STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–13.
Published online in International Academic Press (www.IAPress.org)

Application of Ujlayan-Dixit Fractional Gamma with Two-Parameters Probability Distribution

Iqbal H. Jebril[1], Iqbal M. Batiha[1,2,*], Nadia Allouch[3]

[1] *Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan*
[2] *Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE*
[3] *Laboratory of pure and applied mathematics, University of Mostaganem, Algeria*

Abstract The main goal in this research is to use the Ujlayan-Dixit (UD) fractional derivative to generate a new fractional probability density function for the two-parameters gamma distribution, and develop some applications of this new distribution like cumulative distribution, survival and hazard functions. Additionally, we will establish other concepts and applications for continuous random variables by applying the UD fractional analogues of statistical measures including expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Finally, the UD fractional entropy measures, such as Shannon, Tsallis, and Rényi entropy, are explained.

Keywords Gamma with two parameters distribution, Continuous random variables, Probability distribution, UD fractional derivative, Entropy

AMS 2010 subject classifications 26A33,62E10

DOI: 10.19139/soic-2310-5070-2591

1. Introduction

Fractional calculus is a generalization of traditional calculus, extending the concepts of derivatives and integrals to non-integer (fractional) orders. This broadens the applicability of calculus to a variety of scientific and engineering problems [1, 2, 3]. Fractional integrals and fractional derivatives also provide a mathematical framework for modeling a wide range of phenomena that classical calculus cannot effectively describe. Therefore, several notions of fractional integrals and fractional derivatives have been proposed, such as Riemann-Liouville [4], Caputo [5], Caputo-Hadamard [6], and Conformable derivative [7]. Their versatility makes them increasingly important in modern science and engineering; for more information, consult the sources [4, 5, 6, 8].

Recently, a new type of fractional derivative was proposed by Dixit and Ujlayan [9, 10] known as "Ujlayan-Dixit (UD) fractional derivative" which transforms a fractional derivative into a convex combination of a function and its ordinary derivative. It is a relatively recent development in the field of fractional calculus, introduced as a novel mathematical operator to address some limitations of traditional fractional derivative definitions [11, 12, 13, 14]. The introduction of the UD fractional derivative represents a step towards expanding the versatility of fractional calculus. As researchers continue to explore its properties and improve its applications, it may provide new solutions to existing challenges in science and engineering.

A probability distribution is a mathematical function that describes the likelihood of different outcomes for a random variable. It provides a framework for modeling uncertainty and variability in real-world phenomena,

*Correspondence to: Junfeng Yang (Email: i.batiha@zuj.edu.jo). Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

and forms the backbone of probability theory and statistics. It enables researchers and practitioners to make predictions, test theories, and optimize systems in a variety of fields, from the natural sciences to artificial intelligence; see the references [15, 16, 17]. The gamma with two-parameters distribution is one of the most important continuous probability distributions, widely used in statistics, engineering, biology and other fields. It generalizes the exponential distribution and is particularly useful for modeling waiting times or sums of independent exponential random variables; further details can be found in the papers [18, 19, 20].

Fractional calculus plays a crucial role in understanding and modeling probability distributions, especially in complex, non-local, memory-dependent stochastic systems that are not adequately modeled using classical probability theory. It bridges the gap between classical probability models and more complex stochastic phenomena in the real world. Actually, the link between fractional calculus and probability theory has been the subject of interest for many researchers [21, 22, 23, 24]. Actually, the link between fractional calculus and probability theory has been the subject of interest for many researchers, so several articles have been published on it, some of which can be found in the following sources [25, 26]. In recent times, scientists used fractional derivatives in probability distributions, and the results have been interesting, especially when it came to the conformable fractional derivative, most applications have been established on conformable fractional probability distributions.

Abu Hammad *et al.* in [27] produced a fractional distribution and probability density functions for random variables by applying fractional differential equations. Subsequently, Jebril *et al.* in [28] discovered some properties and applications of the conformable fractional gamma with two-parameters distribution, and also proposed some entropy measures for this distribution. More recently, in [29], Alhribat *et al.* used the UD fractional differential equations to develop novel fractional distributions based on previously existing probability distributions including the exponential, Pareto, Levy, and Lomax distributions.

Motivated by the above mentioned works, in this research, we apply the UD fractional derivative to construct the fractional probability density function for two-parameters gamma distribution and determine certain properties and applications of this new distribution such as cumulative distribution, survival and hazard functions. Furthermore, other notions and applications for continuous random variables are developed using the UD fractional analogues of statistical measures which is expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Lastly, we provide the UD fractional entropy measures including Shannon, Tsallis and Rényi entropy.

2. Basic Concepts

This section will go over the fundamental concepts and properties of the UD fractional integral and UD fractional derivative; for more information, see [9, 10].

