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A New Accelerated Failure Time Model with Censored and Uncensored Real-life Applications: Validation and Different Estimation Methods

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Abstract This study introduces a novel exponential accelerated failure time (AFT) model, detailing its fundamental properties and characterizations. To evaluate the performance of various estimation techniques, we conduct simulation studies that assess the finite-sample behavior of the estimators. Additionally, we propose a modified chi-square goodness-of-fit test tailored for the new model, applicable to both complete and right-censored datasets. The model's validity is examined using the theoretical framework of the Nikulin-Rao-Robson (NRR) statistic, with maximum likelihood estimation employed for parameter estimation. Two separate simulation studies are carried out: one to evaluate the proposed AFT model and another to assess the efficacy of the NRR test statistic. Furthermore, the practical applicability of the test statistic is demonstrated through analyses of three real-life datasets.

Keywords Exponential accelerated failure time model, maximum likelihood estimation, modified chi-square test, Nikulin-Rao-Robson statistic, right-censored data, simulation study, model validation.

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

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1. Introduction

This study addresses significant gaps in survival analysis by introducing a novel quasi Burr-Hatke exponential accelerated failure time (QBHE-AFT) model, which offers greater flexibility compared to traditional AFT models like exponential or Weibull, particularly in handling non-monotonic hazard rates (Yousof et al., 2018). It fills a critical gap in validating parametric AFT models for censored data by proposing a modified chi-square goodness-of-fit (GoF) test based on the NRR statistic, applicable to both complete and right-censored datasets, unlike classical tests that are unsuitable for unknown parameters (Bagdonavicius & Nikulin, 2011; Goual et al., 2020). Furthermore, while many studies rely solely on maximum likelihood estimation, this research evaluates multiple estimation techniques, such as Cramer-von Mises, Anderson-Darling, and L-moments, providing a comprehensive framework for parameter estimation under varying sample sizes and conditions (Goual & Yousof, 2019; Yadav et al., 2020). Despite these advancements, the study highlights the need for further exploration of high-dimensional covariates, robustness to model misspecification, and computational efficiency for large datasets, as noted in similar works (Dupuy, 2014; Voinov et al., 2013).

In order to bridge this research gap, this paper introduces a QBHE-AFT accelerated failure time model, addressing the need for more flexible parametric models in survival analysis. The QBHE-AFT model offers

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advantages over traditional AFT models by accommodating both monotonic and non-monotonic hazard rate functions, making it suitable for diverse real-world applications. Unlike existing models, it combines simplicity with enhanced flexibility through its two-parameter structure, while maintaining computational efficiency. The study fills a gap in the literature by providing a comprehensive validation framework using modified chi-square GoF tests for both complete and censored data scenarios. Our research contributes to the field by proposing multiple estimation methods, including maximum likelihood and various adaptive approaches, allowing for robust parameter estimation across different sample sizes. The model's practical utility is demonstrated through successful applications to real-life datasets from engineering, medical, and reliability studies. This work advances the understanding of AFT models by exploring the asymptotic properties and finite-sample behavior of estimators through extensive simulation studies. The proposed methodology addresses limitations of existing tests by developing a modified Nikulin-Rao-Robson statistic specifically tailored for the QBHE-AFT model. Our findings provide valuable insights into model performance under varying conditions, contributing to more accurate lifetime data analysis. The research offers a comprehensive framework for analyzing survival data, benefiting researchers and practitioners in reliability engineering, medical research, and other fields dealing with time-to-event data.

An appropriate parametric model is often of interest for analyzing survival data since it provides an overview of the failure times characteristics and the risk functions. However, when failure rates of the products or death or remission of patients or any other diseases can have different causes, simple parametric models cannot measure the influence of each cause. In this case, accelerated failure time (AFT) models were proposed in the statistical literature, where the stresses (explanatory variable, temperature, pressure, dose of medicine, etc.) represented by covariates affect directly the functions of interest of the model, such as the failure rate and survival functions. The AFT models are primarily fully parametric, in contrast to proportional hazards models, where Cox's semi-parametric proportional hazards model is more frequently used than parametric models. Also, the regression parameter estimates from AFT models are resistant to omitted covariates, unlike proportional hazards models. Additionally, they are less impacted by the probability distribution of choice. Depending on the values affected to the covariates, by increasing or decreasing them, engineers and practitioners can achieve the desired results, which is why the AFT models are widely used in reliability studies and survival analysis. The objective of this theory is to know the influence of the stresses (covariates) on the life duration of the items. Based on classical distributions called baseline, several AFT models are studied, such as the exponential, Weibull, log-logistic, and log-normal AFT models (Bagdonavicius and Nikulin, 2002; Lawless, 2003; Bagdonavicius et al., 2010), the generalized inverse Weibull AFT model (Goual and Seddik-Ameur, 2014; Bagdonavicius and Nikulin, 2011; Bagdonavicius et al., 2011), which gave chi-squared GoF tests for regression models such as accelerated failure time, proportional hazards, generalized proportional hazards, frailty models, models with cross-effects of survival functions.

The exponential distribution is likely the statistical model that is used most frequently across a variety of fields among parametric distributions. Its significance is due in part to the exponential model's constant failure rate function. Furthermore, this model was the first lifetime model for which extensive statistical tools were created in the literature on life testing. In a random process where events happen at a predetermined pace, the waiting period before the first occurrence is distributed using an exponential function. It is a relatively simple distribution; a random variable having this distribution is necessarily positive, and it is one of the more important distributions among those used for positive random variables. The cumulative distribution function (CDF) of the exponential distribution can be written as $G_\lambda(x) = 1 - \exp(-\lambda x)$, where $\lambda > 0$ and $x \geq 0$. The moments, the moment generating function (MGF), and several other properties of this distribution can be expressed in terms of elementary functions. In the last decades, many new distributions have been developed by adding one or more parameters to classical distributions to increase their flexibility (Yousof et al., 2018).

The most popular AFT model is provided by the log-logistic distribution. It can display a non-monotonic hazard function that rises early and falls later, unlike the Weibull distribution. Although it has heavier tails, it has a form that is relatively comparable to the log-normal distribution. When fitting data with censoring, the log-logistic cumulative distribution function's straightforward closed form plays a crucial computational role. The survival function, which is the complement of the cumulative distribution function, is required for the censored observations. It is unique among distribution families that the Weibull distribution (which includes the exponential distribution as a special example) can be parameterized as either an AFT model or a proportional hazards model. There are two ways to interpret the outcomes of fitting a Weibull model. This model's biological application, however, might be constrained by the hazard function's monotonicity, that is, its ability to be either decreasing or growing. The log-normal, gamma, and inverse Gaussian distributions are additional distributions appropriate for AFT models; however, they are less common than the log-logistic distribution, in part because their cumulative distribution functions do not have a closed form. The Weibull, log-normal, and gamma distributions are special examples of the generalized gamma distribution, a three-parameter distribution.

In this work, we introduce an exponential model dubbed the Burr-Hatke exponential (QBHE) distribution and investigate its mathematical features in the manner of Yousof et al. (2018). The novel model simply has two parameters and can be written as linear combinations of the well-known exponentiated exponential density. Its probability distribution function (PDF) also has a straightforward shape. The asymptotics results can be used to assess how the two parameters affect the QBHE distribution's tails. The novel PDF, CDF, and hazard rate function (HRF) asymptotics results are obtained correspondingly. Using two truncated moments, the HRF, and the conditional expectation of a random variable-based function, various descriptions of the QBHE distribution are provided. The finite sample behavior of the estimators is evaluated using a variety of estimation techniques, such as the maximum likelihood, Cramer-von Mises, Anderson-Darling, right-tail Anderson-Darling, left-tail Anderson-Darling, and method of L-moments. Simulated studies are carried out to compare the estimation techniques. Various sample sizes and parameter values are used to accomplish the simulation experiments. The new QBHE-AFT model can be used in reliability modeling and lifetime testing in many applied fields such as electric insulating, medicine, and lifetime studies. For assessing the estimates of the QBHE-AFT model and depending on using the Barzilai-Borwein (BZB) algorithm, the averages of the simulated values of the maximum likelihood estimators (MLEs) and their corresponding mean squared errors are reported under different sample sizes. The QBHE-AFT model is tested using a novel modified chi-square test in both the complete and right-censored data situations. The theoretical framework of NRR statistics is used to assess the viability of the QBHE-AFT model (see Nikulin, 1973a, 1973b, 1973c; Rao and Robson, 1974). In several validation procedures, the NRR test statistic has recently been enhanced (see, for instance, Goual and Yousof, 2019; Goual et al., 2019, 2020; Yadav et al., 2020, 2022a,b). The modified NRR test statistic for the QBHE-AFT model is evaluated using the maximum likelihood approach at a few empirical levels and equivalent theoretical levels. In order to evaluate the effectiveness of the NRR test statistic in validation, three real datasets are also taken into account.

