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Almost Sure Asymptotic Stability for a Fractional Stochastic Nonlinear Heat Equation In Hilbert Space

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Abstract This paper continues the investigation of a fractional stochastic nonlinear heat equation defined in the Hilbert space $L^2(0,1)$, driven by a fractional power of the Laplacian and perturbed by trace-class noise. In our previous work, we established the well-posedness, the p-th moment exponential stability, and the almost sure exponential stability of the problem under a semigroup framework. In the present study, we extend these results by proving the almost sure asymptotic stability under the same structural assumptions, supplemented with a regularity condition on the initial data. Several examples are provided to illustrate the theoretical findings.

Keywords Almost sure asymptotic stability, nonlinear heat equation, fractional power of the Laplacian, trace-class noise

AMS 2010 subject classifications 60H15, 60G18, 35R60

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1. Introduction

The study of fractional-order partial differential equations (PDEs) has garnered significant attention due to their ability to model complex systems that exhibit memory and hereditary characteristics, commonly observed in viscoelastic materials, diffusion processes, and anomalous transport phenomena. These fractional models, when integrated with stochastic perturbations, offer more realistic descriptions of physical, biological, and engineering systems where uncertainty and noise are inherent.
In practical scenarios, many dynamical systems governed by heat-like processes are influenced not only by spatially varying parameters but also by inherent randomness due to environmental factors or system complexity. When such systems involve long-range dependence or memory, fractional differential operators become essential. The combination of fractional dynamics with stochastic effects gives rise to fractional stochastic partial differential equations (FSPDEs), see [2, 3, 6, 10, 11, 13, 14, 15, 16, 17, 24] and the references therein.
Understanding the stability properties of these equations is crucial for both theoretical insight and practical control. In Particular, almost sure asymptotic stability provides stronger guaranties than moment-based criteria, ensuring that sample paths themselves converge to equilibrium with probability one. In the literature there are many papers concerned with different types of stability of stochastic fractional (partial) differential equations; see for a short list [4, 5, 12, 18, 20, 21, 22, 25, 27]. Despite recent progress, results on asymptotic stability for FSPDEs, especially those perturbed by multiplicative trace-class noise, remain limited. This gap motivates the current work. Fractional

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stochastic PDEs (FSPDEs) are natural extensions of classical stochastic PDEs (SPDEs), incorporating nonlocal diffusion terms via fractional powers of differential operators. The nonlinear stochastic heat equation, driven by a fractional Laplacian and subject to multiplicative noise, serves as a prototypical model in this context. While numerous contributions have addressed the well-posedness and moment stability of such equations, their almost sure asymptotic behavior is still under-explored.

In our recent work [8], we have established the existence, uniqueness, and exponential stability (both in the p-th moment and in the almost sure sense) of the mild solution for a fractional stochastic nonlinear heat equation posed on the Hilbert space $L^2(0,1)$. The equation was governed by a semigroup generated by the fractional power of the Laplacian and perturbed by a trace-class noise. The current paper serves as a natural extension of that study. Our primary goal is to prove the almost sure asymptotic stability of the same system under the previously imposed assumptions, along with an additional regularity condition on the initial state. This form of stability ensures that, with probability one, trajectories of the solution tend to zero as time goes to infinity, a stronger notion than exponential stability in expectation.

We conclude this section by introducing the model under consideration and the notational framework used throughout the paper. Let us consider the following fractional stochastic nonlinear heat equation:

$$\begin{cases} du(t) &= (-A_\alpha u(t) + F(u(t)))\,dt + G(u(t))\,dW(t),\ t \ \in \ (0,T], \\ u(0) &= u_0. \end{cases} \tag{1.1}$$

where:

- For the minus Laplacian $A := -\frac{\partial^2}{\partial^2 x}$ endowed with Dirichlet boundary conditions, the operator $A_\alpha := (-\frac{\partial^2}{\partial^2 x})^{\frac{\alpha}{2}} = A^{\frac{\alpha}{2}}$, $\alpha \in (1,2]$ is its fractional version,
- $F : L^2(0,1) \to L^2(0,1)$ be a nonlinear operator,
- $G : L^2(0,1) \to \mathcal{L}(L^2(0,1))$ be an operator (not necessarily linear),
- W is a $L^2(0,1)$-valued cylindrical Wiener process,
- u_0 is a $L^2(0,1)$-valued $\mathcal{F}_0$-measurable random variable,

and,

- $\mathbb{N}_0 := \mathbb{N} - \{0\}$,
- The Hilbert space $L^2(0,1)$, its norm and its inner product are denoted by H, $|.|_H$ and $\langle .,. \rangle_H$, respectively,
- H_2^η is the fractional Sobolev space of order η,
- For a Banach space E, we denote its norm by $\|.\|_E$,
- $\mathcal{L}(H)$ is the space of all linear and bounded operators defined in H into itself,
- HS is the space of all Hilbert-Schmidt operators defined from H into itself,
- For a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ being a normal filtration, $L^p(\Omega, E)$ is the space of all E-valued p-th integrable random variables on Ω,
- $\Lambda([0,T]; H) := \{v \in C([0,T]; L^p(\Omega, H))$, such that v is $\mathbb{F} - adapted\}$ is a Banach space, its norm is defined by $\|v\|_\Lambda := \sup_{t\in[0,T]} \|v(t)\|_{L^p(\Omega,H)}$,
- $(S_\alpha(t) := e^{-tA_\alpha})_{t\in[0,T]}$ is the semigroup generated by the operator A_α,
- The abbreviations: Prb., Est., RHS, B-D-G, s.t. and ASA are used respectively for: problem, estimate, right hand side, Burkholder-Davis-Gundy, such that and almost sure asymptotic.

The rest of this paper is organized as follows. Section 2 provides the preliminary results and revisits the well-posedness. In Section 3, we present the main result on almost sure asymptotic stability. Section 4 illustrates our findings with examples.

2. Preliminaries

Firstly, we provide the meaning of the spectral fractional Laplacian and the mild solution.

Definition 2.1
(Spectral fractional Laplacian, [1, 7])
Let $\alpha > 1$. The spectral fractional Laplacian A_α is defined for any $u \in D(A_\alpha)$ by

$$A_\alpha u := \sum_{n\in\mathbb{N}_0} \lambda_n^{\frac{\alpha}{2}} \langle u, e_n\rangle_H e_n, \tag{2.1}$$

where $D(A_\alpha)$ is the domain of the definition of A_α given by

$$D(A_\alpha) := \{v \in H, \text{ s.t. } |v|^2_{D(A_\alpha)} := \sum_{n\in\mathbb{N}_0} \lambda_n^\alpha \langle v, e_n\rangle_H^2 < +\infty\},$$

and $(\lambda_n := (n\pi)^2)_{n\in\mathbb{N}_0}$ are the eigenvalues of A corresponding to the set of eigenfunctions $(e_n(.) := \sqrt{2}sin(\pi.))_{n\in\mathbb{N}_0}$.

Definition 2.2
(Mild solution, [7])
A H-valued predictable process $u := (u(t))_{t\in[0,T]}$ is said to be a mild solution of Prb.(1.1) if for $T > 0$ be fixed

- $\forall\, t \in [0, T]$, $S_\alpha(t - .)F(u(.)) : (0, t) \to H$ is Böchner integrable $\mathbb{P}$-a.s.,
- $\forall\, t \in [0, T]$, $1_{[0,t[}(.)S_\alpha(t - .)G(u(.)) : [0, T] \times \Omega \to HS$ is continuous predictable, where

$$\mathbb{P}\left(\int_0^T \|1_{[0,t[}(s)S_\alpha(t - s)G(u(s))\|^2_{HS} ds < \infty\right) = 1, \tag{2.2}$$

- $$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t - s)F(u(s))\, ds + \int_0^t S_\alpha(t - s)G(u(s))\, dW(s), \ \ \mathbb{P} - a.s., \tag{2.3}$$

  $\forall\, t \in [0, T]$.

