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Galerkin Method for the Solvability of a Micropolar Fluid Flow Model with Novel Frictional Boundary Conditions

El-Hassan Benkhira [1], Jawad Chaoui [2], Rachid Fakhar [2,*]

[1] *University Moulay Ismail, ESTM, Laboratory LEM2A, BP 3103, Toulal-Meknes, Morocco*
[2] *University Sultan Moulay Slimane, Laboratory LS2ME, 25000 Khouribga, Morocco*

Abstract We investigate a mathematical model describing the flow of an incompressible micropolar fluid within a bounded domain of $\mathbb{R}^3$. The fluid's behavior is governed by a non-symmetric constitutive law, coupled with a couple stress tensor. Frictional boundary conditions are imposed through homogeneous Neumann conditions for the angular velocity field, along with a friction coefficient $h \in L^\infty(\partial\mathcal{O})$, which depends on the tangential component of the velocity field. To address the problem, we derive a variational formulation leading to a coupled system consisting of a variational equation with nonlinear terms governing the velocity field and a linear one describing the microrotational velocity. By applying the Galerkin method, the Cauchy-Lipschitz theorem, and compactness results, we obtain an approximate weak solution to this system.

Keywords Galerkin method, Incompressible micropolar fluid, Variational methods, Frictional boundary conditions, Weak solution.

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1. Introduction

The Navier-Stokes equations play a key role in fluid mechanics by describing fluid motion under convective forces. However, they fall short when it comes to capturing the dynamics of micropolar fluids, which are composed of randomly oriented or spherical particles dispersed in a viscous medium, without accounting for fluid particle deformation. Micropolar fluids are more intricate than classical fluids due to the presence of micro-rotation effects and micro-rotation inertia [1]. Such fluids include biological substances like blood, liquid crystals, and certain polymers, all of which can be accurately represented using the micropolar fluid theory. Understanding the behavior of these fluids is crucial for various applications in fields like biology and materials science.

The theoretical model for micropolar fluid flow was introduced by Eringen in 1966 [2]. This model is more complex than classical fluid models because of the presence of internal angular momentum in the fluid. In this study, we examine a mathematical model describing an incompressible micropolar fluid, where the velocity field satisfies friction boundary conditions, while the angular velocity field is governed by zero-flux Neumann boundary conditions. Specifically, we consider an open, bounded, and connected domain $\mathcal{O} \subset \mathbb{R}^3$ filled with a viscous, incompressible micropolar fluid, where the boundary $\partial\mathcal{O} = \Upsilon$ is Lipschitz continuous. The time interval of interest is $t \in (0, T)$, where $0 < T < \infty$. We assume that the boundary Υ can be divided into two disjoint and relatively open subsets, Υ_1 and Υ_2. We study a nonlinear system of coupled partial differential equations that governs the conservation of momentum, angular momentum, and mass within the space-time domain $Q = \mathcal{O} \times (0, T)$. For additional information on this topic, we refer the reader to [3].

*Correspondence to: Rachid Fakhar (Email: r.fakhar@usms.ma). University Sultan Moulay Slimane, Laboratory LS2ME, 25000 Khouribga, Morocco.

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The basic equations describing the flow of an incompressible micropolar fluid are:

$$\begin{cases} \rho\left(\frac{\partial\omega}{\partial t}+(\omega\cdot\nabla)\omega\right)+\nabla p-\nu_1\,\mathrm{div}(D(\omega))=2\nu_r\,\mathrm{curl}(\vartheta)+f, \\ \rho\left(\frac{\partial\vartheta}{\partial t}+(\omega\cdot\nabla)\vartheta\right)-\nu_2\Delta\vartheta-\nu_3\nabla(\mathrm{div}(\vartheta))+4\nu_r\vartheta=2\nu_r\,\mathrm{curl}(\omega)+g, \\ \mathrm{div}(\omega)=0, \end{cases}$$

where

$$\nu_1:=2(\nu+\nu_r),\quad \nu_2:=c_d+c_a,\quad \nu_3:=c_0+c_d-c_a.$$

Here, $\omega=\omega(x,t)\in\mathbb{R}^3$ represents the velocity field, $\vartheta=\vartheta(x,t)\in\mathbb{R}^3$ is the micro-rotation velocity, and $p(x,t)\in\mathbb{R}$ is the pressure at the point $(x,t)\in Q$. The term $D(\omega)$ represents the symmetric part of the velocity gradient, given by $D(\omega)=\frac{1}{2}[\nabla\omega+(\nabla\omega)^T]$, and $\mathrm{div}(D)$ is the divergence of D. The functions $f\in L^2(Q)$ and $g\in L^2(Q)$ describe the external forces associated with linear and angular momentum, respectively.

The physical properties of the fluid are characterized by the constants ρ, ν, ν_r, c_0, c_a, and c_d, which account for isotropic behavior. Specifically, ρ represents the density of the fluid (typically taken as $\rho=1$) for general modeling purposes), ν is the Newtonian viscosity, and ν_r corresponds to the microrotation viscosity. The parameters c_0, c_a, and c_d are stress-related viscosities, which satisfy the condition $(c_0+c_d)-c_a>0$ to ensure the physical consistency of the model.

Let n be the outward unit normal vector to Υ. The velocity field ω and the tensor D are decomposed into their normal and tangential components on Υ as follows:

$$\omega_n=\omega\cdot n,\quad \omega_\tau=\omega-\omega_n n,\quad\text{and}\quad D_n=(Dn)\cdot n,\quad D_\tau=Dn-D_n n.$$

For the boundary conditions, we have the following equations:

$$\omega_n=0\quad\text{on}\quad\Upsilon_2\times(0,T),$$
$$D(\omega)_\tau=h(x,\|\omega_\tau\|)\omega_\tau\quad\text{on}\quad\Upsilon_2\times(0,T),$$

where $h\in L^\infty(\Upsilon_2)$ is a non-negative friction coefficient satisfying:

$$(\mathcal{H})\begin{cases} \Upsilon_2 \text{ is an open and bounded subset of } \mathbb{R}^3, \\ h:\Upsilon_2\times[0,\infty)\to[0,\infty) \text{ is a Carathéodory function}, \\ 0<h_0\le h(x,s)\le h_1,\quad (x,s)\in\Upsilon_2\times[0,\infty), \end{cases}$$

where h_0 and h_1 are constants. The friction coefficient h in the slip boundary condition can be defined in various ways, depending on the physical characteristics of the fluid and the specific application. Possible forms include:

- Constant friction: $h=C_{\text{constant}}$, assuming uniform friction along the boundary.
- Velocity-dependent friction: $h(x,\|\omega_\tau\|)=h_1(x)+h_2\|\omega_\tau\|^m$, reflecting a nonlinear dependence on the tangential velocity.
- Velocity and microrotation-dependent friction: $h(x,\|\omega_\tau\|,\|\vartheta\|)=h_0+\alpha\|\vartheta\|$ or $h(x,\|\omega_\tau\|,\|\vartheta\|)=h_1\|\omega_\tau\|^m+\beta\|\vartheta\|^n$, modeling more complex interactions involving both tangential motion and microrotation effects.

The appropriate choice of h depends on the rheological properties of the fluid and the specific modeling objectives of the application.

