations reveal the sensitivity of mean-variance optimization to input parameters under strict constraints. Such instability motivates our proposed Hilbert and Fourier transform methods for better asset allocation.

3.4. Real-based GMV Portfolio Construction

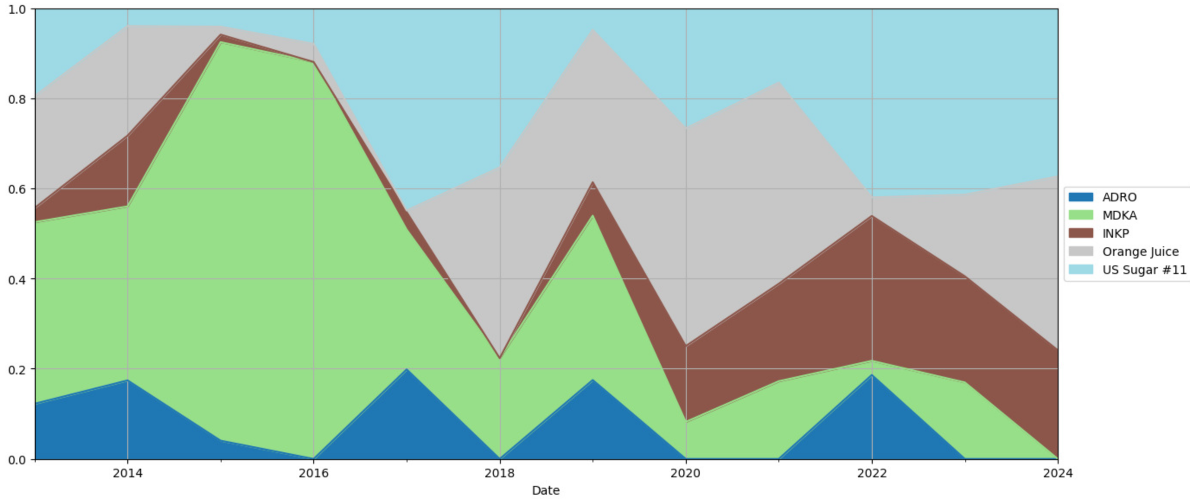

Figure 8. Asset allocation of real-based GMV portfolio

Figure 8 represents the asset allocation differences of the real-based GMV portfolio from 2013 to 2024. This picture is highlighting the varying proportions of ADRO, MDKA, INKP, Orange Juice Futures, and US Sugar #11 Futures. The allocation shows substantial fluctuations over time, with MDKA demonstrating a dominant allocation in 2015 and 2016, whereas Orange Juice Futures and US Sugar #11 Futures show significant weights in multiple years. The assets distribution shows a shift in portfolio combination. This shifts reflect market conditions and risk-return trade-offs. The existence of commodities alongside equities indicates a diversification strategy aimed at optimizing risk-adjusted returns.

Table 9. Result of real-based GMV Portfolio

$Period$	w_{ADRO}	w_{MDKA}	w_{INKP}	$w_{OrangeJuice}$	$w_{USSugar\#11}$
2013	0.1224	0.4028	0.0314	0.248	0.1954
2014	0.1742	0.386	0.1569	0.243	0.04
2015	0.0405	0.8839	0.0167	0.0177	0.0412
2016	0	0.8804	0	0.0403	0.0794
2017	0.1985	0.31	0.0388	0.0037	0.449
2018	0	0.2235	0	0.4231	0.3535
2019	0.175	0.3646	0.0742	0.3379	0.0482
2020	0	0.0821	0.1682	0.4828	0.2669
2021	0	0.1724	0.2161	0.4462	0.1653
2022	0.1862	0.0314	0.3213	0.0413	0.4198
2023	0	0.1697	0.2354	0.1811	0.4137
2024	0	0	0.241	0.3856	0.3734

Table 9 presents the asset allocation of the real-based GMV portfolio under a no-short-sale constraint across the period from 2013 to 2024. The results indicate significant variation in asset weights over time, reflecting shifts in risk-minimizing portfolio composition. Notably, MDKA frequently exhibits a dominant allocation, particularly in 2015 and 2016, while Orange Juice Futures attain substantial weights in multiple years, such as 2018 to 2021. The existence of zero weights of assets in several periods suggests that certain assets were excluded from the optimal portfolio due to their risk-return characteristics relative to the other assets. This phenomenon arises because the no-short-sale constraint prevents negative weights, leading to asset exclusion when their inclusion would not contribute to variance minimization within the GMV framework.