In [8], we have imposed the following assumptions in order to prove the wellposdness of the problem.
$\mathcal{A}_F$- $F : H \to H$ is a nonlinear operator satisfies for all $u, v \in H$:

$$|F(u) - F(v)|_H \le C_F|u - v|_H, \tag{2.4}$$

and

$$|F(u)|_H \le C_F|u|_H, \tag{2.5}$$

for some $C_F > 0$.
$\mathcal{A}_G$- $G : H \to HS$ is an operator satisfies for all $u, v \in H$:

$$\|G(u) - G(v)\|_{HS} \le C_G|u - v|_H, \tag{2.6}$$

and

$$\|G(u)\|_{HS} \le C_G|u|_H, \tag{2.7}$$

for some $C_G > 0$.
$\mathcal{A}_{u_0}$- $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, for $p \ge 2$.

Remark 1
In Prb.(1.1), the Lipschitz conditions (i.e., $\mathcal{A}_F$ and $\mathcal{A}_G$) that have been imposed on its coefficients (drift and diffusion terms) ensure its wellposedness by preventing the rapid growth of its solution, which leads to its explosion. Heuristically, a Lipschitz-continuous coefficient behaves predictably, when the change in the solution is proportional to the change in the initial condition. Hence, this predictable behavior prevents solutions from diverging wildly, making the mathematical framework tractable and ensuring that distinct initial conditions lead to distinct, stable trajectories.

Theorem 2.3
(**Wellposedness**, [8]) Presume that the Assumptions $\mathcal{A}_F$, $\mathcal{A}_G$ and $\mathcal{A}_{u_0}$ are satisfied. If

$$C_F \frac{1}{\pi^\alpha} + C_p^{\frac{1}{p}} C_G \frac{1}{\sqrt{2\pi^\alpha}} < 1,$$

where $C_p^{\frac{1}{p}} := (\frac{p}{2}(p-1))^{\frac{1}{2}}(\frac{p}{p-1})^{(\frac{p}{2}-1)}$ for $p \geq 2$ and $\alpha \in (1,2]$. Then, Prb.(1.1) has an unique mild solution u in the space $\Lambda([0,T];H)$.

Theorem 2.4
(**Temporal regularity**, [9]) Let the mild solution u of Prb.(1.1) with $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H_2^\eta)$ (i.e. $(\mathbb{E}|A^{\frac{\eta}{2}}u_0|_H^p)^{\frac{1}{p}} < +\infty$), for $\eta > \frac{\alpha}{2}$. Then, under the Assumptions $\mathcal{A}_F$ and $\mathcal{A}_G$, u is time Hölder continuous with exponent less than $\frac{1}{2}$.

We require the following helpful lemmas.

Lemma 2.5
([8, Lemma 2]) Let $\alpha \in (1,2]$ and $t \in [0,T]$, for $T > 0$ be fixed. Then,

$$\|S_\alpha(t)\|_{\mathcal{L}(H)} \leq 1. \tag{2.8}$$

Lemma 2.6
([4]) Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an integrable continuous function, and let $\nu > 0$ be an arbitrary constant. Then, $\exists(\varrho_n)_{n=0}^{+\infty}$ be an increasing sequence, which starts from 0 and satisfies

- $\forall\, n \in \mathbb{N},\ \nu > \varrho_{n+1} - \varrho_n$.
- $\lim_{n\to+\infty} \varrho_n = +\infty$.
- $\sum_{n=0}^{+\infty} \varphi(\varrho_n) < +\infty$.

Lemma 2.7
Let $n \in \mathbb{N}_0$, $p \geq 2$ and let $c_i \geq 0$, $i \in \{1, ..., n\}$. It is true that

$$\left(\sum_{i=1}^{n} c_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} c_i^p. \tag{2.9}$$

3. Main result: Almost sure asymptotic stability

In this section, we study the ASA stability of Prb.(1.1), which has the following meaning.

Definition 3.1
([22, Definition 1.4.9]) The null (or trivial) solution of Prb.(1.1) is said to be ASA stable if

$$\lim_{t\to+\infty} |u(t)|_H = 0, \quad \mathbb{P}-a.s.$$

The proof of the ASA stability is based on the p^{th}-moment exponential stability, which has been proved in our work [8] under the following additional assumption:

$\mathcal{A}_{F,G}^S$- The two operators F and G satisfy respectively,

$$F(0_H) \equiv 0_H \quad \text{and} \quad G(0_H) \equiv 0_{\mathcal{L}(H)}, \tag{3.1}$$

where 0_H and $0_{\mathcal{L}(H)}$ are the null function and the null operator in H and $\mathcal{L}(H)$ respectively.