2. The quasi Burr Hatke exponetial model

2.1. Formluation

Based on the Burr-Hatke differential equation, Yousof et al. (2018) introduced a novel family of distributions referred to as the BH-G family. According to Yousof et al. (2018), the cumulative distribution function (CDF) of the quasi Burr-Hatke exponential (QBHE) distribution can be derived as:

$$F_\lambda(x) = 1 - \frac{\exp(-\lambda x)}{\lambda x + 1}. \tag{1}$$

The PDF corresponding to (1) is given by

$$f_\lambda(x) = \lambda(\lambda x + 1)^{-2}\left[(\lambda x + 1) + 1\right]\exp(-\lambda x)\ . \tag{2}$$

The HRF of the QBHE model can be expressed as

$$h_\lambda(x) = \lambda\frac{(\lambda x + 1) + 1}{\lambda x + 1}. \tag{3}$$

Mixture representations for Equations (2) and (3) are obtained. Consider the following expansions,

$$\left(1 - \frac{\zeta_1}{\zeta_2}\right)^{\zeta_3} = \sum_{\zeta_4=0}^{\infty}(-1)^{\zeta_4}\binom{\zeta_3}{\zeta_4}\left(\frac{\zeta_1}{\zeta_2}\right)^{\zeta_3},\ |\frac{\zeta_1}{\zeta_2}|\ < 1 \tag{4}$$

and

$$\log\left(1-\frac{\zeta_1}{\zeta_2}\right)=-\sum_{\zeta_4=0}^{\infty}\frac{1}{1+\zeta_4}\left(\frac{\zeta_1}{\zeta_2}\right)^{1+\zeta_4},\ |\frac{\zeta_1}{\zeta_2}|<1. \tag{5}$$

Firstly, the CDF (2) can be rewriten as

$$F_\lambda(x)=1-\frac{\overbrace{1-[1-\exp(-\lambda x)]}^{A_\lambda(x)}}{\underbrace{1-\log\{1-[1-\exp(-\lambda x)]\}}_{B_\lambda(x)}}.$$

Applying (4) to $A_\lambda(x)$. Then, $A_\lambda(x)=\sum_{k=0}^{\infty}a_k[1-\exp(-\lambda x)]^k$, where $a_k=(-1)^k\binom{1}{k}$. Now, applying (5) to $B_\lambda(x)$, still in Equation (2), we obtain

$$\begin{aligned}B_\lambda(x) &= 1+\sum_{i=0}^{\infty}\frac{1}{i+1}[1-\exp(-\lambda x)]^{i+1}\\ &= \sum_{k=0}^{\infty}b_k[1-\exp(-\lambda x)]^k,\end{aligned}$$

where $b_0=1$, $k\geq 1$ and $b_k=\frac{-1}{k}$. Then, Equation (2) can be writen as

$$F(x;\lambda)=1-\frac{\sum_{k=0}^{\infty}a_k[1-\exp(-\lambda x)]^k}{\sum_{k=0}^{\infty}b_k[1-\exp(-\lambda x)]^k}=1-\sum_{k=0}^{\infty}c_k[1-\exp(-\lambda x)]^k,$$

where $c_0=\frac{a_o}{b_0}$ and, for $k\geq 1$, we have $c_k=\frac{1}{b_0}\left(a_k-\frac{1}{b_0}\sum_{r=1}^{k}b_r c_{k-r}\right)$. At the end, the CDF (2) can be writen as

$$F_\lambda(x)=\sum_{k=0}^{\infty}d_k\mathbf{\Pi}_{1+k}(x;\lambda), \tag{6}$$

where $d_0=1-c_k$, for $k\geq 1$ we have $d_0=-c_k$ and $\mathbf{\Pi}_{1+k}(x;\lambda)=[1-\exp(-\lambda x)]^{1+k}$ is the CDF of the exponentiated exponential model with power parameter $1+k$. By differentiating (6), we obtain the same mixture representation

$$f_\lambda(x)=\sum_{k=0}^{\infty}d_k\pi_{1+k}(x;\lambda), \tag{7}$$

where $\pi_\varsigma(x)=(1+k)\lambda\exp(-\lambda x)[1-\exp(-\lambda x)]^k$ is the PDF of the exponentiated exponential with power parameter (ς). Equation (7) demonstrates that the exponentiated exponential densities are combined linearly to form the QBHE density function. As a result, it is possible to derive some structural characteristics of the new model, including the generating function, ordinary and incomplete moments, and Exp-E distribution, right away. Many authors have recently explored the exponentiated exponential distribution's properties.

2.2. Properties

Let $a=\inf\{x|F_\lambda(x)>0\}$, the asymptotics of CDF, PDF and HRF as $x\to a$ are given by

$$F_\lambda(x)\sim 1-\exp(-\lambda x)|_{x\to a},\ f_\lambda(x)\sim\lambda\exp(-\lambda x)|_{x\to a},$$

and

$$h_\lambda(x) \sim \lambda \exp(-\lambda x)\,|_{x\to a}.$$

The asymptotics of CDF, PDF and HRF as $x \to \infty$ are given by

$$1 - F_\lambda(x) \sim \frac{\exp(-\lambda x)}{\lambda x}|_{x\to\infty},\ f_\lambda(x) \sim \frac{\exp(-\lambda x)}{\lambda x^2}(1 - \lambda x)\,|_{x\to\infty},$$

and

$$h_\lambda(x) \sim \frac{1}{x}(1 - \lambda x)\,|_{x\to\infty}.$$

The effect of the parameters on tails of distribution can be evaluated by means of the above equations.

Theorem 1: Let T be a random variable with the exponentiated exponential distribution with positive parameters λ and ς. Then, for any $r > -1$, the r$^{\text{th}}$ ordinary and incomplete moments of T are given by

$$\mu'_{r,T} = \sum_{w=0}^{\infty} C_w^{(r,\varsigma)}\,\Gamma(1+r)$$

and

$$\mathbf{I}_{r,T}(t) = \sum_{w=0}^{\infty} C_w^{(r,\varsigma)}\,\gamma(1+r,(\lambda t)),$$

respectively, where

$$C_w^{(r,\varsigma)} = \varsigma\lambda^{-r}\frac{(-1)^w}{(w+1)^{(1+r)}}\binom{\varsigma-1}{w}$$

and $\gamma(\zeta_1,\zeta_2)$ is the incomplete gamma function which can be expressed as

$$\gamma(\zeta_1,\zeta_2) = \int_0^{\zeta_2}\exp(-w)\,dw = \frac{1}{\zeta_1}\zeta_2^{\zeta_1}\{{}_1\mathbf{F}_1[\zeta_1;\zeta_1+1;-\zeta_2]\} = \sum_{\kappa=0}^{\infty}\frac{(-1)^\kappa}{\kappa!\,(\zeta_1+\kappa)}\zeta_2^{\zeta_1+\kappa},$$

and ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. Based on Theorem 1**,** the r^{th} ordinary moment of X is given by $\mu'_{r,X} = E(X^r) = \int_{-\infty}^{\infty} x^r\, f(x)\,dx$. Then, we obtain

$$\mu'_{r,X} = \sum_{k,w=0}^{\infty} C_{k,w}^{(1+k,r)}\Gamma(1+r)\,|_{r>-1}, \tag{8}$$

where $C_{k,w}^{(1+k,r)} = d_k C_w^{(r,1+k)}$ and

$$C_w^{(r,1+k)} = (1+k)\,\frac{(-1)^w}{(w+1)^{(1+r)}}\binom{k}{w}$$

The cumulants, central moment, skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Based on Theorem 1**,** the rth incomplete moment of X, say $\mathbf{I}_{r,X}(t) = \int_{-\infty}^{t} x^r\, f(x)dx$, can be determined from (7) and (8) as

$$\mathbf{I}_{r,X}(t) = \int_{-\infty}^{t} x^r\, f(x)dx = \sum_{k,w=0}^{\infty} C_{k,w}^{(1+k,r)}\gamma(1+r,(\lambda t))\,|_{r>-1}. \tag{9}$$

The MGF of X follows from (7) and (8) as

$$M_X(t) = \sum_{k,w,r=0}^{\infty}\frac{t^r}{r!}C_{k,w}^{(1+k,r)}\Gamma(1+r)\,|_{r>-1}.$$

3. Different estimation methods

3.1. Maximum likelihood (ML) method

Let $x_1, x_2, \ldots, x_n$ be a RS from this distribution with parameter vector $(\lambda)^\intercal$. The log-likelihood function for (λ), say $\ell(\lambda)$, is given by

$$\ell(\lambda) = n \log \lambda - 2n\sum_{i=0}^{n} \log\left(\lambda x_{i:n} + 1\right) + \sum_{i=0}^{n} \log[(\lambda x_{i:n} + 1) + 1] - \lambda\sum_{i=0}^{n} \log x_{i:n}$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\lambda)$ by differentiation. The score vector, $\mathbf{U}\left(\lambda\right) = \left(\frac{\partial}{\partial\lambda}\ell(\lambda)\right)^\intercal$, are easily derived.