Also, we assume that:

$$\omega=0,\quad\vartheta=0\quad\text{on}\quad\Upsilon_1\times(0,T),$$

and the Neumann boundary condition for ϑ:

$$\frac{\partial\vartheta}{\partial n}=0,\quad\text{on}\quad\Upsilon_2\times(0,T).$$

The initial conditions are:

$$\omega(x,0)=\omega_0,\quad \vartheta(x,0)=\vartheta_0 \quad \text{in} \quad \mathcal{O}.$$

Under these notations and conditions, the mechanical problem can be formulated as follows.

Problem (P). *Find a velocity field* $\omega : \mathcal{O}\times(0,T)\to\mathbb{R}^3$ *and an angular velocity field* $\vartheta : \mathcal{O}\times(0,T)\to\mathbb{R}^3$ *such that*

$$\frac{\partial\omega}{\partial t}+(\omega\cdot\nabla)\omega+\nabla p-\nu_1\operatorname{div}(D(\omega))=2\nu_r\operatorname{curl}(\vartheta)+f \qquad \text{in } Q, \tag{1}$$

$$\frac{\partial\vartheta}{\partial t}+(\omega\cdot\nabla)\vartheta-\nu_2\Delta\vartheta-\nu_3\nabla(\operatorname{div}(\vartheta))+4\nu_r\vartheta=2\nu_r\ \operatorname{curl}(\omega)+g \qquad \text{in } Q, \tag{2}$$

$$\operatorname{div}(\omega)=0 \qquad \text{in } Q, \tag{3}$$

$$\omega_n=0,\quad -D(\omega)_\tau=h(x,\|\omega_\tau\|)\omega_\tau,\quad \frac{\partial\vartheta}{\partial n}=0 \qquad \text{on } \Upsilon_2\times(0,T), \tag{4}$$

$$\omega=0,\quad \vartheta=0 \qquad \text{on } \Upsilon_1\times(0,T), \tag{5}$$

$$\omega(x,0)=\omega_0,\quad \vartheta(x,0)=\vartheta_0 \qquad \text{on } \mathcal{O}. \tag{6}$$

This system is inspired by previous works on micropolar fluids, notably those by Duarte-Leiva et al. [6], who studied a three-dimensiona micropolar fluid model with Navier boundary conditions but without friction for the velocity field. By applying the Galerkin method, they demonstrated the existence of weak solutions and obtained a Prodi–Serrin-type regularity criterion for the global existence of strong solutions. Micropolar fluid models, such as the one studied here, have important applications in various fields including biological flows (e.g., blood flow with microstructure effects), materials science (e.g., suspensions, liquid crystals), and lubrication theory. The inclusion of friction boundary conditions for the velocity field in our model enhances its ability to capture more realistic interactions at fluid-solid interfaces, which is crucial for accurately describing these practical phenomena. Thus, beyond its mathematical interest, the model is well-suited to contribute to the understanding and simulation of complex micropolar fluid behaviors in applied contexts.

The literature on micropolar fluids is vast. For example, in [4], the authors addressed a comparable system under Dirichlet boundary conditions, while in [5], the focus was on a system with variable density, and the existence of local-in-time strong solutions was demonstrated. Previous works, such as [6] and [7], focused on systems with homogeneous boundary conditions, while numerical methods, as in [8] and [9], were applied to micropolar fluid systems with variable density to establish local-in-time strong solutions and uniqueness results via the semi-Galerkin method. Theoretical studies, such as [10] and [11], further explored global existence, uniqueness, and asymptotic stability using the Duhamel principle and spaces of tempered distributions.

In contrast to these previous studies, our work introduces a novel mathematical model for the flow of an incompressible micropolar fluid within a domain $\mathcal{O}\subset\mathbb{R}^3$. The novelty of this model stems from the incorporation of friction boundary conditions for the velocity field alongside homogeneous Neumann boundary conditions for the angular velocity field. As far as we are aware, this particular model has not yet been explored in the existing literature, and no known results have been established for such a problem. The main goal of this work is to demonstrate the existence and uniqueness of weak solutions for the corresponding variational formulation.

The paper is organized as follows: Section 2 introduces the functional spaces for the various quantities, defines the relevant notations, outlines the assumptions on the given data, and derives the variational formulation of the mathematical model. The main result, concerning the existence of solutions to the weak formulation, is presented in Theorem 2.1. The proof of this theorem, detailed in Section 3, relies on the Galerkin method, the Cauchy-Lipschitz theorem, and compactness arguments. Finally, the long-time behavior of the solutions is investigated, demonstrating that the solutions remain uniformly bounded as time tends to infinity. This reflects the physical balance between energy dissipation and external forcing. Together, these results provide a rigorous mathematical framework for the study of incompressible micropolar fluid flow with friction boundary conditions and Neumann conditions on the angular velocity.

2. Weak Formulation and Main Result

In this section, we derive the weak formulation of Problem (P) and formulate the main theorem regarding the existence of weak solutions. To achieve this, we introduce the functional spaces required for the analysis and outline the assumptions on the given data.

To formulate the problem, consider the following functional spaces:

$$\tilde{U} = \{\omega \in C_0^\infty(\mathcal{O}) \mid \operatorname{div}\omega = 0 \text{ in } \mathcal{O},\, \omega = 0 \text{ on } \Upsilon_1,\, \omega_\nu = 0 \text{ on } \Upsilon_2\},$$

where U is the closure of $\tilde{U}$ in $H^1(\mathcal{O})$, and H is the closure of $\tilde{U}$ in $L^2(\mathcal{O})$. Similarly, we define:

$$W = \left\{\vartheta \in L^2(\mathcal{O}) \mid \operatorname{curl}(\vartheta) \in L^2(\mathcal{O}),\, \vartheta = 0 \text{ on } \Upsilon_1\right\}.$$

The spaces U and W are equipped with the $H^1(\mathcal{O})$ norm. The embedding $U \subset H$ is continuous and dense, establishing an evolution triple $(U, H, U^\star)$. Furthermore, H is identified with its dual space, and the embeddings $U \subset H \subset U^\star$ are dense and continuous.

We also define the functional spaces $\mathcal{U}$ and $\mathcal{W}$ to describe time-dependent functions:

$$\mathcal{U} = \left\{\omega \in L^\infty(H) \cup L^2(U) \mid \omega' \in L^{\frac{4}{3}}(U^\star)\right\}, \quad \mathcal{W} = \left\{\vartheta \in L^\infty(L^2(\mathcal{O})) \cup L^2(W) \mid \vartheta' \in L^{\frac{4}{3}}(W^\star)\right\},$$

and the inner product in $L^2(\mathcal{O})$ and the duality pairing between $\mathcal{U}^\star$ and $\mathcal{U}$ is given, respectively, by

$$(\omega, v) = \int_{\mathcal{O}} \omega(x)\cdot v(x)\mathrm{d}x, \text{ for } \omega \in U^\star, v \in U, \qquad \langle u, v\rangle_{\mathcal{U}^\star\times\mathcal{U}} = \int_{\mathcal{O}} \langle \omega, v\rangle_{U^\star\times U}\, \mathrm{d}t,$$

and also the norm of gradient and function in defined,respectively,by

$$\|\nabla\omega\| = \Big[\int_{\mathcal{O}} |\nabla\omega(x)|^2\mathrm{d}x\Big]^{\frac{1}{2}}, \quad \|\omega\| = \Big[\int_{\mathcal{O}} |\omega(x)|^2\mathrm{d}x\Big]^{\frac{1}{2}}, \quad |\omega|^2 = \sum_{i=1}^{3}\int_{\mathcal{O}} |\omega_i(x)|^2\mathrm{d}x.$$