Theorem 3.2
([8]) Presume that the Assumptions $\mathcal{A}_F$, $\mathcal{A}_G$, $\mathcal{A}_{u_0}$ and $\mathcal{A}^S_{F,G}$ are satisfied. If $C^* > 0$, where $C^* := p\frac{\pi^\alpha}{2} - 3^{p-1}\left(C_F^p\left(\frac{2(p-1)}{\pi^\alpha p}\right)^{p-1} + C_p\, C_G^p\left(\frac{p-2}{p\pi^\alpha}\right)^{\frac{p-2}{2}}\right)$, for $p > 2$. Then, the null solution of Prb.(1.1) is p^{th}-moment exponentially stable, i.e.

$$\mathbb{E}(|u(t)|^p_H) \leq 3^{p-1}\mathbb{E}|u_0|^p_H\, e^{-C^* t}, \quad \forall\, t \geq 0. \tag{3.2}$$

In order to obtain the main result of this paper, we also require the following useful lemma.

Lemma 3.3
Let $p > 2$ and let the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t. $\varphi(t) := \mathbb{E}|u(t)|^p_H$, where u be the mild solution of Prb.(1.1). Then, under the assumptions of Theorem 3.2, the function φ is integrable. Additionally, if the initial condition $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H^\eta_2)$ for $\eta > \frac{\alpha}{2}$, then φ is continuous.

Proof

- The integrability of the function φ is fulfilled directly from Est.(3.2).
- The continuity can be obtained under the temporal regularity of the mild solution, as it has been proved in [4, Lemma 2.1. (iii)].

□

Now, we are in position to state our main result.

Theorem 3.4
Let $\alpha \in (1, 2]$, $p > 2$ and let $u \in \Lambda([0, T]; H)$ be the mild solution of Prb.(1.1), with initial condition $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H^\eta_2)$, for $\eta > \frac{\alpha}{2}$. If the assumptions of Theorem 3.2 are satisfied, then the null solution of Prb.(1.1) is ASA stable, i.e.

$$\lim_{t\to+\infty} |u(t)|_H = 0, \quad \mathbb{P} - a.s.$$

Moreover, it satisfies

$$\lim_{t\to+\infty} \mathbb{E}|u(t)|^p_H = 0.$$

Proof
Let $\alpha \in (1, 2]$, $p > 2$ (we take $p > 2$ in order to estimate (3.6) below by applying Hölder iquality) and the initial condition $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H^\eta_2)$, for $\eta > \frac{\alpha}{2}$. Presume that the assumptions of Theorem 3.2 are satisfied.

To fulfill the ASA stability, let $\varphi(.) := \mathbb{E}|u(.)|^p_H$, $\nu > 0$ and $(\varrho_n)^{+\infty}_{n=0}$ satisfy Lemma 2.6. Then, $\forall\, t \geq 0$, $\exists\, n \in \mathbb{N}$ s.t. $t \in [\varrho_n, \varrho_{n+1}]$. From equation (2.3) with the help of the semigroup property, we have

$$\begin{aligned} u(t) &= S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)F(u(s))\,ds + \int_0^t S_\alpha(t-s)G(u(s))\,dW(s) \\ &= S_\alpha(t-\varrho_n)S_\alpha(\varrho_n)u_0 + \int_0^{\varrho_n} S_\alpha(t-\varrho_n)S_\alpha(\varrho_n - s)F(u(s))\,ds \\ &+ \int_0^{\varrho_n} S_\alpha(t-\varrho_n)S_\alpha(\varrho_n - s)G(u(s))\,dW(s) \\ &+ \int_{\varrho_n}^t S_\alpha(t-s)F(u(s))\,ds + \int_{\varrho_n}^t S_\alpha(t-s)G(u(s))\,dW(s) \\ &= S_\alpha(t-\varrho_n)u(\varrho_n) + \int_{\varrho_n}^t S_\alpha(t-s)F(u(s))\,ds + \int_{\varrho_n}^t S_\alpha(t-s)G(u(s))\,dW(s). \end{aligned}$$