Definition 2.1
[10] The UD fractional derivative of order $\alpha \in [0,1]$ for a function $g:[0,+\infty) \to \mathbb{R}$, is defined by:

$$D^{\alpha} g(x) = \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} g\left(x e^{\frac{\varepsilon \alpha}{x}}\right) - g(x)}{\varepsilon}, \tag{1}$$

if limit exists. Also, if g is UD differentiable in the interval $(0,x)$ and for $x>0$ and $\alpha \in [0,1]$ such that $\lim_{x\to 0^+} g^{\alpha}(x)$ exist, then,

$$g^{\alpha}(0) = \lim_{x \to 0^+} g^{\alpha}(x),$$

Notice that,

$$D^{\alpha} g(x) = \frac{d^{\alpha} g}{dx^{\alpha}}.$$

Theorem 2.2
[9] Let $g:[0,+\infty) \to \mathbb{R}$ be a differentiable function. Then, the function g is UD differentiable, and

$$D^{\alpha} g(x) = (1-\alpha) g(x) + \alpha g'(x), \quad \alpha \in [0,1]. \tag{2}$$

Specifically, for $\alpha = 0$ we have $D^0 g(x) = g(x)$, and if $\alpha = 1$ we have $D^1 g(x) = g'(x)$.

Lemma 2.3
[10] Let $a, b \in \mathbb{R},\ x \geq 0$ and for $\alpha \in [0,1]$. The following lists the UD derivatives of some elementary real-valued differentiable:

- $D^\alpha(a) = (1-\alpha)a$.
- $D^\alpha((ax+b)^n) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}$.
- $D^\alpha(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}$.
- $D^\alpha(e^{ax+b}) = ((1-\alpha)+a\alpha)e^{ax+b}$.

Properties 2.4
[9] Let $g, h : [0,+\infty) \to \mathbb{R}$ be two differentiable functions and for $\alpha, \beta \in [0,1]$. Then, the properties of the UD fractional derivative are given by:

- The UD derivative is a linear operator, such that for all $\lambda, \gamma \in \mathbb{R}$, we have:
$$D^\alpha(\lambda g(x) + \gamma h(x)) = \lambda D^\alpha g(x) + \gamma D^\alpha h(x).$$
- The UD derivative satisfies the following product rule:
$$D^\alpha(g(x).h(x)) = (D^\alpha g(x))h(x) + \alpha(D^\alpha h(x))g(x).$$
Thus, The UD derivative does not satisfy the Leibnitz's rule, i.e.:
$$D^\alpha(g(x).h(x)) \neq h(x)D^\alpha g(x) + g(x)D^\alpha h(x).$$
- The UD derivative satisfies the following quotient rule:
$$D^\alpha(g(x).h(x)) = \frac{(D^\alpha g(x))h(x) - \alpha(D^\alpha g(x))h(x)}{(h(x))^2}, \quad \text{with } \ h(x) \neq 0.$$
- The UD derivative is a commutative operator, such that:
$$D^\alpha(D^\beta g(x)) = D^\beta(D^\alpha g(x)).$$
So, the UD derivative does not satisfy the semi-group property, i.e.:
$$D^\alpha(D^\beta g(x)) \neq D^{\alpha+\beta} g(x).$$

Definition 2.5
[9] The UD fractional integral of order $\alpha \in [0,1]$ for a function $g : [a,b] \to \mathbb{R}$, is defined by:
$$I_a^\alpha g(x) = \frac{1}{\alpha}\int_a^x e^{\frac{(1-\alpha)}{\alpha}(s-x)} g(s)ds.$$

Properties 2.6
[9] Let g, h be two continuous functions and for $\alpha, \beta \in [0,1]$. Then, the properties of the UD fractional integral are given by:

- The UD integral is a linear operator, such that for all $\lambda, \gamma \in \mathbb{R}$, we have:
$$I_a^\alpha(\lambda g(x) + \gamma h(x)) = \lambda I_a^\alpha g(x) + \gamma I_a^\alpha h(x).$$
- The UD integral is a commutative operator, such that:
$$I_a^\alpha(I_a^\beta g(x)) = I_a^\beta(I_a^\alpha g(x)).$$
Thus, the UD integral does not satisfy the semi-group property, i.e.:
$$I_a^\alpha(I_a^\alpha g(x)) \neq I_a^{2\alpha} g(x).$$

3. Main Results

in this section, we use the UD derivative to present the main results on the fractional probability density function of the gamma with two-parameters distribution, as well as developing some applications for this new distribution.