We also introduce the trilinear form $b : X \times X_1 \times X_1 \to \mathbb{R}$ defined by:

$$b(\omega, v, u) = ((\omega\cdot\nabla)v, u),$$

which satisfies the following properties (see [9]):

$$b(\omega, v, u) = -b(\omega, u, v) \quad \forall\omega \in X, \forall v, u \in X_1, \tag{7}$$

$$b(\omega, v, v) = 0 \quad \forall\omega \in X, \forall v \in X_1. \tag{8}$$

Additionally, we use the following notations:

$$B_\omega v = (\omega.\nabla)v.$$

According to this,we have

$$\langle B_\omega v, w\rangle = b(\omega, v, u) \quad \forall\omega \in X, \forall u, v \in X_1.$$

For $p \in \left(\frac{3d}{d+2}, d\right)$, we have the embedding:

$$X, X_1 \subset L^{\frac{dp}{d-p}}(\mathcal{O};\mathbb{R}^d) \subset L^{\frac{2p}{p-1}}(\mathcal{O};\mathbb{R}^d),$$

and thus:

$$\langle B_\omega v, u\rangle \le \|\omega_1\|_{L^{\frac{2p}{p-1}}(\mathcal{O};\mathbb{R}^d)} \|\nabla v\|_{L^{\frac{dp}{d-p}}(\mathcal{O};\mathbb{R}^d)} \|u\|_{L^{\frac{2p}{p-1}}(\mathcal{O};\mathbb{R}^d)}. \tag{9}$$

Thus, we obtain:

$$\langle B_\omega v, u\rangle \leq M\|\omega\|_X \|v\|_{X_1}\|u\|_{X_1} \quad \forall \omega \in X_1, \forall v, u \in X_1 \text{ with } M > 0. \tag{10}$$

Based on the interpolation inequality in three-dimensional domains:

$$\|\omega\|_{L^4} \leq C\|\omega\|^{\frac{1}{4}}\|\omega\|^{\frac{3}{4}}_{H^1} \quad \forall \omega \in H^1(\mathcal{O}),$$

and using the previous inequality, we deduce:

$$B_\omega \omega \in L^{\frac{4}{3}}(X^\star) \quad \text{and} \quad B_\omega v \in L^{\frac{4}{3}}(X_1^\star). \tag{11}$$

Furthermore, we introduce two bilinear forms $R_1 : L^2(\mathcal{O}) \times L^2(\mathcal{O}) \to \mathbb{R}$ and $R_2 : L^2(\mathcal{O}) \times L^2(\mathcal{O}) \to \mathbb{R}$, defined by:

$$R_1(\vartheta, \omega) = \int_{\mathcal{O}} \operatorname{curl}(\vartheta) \cdot \omega \, dx, \quad R_2(\vartheta, \omega) = \int_{\mathcal{O}} \operatorname{curl}(\vartheta) \cdot \operatorname{curl}(\omega) \, dx.$$

For all $\vartheta, \omega \in L^2(\mathcal{O})$, applying Hölder's and Young's inequalities yields:

$$|R_1(\vartheta, \omega)| \leq \|\operatorname{curl}\vartheta\|\|\omega\| \leq \sqrt{2}\|\nabla\vartheta\|\|\omega\| \leq \sqrt{2}\left(\|\vartheta\|^2_{H^1} + \|\omega\|^2\right), \tag{12}$$

$$|R_2(\vartheta, \omega)| \leq \|\operatorname{curl}\vartheta\|\|\operatorname{curl}\omega\| \leq 2\|\nabla\vartheta\|\|\nabla\omega\| \leq \sqrt{2}\left(\|\vartheta\|^2_{H^1} + \|\omega\|^2_{H^1}\right). \tag{13}$$

Finally, we recall the following Green's formulas (see [10, Theorems 2.24 and 2.25]):

$$\int_{\mathcal{O}} D(\omega) : D(v)\, dx + \int_{\mathcal{O}} \operatorname{div}(D) \cdot v \, dx = \int_{\partial\mathcal{O}} Dn \cdot v \, d\Upsilon, \tag{14}$$

where D is the deformation tensor and v is a vector field, and

$$\int_{\mathcal{O}} \vartheta \cdot \nabla\psi \, dx + \int_{\mathcal{O}} \operatorname{div}(\vartheta) \cdot \psi \, dx = \int_{\partial\mathcal{O}} \psi(\vartheta \cdot n)\, d\Upsilon, \tag{15}$$

where ϑ is a vector field and ψ is a scalar function.

We use integration by parts and equations (14)-(15) to derive the following weak formulation of Problem (P), expressed in terms of the velocity field and angular velocity field.

Problem (P_V). *Find a velocity field $\omega \in \mathcal{U}$ and an angular velocity field $\vartheta \in \mathcal{W}$ such that*

$$\begin{aligned} &\langle \partial\omega_t, v\rangle_{U^\star \times U} + \langle B_\omega \omega, v\rangle + \nu_1(D(\omega), D(v)) \\ &= 2\nu_r(\operatorname{curl}(\vartheta), v) + \nu_1 \int_{\Upsilon_2} h(x, |\omega_\tau|)\omega_\tau \cdot v_\tau dx + (f, v), \end{aligned} \tag{16}$$

$$\begin{aligned} &\langle \partial\vartheta_t, \varphi\rangle_{W^\star \times W} + \langle B_\omega \vartheta, \varphi\rangle - \nu_3(\operatorname{curl}(\vartheta), \operatorname{curl}(\varphi)) + (\nu_2 + \nu_3)(\nabla\vartheta, \nabla\varphi) + 4\nu_r(\vartheta, \varphi) \\ &= \nu_3 \int_{\Upsilon_2} (\operatorname{curl}\vartheta \times n) \cdot \varphi d\sigma + 2\nu_r(\operatorname{curl}(\omega), \varphi) + (g, \varphi), \end{aligned} \tag{17}$$

for all $v, \varphi \in W \times U$ and almost every $t(0, T)$.

Our main existence result, established in Section 3, is stated as follows:

Theorem 2.1
Let $(\omega_0, \vartheta_0) \in H \times L^2(\mathcal{O})$ and $(f, g) \in L^2(Q) \times L^2(Q)$. Under assumption $(\mathcal{H})$, Problem (P_v) admits at least one weak solution.

A pair of functions (ω, ϑ) that satisfies equations (1), (2), (16), and (17) is referred to as a weak solution of the incompressible micropolar fluid flow problem (P).

Theorem 2.1 ensures that, under assumption $(\mathcal{H})$ and the condition $c_0 + c_d > c_a$, the variational formulation (16)-(17) admits at least one weak solution (ω, ϑ) such that $\omega \in \mathcal{U}$ and $\vartheta \in \mathcal{W}$.