3.2. Cramér-von-Mises method

The Cramér-von-Mises estimation (CVME) of the parameter λ is obtained via minimizing the following expression with respect to the parameter λ respectively, where

$$CVM_{(\lambda)} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n} \left[F_\lambda(x_{i:n}) - \zeta_{(i,n)}\right]^2,$$

where $\zeta_{\ (i,n)} = \frac{2i-1}{2n}$ and

$$CVM_{(\lambda)} = \sum_{i=1}^{n} \left(1 - \frac{\exp\left(-\lambda x_{i:n}\right)}{\lambda x_{i:n} + 1} - \zeta_{(i,n)}\right)^2.$$

The, CVME of the parameter λ are obtained by solving the following non-linear equation

$$\sum_{i=1}^{n} \left(1 - \frac{\exp\left(-\lambda x_{i:n}\right)}{\lambda x_{i:n} + 1} - \zeta_{(i,n)}\right) \varsigma_{(\lambda)}(x_{i:n}, \lambda) = 0,$$

where $\varsigma_\lambda(x_{i:n}, \lambda)$ are the first derivatives of the CDF of QBHE distribution with respect to λ respectively.

3.3. Method of L-moments

However, linear combinations of the order statistics can be used to estimate the L-moments, which are comparable to ordinary moments. They are relatively resistant to the effects of outliers and exist anytime the distribution's mean does, even if certain higher moments do not. Based on the moments of the order statistics, we can derive explicit expressions for the L-moments of x as infinite weighted linear combinations of the means of suitable QBHE order statistics. The L-moments for the population can be obtained from

$$\gamma_r = \frac{1}{r}\sum\nolimits_{m=0}^{r-1} (-1)^m \binom{r-1}{m} \mathbf{E}\left(x_{r-m:m}\right)|_{\ (r\geq 1)}.$$

The first four L-moments are given by

$$\gamma_1\left(\lambda\right) = \mathbf{E}\left(x_{1:1}\right) = \mu_1' = \mathbb{L}_1,$$

Then, the L-moments estimators $\widehat{\lambda}_{(\mathrm{LME})}$ of the parameters λ can be obtained by solving the following equation numerically

$$\gamma_1\left(\widehat{\lambda}_{(\mathrm{LME})}\right) = \mathbb{L}_1,$$

3.4. Anderson Darling method

The Anderson Darling estimation (ADE) of and $\widehat{\lambda}_{(\text{ADE})}$ are obtained by minimizing the function

$$\text{ADE}\,(\lambda) = -n - n^{-1}\sum_{i=1}^{n}(2i-1)\left\{\begin{array}{c}\log F_{(\lambda)}(x_{i:n})\\ +\log\left[1-F_{(\lambda)}(x_{[-i+1+n:n]})\right]\end{array}\right\}.$$

The parameter estimates of $\widehat{\lambda}_{(\text{ADE})}$ follow by solving the nonlinear equation

$$\frac{\partial}{\partial\lambda}\left[\text{ADE}\,(\lambda)\right] = 0.$$

3.5. Right Tail-Anderson Darling method

The Tail-Anderson Darling estimation (RTADE) of and $\widehat{\lambda}_{(\text{RTADE})}$ are obtained by minimizing

$$\text{RTADE}\,(\lambda) = \frac{1}{2}n - 2\sum_{i=1}^{n}F_{(\lambda)}(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[1-F_{(\lambda)}(x_{[-i+1+n:n]})\right]\right\}.$$

The parameter estimates of and $\widehat{\lambda}_{(\text{RTADE})}$ follow by solving the nonlinear equation

$$\frac{\partial}{\partial\lambda}\left[RTADE\,(\lambda)\right] = 0.$$

3.6. Left Tail-Anderson Darling method

The left Tail-Anderson Darling estimation (LTADE) of and $\widehat{\lambda}_{(\text{LTADE})}$ are obtained by minimizing

$$LTADE\,(\lambda) = -\frac{3}{2}n + 2\sum_{i=1}^{n}F_{(\lambda)}(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F_{(\lambda)}(x_{i:n}).$$

The parameter estimates of and $\widehat{\lambda}_{(\text{LTADE})}$ follow by solving the nonlinear equation

$$\frac{\partial}{\partial\lambda}\left[LTADE\,(\lambda)\right] = 0.$$

4. Simulation studies for comparing estimation methods

Table 1: Simulation results for parameter $\lambda = 0.7$

	n	BIAS(λ)	RMSE(λ)	D_{abs}	D_{max}
MLE	50	0.00948	0.11545	0.00311	0.00453
CVM		0.00858	0.12920	0.00281	0.00411
L-MOMENT		0.01154	0.11700	0.00377	0.00551
ADE		0.00493	0.12157	0.00162	0.00237
RTADE		0.00125	0.11680	0.00041	0.00060
LEADE		0.01543	0.14386	0.00503	0.00735
MLE	100	0.00752	0.08000	0.00247	0.00360
CVM		0.00573	0.08932	0.00189	0.00275
L-MOMENT		0.00906	0.08149	0.00298	0.00434
ADE		0.00408	0.08408	0.00134	0.00196
RTADE		0.00336	0.08128	0.00111	0.00161
LEADE		0.00740	0.09749	0.00243	0.00355
MLE	200	0.00470	0.05543	0.00155	0.00226
CVM		0.00527	0.06260	0.00173	0.00253
L-MOMENT		0.00506	0.05604	0.00166	0.00243
ADE		0.00408	0.05933	0.00134	0.00196
RTADE		0.00318	0.05685	0.00105	0.00153
LEADE		0.00650	0.06885	0.00214	0.00312
MLE	300	0.0005	0.04382	0.00017	0.00024
CVM		0.00069	0.05070	0.00023	0.00033
L-MOMENT		0.00070	0.04390	0.00023	0.00034
ADE		0.00014	0.04782	0.00005	0.00007
RTADE		-0.00056	0.04571	0.00018	0.00027
LEADE		0.0017	0.05591	0.00065	0.00094

To rigorously assess and compare the performance of different parameter estimation techniques, a comprehensive numerical simulation study is conducted. The simulation is based on data generated from the QBHE distribution, with $N = 1000$ independent simulation replications. For each replication, synthetic data sets are generated at varying sample sizes: $n = 50, n = 100, n = 200$, and $n = 300$, to investigate how estimation accuracy behaves under different data volumes. The simulations are performed for different values of the shape parameter λ, reflecting a range of distributional shapes and complexities. Specifically, the simulations explore the following λ values as λ = 0.7 (see Table 1), λ = 2 (see Table 2) and λ = 5 (see Table 2). To comprehensively compare the performance of the estimation methods, several evaluation metrics are simultaneously considered. These include the bias, which reflects the average deviation of the estimator from the true parameter value; the root mean squared error (RMSE), which accounts for both the bias and variability of the estimates; the mean absolute deviation in distribution (M-AD), measuring the average difference between the theoretical and empirical cumulative distribution functions; and the maximum absolute deviation in distribution (Max-AD), which captures the largest observed discrepancy between the estimated and true distributions across all data points. Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study where:

1-BIAS(λ) $= \frac{1}{B}\sum\limits_{i=1}^{B}\left(\widehat{\lambda}_i - \lambda\right)$, 2-RMSE$(\lambda)=\sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\lambda}_i-\lambda\right)^2}$,

3-The M-AD $\left(D_{(\text{abs})}\right)$:D$_{(\text{abs})}=\frac{1}{nB}\sum_{i=1}^{B}\sum_{j=1}^{n}|F_{(\lambda)}(x_{ij})-F_{(\widehat{\lambda})}(t_{ij})|$ and

4-The Max-AD $\left(D_{(\max)}\right)$:D$_{(\max)}=\frac{1}{B}\sum_{i=1}^{B}\max_j|F_{(\lambda)}(x_{ij})-F_{(\widehat{\lambda})}(w_{ij})|$.