3.1. The UD Fractional Gamma Distribution (UDFGD)

[18] A continuous random variable X is said to have a gamma with two-parameters distribution if its probability density function is defined by:

$$g(x,k,\theta)=\frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\Gamma(k)};\quad x>0, \tag{3}$$

where $k>0$ is the shape parameter and $\theta>0$ is the scale parameter, $\Gamma(x)=\int_0^{+\infty}s^{x-1}e^{-s}ds$ is the gamma function.
Now, we take $y=\frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\Gamma(k)}$. Hence, the first derivative of y is provided by:

$$y'=\frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\Gamma(k)}\left(\frac{k-1}{x}-\frac{1}{\theta}\right)=\left(\frac{k-1}{x}-\frac{1}{\theta}\right)y.$$

So, it can be written as follows:

$$y'-\left(\frac{k-1}{x}-\frac{1}{\theta}\right)y=0. \tag{4}$$

Thus, the equation (4) is a first-order ordinary differential equation.
Next, we consider the α-order differential equation with respect to the UD derivative in the following manner:

$$\begin{aligned} y^{(\alpha)}-\left(\frac{k-1}{x}-\frac{1}{\theta}\right)y &= 0,\\ (1-\alpha)y+\alpha y'-\left(\frac{k-1}{x}-\frac{1}{\theta}\right)y &= 0,\\ \alpha y'+\left(1-\alpha-\frac{k-1}{x}+\frac{1}{\theta}\right)y &= 0,\\ y'+\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}-\frac{k-1}{\alpha x}\right)y &= 0. \end{aligned} \tag{5}$$

Thus, the equation (5) is a linear first-order differential equation with an integrating factor

$$\begin{aligned} \psi(x) &= e^{\int\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}-\frac{k-1}{\alpha x}\right)dx},\\ &= e^{\frac{\theta(1-\alpha)+1}{\alpha\theta}x-\frac{k-1}{\alpha}lnx},\\ &= x^{-\frac{k-1}{\alpha}}e^{\frac{\theta(1-\alpha)+1}{\alpha\theta}x}. \end{aligned}$$

Therefore, the general solution to the equation (5) can be expressed by:

$$\begin{aligned} y &= \frac{\mathcal{C}}{\psi(x)},\\ &= \mathcal{C}x^{\frac{k-1}{\alpha}}e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}. \end{aligned}$$

Consequently, we give the new probability distribution as:

$$g_\alpha(x)=\mathcal{C}x^{\frac{k-1}{\alpha}}e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}. \tag{6}$$

To determine the normalization constant $\mathcal{C}$, the following equation can be solved:

$$\int_0^{+\infty} g_\alpha(x)dx = 1.$$

This implies that,

$$\int_0^{+\infty} \mathcal{C}\, x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x} dx = 1,$$

$$\mathcal{C}\, \frac{\Gamma\left(\frac{k-1}{\alpha}+1\right)}{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}} = 1.$$

As a result, the normalization constant $\mathcal{C}$ will be:

$$\mathcal{C} = \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}.$$

The α-gamma with two-parameters distribution has the UD fractional probability density function (UDFPDF), which can be written in the following format:

$$g_\alpha(x) = \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}, \quad x > 0,\ k > 0,\ \theta > 0,\ 0 < \alpha < 1. \tag{7}$$

Note that, for $\alpha \to 1^-$, we have:

$$\lim_{\alpha\to 1^-} g_\alpha(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\Gamma(k)} = g(x,k,\theta). \tag{8}$$

A comparison between the classical case of the probability density function (PDF) for the gamma distribution and the UD fractional probability density function (UDFPDF) of the α-gamma distribution for $\alpha = 1$ and $k = 2,\ \theta = 5$ can be illustrated in Fig. 1. Then, the UD fractional probability density function (UDFPDF) for the α-gamma distribution can be plotted by taking different values of α according to $k = 2$ and $\theta = 5$ in Fig. 2.

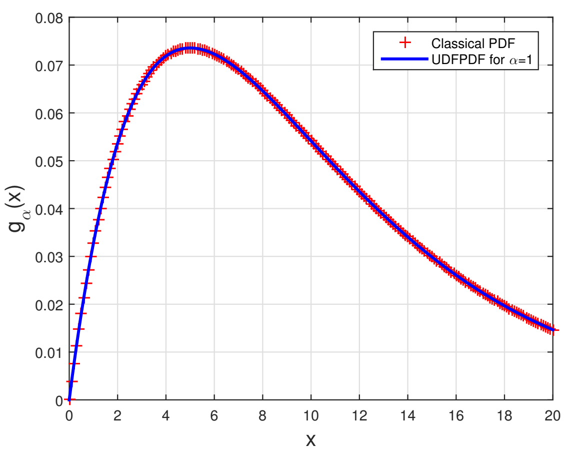

Figure 1. Comparison of the classical PDF of gamma distribution with the UDFPDF of α-gamma distribution for $\alpha = 1$ and $k = 2,\ \theta = 5$.

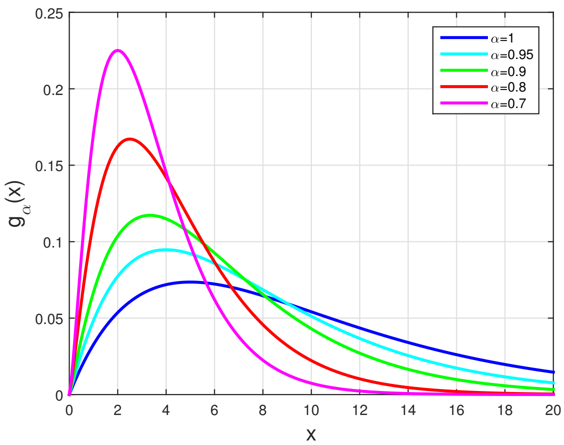

Figure 2. the UDFPDF of α-gamma distribution for different values of α according to $k = 2$ and $\theta = 5$.