3.5. *Complex-based GMV Portfolio Construction with Hilbert Transform Approach*

After transforming the return data of the selected asset using Hilbert transform into complex valued return, then the result of complex-based GMV portfolio construction can be seen in Table 10 and Figure 9.

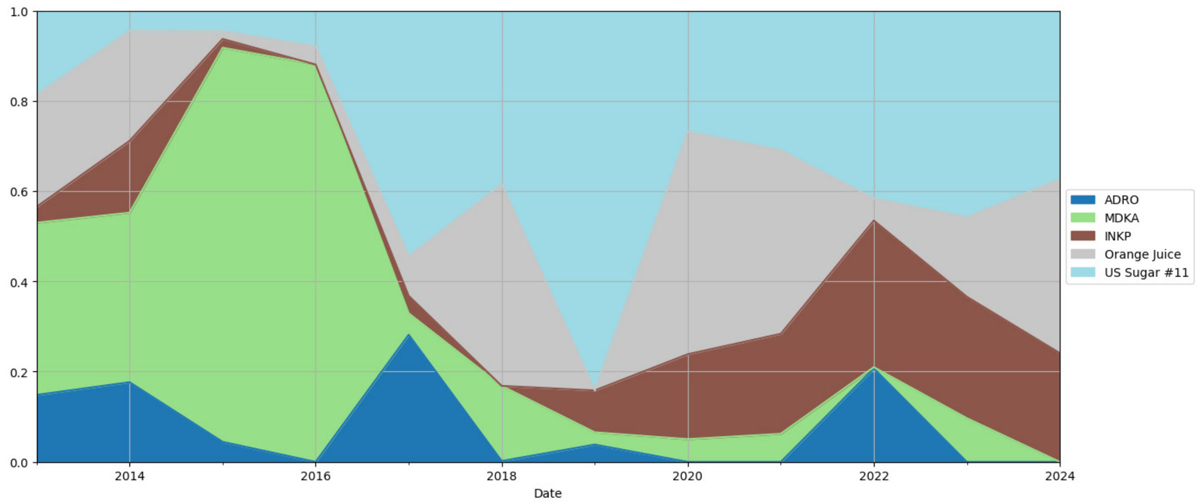

Figure 9. Asset allocation of complex-based GMV portfolio with Hilbert transform approach

Figure 9 represents the asset allocation differences of the complex-based GMV portfolio with Hilbert transform from 2013 to 2024. The graph describes the temporal dynamics of the optimal weight allocated to each asset in portfolio (ADRO, MDKA, INKP, Orange Juice Futures, and US Sugar #11 Futures). This visualization highlights significant switches in the portfolio assets allocation over time, which is explained as the portfolio's response to dynamic market conditions and the non-stationary nature of the financial data analyzed in the complex domain.

Generally, Figure 9 illustrates that asset weights are unevenly distributed throughout the observation period. MDKA, for instance, dominated the portfolio asset allocation in the early years such as 2015 and 2016 with a weight close to 0.9, before having a sharp decline in its contribution after 2017. In contrast, commodities such as US Sugar #11 Futures and Orange Juice Futures started to present a significant increment in weight allocation from 2017 on wards, with relatively more stable fluctuations than other assets. This proposes that the Hilbert Transformation approach has ability to capture phase changes as well as amplitude relations across time series. Thereby asset allocation strategy resulted are more adaptive to hidden market patterns.

Furthermore, the weight allocation of ADRO and INKP shows a more moderate performance in the portfolio, with an increasing contribution trend in the middle to the end of the observation period, especially from 2021 to 2024. This indicates a change in portfolio strategy from reliance on a single asset to broader diversification, which is in line with the principle of risk reduction. These results express that the Hilbert Transform approach has a role to construct more adaptive portfolios, because it has ability to recognize recurring patterns and irregularities in financial data.

Table 10. Result of complex-based GMV Portfolio using Hilbert Transform

$Period$	w_{ADRO}	w_{MDKA}	w_{INKP}	$w_{OrangeJuice}$	$w_{USSugar\#11}$
2013	0.1484	0.3816	0.0353	0.2486	0.1861
2014	0.1766	0.3756	0.1592	0.2434	0.0452
2015	0.0444	0.8733	0.0195	0.0172	0.0456
2016	0	0.8804	0	0.0403	0.0794
2017	0.2818	0.0475	0.0387	0.0876	0.5443
2018	0.002	0.1664	0	0.447	0.3847
2019	0.0384	0.0271	0.0929	0	0.8416
2020	0	0.0504	0.1882	0.4933	0.2682
2021	0	0.0623	0.2216	0.4069	0.3092
2022	0.2071	0.0028	0.3249	0.0488	0.4164
2023	0	0.0969	0.269	0.1765	0.4576
2024	0	0	0.241	0.3856	0.3734