The use of the basic inequality (2.9) enables us to write

$$\begin{aligned}\mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|u(t)|_H^p\right) &\leq 3^{p-1}\mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|S_\alpha(t-\varrho_n)u(\varrho_n)|_H^p\right)\\ &+ 3^{p-1}\mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|\int_{\varrho_n}^t S_\alpha(t-s)F(u(s))\,ds|_H^p\right)\\ &+ 3^{p-1}\mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|\int_{\varrho_n}^t S_\alpha(t-s)G(u(s))\,dW(s)|_H^p\right)\\ &:= 3^{p-1}(I_1+I_2+I_3). \end{aligned} \tag{3.3}$$

To estimate I_1, we use Est.(2.8) as follows

$$\begin{aligned} I_1 &:= \mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|S_\alpha(t-\varrho_n)u(\varrho_n)|_H^p\right)\\ &\leq \left(\max_{t\in[\varrho_n,\varrho_{n+1}]}\|S_\alpha(t-\varrho_n)\|_{\mathcal{L}(H)}^p\right)\mathbb{E}|u(\varrho_n)|_H^p \leq \mathbb{E}|u(\varrho_n)|_H^p. \end{aligned} \tag{3.4}$$

To estimate I_2, we use Est.(2.8), the Assumption $\mathcal{A}_F$ and Hölder inequality as follows

$$\begin{aligned} &|\int_{\varrho_n}^t S_\alpha(t-s)F(u(s))\,ds|_H^p \leq \left(\int_{\varrho_n}^t |S_\alpha(t-s)F(u(s))|_H\,ds\right)^p\\ &\leq \left(\int_{\varrho_n}^t \|S_\alpha(t-s)\|_{\mathcal{L}(H)}|F(u(s))|_H\,ds\right)^p \leq \left(\int_{\varrho_n}^t |F(u(s))|_H\,ds\right)^p\\ &\leq C_F^p\left(\int_{\varrho_n}^t |u(s)|_H\,ds\right)^p \leq C_F^p\left(\int_{\varrho_n}^t ds\right)^{p-1}\int_{\varrho_n}^t |u(s)|_H^p\,ds. \end{aligned}$$

Subsequently,

$$\begin{aligned} I_2 &:= \mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|\int_{\varrho_n}^t S_\alpha(t-s)F(u(s))\,ds|_H^p\right)\\ &\leq C_F^p(\varrho_{n+1}-\varrho_n)^{p-1}\int_{\varrho_n}^{\varrho_{n+1}}\mathbb{E}|u(s)|_H^p\,ds \leq C_F^p\nu^{p-1}\int_{\varrho_n}^{\varrho_{n+1}}\mathbb{E}|u(s)|_H^p\,ds. \end{aligned} \tag{3.5}$$

It remains to estimate I_3. To do this, we use the B-D-G inequality as follows

$$\begin{aligned} I_3 &:= \mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]}|\int_{\varrho_n}^t S_\alpha(t-s)G(u(s))\,dW(s)|_H^p\right)\\ &\leq c_p\left(\int_{\varrho_n}^{\varrho_{n+1}}(\mathbb{E}\|S_\alpha(t-s)G(u(s))\|_{HS}^p)^{\frac{2}{p}}\,ds\right)^{\frac{p}{2}}, \end{aligned}$$

where $c_p := \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$.

The fact that for every $A\in\mathcal{L}(H)$ and every $B\in HS$, $\|AB\|_{HS}\leq\|A\|_{\mathcal{L}(H)}\|B\|_{HS}$ helps us to write

$$I_3 \leq c_p\left(\int_{\varrho_n}^{\varrho_{n+1}}\|S_\alpha(t-s)\|_{\mathcal{L}(H)}^2(\mathbb{E}\|G(u(s))\|_{HS}^p)^{\frac{2}{p}}\,ds\right)^{\frac{p}{2}}.$$