3.2. Applications of The UD Fractional Gamma Distribution

Based on these sources [28, 30], in this part we establish novel applications of the UD fractional probability for the α-gamma distribution to the probabilistic random variables.

3.2.1. The UD fractional cumulative distribution function For the α-gamma distribution, we find the UD fractional cumulative distribution function (UDFCDF) as follows:

$$\mathcal{G}_\alpha(x) = \frac{(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-1}{\alpha}+1, \frac{x}{\alpha\theta}\right), \tag{9}$$

where $\gamma(x, y) = \int_0^y s^{x-1}e^{-s}ds$ is lower incomplete gamma function. In actuality,

$$\begin{aligned}
\mathcal{G}_\alpha(x) &= \mathcal{P}_\alpha(X \leq x), \\
&= I_0^\alpha g_\alpha(x), \\
&= \frac{1}{\alpha}\int_0^x e^{\frac{(1-\alpha)}{\alpha}(s-x)} g_\alpha(s)ds, \\
&= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} \int_0^x e^{\frac{(1-\alpha)}{\alpha}(s-x)} s^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}s} ds, \\
&= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x} \int_0^x s^{\frac{k-1}{\alpha}} e^{-\frac{s}{\alpha\theta}} ds,
\end{aligned}$$

By using the variable change $v = \frac{s}{\alpha\theta}$, we obtain:

$$\begin{aligned}
\mathcal{G}_\alpha(x) &= \frac{(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x} \int_0^{\frac{x}{\alpha\theta}} v^{\frac{k-1}{\alpha}} e^{-v} dv, \\
&= \frac{(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-1}{\alpha}+1, \frac{x}{\alpha\theta}\right).
\end{aligned}$$

In particular case,

$$\lim_{\alpha\to 1^-}\mathcal{G}_\alpha(x)=\frac{\gamma\left(k,\frac{x}{\theta}\right)}{\Gamma\left(k\right)}=\mathcal{G}(x). \tag{10}$$

where $\mathcal{G}$ is the classical cumulative distribution function (CDF) for the gamma distribution.
In Fig. 3, we compare the classical case of the cumulative distribution function (CDF) for the gamma distribution with the UD fractional cumulative distribution function (UDFCDF) of the α-gamma distribution for $\alpha=1$ and $k=2,\ \theta=5$. In Fig. 4, we display the UD fractional cumulative distribution function (UDFCDF) for the α-gamma distribution under different values of α according to $k=2$ and $\theta=5$.

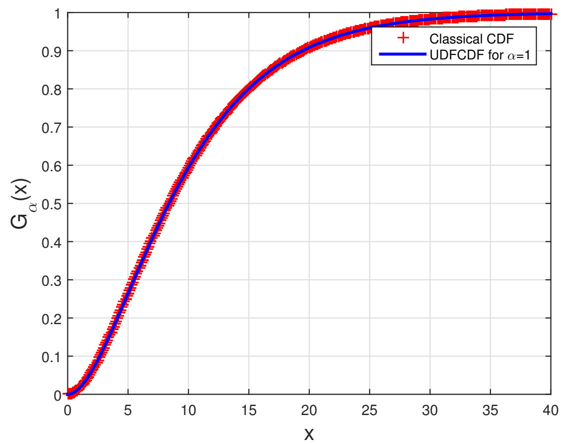

Figure 3. Comparison of the classical CDF of gamma distribution with the UDFCDF of α-gamma distribution for $\alpha=1$ and $k=2,\ \theta=5$.

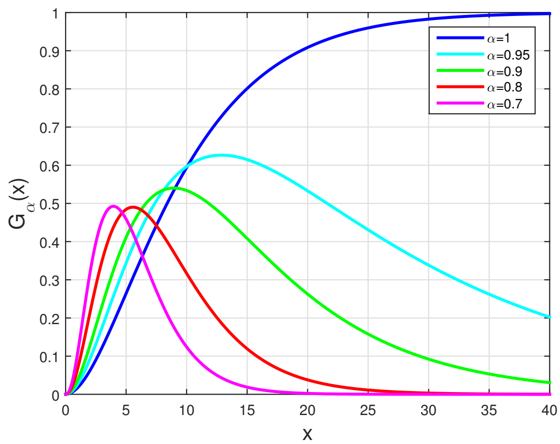

Figure 4. the UDFCDF of the α-gamma distribution for various values of α according to $k=2$ and $\theta=5$.