3. Proof of theorem 2.1

To prove the existence of a solution to the variational problem (16)-(17), we use the Galerkin method. For this purpose, we represent by $V_m = Vect(v_1, \ldots, v_m) \subset U$ the subspace generated by the basis functions $\{\varphi_1, \ldots, \varphi_m\}$, and by $W_m = Vect(\varphi_1, \ldots, \varphi_m) \subset L^2(\mathcal{O})$ the subspace generated by $\{\varphi_1, \ldots, \varphi_m\}$. Let us consider the pair (ω^m, ϑ^m), defined as:

$$\omega^m(x,t) := \sum_{i=0}^{m} \alpha_i^m(t) v_i(x), \quad \vartheta^m(x,t) := \sum_{i=0}^{m} \beta_i^m(t)\varphi_i(x),$$

which satisfies the following approximate problem:

Find $\omega^m \in C^1(0,T;U^m)$ *and* $\vartheta^m \in C^1(0,T;W^m)$ *such that*

$$\begin{aligned}
&\langle \partial\omega_t^m, v\rangle_{U^\star\times U} + \langle B_{\omega^m}\omega^m, v\rangle + \nu_1(D(\omega^m), D(v)) \\
&= 2\nu_r(\mathrm{curl}(\vartheta^m), v) + \nu_1 \int_{\Upsilon_2} h(x, |\omega_\tau^m|)\omega_\tau^m \cdot v_\tau dx + (f^m, v), \\
&\quad \langle \partial\vartheta_t^m, \varphi\rangle_{W^\star\times W} + \langle B_{\omega^m}\vartheta^m, \varphi\rangle - \nu_3(\mathrm{curl}(\vartheta^m), \mathrm{curl}(\varphi)) + (\nu_2+\nu_3)(\nabla\vartheta^m, \nabla\varphi) + 4\nu_r(\vartheta^m, \varphi) \\
&= \nu_3 \int_{\Upsilon_2} (\mathrm{curl}\,\vartheta^m \times n)\cdot\varphi d\sigma + 2\nu_r(\mathrm{curl}(\omega^m), \varphi) + (g^m, \varphi), \\
&\int_{\mathcal{O}} (\omega^m(x,0) - \omega_0(x))\cdot v, \mathrm{d}x = 0, \\
&\int_{\mathcal{O}} (\vartheta^m(x,0) - \vartheta_0(x))\cdot\varphi \mathrm{d}x = 0.
\end{aligned} \tag{18}$$

where $f^m \in\ C(0,T;U^*)$ and $g^m \in C(0,T;W^*)$ such that

$$\begin{aligned}
&\|f^m\|_{L^2(0,T;U^*)} \le \|f\|_{L^2(0,T;U^*)}, \qquad && f^m \to f \in L^2(0,T;U^*) \text{ as } m\to\infty, \\
&\|g^m\|_{L^2(0,T;W^*)} \le \|g\|_{L^2(0,T;W^*)}, && g^m \to g \in L^2(0,T;W^*) \text{ as } m\to\infty.
\end{aligned}$$

It can be noted that system (18) is equivalent to a Cauchy problem for a nonlinear first-order ordinary differential system, where the unknown functions are $\alpha_i^m(\cdot)$ and $\beta_i^m(\cdot)$.

Find $\alpha^m \in C^1(0,T;U^m)$ *and* $\beta^m \in C^1(0,T;W^m)$ *verifying: for j=0,...,m*

$$\begin{aligned}
&\sum_{i=0}^{m} \frac{d\alpha_i^m(t)}{dt}(v_i, v_j)dx + 2\nu_1 \sum_{i=0}^{m} \alpha_i^m(t)(D(v_i), D(v_j)) - 2\nu_r \sum_{i=0}^{m} \beta_i^m(t)(\mathrm{curl}(\varphi_i), v_j) \\
&+ \sum_{i,k=0}^{m} \alpha_i\alpha_k((v_i.\nabla)v_k, v_j) = 2\nu_1 \sum_{i=0}^{m} \alpha_i^m(t) \int_{\Upsilon_2} h(x, |\omega_\tau^m|)((v_i)_\tau, (v_j))_\tau \mathrm{d}x + (f^m.v_j), \\
&\sum_{i=0}^{m} \frac{d\beta_i(t)}{dt}(\varphi_i.\varphi_j) + \sum_{i,k=0}^{m} \alpha_i^m\beta_k^m((v_i\cdot\nabla)\varphi_k, \varphi_j) + 4\nu_r \sum_{i=0}^{m} \beta_i^m(\varphi_i, \varphi_j) + \nu_2 \sum_{i=0}^{m} \beta_i^m(\nabla\varphi_i, \nabla\varphi_j) \\
&- \nu_3 \sum_{i=0}^{m} \beta_i^m(\mathrm{curl}(\varphi_i), \mathrm{curl}(\varphi_j)) - 2\nu_r \sum_{i=0}^{m} \alpha_i^m(\mathrm{curl}(v_i), \varphi_j) = \nu_3 \sum_{i=0}^{m} \beta_i^m \int_{\Upsilon_2} (\mathrm{curl}\,\varphi_i \times n)\cdot\varphi_j d\sigma + (g^m\,\varphi_j), \\
&\sum_{i=0}^{m} \alpha_i^m \int_{\mathcal{O}} v_i\cdot v_j, \mathrm{d}x = \int_{\mathcal{O}} \omega_0\cdot v_j, \mathrm{d}x, \\
&\sum_{i=0}^{m} \beta_i^m \int_{\mathcal{O}} \varphi_i\cdot\varphi_j, \mathrm{d}x = \int_{\mathcal{O}} \vartheta_0\cdot\varphi_j, \mathrm{d}x.
\end{aligned}$$

By multiplying these equations by the inverses of the matrices $\int_{\mathcal{O}} v_i \cdot v_j \, dx$ and $\int_{\mathcal{O}} \varphi_i \cdot \varphi_j \, dx$, we derive an equivalent system, which can be written as:

Find $\alpha^m \in C^1(0,T;V_m)$ and $\beta^m \in C^1(0,T;W_m)$ verifying: for $i = 0,\dots,m$,

$$
\begin{aligned}
&\frac{d\alpha_i^m(t)}{dt} + \sum_{j=0}^{m} a_{ij}\alpha_j^m(t) - 2\nu_r \sum_{j=0}^{m} b_{ij}\beta_j(t) + \sum_{j,k=0}^{m} c_{ijk}\alpha_j\alpha_k = \sum_{j=0}^{m} d_{ij}\int_{\mathcal{O}} f^m \cdot v_j \mathrm{d}x,\\
&\frac{d\beta_i(t)}{dt} + \sum_{j,k=0}^{m} \alpha_j\beta_k A_{ijk} + 4\nu_r \sum_{j=0}^{m} \beta_j B_{ij} + \sum_{j=0}^{m} \beta_j C_{ij} - \sum_{j=0}^{m} \beta_j D_{ij}\\
&- 2\nu_r \sum_{J=0}^{m} \alpha_i E_{ij} = \sum_{j=0}^{m} \beta_j F_{ij} + \sum_{J=0}^{m} \int_{\mathcal{O}} g^m \varphi_j \mathrm{d}x,\\
&\alpha_i(0) = \sum_{j=0}^{m} e_{ij}\int_{\mathcal{O}} \omega_0 \cdot v_j dx,\\
&\beta_i(0) = \sum_{j=0}^{m} \int_{\mathcal{O}} \vartheta_0 \cdot \varphi_j dx.
\end{aligned}
\tag{19}
$$

As a result, the standard existence and uniqueness theory for ordinary differential systems can be applied. Therefore, for each $m \in \mathbb{N}$, there exists a unique pair (ω^m, ϑ^m) that solves this system over the time interval $[0, T_m]$ (for further details, refer to [11, Chapter 3]).