Now, the use of Est.(2.8) and the Assumption $\mathcal{A}_G$, besides an application of Hölder inequality yield

$$\begin{aligned} I_3 &\leq c_p C_G^p \left(\int_{\varrho_n}^{\varrho_{n+1}} (\mathbb{E}|u(s)|_H^p)^{\frac{2}{p}} \, ds \right)^{\frac{p}{2}} \\ &\leq c_p C_G^p \left(\int_{\varrho_n}^{\varrho_{n+1}} ds \right)^{\frac{p-2}{2}} \left(\int_{\varrho_n}^{\varrho_{n+1}} \mathbb{E}|u(s)|_H^p \, ds \right) \\ &= c_p C_G^p (\varrho_{n+1} - \varrho_n)^{\frac{p-2}{2}} \int_{\varrho_n}^{\varrho_{n+1}} \mathbb{E}|u(s)|_H^p \, ds \\ &\leq c_p C_G^p \nu^{\frac{p-2}{2}} \int_{\varrho_n}^{\varrho_{n+1}} \mathbb{E}|u(s)|_H^p \, ds. \end{aligned} \tag{3.6}$$

We replace Est.(3.4), Est.(3.5) and Est.(3.6) in Est.(3.3), to get

$$\mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]} |u(t)|_H^p \right) \leq \mathcal{C}_1 \mathbb{E}|u(\varrho_n)|_H^p + (\mathcal{C}_2 + \mathcal{C}_3) \int_{\varrho_n}^{\varrho_{n+1}} \mathbb{E}|u(s)|_H^p \, ds,$$

where $\mathcal{C}_1 := 3^{p-1}$, $\mathcal{C}_2 := 3^{p-1} C_F^p \nu^{p-1}$ and $\mathcal{C}_3 := 3^{p-1} c_p C_G^p \nu^{\frac{p-2}{2}}$.

From the fact that $\lim_{n\to+\infty} \varrho_n = +\infty$ in Lemma 2.6, we have

$$\begin{aligned} \sum_{n=0}^{+\infty} \mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]} |u(t)|_H^p \right) &\leq \mathcal{C}_1 \sum_{n=0}^{+\infty} \mathbb{E}|u(\varrho_n)|_H^p + (\mathcal{C}_2 + \mathcal{C}_3) \sum_{n=0}^{+\infty} \int_{\varrho_n}^{\varrho_{n+1}} \mathbb{E}|u(s)|_H^p \, ds \\ &= \mathcal{C}_1 \sum_{n=0}^{+\infty} \mathbb{E}|u(\varrho_n)|_H^p + (\mathcal{C}_2 + \mathcal{C}_3) \int_0^{+\infty} \mathbb{E}|u(s)|_H^p \, ds. \end{aligned}$$

Since we have $\sum_{n=0}^{+\infty} \varphi(\varrho_n) < +\infty$ and $\mathbb{E}|u(.)|_H^p \in L^1(\mathbb{R}^+)$ from Lemma 2.6 (with $\varphi(.) := \mathbb{E}|u(.)|_H^p$), then it is easy to see that

$$\sum_{n=0}^{+\infty} \mathbb{E}\left(\max_{t\in[\varrho_n,\varrho_{n+1}]} |u(t)|_H^p \right) < +\infty.$$

Consequently,

$$\sum_{n=0}^{+\infty} \max_{t\in[\varrho_n,\varrho_{n+1}]} |u(t)|_H^p < +\infty, \quad \mathbb{P} - a.s.$$

This implies

$$\lim_{n\to+\infty} \max_{t\in[\varrho_n,\varrho_{n+1}]} |u(t)|_H^p = 0, \quad \mathbb{P} - a.s.,$$

and so, $\lim_{t\to+\infty} |u(t)|_H^p = 0, \quad \mathbb{P} - a.s.$ as $\lim_{n\to+\infty} \varrho_n = +\infty$.

The proof of $\lim_{t\to+\infty} \mathbb{E}|u(t)|_H^p = 0$ has been obtained directly from Est.(3.2). Thus, the proof of our main result has been completed. □

4. Examples

In this section, we focus on supporting our theoretical result Theorem 3.4, by introducing some examples on the two operators F and G, which have been proposed in our work [8].