Table 10 represents the asset allocation of the complex-based GMV portfolio with Hilbert transform method under a no-short-sale constraint across the period from 2013 to 2024. The portfolio is re-balanced annually from 2013 to 2024, with the aim of minimizing total risk (variance). The weight allocation values in the table exhibits the proportion of funds invested in each asset each year. The INKP and US Sugar #11 Futures tend to gain more weight allocation in the period after 2018. This condition indicating that the model considers these two assets to be more effective in reducing portfolio risk. Meanwhile, ADRO exhibits a variable role. In some years such as 2017 and 2022, ADRO receives a high weight, but in other years such as 2020 and 2024, ADRO is not used at all. This pattern describes how the Hilbert approach captures the changing structure of the relationship between assets dynamically.

In general, Table 10 exhibits that the Hilbert Transform approach have ability in producing portfolio allocations that are flexible and reactive to market changes. This approach supports the model to recognize invisible patterns in financial data, such as cycles or trend shifts, that are not easily captured by standard methods. Thereby, the model is able to construct portfolios that are not only risk-efficient, but also adaptive to volatile market conditions.

3.6. Complex-based GMV Portfolio Construction with Discrete Fourier Transform Approach

After transforming the return data of the selected asset using Fourier transform into complex valued return, then the result of complex-based GMV portfolio construction with Fourier transform can be seen in Table 11 and Figure 10. Figure 10 describes the asset allocation changing of the complex-based GMV portfolio constructed with the discrete Fourier transform approach. The portfolio is constructed without short-selling method and rebalancing every year during 2013-2024. Each bar represents the weights of the five assets in the portfolio in each year from 2013 to 2024. The allocation structure exposes significant variability across different time periods, reflecting the impact of frequency-domain information on portfolio optimization.

The allocation patterns exhibited in Figure 10 indicate significant variations in the weighting of each asset every year during 2013-2024. MDKA, for instance, occurs to be highly dominant in 2013 and 2016. This condition indicates that MDKA made a substantial contribution to portfolio risk minimization in 2013-2024 in the context of low frequency or dominant signals. Conversely, assets such as US Sugar #11 Futures and INKP tend to express more consistent and dispersed allocation in the portfolio. This condition reflecting that US Sugar #11 Futures and INKP have stable contributions in the mid to high frequency spectrum.

The allocation patterns in Figure 10 describes that the discrete Fourier transform approach in portfolio construction enables the transformation of asset return data into the frequency domain, which is then used to identify hidden cyclical or periodic patterns in financial data that are not easily recognized in the time domain. Unlike real based GMV portfolio that only consider static correlations between assets, the Fourier transform provides an alternative framework that enables portfolio optimization based on complex frequency information. This result support the role of the Fourier Transform in expanding the paradigm of risk management and portfolio diversification, especially in volatile and uncertain market environments.

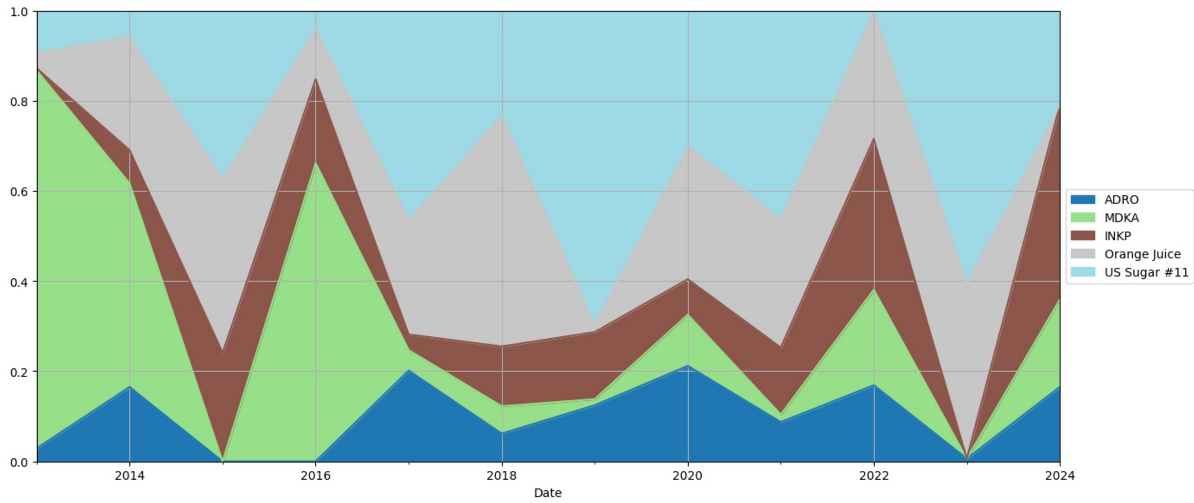

Figure 10. Asset allocation of complex-based GMV portfolio with discrete Fourier transform approach