3.2.2. The UD fractional survival distribution function For the α-gamma distribution, we define the UD fractional survival distribution function (UDFSDF) of X by:

$$\begin{aligned}\mathcal{S}_\alpha(x) &= 1-\mathcal{G}_\alpha(x),\\ &= 1-\frac{(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)}e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-1}{\alpha}+1,\frac{x}{\alpha\theta}\right).\end{aligned} \tag{11}$$

If $\alpha\to 1^-$ in the formula (11), then we get the classical survival distribution function (SDF) for the gamma distribution, i.e.:

$$\lim_{\alpha\to 1^-}\mathcal{S}_\alpha(x)=1-\frac{\gamma\left(k,\frac{x}{\theta}\right)}{\Gamma\left(k\right)}=\mathcal{S}(x). \tag{12}$$

Fig. 5 shows the comparison between the classical case of the survival distribution function (SDF) for the gamma distribution and the UD fractional survival distribution function (UDFSDF) of the α-gamma distribution for $\alpha=1$ according to $k=2$ and $\theta=5$. Fig. 6 also shows survival distribution function (UDFSDF) for the α-gamma distribution by taking various values of α according to $k=2$ and $\theta=5$.

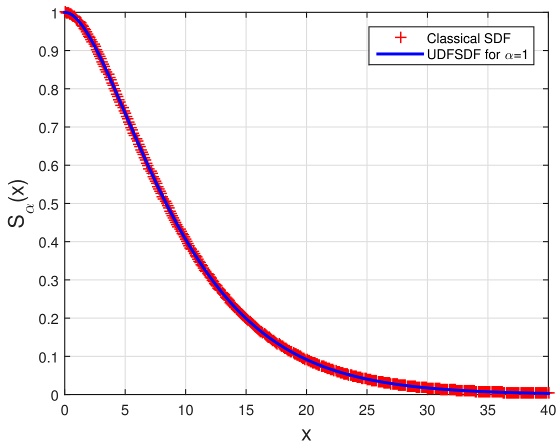

Figure 5. Comparison of the classical SDF of the gamma distribution with the UDFSDF of the α-gamma distribution for $\alpha = 1$ and $k = 2,\ \theta = 5$.

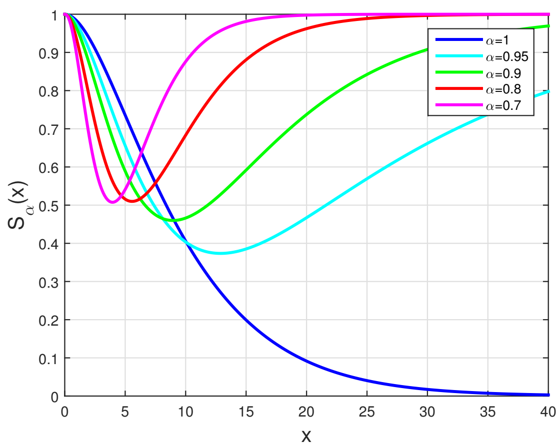

Figure 6. the UDFSDF of the α-gamma distribution for different values of α according to $k = 2$ and $\theta = 5$.

3.2.3. The UD fractional hazard distribution function For the α-gamma distribution, we give the UD fractional hazard distribution function (UDFHDF) of X as:

$$\begin{aligned} h_\alpha(x) &= \frac{g_\alpha(x)}{S_\alpha(x)}, \\ &= \frac{\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}}{1-\frac{(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-1}{\alpha}+1, \frac{x}{\alpha\theta}\right)}, \\ &= \frac{\alpha\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1} x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}}{\alpha\Gamma\left(\frac{k-1}{\alpha}+1\right)-(\theta(1-\alpha)+1)^{\frac{k-1}{\alpha}+1} e^{\frac{(\alpha-1)}{\alpha}x}\,\gamma\left(\frac{k-1}{\alpha}+1, \frac{x}{\alpha\theta}\right)}. \end{aligned} \tag{13}$$

Notice that, for $\alpha \to 1^-$ in the formula (13), we find the classical hazard distribution function (HDF) for the gamma distribution , i.e.:

$$\lim_{\alpha\to 1^-} h_\alpha(x) = \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\theta^k\left(\Gamma\left(k\right) - \gamma\left(k, \frac{x}{\theta}\right)\right)} = h(x), \tag{14}$$

Actually, we can explain the graphical comparison between the UD fractional hazard distribution function (UDFHDF) of the α-gamma distribution for $\alpha = 1$ and the classical case of the hazard distribution function (HDF) of gamma distribution according to $k = 2$ and $\theta = 5$ in Fig. 3. Also, we can plot the UD fractional hazard distribution function (UDFHDF) of the α-gamma distribution for different values of α according to $k = 2$ and $\theta = 5$ as shown in Fig. 8.