1. **Example of F**
   We define the operator $F : H \to H$ as a Nemytskii operator corresponding to the real function $f : \mathbb{R} \ni y \mapsto |y|$, i.e.

$$\forall v \in H, \quad F(v)(x) := f(v(x)), \quad \forall x \in (0,1).$$

   The nonlinearity of f ensures the nonlinearity of F.
2. **Example of G**
   We define the operator $G : H \to \mathcal{L}(H)$, $\forall\, u, v \in H$, as follows

$$(G(u)(v))(x) := (B(u))(x).(Q(v))(x), \;\; \forall x \in (0,1), \tag{4.1}$$

   where B and Q are defined by:

   - The operator $B : H \to H$ is a Nemytskii operator corresponding to the real function f or to the function $g : \mathbb{R} \ni y \mapsto y$.
   - The operator $Q : H \to H$ is defined by

$$Q(v) := \sum_{n\in\mathbb{N}_0} \kappa_n \langle \mathcal{E}_n, v\rangle_H \mathcal{E}_n,$$

     for real numbers $(\kappa_n)_{n\in\mathbb{N}_0}$ satisfy $\sum_{n\in\mathbb{N}_0} \kappa_n^2 < \infty$, where $(\mathcal{E}_n := \sqrt{2}\sin(n\frac{\pi}{2}))_{n\in\mathbb{N}_0}$ be another orthogonormal basis of the space H.

   In [8] and [9], we have proved that the above examples satisfy the assumptions of Theorem 2.3, Theorem 3.2 and Theorem 2.4 respectively. Consequently, they also satisfy Theorem 3.4.

Simulation

We implement a numerical simulation of the fractional stochastic nonlinear heat equation (1.1) with the initial condition $u_0(\theta) = \sin(\pi\theta)$. Let us take in this simulation $\kappa_n = \frac{1}{n}$, the time range $t \in [0, 10]$, spatial domain $x \in [0,1]$ and the parameters $\alpha = 1.4, C_F = 2, C_G = 1$. It is easy to verify that the parameters $\alpha = 1.4$, $C_F = 2, C_G = 1$, and $p = 3$ ensure the existence of the unique mild solution of the equation that is

$$C_F \frac{1}{\pi^\alpha} + C_p^{\frac{1}{p}} C_G \frac{1}{\sqrt{2\pi^\alpha}} < 1.$$

Now, we simulate the equation by using Python program, we obtain the following figures:

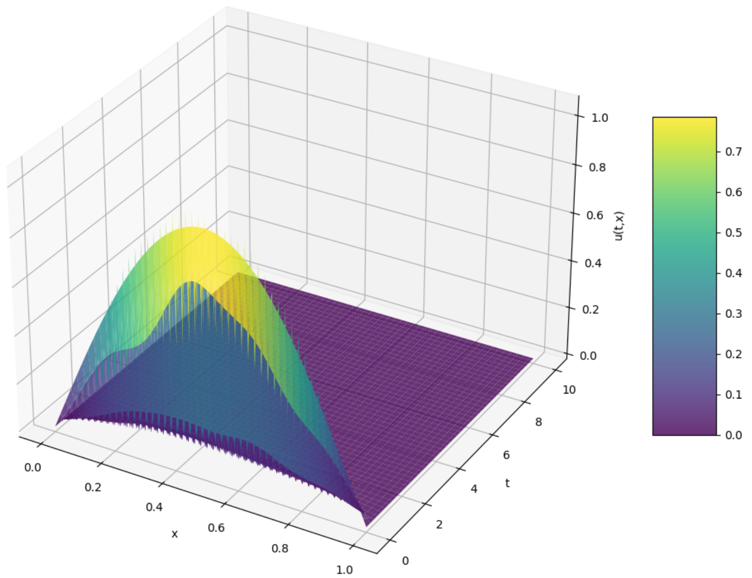


Figure 1. Evolution of the solution of the equation over time with $\alpha = 1.4, C_F = 2$, and $C_G = 1$.

Remark 2
Observe that the solution starts from an initial data, approximately to $sin(\pi x)$. When t increases, the solution decreases steadily towards zero. By the end $(t = 10)$, the surface is nearly flat at $u = 0$.