Table 2: Simulation results for parameter $\lambda=2$

	n	BIAS λ	RMSE λ	Dabs	Dmax
MLE	50	0.03905	0.35070	0.00447	0.00652
CVM		0.02128	0.37929	0.00246	0.00357
L-MOMENT		0.04869	0.34453	0.00558	0.00810
ADE		0.01516	0.36006	0.00175	0.00254
RTADE		0.01279	0.34892	0.00148	0.00215
LEADE		0.07173	0.42749	0.00815	0.01187
MLE	100	0.03221	0.23292	0.00369	0.00539
CVM		0.01140	0.26130	0.00131	0.00192
L-MOMENT		0.02008	0.23640	0.00231	0.00337
ADE		0.00657	0.24693	0.00076	0.00111
RTADE		0.00266	0.23864	0.00031	0.00045
LEADE		0.01392	0.28227	0.00161	0.00234
MLE	200	0.01450	0.15403	0.00167	0.00244
CVM		0.00928	0.17764	0.00107	0.00156
L-MOMENT		0.01599	0.16415	0.00184	0.00269
ADE		0.00712	0.16985	0.00082	0.00120
RTADE		0.00817	0.16539	0.00094	0.00137
LEADE		0.01345	0.19161	0.00155	0.00226
MLE	300	0.00392	0.13041	0.00045	0.00066
CVM		0.00790	0.14843	0.00091	0.00133
L-MOMENT		0.00948	0.13605	0.00109	0.00159
ADE		0.00692	0.14175	0.00080	0.00117
RTADE		0.00515	0.13755	0.00059	0.00087
LEADE		0.00723	0.16183	0.00083	0.00122

From the simulation results displayed in Tables 1, 2 and 3, a consistent pattern emerges across all three values of the parameter λ (0.7, 2, and 5): as the sample size nn increases, the performance of all estimators significantly improves. The BIAS(λ) systematically decreases toward zero for all methods, confirming that each estimator is asymptotically unbiased. Similarly, the RMSE also declines steadily with increasing nn, indicating stronger consistency, that is, estimates become more accurate and tightly clustered around the true parameter value. These trends are evident even in the more challenging scenario of λ=5, where estimation is inherently more difficult due to the heavier tail behavior of the distribution. Despite higher initial error at small sample sizes (e.g., $n=50$), most estimators exhibit rapid improvement as nn grows. Among the methods, RTADE and ADE consistently deliver superior performance, particularly at lower sample sizes, as reflected in their lower bias, RMSE, and deviation measures (M-AD and Max-AD). The LEADE method, while competitive in some scenarios, tends to show higher variability and error, especially in small samples. Overall, the simulation results underscore the reliability and robustness of the estimators, particularly the adaptive ones, with increasing data, and highlight their suitability across a range of parameter settings.

Table3: Simulation results for parameter $\lambda = 5$					
	n	BIAS λ	RMSE λ	Dabs	Dmax
MLE	50	0.12005	0.89825	0.00548	0.00799
CVM		0.14212	0.89758	0.00648	0.00944
L-MOMENT		0.17317	0.82756	0.00787	0.01147
ADE		0.11571	0.84090	0.00529	0.00771
RTADE		0.09747	0.81261	0.00447	0.00651
LEADE		0.16405	1.09597	0.00745	0.01088
MLE	100	0.04262	0.56296	0.00196	0.00286
CVM		0.02053	0.67329	0.00095	0.00138
L-MOMENT		0.04685	0.59259	0.00216	0.00314
ADE		0.00954	0.63367	0.00044	0.00064
RTADE		0.00445	0.60647	0.00021	0.00030
LEADE		0.07823	0.72400	0.00359	0.00523
MLE	200	0.04194	0.39887	0.00192	0.00282
CVM		0.02613	0.45062	0.00120	0.00176
L-MOMENT		0.02922	0.40273	0.00135	0.00196
ADE		0.02005	0.42645	0.00092	0.00135
RTADE		0.01498	0.40987	0.00069	0.00101
LEADE		0.02891	0.49821	0.00133	0.00194
MLE	300	0.00782	0.31943	0.00036	0.00053
CVM		0.00386	0.35260	0.00018	0.00026
L-MOMENT		-0.00196	0.31932	0.00009	0.00013
ADE		-0.00217	0.33414	0.00010	0.00015
RTADE		-0.00721	0.32456	0.00033	0.00049
LEADE		0.02934	0.39068	0.00135	0.00197

5. The QBHE-AFT model

In this section, we introduce a novel accelerated failure time (AFT) model based on the Burr-Hatke exponential distribution. This model is designed to provide greater flexibility in modeling survival data while accommodating both monotonic and non-monotonic hazard rate functions. To construct this model, we assume that n independent failure time variables are observed, and we consider the hypothesis H_0, which specifies the survival function given a vector of explanatory variables $z(t) = (z_0(t), z_1(t), ..., z_m(t)),\quad z_0(t) = 1$ represents the baseline covariate (e.g., temperature, stress, or other external factors). The survival function under this hypothesis takes the form:

$$S(t|z) = S_0\left(\int_0^t e^{-\beta^T z(u)} du; \zeta\right),$$

where $\beta = (\beta_0, \beta_1, ..., \beta_m)^T$ is a vector of unknown regression parameters, the function S_0 is a specified functional of time and does not depend on z_i. If explanatory variables are constant over time, the parametric accelerated failure time (AFT) model has the form

$$S(t|z) = S_0\left[\exp\left(-\beta^T z\right) t; \zeta\right].$$

Consider the QBHE distribution as baseline distribution where

$$H_0 = F(t) = F_{\text{AFT}}(t, \lambda, \beta) = F_{\text{AFT}}.$$

So, the CDF of the AFT model can be expressed as

$$F_{\text{AFT}} = 1 - \frac{\exp\left[-\lambda t \exp\left(-\beta^T z\right)\right]}{1 + \lambda t \exp\left(-\beta^T z\right)}, t > 0; \lambda > 0,$$

and then, the PDF of the AFT model can be re-expressed as

$$\begin{aligned} f_{\text{AFT}} &= \frac{\lambda \exp\left(-\beta^T z\right) \exp\left[-\lambda t \exp\left(-\beta^T z\right)\right]}{1 + \lambda t \exp\left(-\beta^T z\right)} + \lambda \frac{\exp\left(-\beta^T z\right) \exp\left[-\lambda t \exp\left(-\beta^T z\right)\right]}{\left(1 + \lambda t \exp\left(-\beta^T z\right)\right)^2} \\ &= \frac{\lambda \exp\left(-\beta^T z\right) \exp\left[-\lambda t \exp\left(-\beta^T z\right)\right] \left[\left(1 + \lambda t \exp\left(-\beta^T z\right)\right) + 1\right]}{\left(1 + \lambda t \exp\left(-\beta^T z\right)\right)^2}. \end{aligned}$$

Analogously, the corresponding survival function (SF), HRF and cumulative HRF of the AFT model are given by

$$S_{\text{AFT}} = S_0\left[t \exp\left(-\beta^T z\right)\right] = \frac{\exp\left[-\lambda t \exp\left(-\beta^T z\right)\right]}{1 + \lambda t \exp\left(-\beta^T z\right)}$$

$$h_{\text{AFT}} = \lambda \exp\left(-\beta^T z\right) \frac{\left\{\left[1 + \lambda t \exp\left(-\beta^T z\right)\right] + 1\right\}}{1 + \lambda t \exp\left(-\beta^T z\right)},$$

and

$$H_{\text{AFT}} = -\log\left\{\frac{\exp\left[-\lambda t \exp\left(-\beta^T z\right)\right]}{1 + \lambda t \exp\left(-\beta^T z\right)}\right\}.$$

These expressions provide a comprehensive framework for analyzing the effects of covariates on survival times, enabling researchers to estimate key quantities such as the likelihood of failure at a given time and the instantaneous risk of failure. The proposed QBHE-AFT model is particularly well-suited for applications in fields such as engineering, medicine, and reliability studies, where understanding the impact of stresses or treatments on system lifetimes is critical. By incorporating the Burr-Hatke structure, the model achieves enhanced flexibility, allowing for accurate modeling of complex hazard rate patterns observed in real-world datasets.

6. The MLE for the QBHE-AFT model

In this section, we apply the maximum likelihood method to estimate the parameters of the AFT for the QBHE distribution. We give a detailed description of the method as well as the score functions and the elements of the FIM.