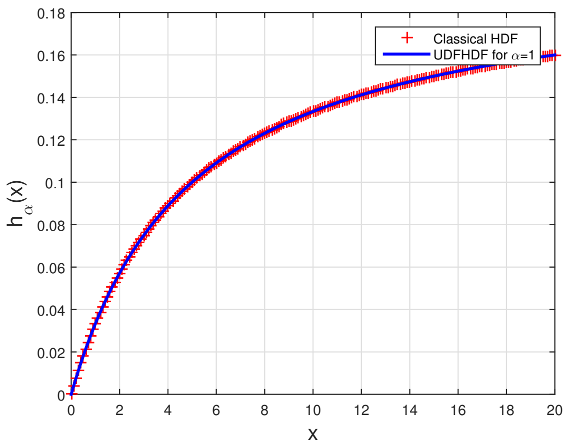

Figure 7. Comparison of the classical HDF of the gamma distribution with the UDFHDF of the α-gamma distribution for $\alpha = 1$ and $k = 2,\ \theta = 5$.

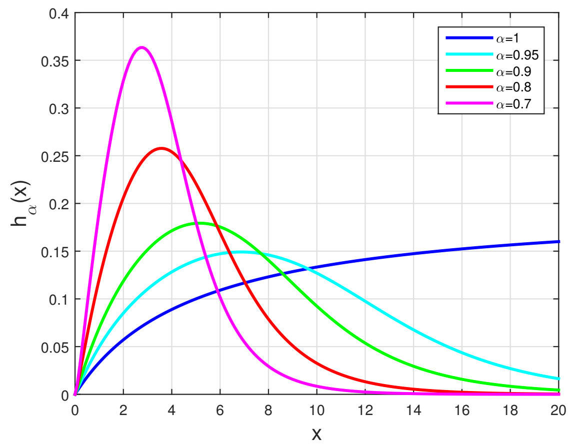

Figure 8. the UDFHDF of the α-gamma distribution for different values of α according to $k = 2$ and $\theta = 5$.

3.2.4. The UD fractional expectation

• *The r^{th} UD fractional moment* $\mathbb{E}_\alpha[X^r]$ For α-gamma distribution, the UD fractional moment of order r denote by $\mathbb{E}_\alpha[X^r]$ of continuous random variable X whose $g_\alpha(x)$ is given by:

$$\begin{aligned}
\mathbb{E}_\alpha[X^r] &= \int_0^{+\infty} x^r g_\alpha(x)dx, \\
&= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} \int_0^{+\infty} x^r x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}dx, \\
&= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} \int_0^{+\infty} x^{r+\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}dx, \\
&= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} \frac{\Gamma\left(\frac{k-1}{\alpha}+r+1\right)}{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+r+1}}, \\
&= \frac{\left(\frac{\alpha\theta}{\theta(1-\alpha)+1}\right)^r \Gamma\left(\frac{k-1}{\alpha}+r+1\right)}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}.
\end{aligned} \tag{15}$$

If $r = 1$, then we get the UD fractional expectation $\mathbb{E}_\alpha[X]$:

$$\begin{aligned}\mathbb{E}_\alpha[X] &= \frac{\left(\frac{\alpha\theta}{\theta(1-\alpha)+1}\right)\Gamma\left(\frac{k-1}{\alpha}+2\right)}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}, \\ &= \frac{\theta(k-1+\alpha)}{\theta(1-\alpha)+1}, \end{aligned} \tag{16}$$

we notice that, for $\alpha \to 1^-$ we have the classical expectation $\mathbb{E}[X]$, i.e.:

$$\lim_{\alpha\to 1^-} \mathbb{E}_\alpha[X] = k\theta = \mathbb{E}[X]. \tag{17}$$

If $r = 2$, then we have:

$$\begin{aligned}\mathbb{E}_\alpha[X^2] &= \frac{\left(\frac{\theta\alpha}{\theta(1-\alpha)+1}\right)^2\Gamma\left(\frac{k-1}{\alpha}+3\right)}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}., \\ &= \frac{\theta^2(k-1+2\alpha)(k-1+\alpha)}{(\theta(1-\alpha)+1)^2}. \end{aligned} \tag{18}$$

For $\alpha \to 1^-$ in the formula (15), we find the classical the r^{th} moment $\mathbb{E}[X^r]$, i.e.:

$$\lim_{\alpha\to 1^-} \mathbb{E}_\alpha[X^r] = \frac{\theta^r\Gamma(k+r)}{\Gamma(k)} = \mathbb{E}[X^r]. \tag{19}$$

• *The r^{th} UD fractional central moment $\mathbb{E}_\alpha(X-\mu)^r$* Let us take:

$$\mu = \mathbb{E}_\alpha[X] = \frac{\theta(k-1+\alpha)}{\theta(1-\alpha)+1}. \tag{20}$$

For the α-gamma distribution, the r^{th} UD fractional central moment $\mathbb{E}_\alpha(X-\mu)^r$ of X is defined by:

$$\mathbb{E}_\alpha(X-\mu)^r = \int_0^{+\infty} (x-\mu)^r g_\alpha(x)dx. \tag{21}$$