Taking $v = \omega^m$ and $\varphi = \vartheta^m$ in system (18), and considering the Young inequalities and the properties $< B_{\omega^m}\omega^m, \omega^m >= 0$ and $< B_{\vartheta^m}\vartheta^m, \vartheta^m >= 0$, we obtain the following inequalities:

$$\frac{1}{2}\frac{d}{dt}\|\omega^m\|^2 \le 2\sqrt{2}\nu_r\|\nabla\vartheta^m\|^2 + (2\sqrt{2}\nu_r + 2\nu_1\|h(x,\|\omega_\tau\|)\|_{L^\infty(\mathcal{O})} + 1)\|\omega^m\|^2 + \|f\|^2, \tag{20}$$

$$\frac{1}{2}\frac{d}{dt}\|\vartheta^m\|^2 \le (1 + \sqrt{2}\nu_3 - 2\nu_r)\|\vartheta^m\|^2 + (3\nu_3 - \nu_2)\|\nabla\vartheta^m\|^2 + \nu_3\sqrt{2}\|\nabla\omega^m\|^2 + \|g\|^2, \tag{21}$$

$$\|\omega^m(x,0)\|^2_= \int_{\mathcal{O}} |\omega^m(x,0)|^2 dx \le \int_{\mathcal{O}} \omega_0(x)\cdot\omega^m(x,0)dx \le \int_{\mathcal{O}} |\omega_0(x)|^2 dx = \|\omega_0(x)\|^2, \tag{22}$$

$$\|\vartheta^m(x,0)\|^2 = \int_{\mathcal{O}} |\vartheta^m(x,0)|^2 dx \le \int_{\mathcal{O}} \vartheta_0(x)\cdot\vartheta^m(x,0)dx \le \int_{\mathcal{O}} |\vartheta_0(x)|^2 dx = \|\vartheta_0(x)\|^2. \tag{23}$$

By adding inequalities (20) and (21), we obtain the following result with a constant $C_1 > 0$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\omega^m\|^2 + \|\vartheta^m\|^2)\\
&\le (1 + \sqrt{2}\nu_3 - 2\nu_r)\|\vartheta^m\|^2 + (2\sqrt{2}\nu_r + 2\nu_1\|h(x,\|\omega_\tau\|)\|_{L^\infty(\mathcal{O})} + 1)\|\omega^m\|^2\\
&\quad + C_1(\|\omega^m\|^2_{H^1} + \|\vartheta^m\|^2_{H^1} + \|f\|^2 + \|g\|^2).
\end{aligned}
\tag{24}
$$

From inequalities (20)–(21) and applying Gronwall's lemma, we derive the following estimate:

$$\|\omega^m\|^2_{L^\infty(H)} + \|\vartheta^m\|^2_{L^\infty(L^2(\mathcal{O}))} \le C_2 exp(C_3 T)(\|\omega(0)\|^2 + \|\vartheta(0)\|^2 + \|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(Q)}), \tag{25}$$

where $C_2, C_3 > 0$.

Next, integrating inequality (24) over the interval $[0,T]$ and using (25), we deduce the existence of a constant $C_4 > 0$, independent of m, such that:

$$\|\omega^m\|^2_{L^\infty(H)\cap L^2(U)} + \|\vartheta^m\|^2_{L^\infty(L^2(\mathcal{O}))\cap L^2(W)} \le C_4.$$

Consequently, we conclude:

$$
\begin{aligned}
&\{\omega^m\}_{m\ge1} \text{ is bounded in } L^\infty(H)\cap L^2(U),\\
&\{\vartheta^m\}_{m\ge1} \text{ is bounded in } L^\infty(L^2(\mathcal{O}))\cap L^2(W(\mathcal{O})).
\end{aligned}
\tag{26}
$$

Now consider the following system:

$$\begin{aligned}\langle \partial_t \omega^m, v\rangle_{U^*\times U} &= -(B_\omega \omega, v) + 2\nu_1 (D(\omega), D(v)) \\ &2\nu_1 \int_{\Upsilon_2} h(x, |\omega^m_\tau|)\omega^m_\tau . v_\tau, \mathrm{d}x + 2\nu_r(\operatorname{curl}(\vartheta), v) + (f, v) \quad \forall v \in U,\end{aligned} \tag{27}$$

$$\begin{aligned}\langle \partial_t \vartheta^m, v\rangle_{W^*\times W} &= -(B_\omega \vartheta, \varphi) + \nu_3(\operatorname{curl}(\vartheta), \operatorname{curl}(\varphi)) - (\nu_3 + \nu_2)(\nabla\vartheta, \nabla\varphi) \\ &- 4\nu_r(\vartheta, \varphi) - 2\nu_r(\operatorname{curl}(\omega), \varphi) + \nu_3 \int_{\Upsilon_2} (\operatorname{curl}\varphi \times n)\cdot \varphi d\sigma + (g, \varphi) \quad \forall \varphi \in W.\end{aligned} \tag{28}$$

From (27) and (28), we get the following bounds:

$$\begin{aligned}&|\langle \partial_t \omega^m, v\rangle_{U^\star\times U}| \le \|\omega^m\|_{L^4(\mathcal{O},\mathbb{R}^3)} \|\nabla\omega^m\| \|v\|_{L^4(\mathcal{O},\mathbb{R}^3)} + 2\nu_1 \|D(\omega^m)\|_{L^2(\mathcal{O},\mathbb{S}^d)} \|D(v)\|_{L^2(\mathcal{O},\mathbb{S}^d))} \\ &+ 2\nu_r \|\operatorname{curl}(\vartheta)\| \|v\| + 2\nu_1 h_1 \|\omega^m\| \|v\| + f\| \|v\|. \\ &|\langle \partial_t \vartheta^m, \varphi\rangle_{W^\star\times W}| \le \|\omega^m\|_{L^4(\mathcal{O},\mathbb{R}^3)} \|\nabla\vartheta^m\| \|\varphi\|_{L^4(\mathcal{O},\mathbb{R}^3)} + \nu_3 \|\operatorname{curl}(\vartheta^m)\| \|\operatorname{curl}(\varphi)\| \\ &+ (\nu_3+\nu_2)\|\nabla(\vartheta^m)\| \|\nabla\varphi\| + 4\nu_r \|\vartheta^m\| \|\varphi\| + 2\nu_r \|\operatorname{curl}(\omega^m)\| \|\varphi\| + \nu_3 \|\operatorname{curl}(\vartheta^m)\| \|\varphi\| + \|g\| \|\varphi\|.\end{aligned}$$

By using (11) and (26), we obtain the following bounds:

$$\|\partial_t \omega\|_{L^{\frac{4}{3}}(U^\star)} \le C_5, \text{ and } \quad \|\partial_t \vartheta\|_{L^{\frac{4}{3}}(W^\star)} \le C_6, \qquad C_5 > 0, C_6 > 0. \tag{29}$$

Consequently, we conclude:

$$\{\partial_t \omega^m\}_{m\ge 1} \text{ is bounded in } L^{\frac{4}{3}}(U^\star) \text{ and } \{\partial_t \vartheta^m\} \text{ is bounded in } L^{\frac{4}{3}}(W^\star). \tag{30}$$

From the bounds obtained in (30) and (26), we deduce that $(\partial_t \omega^m, \partial_t \vartheta^m)m \ge 1$ and $(\omega^m, \vartheta^m)m \ge 1$ are bounded in reflexive separable Hilbert spaces. Therefore, possibly for a subsequence, we obtain:

$$\omega^m \to \omega \text{ weakly in } L^2(U) \text{ and weakly}^\star \text{ in } L^\infty(H), \tag{31}$$

$$\vartheta^m \to \vartheta \text{ weakly in } L^2(W) \text{ and weakly}^\star \text{ in } L^\infty(L^2(\vartheta)), \tag{32}$$

$$(\partial_t \omega^m, \partial_t \vartheta^m) \to (\partial_t \omega, \partial_t \vartheta) \text{ weakly}^\star \text{ in } L^{\frac{4}{3}}(U) \times L^{\frac{4}{3}}(W). \tag{33}$$