6.1. The MLE derivations

Let $x_1, \ldots, x_n$ be a RS from the AFT for the QBHE model with parameters λ and β. Let $\underline{\mathbf{V}} = (\lambda, \beta_0, \beta_1)^\intercal$ be the 4×1 parameter vector. For determining the MLE of $\underline{\mathbf{V}}$, we have the log-likelihood function

$$\begin{aligned} \ell &= \ell\left(x; \underline{\mathbf{V}}\right) = \sum_{i=1}^{n} \log\left[\lambda \exp\left(-\beta^T z_i\right)\right] - \lambda \sum_{i=1}^{n} x_i \exp\left(-\beta^T z\right) \\ &\quad + \sum_{i=1}^{n} \log\left[1 + \left(1 + \lambda x_i \exp\left(-\beta^T z\right)\right)\right] - 2 \sum_{i=1}^{n} \log\left[1 + \lambda x_i \exp\left(-\beta^T z\right)\right]. \end{aligned}$$

The score vector $\mathbf{I}_{(\underline{\mathbf{V}})} = \frac{\partial \ell}{\partial \underline{\mathbf{V}}} = \left(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \beta_0}, \frac{\partial \ell}{\partial \beta_1}\right)^{\intercal}$ is given by

$$\begin{aligned}
\mathbf{I}_{(\lambda)} &= \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \exp\left(-\beta^T z\right) \\
&\quad + \sum_{i=1}^{n} \frac{x_i \exp\left(-\beta^T z\right)}{1 + (1 + \lambda x_i \exp\left(-\beta^T z\right))} - 2\sum_{i=1}^{n} \frac{x_i \exp\left(-\beta^T z\right)}{1 + \lambda x_i \exp\left(-\beta^T z\right)},
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(\beta_0)} &= \lambda \sum_{i=1}^{n} x_i \exp\left(-\beta^T z\right) \\
&\quad -\lambda \sum_{i=1}^{n} \frac{x_i \exp\left(-\beta^T z\right)}{1 + [1 + \lambda x_i \exp\left(-\beta^T z\right)]} + 2\lambda \sum_{i=1}^{n} \frac{x_i \exp\left(-\beta^T z\right)}{1 + \lambda x_i \exp\left(-\beta^T z\right)} - 1,
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_{(\beta_1)} &= -\sum_{i=1}^{n} z_i + \lambda \sum_{i=1}^{n} z_i x_i \exp\left(-\beta^T z\right) \\
&\quad +2\lambda \sum_{i=1}^{n} \frac{z_i x_i \exp\left(-\beta^T z\right)}{1 + \lambda x_i \exp\left(-\beta^T z\right)} - \lambda \sum_{i=1}^{n} \frac{z_i x_i \exp\left(-\beta^T z\right)}{1 + [1 + \lambda x_i \exp\left(-\beta^T z\right)]}.
\end{aligned}$$

Setting the nonlinear system of equations $\mathbf{I}_{(\lambda)} = 0, \mathbf{I}_{(\beta_0)} = 0$ and $\mathbf{I}_{(\beta_1)} = 0$ and solving them simultaneously yields the MLE $\widehat{\underline{\mathbf{V}}} = (\widehat{\lambda}, \widehat{\beta}_0, \widehat{\beta}_1)^{\intercal}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. Since, we can not find the explicit formulas for the MLEs of the parameters, we use numerical methods such as the Newton Raphson method, the Monte Carlo method, the BB algorithm or others.

6.2. *Assessing the QBHE-AFT model via a simulation study*

We conduct a comprehensive simulation study using the R programming software to evaluate the performance of the maximum likelihood estimators (MLEs) for the parameters of the QBHE-AFT model. This study is crucial for assessing the accuracy, consistency, and efficiency of the estimation methods under varying sample sizes and parameter settings. The results are obtained using a numerical optimization approach, specifically the Newton-Raphson method , which is widely recognized for its robustness and efficiency in solving nonlinear equations.

In this simulation setup, we assume that the data follows the QBHE-AFT distribution. To ensure reliability and generalizability of the results, the data generation process is repeated N = 5000 times . The true parameter values used in the simulation are set as follows: $\lambda = 2.50, \beta_0 = 1.96, \beta_1 = 1.55$. These values represent the baseline hazard rate and the regression coefficients associated with the explanatory variables in the AFT model.

For computational purposes, we utilize the BB algorithm (Barzilai-Borwein algorithm), as described in Ravi (2009), implemented in the R software. This algorithm is particularly well-suited for optimizing high-dimensional nonlinear objective functions and provides efficient computation of the MLEs. Using this approach, we calculate the averages of the simulated MLEs for the parameters λ , β_0 , and β_1 , along with their respective mean squared errors (MSE) . The MSE serves as a key metric to evaluate the precision and bias of the estimators across different sample sizes.

The simulation study considers six distinct sample sizes: $n = 15, n = 30, n = 50, n = 150, n = 300$, and $n = 500$. These sample sizes span a wide range, from small to large datasets, enabling us to examine how the performance of the estimators evolves as the sample size increases. For each sample size, the simulation generates synthetic datasets based on the specified parameter values and computes the MLEs iteratively over the 5000 replications. The results are then aggregated to compute the average estimates and their corresponding MSEs. The outcomes of the simulation study are summarized in Table 4 , which lists the MSE for the MLEs of the parameters $\lambda, \beta_0, \beta_1$ across the different sample sizes. The table provides a clear depiction of how the MSE decreases as the sample size grows, reflecting the asymptotic properties of the MLEs. Specifically for $\lambda = 2.50$, the MSE decreases significantly as the sample size increases from $n = 15$ to $n = 500$, indicating improved accuracy and reduced variability in the estimates. Similarly, for $\beta_0 = 1.96$ and $\beta_1 = 1.55$, the MSEs also exhibit a consistent downward trend, confirming the consistency of the estimators. These findings underscore the reliability of the MLEs for the QBHE-AFT model, particularly for larger sample sizes. The results demonstrate that the proposed estimation method performs well under the specified simulation conditions, providing accurate and stable parameter estimates. Overall, this simulation study validates the practical applicability of the QBHE-AFT model and highlights its potential for real-world survival analysis applications.

Table 4: MLEs $\left(\widehat{\lambda}, \widehat{\beta}_0, \widehat{\beta}_1\right)$ of the parameters and their mean squared errors.

$N = 5000$	$n = 15$	$n = 30$	$n = 50$	$n = 150$	$n = 300$	$n = 500$
$\widehat{\lambda}$	2.5553	2.5545	2.5427	2.5294	2.5150	2.5079
MSE	1.6271×10^{-2}	8.0192×10^{-3}	4.7314×10^{-3}	2.0100×10^{-3}	9.6088×10^{-4}	6.5547×10^{-4}
$\widehat{\beta}_0$	1.9909	1.9854	1.9767	1.9738	1.9674	1.9601
MSE	9.2760×10^{-3}	5.6049×10^{-3}	3.0585×10^{-3}	7.3168×10^{-4}	3.6641×10^{-4}	1.2534×10^{-4}
$\widehat{\beta}_1$	1.6011	1.5819	1.5517	1.5556	1.5501	1.5498
MSE	2.1224×10^{-2}	1.7578×10^{-2}	8.7629×10^{-3}	5.2243×10^{-3}	4.7595×10^{-3}	1.1659×10^{-3}

The results obtained from the proposed methods are compelling and statistically meaningful, as demonstrated in the accompanying table. The performance of the models is not only evaluated through GoF measures but also supported by precise parameter estimation, with relatively low standard errors even under small sample conditions. To further validate the consistency of the MLEs, we refer to the simulation results illustrated in Figure 1. These results clearly show that the estimators exhibit convergence rates exceeding the classical $O(n^{-1/2})$ benchmark. Specifically, the bias and standard deviation of the MLEs decrease more rapidly than $n^{-1/2}$ as the sample size increases, providing strong empirical evidence for their $\sqrt{n}$-consistency. This rapid convergence highlights the robustness and efficiency of the estimation procedure across a range of sample sizes, thereby reinforcing the practical applicability of the proposed models in real-world settings.

7. Validation of the QBHE-AFT model

The GoF testing has witnessed significant development across recent literature, with adaptations tailored to new lifetime and reliability distributions under both complete and censored data. Goual et al. (2019) introduced modified GoF tests for odd Lindley exponentiated exponential and composite transmuted models. Goual and Yousof (2020) developed a modified chi-square test for Burr XII-Inverse Rayleigh models, while Yadav et al. (2020) applied the Nikulin-Rao-Robson (NRR) test to the Topp-Leone-Lomax model. Ibrahim et al. (2020) focused on censored Burr XII data, and Yousof et al. (2021a) proposed a chi-square-based test for right-censored data. The NRR test framework has been central to works like Mansour et al. (2020a,b,c), who modeled exponential Lomax, Poisson Lomax, and exponential gamma models, and Mansour et al. (2020d,e,f), who used modified Bagdonavičius–Nikulin (BN) tests for new life models including acute bone cancer data. Ibrahim et al. (2020) validated a flexible log-logistic gamma model using GoF techniques. Goual et al. (2020) and Yousof et al. (2021b) used NRR testing for the exponential Weibull and Poisson exponential models, respectively. Ibrahim et al. (2021) applied the BN test to exponential generalized log-logistic models, while Yousof and Ahsan (2021) proposed a new GoF method for right-censored Poisson exponential data. Yadav et al. (2021) validated the Topp-Leone q-exponential model with NRR testing. Ibrahim et al. (2022) advanced the NRR test under both Bayesian and classical settings for the Double Burr XII model. Yadav et al. (2022) assessed the xgamma exponential model with NRR tests under censoring.