Table 11. Result of complex-based GMV Portfolio using Fourier Transform

$Period$	w_{ADRO}	w_{MDKA}	w_{INKP}	$w_{OrangeJuice}$	$w_{USSugar\#11}$
2013	0.0289	0.8386	0.0046	0.0347	0.0933
2014	0.165	0.4516	0.0737	0.2516	0.058
2015	0	0	0.241	0.3856	0.3734
2016	0	0.6616	0.1861	0.1128	0.0394
2017	0.2015	0.0454	0.0347	0.2539	0.4645
2018	0.0617	0.0607	0.1326	0.5128	0.2322
2019	0.1246	0.0134	0.1488	0.0165	0.6966
2020	0.2118	0.1136	0.0787	0.2959	0.3
2021	0.0874	0.0164	0.1488	0.2836	0.4638
2022	0.169	0.2117	0.3345	0.2849	0
2023	0.0061	0	0	0.3875	0.6064
2024	0.1653	0.1937	0.4311	0	0.2099

Table 11 represents the GMV portfolio asset allocations without short-selling constructed using a complex-based approach with Fourier Transform on five financial assets over the 2013-2024. The estimation results show a variation in the weights distribution across periods, reflecting the volatility dynamics and asset correlations captured through the spectral approach. Particularly, INKP exhibits a significant weight increase in 2016 and 2024, while the Orange Juice Futures has more volatile weight, indicating sensitivity to structural changes in the data. The existence of zero weights in several periods of complex-based GMV portfolio with Fourier transform is less than complex-based GMV portfolio with Hilbert transform approach. This is due to the differences of both transformations capture spectral information from the asset price data. The Fourier transform transforms asset price data from the time domain to the frequency domain, therefore it captures spectral information from the entire time period at once. This allows complex-based GMV with Fourier transform to consider long-term patterns in the relationships between assets. This result more assets gain positive weight in the portfolio. On the other hand, the Hilbert transform concentrates more on local analysis through phase and amplitude components, This lead to the exclusion of certain assets if their contribution to risk reduction is less significant.

3.7. Comparison between real-based MV, real-based GMV, and complex-based GMV Portfolio

This section will discuss the comparison of real-based MV, real-based GMV and complex-based GMV portfolios based on the expected return, standard deviation, and Sharpe ratio values. Visually, the comparison of both portfolios describes in Figure 11, 12, and 13.

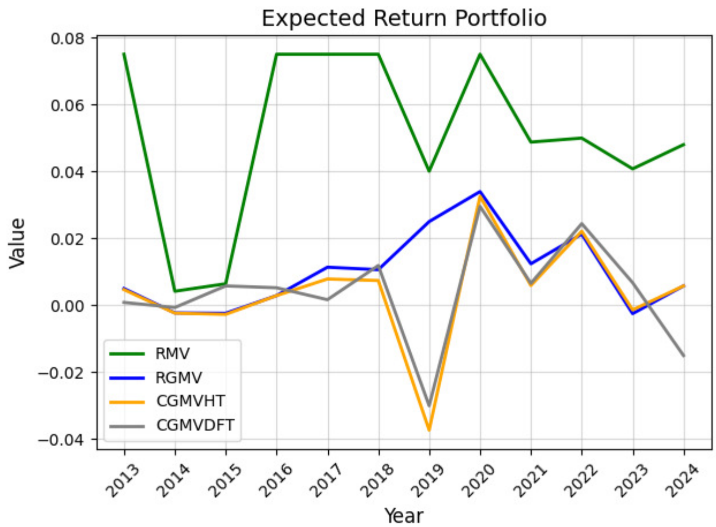

Figure 11. Expected return of real-based MV, real-based GMV, and complex-based GMV Portfolio