Moreover, by applying the Aubin–Lions lemma (see [12, Theorem 5.1, p. 58]) and ([13, Corollary 4]) to the relations (31)-(33), we obtain the strong convergence results:

$$\omega^m \to \omega \text{ in } L^2(H) \cap C(0,T;U), \tag{34}$$

$$\vartheta^m \to \vartheta \text{ in } L^2(0,T,L^2(\mathcal{O})) \cap C(0,T;W^\star). \tag{35}$$

The function h is continuous, so

$$h(x, |\omega^m_\tau|) \to h(x, |\omega_\tau|) \text{ in } \quad [0,\infty) \times L^2(\Upsilon_2). \tag{36}$$

Let $v \in U, \varphi \in W$ such that:

$$v = \sum_{j=1}^{\infty} a_j S_j, \qquad \varphi = \sum_{j=1}^{\infty} b_j N_j,$$

where S_j and N_j are some basis elements, and a_j, b_j are the corresponding coefficients. Let ψ be a continuous function, differentiable on $[0;T]$ with boundary conditions:

$$\psi(T) = 0 \text{ and } \psi(0) \ne 0.$$

We multiply the equations of system (18) by $\psi(t)$, and using integration by parts, we have:

$$\begin{aligned}
&-\int_0^T \langle \omega^m, \psi(t)' S_j\rangle_{U^*\times U}, \mathrm{d}t + \int_0^T (B_{\omega^m}\omega^m, S_j\psi(t))dt + 2\nu_1 \int_0^T (D(\omega^m), D(S_j)\psi(t))dt \\
&= \langle \omega^m(0), S_j\psi(t)\rangle_{U^*\times U} + 2\nu_r \int_0^T (\mathrm{curl}(\vartheta^m), S_j\psi(t))dt + 2\nu_1 \int_0^T \int_{\Upsilon_2} h(x, |\omega_\tau^m|)\omega_\tau^m \cdot v_\tau \psi(t)\mathrm{d}x \\
&\quad + \int_0^T (f^m, N_j\psi(t))dt,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&-\int_0^T \langle \vartheta^m, \psi(t)' N_j\rangle_{W^*\times W} \mathrm{d}x dt + \int_0^T (B_{\vartheta^m}\omega^m, \psi(t)N_j)dt + (\nu_2+\nu_3)\int_0^T (\nabla\vartheta^m, \nabla N_j\psi(t))dt \\
&\quad - \nu_3 \int_0^T (\mathrm{curl}(\vartheta^m), \mathrm{curl}(N_j\psi(t)))dt + 4\nu_r \int_0^T (\vartheta^m, N_j\psi(t))dt \\
&= \langle \vartheta^m(0), N_j\psi(0)\rangle_{W^*\times W} + 2\nu_r \int_0^T (\mathrm{curl}(\omega^m), N_j\psi(t))dt. + \nu_3 \int_0^T \int_{\Upsilon_2} (\mathrm{curl}\,\vartheta^m \times n)\cdot N_j\psi(t)\mathrm{d}\sigma \\
&\quad + \int_0^T (g^m, N_j\psi(t))dt.
\end{aligned} \tag{38}$$

We let $m \to +\infty$. Due to the completeness of the sequences $S_j{}_{j=1}^{\infty}$ and $N_j{}_{j=1}^{\infty}$, we obtain:

$$\begin{aligned}
&-\int_0^T \langle \omega, \psi(t)' v\rangle_{U^*\times U} \cdot v\mathrm{d}t + \int_0^T (B_\omega\omega, \psi(t)v)\mathrm{d}t + 2\nu_1 \int_0^T (D(\omega) : D(v)\psi(t))\mathrm{d}t \\
&= 2\nu_r \int_0^T (\mathrm{curl}(\vartheta), \psi(t)v)\mathrm{d}t + \langle \omega_0(x), \psi(0)v\rangle_{U^*\times U} + 2\nu_1 \int_0^T \int_{\Upsilon_2} h(x, |\omega_\tau|)\omega_\tau \cdot v_\tau\psi(t), \mathrm{d}x \\
&\quad + \int_0^T (f, \psi(t)v)\mathrm{d}t,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&-\int_0^T \langle \vartheta, \varphi\psi(t)\rangle_{W^*\times W}, \mathrm{d}x\mathrm{d}t + \int_0^T (B_\omega\vartheta, \varphi\psi(t))\mathrm{d}t - \nu_3 \int_0^T (\mathrm{curl}(\vartheta), \mathrm{curl}(\varphi\psi(t)))\mathrm{d}t \\
&\quad + (\nu_2+\nu_3)\int_0^T (\nabla\vartheta, \nabla\varphi\psi(t))\mathrm{d}t + 4\nu_r \int_0^T (\vartheta, \varphi\psi(t)) - 2\nu_r \int_0^T (\mathrm{curl}(\omega), \varphi\psi(t))\mathrm{d}t \\
&= \langle \vartheta_0(x), \varphi\psi(0)\rangle_{W^*\times W} + \nu_3 \int_0^T (\mathrm{curl}\,\vartheta \times n)\cdot \varphi\psi(t)\mathrm{d}\sigma + \int_0^T (g, \varphi\psi(t))\mathrm{d}t.
\end{aligned} \tag{40}$$

We multiply the system (1)-(2) by $\psi(t)$, and using integration by parts, we obtain:

$$\begin{aligned}
&-\int_0^T \langle \omega, \psi(t)' v\rangle_{U^*\times U} \cdot v\mathrm{d}t + \int_0^T (B_\omega\omega,\ v\psi(t))\mathrm{d}t + 2\nu_1 \int_0^T (D(\omega), D(v)\psi(t),)\mathrm{d}t \\
&\quad - 2\nu_r \int_0^T (\mathrm{curl}(\vartheta), v\psi(t))\mathrm{d}t \\
&= \langle \omega(x,0), \psi(0)v\rangle_{U^*\times U} + 2\nu_1 \int_0^T h(x, |\omega_\tau|)\omega_\tau \cdot v_\tau\psi(t), \mathrm{d}x + \int_0^T (f, v\psi(t))\mathrm{d}t,
\end{aligned} \tag{41}$$

$$
\begin{aligned}
&-\int_0^T \langle\vartheta,\psi(t)'\varphi\rangle_{W^*\times W}\mathrm{d}t+\int_0^T(B_\omega\vartheta,\varphi\psi(t))\mathrm{d}t-\nu_3\int_0^T(\mathrm{curl}(\vartheta),\mathrm{curl}(\varphi\psi(t)))\mathrm{d}t\\
&\quad+\nu_2\int_0^T(\nabla\vartheta,\nabla\varphi\psi(t))dt+4\nu_r\int_0^T(\vartheta,\varphi\psi(t)')\mathrm{d}t-2\nu_r\int_0^T(\mathrm{curl}(\omega),\varphi\psi(t))dt\\
&=\langle\vartheta(x,0),\varphi\psi(0)\rangle_{W^*\times W}+\nu_3\int_0^T(\mathrm{curl}\,\vartheta\times n).\varphi\psi(t)\mathrm{d}\sigma+\int_0^T(g,\varphi\psi(t))dt.
\end{aligned}
\tag{42}
$$