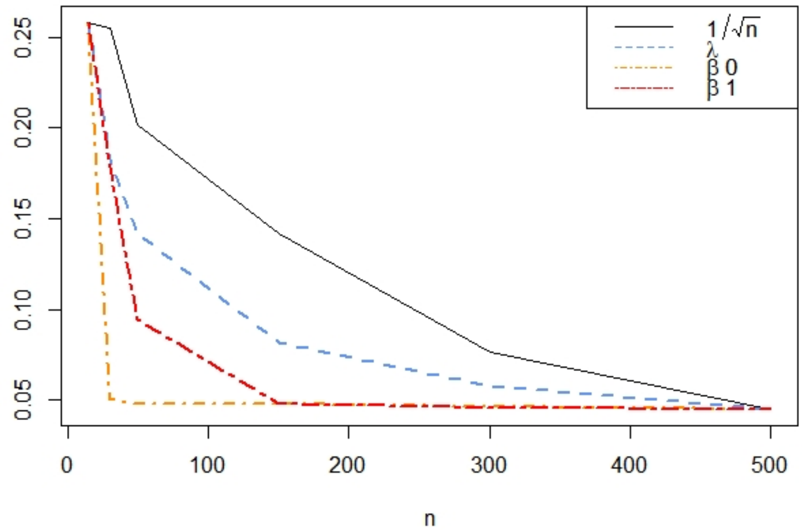

Figure 1. The $\sqrt{n}$-convergence of the parameters $\widehat{\lambda}$, $\widehat{\beta_0}$ and $\widehat{\beta_1}$ of the $QBHE - AFT$ distribution.

Ibrahim et al. (2022) offered chi-square-based validation for odd Fréchet inverse exponential models. Yousof et al. (2022) and Goual et al. (2022) explored Bayesian vs. frequentist GoF techniques for Burr Type XII-based models. Emam et al. (2023) examined inference and GoF testing for exponential type models under censoring. Hashem et al. (2023) proposed GoF validation for accelerated life testing with hybrid censoring. Yousof et al. (2023) and Hashem et al. (2024) extended BN and Bayesian methods to real data with censoring and acceleration schemes. Loubna et al. (2024) introduced the quasi-xgamma frailty model with appropriate GoF tools, and Teghri et al. (2024) validated a two-parameter Lindley frailty model. Shehata et al. (2024) explored Bayesian and classical validation for Burr Type XII-based models using NRR. Hashem et al. (2024) analyzed inference and GoF in nonlinear hybrid accelerated failure models. Salem et al. (2023) presented validation for a Lomax variant applied to insurance and medical data, while Salem et al. (2024) developed a right-skewed model validated through NRR testing for actuarial and reliability applications. Collectively, these studies highlight the evolution of GoF testing into a flexible, distribution-specific, and censor-aware methodology fundamental to modern statistical validation.

In the case of well-specified distributions, classical GoF tests such as Pearson's chi-square, the Kolmogorov-Smirnov (K–S) statistic, and the Anderson-Darling (A–D) statistic are commonly employed to assess the adequacy of the proposed model. However, when the distribution's parameters are unknown and must be estimated from the observed data, these traditional tests lose their theoretical rigor. The resulting test statistics no longer follow their classical distributions, as their asymptotic behavior becomes dependent on both the underlying model and the estimation procedure used. When complete data are available, several techniques exist to assess the fit of probabilistic models to empirical data. Among these, Pearson's chi-square test is one of the most widely applied. Nevertheless, its applicability becomes limited in cases involving censored data or models with unknown parameters. To address these challenges, Nikulin (1973) and independently Rao and Robson (1974) developed a statistic, now commonly referred to as the NRR statistic, specifically designed for use with complete samples. This statistic is constructed based on the maximum likelihood estimators (MLEs) derived from the original data, and under regularity conditions, it converges in distribution to a chi-square distribution. For a detailed exposition of the construction and properties of this class of test statistics, readers are referred to Voinov et al. (2013) and Goual et al. (2019). The NRR-based methods have been successfully applied in various contexts, including model adequacy testing for the Lomax Inverse Weibull distribution (see Goual et al., 2020), the Burr XII Inverse Rayleigh model (Goual et al., 2019), and the Lindley Exponentiated distribution (see Goual et al., 2019). Building upon this framework, the current section proposes a modified chi-square-type test based on the NRR statistic, tailored specifically to evaluate the fit of the newly introduced QBHE model.

7.1. The NRR statistic test for the QBHE-AFT model

To test the hypothesis H_0 according to which $T_1, T_2, \cdots, T_n$, an n-sample comes from a parametric family $F_{\underline{\mathbf{V}}}(t)$

$$H_0 : \Pr\{T_i \le t\} = F_{\underline{\mathbf{V}}}(t), \quad t \in \mathbb{R},$$

where $\underline{\mathbf{V}} = (\underline{\mathbf{V}}_1, \underline{\mathbf{V}}_2, \cdots, \underline{\mathbf{V}}_s)^T$ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson (1974) proposed K^2 the NRR statistic defined as below. Observations $T_1, T_2, \cdots, T_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_r$ mutually disjoint $\mathbf{I}_j =]a_j\text{-}1; a_j]$; where $j = \overline{1; r}$. The limits a_j of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\underline{\mathbf{V}}) = p_j(\underline{\mathbf{V}}; a_{j-1}, a_j) = \int_{a_{j-1}}^{a_j} f_{\underline{\mathbf{V}}}(t)dt|_{(\ j=1,2,\cdots,r)},$$

so

$$a_j = F^{-1}\left(\frac{j}{r}\right)|_{(j=1,\cdots,r-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_r)^T$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{t_i \in \mathbf{I}_j\}}\ |_{(j=1,\ldots,r)}.$$

The NRR statistic is given by

$$K^2(\widehat{\underline{\mathbf{V}}}_n) = X_n^2(\widehat{\underline{\mathbf{V}}}_n) + \frac{1}{n}\mathbf{L}^T(\widehat{\underline{\mathbf{V}}}_n)(\mathbf{I}(\widehat{\underline{\mathbf{V}}}_n) - \mathbf{J}(\widehat{\underline{\mathbf{V}}}_n))^{-1}\mathbf{L}(\widehat{\underline{\mathbf{V}}}_n),$$

where

$$X_n^2(\underline{\mathbf{V}}) = \left(\frac{\nu_1 - np_1(\underline{\mathbf{V}})}{\sqrt{np_1(\underline{\mathbf{V}})}}, \frac{\nu_2 - np_2(\underline{\mathbf{V}})}{\sqrt{np_2(\underline{\mathbf{V}})}}, \cdots, \frac{\nu_r - np_r(\underline{\mathbf{V}})}{\sqrt{np_r(\underline{\mathbf{V}})}}\right)^T$$

and $\mathbf{J}(\underline{\mathbf{V}})$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\underline{\mathbf{V}}) = B(\underline{\mathbf{V}})^T B(\underline{\mathbf{V}}),$$

with

$$B(\underline{\mathbf{V}}) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial p_i(\underline{\mathbf{V}})}{\partial \mu}\right]_{r \times s} |_{(i=1,2,\cdots,r \text{ and } k=1,\cdots,s)},$$

then

$$\mathbf{L}(\underline{\mathbf{V}}) = (\mathbf{L}_1(\underline{\mathbf{V}}), ..., \mathbf{L}_s(\underline{\mathbf{V}}))^T \text{ with } \mathbf{L}_k(\underline{\mathbf{V}}) = \sum_{i=1}^{r} \frac{\nu_i}{p_i}\frac{\partial}{\partial \underline{\mathbf{V}}_k} p_i(\underline{\mathbf{V}}),$$

where $\mathbf{I}_n(\widehat{\underline{\mathbf{V}}_n})$ represents the estimated FIM and $\widehat{\underline{\mathbf{V}}_n}$ is the maximum likelihood estimator of the parameter vector. The K^2 statistic follows a distribution of chi-square χ^2_{r-1} with $(r-1)$ degrees of freedom.