Figure 11 represents the expected return comparison of real-based MV, real-based GMV, and complex-based GMV portfolios with Hilbert and Fourier transform for 12 years. In Figure 11, RMV is the abbreviation of real-based mean variance portfolio. RGMV is the abbreviation of real-based GMV portfolio. CGMVHT refers to complex-based GMV portfolio with Hilbert transform approach, while CGMVDFT refers to complex-based GMV portfolio with Fourier transform approach. The CGMVHT portfolio has the most negative return in 2019. It shows a heightened sensitivity to instantaneous frequency components. Meanwhile, the CGMVDFT portfolio also shows negative return in 2019, but it is less than CGMVHT. The RGMV portfolio, denoted by the blue line, consistently aligns closely with the complex-based approaches, even with slightly higher peaks and troughs in certain periods. Conversely, the RMV portfolio shows a higher return than all GMV portfolios because RMV has a target return 0.75.The drastic decrease in expected returns on complex-based portfolios in 2019, as seen in CGMVHT (-0.03740884) and CGMVDFT (-0.03011784), is most likely influenced by the sensitivity of the transformation method to significant market structure changes. Hilbert and Fourier transforms rely on spectral and phase information which magnifies the impact of high volatility changes or market shocks. These findings highlight the impact of different mathematical transformations on expected return portfolio.

Table 12 presents the expected return comparison of the real-based mean variance portfolio (MV), real-based GMV portfolio (RGMV), the complex-based portfolio with Hilbert transform (CGMVHT), and the complex-based portfolio with Fourier transform (CGMVDFT) over the period 2013-2024. Generally, the RGMV portfolio exhibits a consistent expected return pattern with positive values in most periods, such as 2016 (0.002819405) and 2024 (0.005717458), although there are some years with negative values, such as 2014 (-0.002352861). The RMV portfolio shows positive returns throughout the period due to the determination of the target return in the constraint function. The CGMVHT portfolio shows a similar expected return pattern with the RGMV, but with a slight difference value in some periods, such as 2013 to 2016. Meanwhile, the CGMVDFT portfolio exposes more significant fluctuations with higher expected return values in some periods, such as 2015 (0.005717458), but also larger negative values in certain years, such as 2024 (-0.01509264). These results suggest that the Fourier transform develops more varying expected returns than the Hilbert transform and real number-based portfolios, which may reflect the sensitivity of the Fourier transform to asset data fluctuations.

Table 12. Expected Return of Portfolio

$Period$	RMV	$RGMV$	$CGMVHT$	$CGMVFT$
2013	0.075	0.004925226	0.004605696	0.000785084
2014	0.004122	-0.002352861	-0.002422432	-0.000744575
2015	0.006363	-0.00246456	-0.002791151	0.005717458
2016	0.075	0.002819405	0.002820152	0.005167371
2017	0.075	0.01130866	0.007784864	0.001599171
2018	0.075	0.01057482	0.00733902	0.01181949
2019	0.040017	0.02495313	-0.03740884	-0.03011784
2020	0.075	0.03391949	0.03253186	0.02950533
2021	0.048731	0.01233831	0.005843487	0.006490994
2022	0.04992	0.02121965	0.02207883	0.02433837
2023	0.040729	-0.002568482	-0.001450369	0.006678285
2024	0.04796	0.005717458	0.005717458	-0.01509264

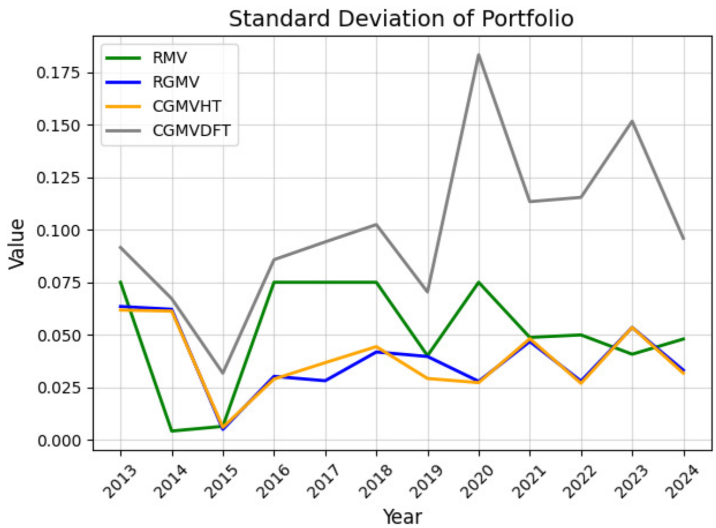

Figure 12. Standard deviation of real-based MV, real-based GMV and complex-based GMV Portfolio

Figure 12 represents the standard deviation of real-based MV (RMV), real-based GMV (RGMV), and complex-based GMV portfolios with Hilbert transform (CGMVHT) and Fourier transform (CGMVDFT) approach. The results reveal that the CGMVDFT portfolio exposes significantly higher volatility across most periods, particularly in 2020 and 2023. It shows greater sensitivity to frequency-domain transformations. The RMV portfolio has a slightly higher standard deviation than the RGMV and CGMVHT portfolios. This rather high standard deviation of RMV is in line with the target return set. Conversely, the RGMV and CGMVHT portfolios maintain relatively lower and stable standard deviations. It indicates more constrained risk profile. The alignment between RGMV and CGMVHT shows that the Hilbert transform preserves key temporal characteristics of asset returns while mitigating excess variance. These findings describes the influence of complex-valued transformations on portfolio optimization.