To find the first and second central moments, we can apply the formula (21), such that:

1) First central moment:

$$\mathbb{E}_\alpha(X-\mu) = 0. \tag{22}$$

2) Second central moment:

$$\mathbb{E}_\alpha(X-\mu)^2 = \frac{\alpha\theta^2(k-1+\alpha)}{(\theta(1-\alpha)+1)^2}. \tag{23}$$

• *The UD fractional variance Var_α* For the α-gamma distribution, the UD fractional variance Var_α of X is given by:

$$\begin{aligned}Var_\alpha(X) &= \mathbb{E}_\alpha(X^2) - (\mathbb{E}_\alpha(X))^2, \\ &= \frac{\theta^2(k-1+2\alpha)(k-1+\alpha)}{(\theta(1-\alpha)+1)^2} - \left(\frac{\theta(k-1+\alpha)}{\theta(1-\alpha)+1}\right)^2, \\ &= \frac{\alpha\theta^2(k-1+\alpha)}{(\theta(1-\alpha)+1)^2}. \end{aligned} \tag{24}$$

If $\alpha \to 1^-$ in the above formula, then we get the classical variance of X; i.e.:

$$\lim_{\alpha\to 1^-} Var_\alpha(X) = k\theta^2 = Var(X). \tag{25}$$

• *The UD fractional standard deviation* σ_α For the α-gamma distribution, we give the UD fractional standard deviation σ_α of X by:

$$\begin{aligned}\sigma_\alpha &= \sqrt{Var_\alpha(X)},\\ &= \sqrt{\frac{\alpha\theta^2(k-1+\alpha)}{(\theta(1-\alpha)+1)^2}},\\ &= \frac{\theta\sqrt{\alpha(k-1+\alpha)}}{\theta(1-\alpha)+1}.\end{aligned} \tag{26}$$

For $\alpha \to 1^-$, we have the classical standard deviation of X, such that:

$$\lim_{\alpha\to 1^-} \sigma_\alpha = \theta\sqrt{k} = \sigma. \tag{27}$$

3.2.5. The UD fractional Entropy Measures

• *The UD fractional Shannon entropy* $\alpha\mathcal{H}$ For the α-gamma distribution, we define the UD fractional Shannon entropy $\alpha\mathcal{H}$ of a random variable X as follows:

$$\alpha\mathcal{H}(X) = -\int_0^{+\infty} g_\alpha(x)\log(g_\alpha(x))dx. \tag{28}$$

First, we start by calculating the following quantity:

$$\begin{aligned}\log(g_\alpha(x)) &= \log\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)} x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}\right),\\ &= \log\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right) + \frac{k-1}{\alpha}\log x - \frac{\theta(1-\alpha)+1}{\alpha\theta}x.\end{aligned}$$

Thus,

$$\begin{aligned}\alpha\mathcal{H}(X) &= -\int_0^{+\infty} g_\alpha(x)\log(g_\alpha(x))dx,\\ &= -\int_0^{+\infty} g_\alpha(x)\left[\log\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right) + \frac{k-1}{\alpha}\log x - \frac{\theta(1-\alpha)+1}{\alpha\theta}x\right]dx,\\ &= \frac{\theta(1-\alpha)+1}{\alpha\theta}\int_0^{+\infty} x\, g_\alpha(x)dx - \log\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right)\int_0^{+\infty} g_\alpha(x)dx\\ &- \frac{k-1}{\alpha}\int_0^{+\infty}\log x\, g_\alpha(x)dx,\end{aligned}$$

Then, we simplify the following terms:

$$\int_0^{+\infty} g_\alpha(x)dx = 1,$$

$$\int_0^{+\infty} x\, g_\alpha(x)dx = \mathbb{E}_\alpha[X] = \frac{\theta(k-1+\alpha)}{\theta(1-\alpha)+1},$$

and

$$\begin{aligned}\int_0^{+\infty} \log x\, g_\alpha(x)dx &= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\int_0^{+\infty} \log x\, x^{\frac{k-1}{\alpha}} e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}dx,\\ &= \frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\frac{\Gamma\left(\frac{k-1}{\alpha}+1\right)}{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}\left[\psi\left(\frac{k-1}{\alpha}+1\right)-\log\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)\right],\\ &= \psi\left(\frac{k-1}{\alpha}+1\right)-\log\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right),\end{aligned}$$

where $\psi(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)})$ is digamma function.
Consequently, the UD fractional Shannon entropy $\alpha\mathcal{H}$ can be expressed by:

$$\alpha\mathcal{H}(X) = \frac{k-1+\alpha}{\alpha}+\log\left(\frac{\alpha\theta}{\theta(1-\alpha)+1}\Gamma\left(\frac{k-1}{\alpha}+1\right)\right)-\frac{k-1}{\alpha}\psi\left(\frac{k-1}{\alpha}+1\right). \tag{29}$$

Notice that,

$$\lim_{\alpha\to 1^-}\alpha\mathcal{H}(X) = k+\log\left(\theta\Gamma\left(k\right)\right)+(1-k)\,\psi\left(k\right) = \mathcal{H}(X). \tag{30}$$

where $\mathcal{H}$ the classical Shannon entropy of X for the gamma distribution.