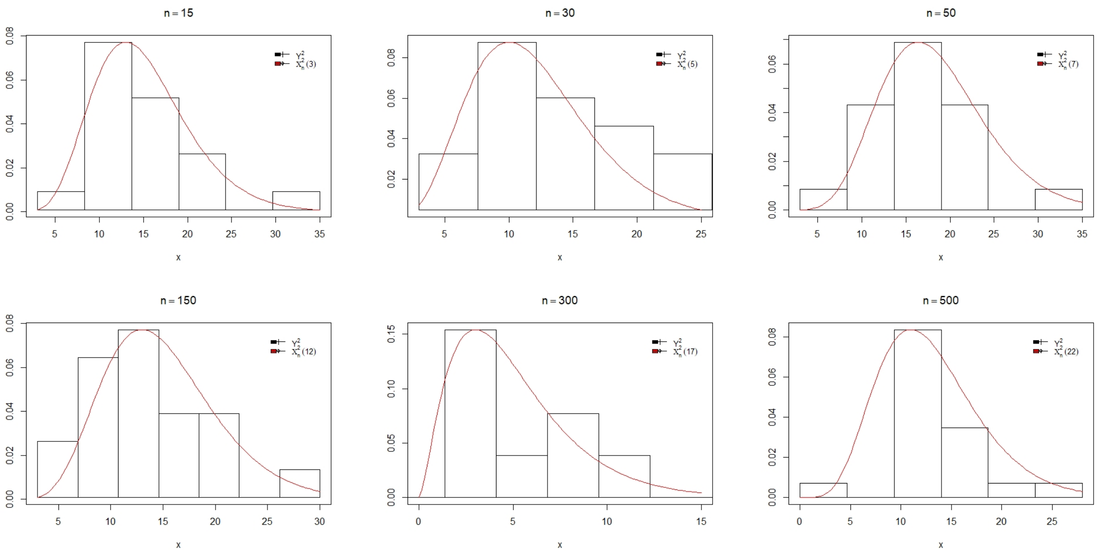

Figure 2. the simulated K^2 histograms compared to the theoretical chi-square with the corresponding degree of freedom k

7.2. *Simulation studies under the NRR statistic* K^2

Consider a sample $T_{1:n}$ where $T = T_{1:n} = (T_1, T_2, \cdots, T_n)^T$. If these data are distributed in accordance with the QBHE-AFT model, then $P\{T_{1:n} \leq t\} = F_{\underline{\mathbf{V}}}(t)$; with unknown parameters $\underline{\mathbf{V}} = (\lambda, \beta_0, \beta_1)^T$, by fitting the NRR statistic created in the preceding section, a chi-square GoF test is created. The MLEs $\widehat{\underline{\mathbf{V}}}_n$ of the unknown parameters of the AFT-QBHE model are computed on the initial data. Since, the statistic K^2 not dependent on the parameters, we can therefore use the estimated Fisher information matrix (FIM) $I_n(\widehat{\underline{\mathbf{V}}}_n)$. All the components of the statistic K^2, for the AFT-QBHE distribution are provided, therefore K^2 can be deduced easily.

In order to support the results obtained in this work, a numerical simulation is performed. Therefore, in order to test the null hypothesis H_0 of the AFT-QBHE model, we calculated 5000 sample data simulations ($n = 15$, $n = 30$, $n = 50$, $n = 150$, $n = 300$ and $n = 500$) from AFT-QBHE distribution, after calculating the value of the criterion statistic K^2, we count the number of rejected cases of the null hypothesis H_0. When $K^2 > \chi^2\,(k = r - 1)$, the significance is different level $\alpha\,(1\%, 5\%, 10\%)$. The simulation results of the rejected level of K^2 and its theoretical value are shown in Table 5 below.

Table 5: Empirical levels K^2 and corresponding theoretical levels.

$N = 5000$	$n = 15$	$n = 30$	$n = 50$	$n = 150$	$n = 300$	$n = 500$
$\alpha = 0.01$	0.019	0.018	0.024	0.0098	0.011	0.012
$\alpha = 0.05$	0.055	0.045	0.046	0.048	0.0498	0.0502
$\alpha = 0.1$	0.178	0.089	0.092	0.095	0.098	0.0997

The results show that the computed empirical level closely matches the corresponding theoretical level. Based on this, we conclude that the proposed test is highly appropriate for the AFT-QBHE distribution. This finding supports the claim that the K^2 statistic asymptotically follows a chi-squared distribution with degrees of freedom given by $k = r - 1$. To validate this, we performed N = 5,000 simulations under the null hypothesis H_0, using various parameter estimates of the AFT-QBHE model $\underline{\mathbf{V}} = (\lambda, \beta_0, \beta_1)^T$ and different values of r intervals. The results were compared against the chi-squared distribution with k degrees of freedom. Their histograms are shown in Figure 2, compared with the chi-square distribution with k degrees of freedom.

Figure 2 shows the statistical distribution of K^2 for various parameter values and r grouping units. The restriction is based on the chi-square with k degrees of freedom within the simulated statistical error. The same findings are achieved for various parameter values and various intervals of equal probability grouping. As a result, the NRR K^2 statistic's limit distribution is chi-square.

7.3. Applications to real data

We take into account the following real data sets and confirm the presumption that their distribution is consistent with the AFT-QBHE model in order to demonstrate the applicability of the proposed modified chi-square GoF test.

7.3.1. Electric insulating fluid data The failure times of 76 electrical insulating fluids, which were tested under varying voltages ranging from 26 to 38 kilovolts, are documented in the work of Lawless (2003). This dataset has been widely utilized in reliability studies due to its relevance in understanding the impact of voltage stress on the durability and performance of insulating materials. Bagdonavicius and Nikulin (2011) further analyzed this dataset to assess its compatibility with the exponential and Weibull accelerated failure time (AFT) power-rule models. Their study aimed to determine whether these traditional parametric models could adequately describe the relationship between voltage levels and the failure times of the insulating fluids. In this section, we extend their analysis by evaluating how well the data fits our proposed quasi Burr-Hatke exponential accelerated failure time (AFT-QBHE) model. The AFT-QBHE model offers a more flexible framework compared to the conventional exponential and Weibull models, as it can accommodate both monotonic and non-monotonic hazard rate functions. This flexibility makes it particularly suitable for modeling complex datasets like the one under consideration, where the influence of voltage stress on failure times may exhibit nonlinear patterns. By applying the AFT-QBHE model to this dataset, we aim to demonstrate its ability to provide a better fit and capture the underlying failure mechanisms more accurately. The results of this evaluation will not only validate the practical utility of the AFT-QBHE model but also highlight its advantages over existing models in analyzing real-world reliability data.The data observations are given as:

Voltage level (z_i)	n_i	Breakdown time x_i
26	3	5.79,1579.52,2323.7
28	5	68.85,426.07,110.29,108.29,1067.6
30	11	17.05,22.66,21.01,175.88,139.07,144.12,
		20.46,43.40,194.90,47.30,7.74
32	15	0.40,82.85,9.88,89.29,215.10,2.75,0.79,
		15.93,3.91,0.27,0.69,100.58,27.80,13.95,53.24
34	19	0.96,4.15,0.19,0.78,8.01,31.75,7.35,6.50,8.27,33.91,
		32.52,3.16,4.85,2.78,4.67,1.31,12.06,36.71,72.89
36	15	1.97,0.59,2.58,1.69,2.71,25.50,0.35,0.99,
		3.99,3.67,2.07,0.96,5.35,2.90,13.77
38	8	0.47,0.73,1.40,0.74,0.39,1.13,0.09,2.38

In case of $\varphi(z) = z$ log-linear assumption: Using R statistical software (the BB package) we find the values of the MLEs of AFT-QBHE distribution parameters :

$$\widehat{\lambda} = 0.1641, \widehat{\beta}_0 = 0.1854, \widehat{\beta}_1 = 0.0306,$$

we choose $r = 8$ intervals and the estimated FIM can be expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 5.1810 & -35.8331 & -994.9820 \\ & 5.8830 & 163.3554 \\ & & 4570.9871 \end{pmatrix},$$

and then the NRR statistic : $K^2 = 28.7132$. For the critical value : $\alpha = 0.01$, we find $K^2 > \chi^2_{0.01}(7) = 18.4753$. 2-

In case of $\varphi(z) = \log(z)$ power-rule assumption: We find the values of the MLEs of the AFT-QBHE distribution parameters:

$$\widehat{\lambda} = 1.4331, \widehat{\beta}_0 = 20.7564, \widehat{\beta}_1 = -4.64356,$$

we take $r = 8$ intervals and the estimated FIM can be:

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.41066 & -0.6121 & -2.0677 \\ -0.6121 & 0.8772 & 2.9633 \\ -2.0677 & 2.9633 & 10.0196 \end{pmatrix},$$

the NRR statistic is $K^2 = 9.8516$. For the critical values : $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$, we find

$$K^2 < \chi^2_{0.01}(7) = 18.4753$$

$$K^2 < \chi^2_{0.05}(7) = 14.0671$$

$$K^2 < \chi^2_{0.1}(7) = 12.0170$$

respectively. So, we can assume that electric insulating fluid data of Lawless (2003) correspond appropriately to the AFT-QBHE model. 3- In case of $\varphi(z) = 1/z$ arrehnius model: We fit these data by the AFT-QBHE model. Using

R statistical software (the BB package) we find the values of the MLEs of the AFT-QBHE distribution parameters :

$$\widehat{\lambda} = 22.1641, \widehat{\beta}_0 = 6.9938, \widehat{\beta}_1 = 31.4706,$$

we take $r = 8$ intervals and the estimated FIM expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.0018 & -0.0338 & -0.0011 \\ & 0.7509 & 0.0265 \\ & & 0.00094 \end{pmatrix},$$

the NRR statistic is: $K^2 = 91.9240$. For the critical value : $\alpha = 0.01$, we find $K^2 > \chi^2_{0.01}(7) = 18.4753$.