Then, we show the evolution of the L^3-norm over time:

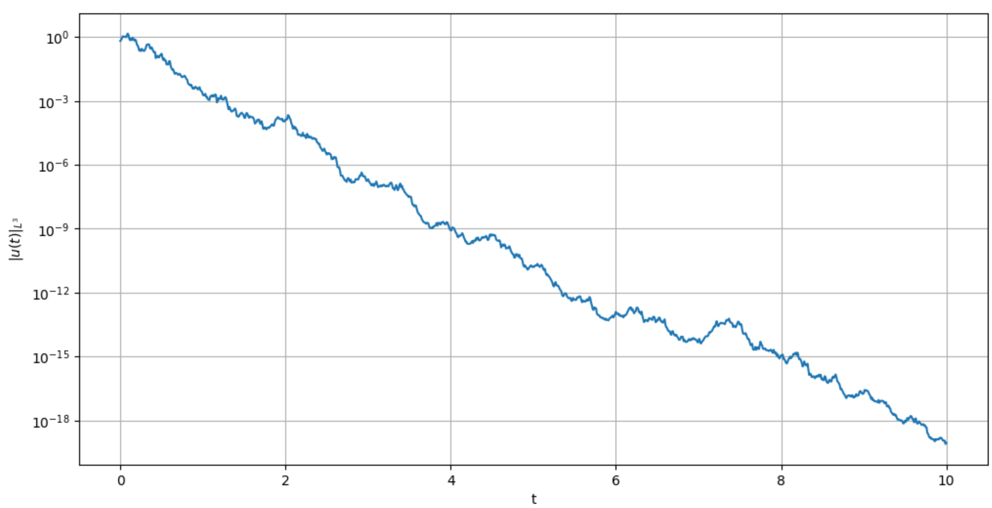


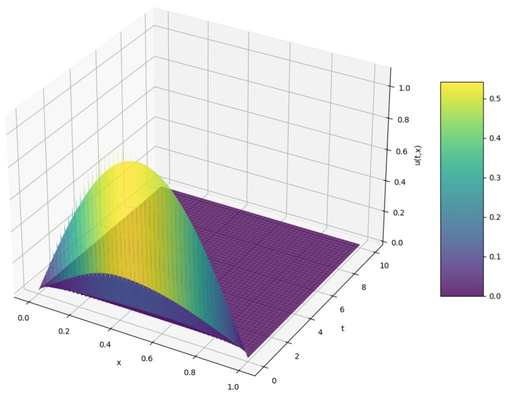
Figure 2. Evolution of the L^3-norm of the solution over time with $\alpha = 1.4, C_F = 2$, and $C_G = 1$.

Remark 3
The figure confirms that the solution decays to the equilibrium $u = 0$ as time grows. That is precisely the meaning of almost sure asymptotic stability:

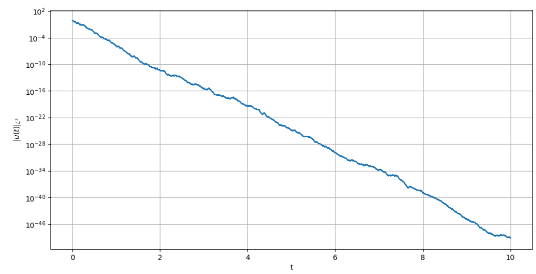
$$P\left(\lim_{t\to+\infty} |u(t)|_H = 0\right) = 1.$$

Thus, despite randomness in the evolution, the trajectory shown decays toward zero, confirming the theoretical results of the paper.

We know that for $\alpha = 2$, we have the classical stochastic nonlinear heat equation. Then, by maintaining the same values of C_F, C_G, and p, we obtain the following figures:



(a) Evolution of the solution of the equation over time



(b) Evolution of the L^3-norm of the solution over time

5. Conclusion

There are very important real-world applications of the FSPDEs. For example, in the heterogeneous materials (e.g. porous or fractal materials) and in nano-systems, the heat flow or the energy transfer is non-local, and the fractional Laplacial is more suitable to describe this long-range thermal energy transfer instead the classical Laplacian. This work has dealt with this kind of equations, which is driven by the fractional Laplacian (defined in the spectral context) and perturbed by trace-class noise. Precisely, under the linear growth and the Lipschitz conditions we have proved the almost sure asymptotic stability with respect to the initial condition and to the coefficients. In order to support and show the effectiveness of our theoretical result we have employed some examples with their numerical simulations, which include stability plots. .
Let us mention that, the obtained result can be extended by considering alternative definitions of the fractional Laplacian, for instance in terms of Riesz operator. Moreover, in the case of spectral Laplacian, it would be intriguing to consider Neumann or Robin boundary conditions. Additionally, it can be extended to the d-dimensional scenario; for $d > 1$.

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