Based on Table 13, the portfolio standard deviation comparison represents that the real-based portfolio (RGMV) has slightly higher value of standard deviation than the complex-based portfolio using Hilbert transform (CGMVHT). This result is in line with [2] that shows standard deviation of the complex-based mean variance portfolio is slightly lower than the real number-based mean variance portfolio. Meanwhile, RMV and CGMVDFT have standard deviations above RGMV and CGMVHT.The standard deviation of the CGMVHT portfolio has a

Table 13. Standard Deviation of Portfolio

$Period$	RMV	$RGMV$	$CGMVHT$	$CGMVFT$
2013	0.238087	0.06346789	0.06177172	0.09157376
2014	0.064806	0.06212857	0.06128907	0.06720342
2015	0.11945	0.06212857	0.06128907	0.06720342
2016	0.063195	0.03017037	0.02888912	0.08576849
2017	0.071638	0.02809034	0.03669535	0.09424182
2018	0.164381	0.04179337	0.04433628	0.1024952
2019	0.096984	0.03964727	0.02916722	0.07035159
2020	0.127107	0.02782324	0.02717544	0.1834032
2021	0.109602	0.04677164	0.04813961	0.1134264
2022	0.097467	0.02792201	0.0268544	0.1154797
2023	0.111054	0.05349322	0.053474	0.1517768
2024	0.076172	0.03310281	0.03169352	0.09594338

lower variation than the CGMVFT, but in 2016, 2022, and 2023, the standard deviation value is higher than the RGMV portfolio. It is caused by the nature of Hilbert transform that preserves the phase component of the signal. Therefore, it results larger fluctuations when there is high volatility in the market data. Meanwhile, the CGMVDFT portfolio has a higher standard deviation than the RMV, RGMV, and CGMVHT because the Fourier transform use full frequency component which amplifies variations and sensitivity of sharp changes in asset prices. It reflects greater risk in the CGMVDFT portfolio.

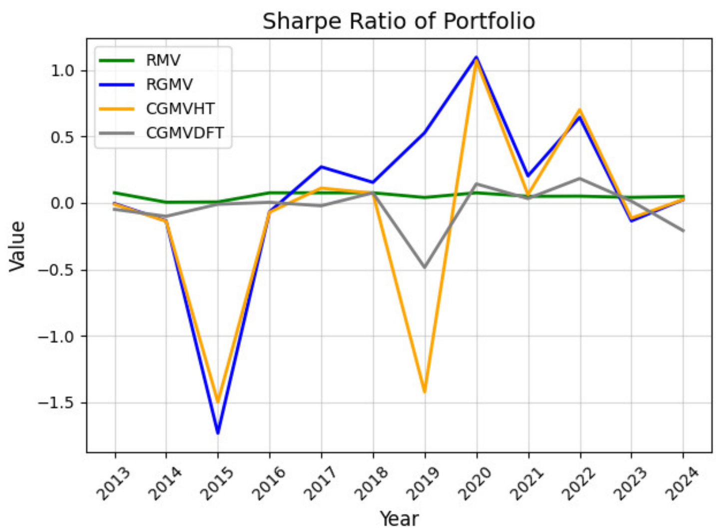

Figure 13. Sharpe ratio of real-based GMV and complex-based GMV portfolio

Figure 13 presents the Sharpe ratio dynamics of the real-based MV (RMV), real-based GMV (RGMV) and complex-based GMV portfolios, incorporating Hilbert (CGMVHT) and discrete Fourier (CGMVDFT) transform. The RGMV portfolio shows substantial fluctuations. The peak is in 2020 and the lowest Sharpe ratio of RGMV is in 2015. The peak indicates a period of exceptionally high risk-adjusted returns. Conversely, the CGMVDFT portfolio exhibits greater stability during the timeline, with less pronounced variations. It means that frequency-domain transformations contribute to return smoothing. RMV shows a very stable Sharpe ratio value that does not even show significant fluctuations in any period. Meanwhile, the CGMVHT portfolio that has more volatile sharpe ratio value than CGMVDFT, shows instances of strong performance, particularly in 2020. These results highlight the impact of complex-valued transformations on the portfolio optimization.