By comparing the two systems (39)-(40) and (41)-(42), we find that:

$$
\langle\omega(x,0)-\omega_0(x),v\rangle_{U^*\times U}\psi(0)=0\quad\text{ and }\quad\langle\vartheta(x,0)-\vartheta_0(x),\varphi\rangle_{W^*\times W}\psi(0)=0.
$$

Thus:

$$
\omega(x,0)=\omega_0(x)\quad\text{ and }\quad\vartheta(x,0)=\vartheta_0(x).\tag{43}
$$

Therefore, from the convergences (31)-(36) and (43), we conclude that (ω^m,ϑ^m) converges to (ω,ϑ) and $(\omega^m(x,0),\vartheta^m(0,x))$ converges to $(\omega(x,0),\vartheta(x,0))$ as $m\to+\infty$. Consequently, we conclude that (ω,ϑ) is a weak solution of system (1)-(6). Let (ω_1,ϑ_1) and (ω_2,ϑ_2) be weak solutions to Problem (1)-(3). Then:

$$
\begin{aligned}
&\omega_1,\omega_2\in L^\infty((0,T),H)\cap L^2((0,T),U),\\
&\vartheta_1,\vartheta_2\in L^\infty((0,T),L^2(\mathcal{O}))\cap L^2((0,T),W(\mathcal{O})).
\end{aligned}
$$

Consider $\tilde\omega:=\omega_1-\omega_2$ and $\tilde\vartheta:=\vartheta_1-\vartheta_2$. Then, $(\tilde\omega,\tilde\vartheta)$ satisfies:

$$
\frac{\partial\tilde\omega}{\partial t}+(\omega_1\cdot\nabla)\tilde\omega+(\omega\cdot\nabla)\omega_2+\nu_1\,\mathrm{div}(D(\tilde\omega))=2\nu_r\,\mathrm{curl}(\tilde\vartheta),\tag{44}
$$

$$
\begin{aligned}
&\frac{\partial\tilde{\tilde\vartheta}}{\partial t}+(\omega_1\cdot\nabla)\tilde\vartheta+(\tilde\omega\cdot\nabla)\vartheta_2-\nu_2\Delta\tilde\vartheta-\nu_3\nabla(\mathrm{div}(\tilde\vartheta))+4\nu_r\vartheta=2\nu_r\ \mathrm{curl}(\tilde\omega),\\
&\tilde\omega(x,0)=\tilde\vartheta(x,0)=0.
\end{aligned}
\tag{45}
$$

Testing the first equation of system (44)-(45) by $\tilde\omega$ and the second equation by $\tilde\vartheta$, using (8) and (10), we get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde\omega\|^2_{L^2(\mathcal{O})}+\nu_1\|D(\tilde\omega)\|^2_{L^2(\mathcal{O},\mathbb{S}^d)}\le 2\nu_r\|\,\mathrm{curl}(\tilde\vartheta)\|_{L^2(\mathcal{O},\mathbb{R}^d))}\|\tilde\omega\|_{L^2(\mathcal{O},\mathbb{R}^d)}\\
&+\|\tilde\omega\|_{L^4(\mathcal{O},\mathbb{R}^d)}\|\nabla\tilde\omega\|_{L^2(\mathcal{O},\mathbb{R}^d)}\|\omega_2\|_{L^4(\mathcal{O},\mathbb{R}^d)}+2\nu_1h_1\|\tilde\omega\|^2_{L^2(\mathcal{O})},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde\vartheta\|^2_{L^2(\mathcal{O})}+4\nu_r\|\tilde\vartheta\|^2_{L^2(\mathcal{O})}+\nu_2\|\nabla(\tilde\vartheta)\|^2_{L^2(\mathcal{O})}\|+\nu_3\|\,\mathrm{curl}(\tilde\vartheta)\|^2_{L^2(\mathcal{O})},\\
&\le\|\tilde\omega\|_{L^4(\vartheta)}\|\nabla\tilde\vartheta\|_{L^2(\mathcal{O})}\|\vartheta_2\|_{L^4(\mathcal{O})}+\nu_3\|\,\mathrm{curl}(\tilde\vartheta)\|_{L^2(\mathcal{O})}\|\tilde\vartheta\|_{L^2(\mathcal{O})},\\
&+2\nu_r\|\,\mathrm{curl}(\tilde\omega)\|_{L^2(\mathcal{O})}\|\tilde\vartheta\|_{L^2(\mathcal{O})}.
\end{aligned}
\tag{47}
$$

Using Hölder's inequality, Young's inequality, and the classical interpolation inequality in 3D domains:

$$
\|\tilde\omega\|_{L^4(\mathcal{O},\mathbb{R}^d)}\le C_K\|\tilde\omega\|^{\frac14}_{L^2(\mathcal{O},\mathbb{R}^d)}\|\tilde\omega\|^{\frac34}_{H^1(\mathcal{O},\mathbb{R}^d)},
$$

we obtain the following estimates:

$$
2\nu_r\|\,\mathrm{curl}(\tilde\vartheta)\|_{L^2(\mathcal{O})}\|\tilde\vartheta\|_{L^2(\mathcal{O})}\le\frac{\nu_3}{2}\|\,\mathrm{curl}(\tilde\vartheta)\|_{L^2(\mathcal{O})}+C_1\|\tilde\vartheta\|^2_{L^2(\mathcal{O})},\tag{48}
$$

$$
\|\tilde\omega\|_{L^4(\mathcal{O},\mathbb{R}^d)}\|\nabla\tilde\omega\|_{L^2(\mathcal{O},\mathbb{R}^d)}\|\omega_2\|_{L^4(\mathcal{O},\mathbb{R}^d)}\le C_K\|\tilde\omega\|^2_{L^2(\mathcal{O},\mathbb{R}^d)}\|\tilde\omega\|_{H^1(\mathcal{O},\mathbb{R}^d)}\|\omega_2\|_{L^4(\mathcal{O},\mathbb{R}^d)},\tag{49}
$$

$$
\|\tilde\omega\|_{L^4(\mathcal{O},\mathbb{R}^d)}\|\nabla\tilde\vartheta\|_{L^2(\mathcal{O},\mathbb{R}^d)}\|\vartheta_2\|_{L^4(\mathcal{O},\mathbb{R}^d)}\le C_K\|\tilde\omega\|^2_{L^2(\vartheta,\mathbb{R}^d)}\|\tilde\vartheta\|_{H^1(\mathcal{O},\mathbb{R}^d)}\|\vartheta_2\|_{H^1(\mathcal{O},\mathbb{R}^d)},\tag{50}
$$

$$
\nu_3\|\,\mathrm{curl}(\tilde\vartheta)\|_{L^2(\mathcal{O})}\|\tilde\vartheta\|_{L^2(\mathcal{O})}\le\frac{\nu_3}{2}\|\,\mathrm{curl}(\tilde\vartheta)\|^2_{L^2(\mathcal{O})}+C_2\|(\tilde\vartheta)\|^2_{L^2(\mathcal{O})},\tag{51}
$$

$$
2\nu_r\|\,\mathrm{curl}(\tilde\omega)\|_{L^2(\mathcal{O})}\|\tilde\vartheta\|_{L^2(\mathcal{O})}\le 2\nu_r\sqrt{2}\|\tilde\omega\|_{H^1(\mathcal{O})}\|\tilde\vartheta\|^2_{L^2(\mathcal{O})}.\tag{52}
$$