• *The UD fractional Tsallis entropy $\alpha\mathcal{T}_q$* For the α-gamma distribution, we give the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$ of a random variable X by:

$$\alpha\mathcal{T}_q(X) = \frac{1}{q-1}\left[1-\int_0^{+\infty}\left(g_\alpha(x)\right)^q dx\right]. \tag{31}$$

Now, we calculate the integral $\int_0^{+\infty}\left(g_\alpha(x)\right)^q dx$:

$$\begin{aligned}\int_0^{+\infty}\left(g_\alpha(x)\right)^q dx &= \int_0^{+\infty}\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}x^{\frac{k-1}{\alpha}}e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}x}\right)^q dx,\\ &= \left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right)^q\int_0^{+\infty} x^{\frac{k-1}{\alpha}q}e^{-\frac{\theta(1-\alpha)+1}{\alpha\theta}qx},\\ &= \left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right)^q\frac{\Gamma\left(q\frac{k-1}{\alpha}+1\right)}{\left(q\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{q\frac{k-1}{\alpha}+1}}.\end{aligned}$$

Then, we find the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$:

$$\alpha\mathcal{T}_q(X) = \frac{1}{q-1}\left[1-\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right)^q\frac{\Gamma\left(q\frac{k-1}{\alpha}+1\right)}{\left(q\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{q\frac{k-1}{\alpha}+1}}\right]. \tag{32}$$

As $q \to 1$, the UD fractional Tsallis entropy $\alpha\mathcal{T}_q$ reduces to the UD fractional Shannon entropy $\alpha\mathcal{H}$, i.e:

$$\lim_{q\to 1} \alpha\mathcal{T}_q(X) = \alpha\mathcal{H}(X). \tag{33}$$

If $\alpha \to 1^-$ in the formula (32), then we get the classical Tsallis entropy $\mathcal{T}_q$ of X for the gamma distribution.

$$\lim_{\alpha\to 1^-} \alpha\mathcal{T}_q(X) = \frac{1}{q-1}\left[1 - \frac{\left(\frac{\theta}{q}\right)^{q(k-1)+1}\Gamma\left(q\left(k-1\right)+1\right)}{\left(\theta^k\Gamma\left(k\right)\right)^q}\right] = \mathcal{T}_q(X), \tag{34}$$

• *The UD fractional Rényi entropy* $\alpha\mathcal{R}_q$ For the α-gamma distribution, we obtain the UD fractional Rényi entropy $\alpha\mathcal{R}_q$ of a random variable X as follows:

$$\begin{aligned} \alpha\mathcal{R}_q(X) &= \frac{1}{1-q}\log\left(\int_0^{+\infty}[f_\alpha(x)]^q dx\right), \\ &= \frac{1}{1-q}\log\left(\left(\frac{\left(\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{\frac{k-1}{\alpha}+1}}{\Gamma\left(\frac{k-1}{\alpha}+1\right)}\right)^q \frac{\Gamma\left(q\frac{k-1}{\alpha}+1\right)}{\left(q\frac{\theta(1-\alpha)+1}{\alpha\theta}\right)^{q\frac{k-1}{\alpha}+1}}\right). \end{aligned} \tag{35}$$

As $q \to 1$, the UD fractional Rényi entropy $\alpha\mathcal{R}_q$ reduces to the UD fractional Shannon entropy $\mathcal{H}_q$, i.e:

$$\lim_{q\to 1} \alpha\mathcal{R}_q(X) = \alpha\mathcal{H}(X). \tag{36}$$

Particularly, for $\alpha \to 1^-$ in the formula (35), we have the classical Rényi entropy $\mathcal{R}_q$ of X for the gamma distribution, such that:

$$\lim_{\alpha\to 1^-} \alpha\mathcal{R}_q(X) = \frac{1}{1-q}\log\left(\frac{\left(\frac{\theta}{q}\right)^{q(k-1)+1}\Gamma\left(q\left(k-1\right)+1\right)}{\left(\theta^k\Gamma\left(k\right)\right)^q}\right) = \mathcal{R}_q(X). \tag{37}$$

4. Conclusion

In this research, we have apply the UD fractional derivative to provide a new fractional probability density function (FPDF) for the two-parameters gamma distribution, and we also create certain applications of the α-gamma distribution such as cumulative distribution, survival and hazard functions with graphical representation of each application. In addition, we have develop notions and applications for continuous random variable using the UD fractional analogues of statistical measures which is expectation, r^{th}-moments, r^{th}-central moments, variance and standard deviation. Lastly, we have establish the UD fractional entropy measures like Shannon, Tsallis and Rényi entropy.

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