Table 14. Sharpe Ratio of Portfolio

$Period$	RMV	$RGMV$	$CGMVHT$	$CGMVFT$
2013	0.292768431	-0.005836275	-0.01116928	-0.04925601
2014	-0.030172815	-0.1356902	-0.138684	-0.101512
2015	0.002527005	-1.731575	-1.498528	-0.01084419
2016	1.109828081	-0.06778205	-0.07076234	0.003532265
2017	0.994937993	0.2699869	0.110647	-0.02255327
2018	0.430977729	0.1535978	0.07180508	0.07477466
2019	0.370606247	0.52661901	-1.422246	-0.4860155
2020	0.562722893	1.094247	1.06927	0.1419351
2021	0.418270631	0.2020586	0.06140028	0.03176766
2022	0.478604007	0.6427815	0.7003294	0.1824257
2023	0.324254263	-0.1362369	-0.1153764	0.01290723
2024	0.564646427	0.02319157	0.02422281	-0.2088981

Table 14 displays the Sharpe ratio comparison of the four portfolio strategies over the period 2013-2024 (RMV, RGMV, CGMVHT, and CGMVFT). In general, the RMV portfolio exhibits the most consistent performance with relatively high Sharpe ratio values, especially in 2016, 2017, and 2024. The superior performance of RMV can be attributed to the target return constraint in the optimization model, which helps maintain a balance between risk and expected return. On the other hand, complex -based GMV portfolios, especially CGMVHT, show promising potential in capturing the phase and frequency information structure of market data, which cannot be captured by conventional approaches. This is reflected in the spike in the Sharpe ratio in 2022, where CGMVHT outperforms all other models. Thus, although the transformation approach has not consistently outperformed real-based portfolios, its superiority in identifying hidden patterns and temporal dynamics of the market makes it a prospective tool for the formulation of more adaptive and sophisticated portfolio strategies.

Based on the analysis of the expected return, Sharpe ratio, and standard deviation of the three types of portfolios (RMV, RGMV, CGMVHT, and CGMVDFT), the appropriate portfolio recommendations for the three types of investor risk profiles can be adjusted as follows:

1. Conservative investors
   RGMV is the most suitable portfolio for conservative investors. This portfolio has a lower standard deviation or small risk volatility, although its expected return is relatively moderate. The Sharpe ratio value of RGMV also exposes more stable performance than other portfolios. It provides a better risk and return balance for investors who prioritize capital safety.

2. Moderate investors
   CGMVHT is a better choice for investors with moderate risk tolerance. This portfolio has a slightly higher risk than RGMV, but provides diversification opportunities with comparable or better expected returns in some periods. Although CGMVHT's Sharpe ratio is not always higher than RGMV, this portfolio provides greater flexibility in capturing market dynamics.

3. Aggressive investors
   CGMVDFT is the most suitable for aggressive investors who pursue high returns despite facing huge risks. Even though this portfolio exhibits the highest volatility and standard deviation, as well as a negative Sharpe ratio in some periods, its potential returns are significant, especially in certain years with high market price swings. This portfolio is suitable for investors who are willing to accept large fluctuations in pursuit of maximum returns.

Meanwhile, RMV portfolios are very flexible because they offer a target return on the constraint function, so that each investor can determine the target return that suits the risk he can bear. By considering risk, return and Sharpe ratio value, these portfolio selections are customized to meet the risk preferences and investment objectives of each type of investor.

Based on the findings of this study, several recommendations can be made for further development. First, future research needs to expand the data coverage by covering a wider range of asset types and market periods. Second, the integration of transaction costs and liquidity factors into the optimization model will make the results more practically relevant. Third, sensitivity analysis is needed to understand the impact of model parameters on portfolio stability. Fourth, the development of a clearer interpretation framework for transformation-based methods will facilitate application by practitioners. Fifth, ethical aspects of applying these methods, such as potential bias in signal processing, need to be addressed. Finally, the exploration of hybrid approaches that combine Hilbert and Fourier transforms with machine learning techniques could be an innovative solution to future portfolio optimization challenges.

4. Conclusion

The conclusion of this study confirms that complex-based portfolios with Fourier transform (CGMVDFT) and Hilbert transform (CGMVHT) approaches offer an attractive diversification alternative compared to real-based portfolios (RGMV). Standard deviation analysis shows that CGMVDFT has the highest volatility. Meanwhile, CGMVHT consistently has the lowest risk. In terms of performance, the expected return of CGMVDFT tends to be higher in certain periods, but the Sharpe ratio value is often lower or even negative. This means that the risk is not worth the return. In contrast, RGMV provides more consistent risk and return stability. The Sharpe ratio value of RMV shows stability due to the explicit target return constraint. The empirical results show that CGMVHT exhibits superior Sharpe ratios in certain periods, especially under dynamic market conditions, indicating its effectiveness in capturing localized phase and amplitude information. This research opens up opportunities for further exploration of complex-based portfolio optimization to improve risk and return efficiency under dynamic market conditions.