Substituting (48)-(52) into (46) and (47), and adding them together, we get the following estimate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\tilde{\omega}\|^2_{L^2(\mathcal{O})}+\|\tilde{\vartheta}\|^2_{L^2(\mathcal{O})}) &\leq (2\nu_r\sqrt{2}\|\tilde{\omega}\|_{H^1(\mathcal{O})}+C_1+C_2)\|\tilde{\vartheta}\|^2_{L^2(\mathcal{O})}\\ +(C_K\|\tilde{\omega}\|_{H^1(\mathcal{O},\mathbb{R}^d)}\|\omega_2\|_{L^4(\mathcal{O},\mathbb{R}^d)} &+ C_K\|\tilde{\vartheta}\|_{H^1(\mathcal{O},\mathbb{R}^d)}\|\vartheta_2\|_{H^1(\mathcal{O},\mathbb{R}^d)}+2\nu_1 h_1)\|\tilde{\omega}\|^2_{L^2(\mathcal{O})}.\end{aligned} \tag{53}$$

Applying Gronwall's lemma to (53) and using the initial conditions $(\omega_0, w_0) = (0, 0)$, we deduce that $\tilde{\omega} = \tilde{\vartheta} = 0$, which proves the uniqueness.

3.1. Regularity Estimates

Testing equation (1.1) by ω and equation (1.2) by ϑ, we obtain the following energy inequalities:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2(\Omega)} \leq 2\sqrt{2}\nu_r\|\nabla\vartheta\|^2_{L^2(\Omega)} + \big(2\sqrt{2}\nu_r + 2\nu_1\|h(x,\|\omega_\tau\|)\|_{L^\infty(\Omega)}+1\big)\|\omega\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}, \tag{54}$$

$$\frac{1}{2}\frac{d}{dt}\|\vartheta\|^2_{L^2(\Omega)} \leq (1+\sqrt{2}\nu_3 - 2\nu_r)\|\vartheta\|^2_{L^2(\Omega)} + (3\nu_3-\nu_2)\|\nabla\vartheta\|^2_{L^2(\Omega)} + \nu_3\sqrt{2}\|\nabla\omega\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}. \tag{55}$$

Adding inequalities (54) and (55), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\omega\|^2_{L^2(\Omega)}+\|\vartheta\|^2_{L^2(\Omega)}\Big) \leq{}& (1+\sqrt{2}\nu_3-2\nu_r)\|\vartheta\|^2_{L^2(\Omega)}\\ &+ \big(2\sqrt{2}\nu_r+2\nu_1\|h(x,\|\omega_\tau\|)\|_{L^\infty(\Omega)}+1\big)\|\omega\|^2_{L^2(\Omega)}\\ &+ C_1\big(\|\omega\|^2_{H^1(\Omega)}+\|\vartheta\|^2_{H^1(\Omega)}+\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)}\big), \quad C_1>0.\end{aligned} \tag{56}$$

By applying Gronwall's lemma to (56), we deduce the estimate

$$\|\omega\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\vartheta\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_2 e^{C_3T}\Big(\|\omega(0)\|^2_{L^2(\Omega)}+\|\vartheta(0)\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(Q)}+\|g\|^2_{L^2(Q)}\Big), \tag{57}$$

where $C_2, C_3 > 0$ are constants independent of time.
Integrating (56) over $[0, T]$ and using (57), there exists a positive constant C_4 such that

$$\|\omega\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))}+\|\vartheta\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C_4.$$

Hence, we conclude that

$$\omega \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \quad \vartheta \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)). \tag{58}$$

For the time derivatives $\partial_t\omega$ and $\partial_t\vartheta$, we consider the variational formulations:

$$\begin{aligned}\langle\partial_t\omega, v\rangle_{U^*,U} ={}& -\int_\Omega(\omega\cdot\nabla)\omega\cdot v\,dx + 2\nu_1\int_\Omega D(\omega):D(v)\,dx + 2\nu_1\int_{\Gamma_2} h(x,|u_\tau|)\omega_\tau\cdot v_\tau\,d\sigma\\ &+ 2\nu_r\int_\Omega \mathrm{curl}(\omega)\cdot v\,dx + \int_\Omega f\cdot v\,dx, \quad \forall v\in U,\end{aligned} \tag{59}$$

$$\begin{aligned}\langle\partial_t\vartheta, \varphi\rangle_{W^*,W} ={}& -\int_\Omega(\omega\cdot\nabla)\vartheta\cdot\varphi\,dx + \nu_3\int_\Omega \mathrm{curl}(\vartheta)\cdot\mathrm{curl}(\varphi)\,dx - (\nu_3+\nu_2)\int_\Omega\nabla\vartheta\cdot\nabla\varphi\,dx\\ &- 4\nu_r\int_\Omega\vartheta\cdot\varphi\,dx - 2\nu_r\int_\Omega\mathrm{curl}(\omega)\cdot\varphi\,dx + \nu_3\int_{\Gamma_2}(\mathrm{curl}\varphi\times n)\cdot\varphi\,d\sigma + \int_\Omega g\cdot\varphi\,dx, \quad \forall\varphi\in W.\end{aligned} \tag{60}$$

Using the above, the following estimates hold:

$$\begin{aligned}|\langle \partial_t \omega, v\rangle_{U^*,U}| \leq \|\omega\|_{L^4(\Omega)}\|\nabla\omega\|_{L^2(\Omega)}\|v\|_{L^4(\Omega)} + 2\nu_1\|D(\omega)\|_{L^2(\Omega)}\|D(v)\|_{L^2(\Omega)} \\ + 2\nu_r\|\mathrm{curl}(\omega)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + 2\nu_1 h_1\|\omega\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)},\end{aligned} \tag{61}$$

$$\begin{aligned}|\langle \partial_t \vartheta, \varphi\rangle_{W^*,W}| \leq \|\omega\|_{L^4(\Omega)}\|\nabla\vartheta\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)} + \nu_3\|\mathrm{curl}(\vartheta)\|_{L^2(\Omega)}\|\mathrm{curl}(\varphi)\|_{L^2(\Omega)} \\ + 2\nu_r\|\mathrm{curl}(\omega)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \nu_2\|\nabla\vartheta\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\ + 4\nu_r\|\vartheta\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \nu_3\|\mathrm{curl}(\vartheta)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.\end{aligned} \tag{62}$$

Consequently, the time derivatives satisfy the following bounds:

$$\|\partial_t \omega\|_{L^{\frac{4}{3}}(0,T;U^*(\Omega))} \leq C_5, \quad \|\partial_t \vartheta\|_{L^{\frac{4}{3}}(0,T;W^*(\Omega))} \leq C_6,$$

where $C_5, C_6 > 0$ are constants independent of time.

3.2. Long-time Behavior of the Solutions

Testing (1.1) by ω and (1.2) by ϑ, and applying Young's inequality, we deduce the following estimate:

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega(t)\|^2 + \|\vartheta(t)\|^2\right) \leq M, \quad M > 0,$$

which implies the uniform boundedness of the solutions as $t \to +\infty$:

$$\sup_{t\geq 0}\|\omega(t)\| \leq C, \quad \sup_{t\geq 0}\|\vartheta(t)\| \leq C,$$

where C is a positive constant. Hence, the energy of the system remains bounded in time, reflecting a balance between energy dissipation and input.

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