Acknowledgement

The authors would like to express sincere gratitude to all colleagues and peers who provided priceless insights and discussions that enriched this work. Although this study was self-funded, the authors intensely appreciate the academic resources and support from Hasanuddin University and Universitas Islam Negeri Alauddin Makassar, which facilitated the development of this study. Finally, the authors present appreciation to family and friends for their unwavering encouragement throughout the research, writing, and publication process.

REFERENCES

1. A. Ahuja, *Portfolio diversification in the Karachi stock exchange*, Pakistan Journal of Engineering, Technology & Science, vol.1, no.1, 2015.
2. I. Anis, A. Rahim, and M. Bahri, *Mean Variance Complex-Based Portfolio Optimization*, Statistics Optimization & Information Computing. vol. 12, no.5, 2024.
3. T. Bodnar, N. Parolya, and W. Schmid, *Estimation of the global minimum variance portfolio in high dimensions*, European Journal of Operational Research, vol.266, no.1 pp. 371–390, 2018.
4. S.E. Chaudhuri, and A. W. Lo, *Spectral portfolio theory*, Social Science Research Network (SSRN), 2016.
5. R. Y. Chen, *The Fourier Transform Method for Volatility Functional Inference by Asynchronous Observations*, ArXiv (Cornell University), 2019.
6. C. Drummond, P. Davy, and C. Gulati, *Hilbert-Huang based volatility forecasts for high frequency data and simulated option markets*, Social Science Research Network (SSRN), 2014.
7. E. Elton, M. Gruber, S. Brown, and W. Goetzmann, *Modern portfolio theory and investment analysis, 9th Edition*. John Wiley and Sons, Inc. 2014.
8. X. Jin, and A. X. Zhang, *Reclaiming quasi–monte carlo efficiency in portfolio value-at-risk simulation through Fourier transform*, Management Science, vol.52, no.6, pp. 925–938, 2006.

9. A. Kourtis, *The Sharpe ratio of estimated efficient portfolios*, Finance Res. Lett., vol. 17, pp.72–78. 2016.
10. V. Kurbatsky, D. Sidorov, V. Spiryaev, and N. Tomin, *Forecasting nonstationary time series based on Hilbert-Huang transform and machine learning*, Autom. Remote Control, vol.75, no.5, pp. 922-934, 2014.
11. L. Gubu, D. Rosadi, and Abdurakhman. *Classical portfolio selection with cluster analysis: Comparison between hierarchical complete linkage and Ward algorithm*. AIP Conference Proceedings, vol.2192, 2019.
12. H. Markowitz, *Portfolio selection*, The Journal of Finance, vol. 7, no.1, pp. 77-91, 1952.
13. N. Nurwahidah, M.Bahri, and A. Rahim, *Two-Dimensional Quaternion Fourier Transform Method in Probability Modeling* Symmetry, vol.16, no.3 2024.
14. J. Park, *Clustering approaches for global minimum variance portfolio*. Master thesis, Graduate School of Convergence Science and Technology, Seoul National University, 2020.
15. S. Somarajan, M. Shankar, T. Sharma, and J. Ramasamy, *Modelling and analysis of volatility in time series data*. Soft Computing and Signal Processing . Advances in Intelligent Systems and Computing, vol. 898, pp. 1-10, 2019.
16. Y. Uchiyama, T. Kadoya, and K. Nakagawa, *Complex valued risk diversification*, Entropy, vol.21, no.2, pp 1–11, 2019.
17. D. Wu, X. Wang, and S. Wu, *Construction of stock portfolios based on k-means clustering of continuous trend features* Knowl.-Based Syst., vol. 252, 2022.
18. M.P. Yadav, N. Puri, P. Bhatia, and A. Shore, *Crude oil and quad financial markets: an insight into dynamic connectedness and portfolio diversification*, SSRN Electronic Journal, 2024.
19. L. Yang, R. Couillet, and M.R. McKay, *A robust statistics approach to minimum variance portfolio optimization* IEEE Trans Signal Process, vol. 63, no.24, pp. 6684– 6697, 2015.
20. A. Yousfat, *The portfolio selection by using quadratic programming approach case study of Malaysia stock exchange*. Int. J. Eng. Sci. Technol, vol. 7, no. 4, pp. 1361–1369, 2015.