7.3.2. Body fat data set The data of Neter et al. (1996) provides information on ($n = 20$) body fat, triceps skinfold thickness, thigh circumference, and mid-arm circumference for twenty healthy females aged 20 to 34. The data are :

z_{i1} (triceps skinfold measurement); z_{i2} (thigh circumference); x_i(body-fat)
$19.5, 24.7, 30.7; 43.1, 49.8, 51.9; 11.9, 22.8, 18.7$
$29.8, 19.1, 25.6; 54.3, 42.2, 53.9; 20.1, 12.9, 21.7$
$31.4, 27.9, 22.1; 58.5, 52.1, 49.9; 27.1, 25.4, 21.3$
$25.5, 31.1, 30.4; 53.5, 56.6, 56.7; 19.3, 25.4, 27.2$
$18.7, 19.7, 14.6, 29.5; 46.5, 44.2, 42.7, 54.4; 11.7, 17.8, 12.8, 23.9$
$27.7, 30.2, 22.7, 25.2; 55.3, 58.6, 48.2, 51.0; 22.6, 25.4, 14.8, 21.1$

For $\varphi(z) = z$ as a log-linear assumption: We fit these data by the AFT-QBHE model. Using R statistical software (the BB package) we find the values of the MLEs of AFT-QBHE distribution parameters :

$$\widehat{\lambda} =: 12.5501, \widehat{\beta}_0 = -1.1, \widehat{\beta}_1 = 0.0048, \widehat{\beta}_2 = 0.0098.$$

we take $r = 4$ intervals and the estimated FIM expressed as

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 2.6666 \times 10^{-7} & -32.1072 & -835.5879 & -1667.993 \\ -32.1072 & 402.9463 & 10486.6278 & 20933.314 \\ -835.5879 & 10486.6277 & 281495.1660 & 553116.651 \\ -1667.993 & 20933.31379 & 553116.651 & 1097012.518 \end{pmatrix},$$

and then the NRR statistic : $K^2 = 6.1991$. For different critical values : $\alpha = 1\%, \alpha = 5\%$ and $\alpha = 10\%$, we find

$$K^2 < \chi^2_{0.01}(3) = 11.3448, K^2 < \chi^2_{0.05}(3) = 7.8147$$
$$K^2 < \chi^2_{0.1}(3) = 6.2513,$$

respectively.

7.3.3. Johnson's data set Johnson (1996) utilized a comprehensive dataset to explore challenges associated with multiple regression analysis. The dataset consisted of a response variable, which was the estimated percentage of body fat, alongside 13 continuous covariates measured for $n = 252$ males . These covariates included demographic and anthropometric measurements such as age, weight, height, and 10 distinct body circumference measurements. The primary objective of the analysis was to predict the percentage of body fat using these covariates, demonstrating potential issues that arise in multiple regression models when dealing with complex, multivariate data. The dataset is widely accessible and can be found in the 'mfp' package within the R statistical software, making it a valuable resource for researchers and practitioners working on regression modeling and related statistical analyses. Below is a detailed breakdown of the variables included in the dataset:

Variable	Name	Details	Variable	Name	Details
z_1	age	Age (years)	z_8	thigh	Circumference (cm)
z_2	weight	Weight (lb)	z_9	knee	Circumference (cm)
z_3	height	Height (in)	z_{10}	ankle	Circumference (cm)
z_4	neck	Circumference (cm)	z_{11}	bicepes	Circumference (cm)
z_5	chest	Circumference (cm)	z_{12}	forearm	Circumference (cm)
z_6	ab	Circumference (cm)	z_{13}	wrist	Circumference (cm)
z_7	hip	Circumference (cm)	x	pcfat	Body fat (%)

This rich dataset allows for an in-depth examination of how various physical and demographic factors influence body fat percentage. By leveraging this data, Johnson (1996) highlighted the complexities and potential pitfalls of multiple regression analysis, particularly when dealing with multicollinearity, nonlinearity, and interactions among predictors. Such datasets remain instrumental in advancing methodologies for predictive modeling and understanding the relationships between anthropometric measures and health outcomes. In our case, we used two covariates density (Density determined from underwater weighing gm/cm^3) and age (years). We consider the log linear assumption $(\varphi(z) = z)$ and we fit this data by the AFT-QBHE model. The values of the MLEs parameters:

$$\widehat{\lambda} = 9.6, \widehat{\beta}_0 = -1.50, \widehat{\beta}_1 = 0.0011, \widehat{\beta}_2 = 0.26,$$

We take $r = 15$ intervals and the estimated FIM $I\left(\underline{\widehat{\mathbf{V}}}\right)$ expressed as

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.0103 & -0.02163 & -0.0228 & -0.5937 \\ -0.0216 & 0.2077 & 0.2197 & 5.6999 \\ -0.0228 & 0.2197 & 0.2326 & 6.0276 \\ -0.5937 & 5.6999 & 6.0276 & 162.1033 \end{pmatrix},$$

The NRR statistic test: $K^2 = 13.4835$. For different critical values : $\alpha = 0.01, \alpha = 0.05$ and $\alpha = 0.1$, we find

$$\begin{aligned} K^2 &< \chi^2_{0.01}(14) = 29.1412, \\ K^2 &< \chi^2_{0.05}(14) = 23.6847, \\ K^2 &< \chi^2_{0.1}(14) = 21.06414, \end{aligned}$$

respectively. This data can be fitted by our proposed AFT-QBHE model with the log-linear assumption $(\varphi(z) = z)$. One can affirm that our proposed AFT-QBHE model can be an appropriate distribution of this data.

8. Censored case

8.1. Maximum likelihood estimation

The likelihood function of the right-censored AFT-QBHE is

$$\begin{aligned} L\left(x;\vartheta\right) &= \prod_{i=1}^{n} f\left(x_i;\vartheta\right) = \prod_{i=1}^{n} \lambda^{\delta_i}\left(x_i;\vartheta\right) S\left(x_i;\vartheta\right), \delta_i = 1_{\{T_i \leq C_i\}} \\ &= \prod_{i=1}^{n} \left\{ \frac{\lambda e^{-\beta^T z}\left[\theta\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)+1\right]}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} \right\}^{\delta_i} \times \frac{\exp\left(-\lambda x_i \exp\left(-\beta^T z_i\right)\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)}. \end{aligned}$$

The log-likelihood function is

$$\begin{aligned} \ell\left(x;\vartheta\right) &= \sum_{i=1}^{n} \delta_i \log\left(\lambda \exp\left(-\beta^T z_i\right)\right) + \sum_{i=1}^{n} \delta_i \log\left[\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)+1\right] \\ &\quad -\sum_{i=1}^{n} \delta_i \log\left[1+\lambda x_i \exp\left(-\beta^T z_i\right)\right] - \lambda \sum_{i=1}^{n} x_i \exp\left(-\beta^T z_i\right) - \sum_{i=1}^{n} \log\left[1+\lambda x_i \exp\left(-\beta^T z_i\right)\right] \\ &= \sum_{i\in F} \log\left(\lambda \exp\left(-\beta^T z_i\right)\right) + \sum_{i\in F} \left[\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)+1\right] - 2\sum_{i\in F} \log\left[1+\lambda x_i \exp\left(-\beta^T z_i\right)\right] \\ &\quad -\lambda \sum_{i\in F} x_i \exp\left(-\beta^T z_i\right) - \sum_{i\in C} \log\left[1+\lambda x_i \exp\left(-\beta^T z_i\right)\right] - \lambda \sum_{i\in C} x_i \exp\left(-\beta^T z_i\right), \end{aligned}$$

where F and C denote the sets of uncensored and censored observations, respectively. The score functions for the parameters λ, β_0 and β_1 are given by

$$\begin{aligned} \frac{\partial \ell\left(x_i;\lambda,\beta\right)}{\partial \lambda} &= \frac{r}{\lambda} + \sum_{i\in F} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)} - 2\sum_{i\in F} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} - \sum_{i\in F} x_i \exp\left(-\beta^T z_i\right) \\ &\quad -\sum_{i\in C} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} - \sum_{i\in C} x_i \exp\left(-\beta^T z_i\right) \\ \frac{\partial \ell\left(x_i;\lambda,\beta\right)}{\partial \beta_0} &= \lambda \sum_{i\in F} x_i \exp\left(-\beta^T z_i\right) - \lambda \sum_{i\in F} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)} + 2\lambda \sum_{i\in F} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} \\ &\quad +\lambda \sum_{i\in C} x_i \exp\left(-\beta^T z_i\right) + \lambda \sum_{i\in C} \frac{x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} - 1, \\ \frac{\partial \ell\left(x_i;\lambda,\beta\right)}{\partial \beta_1} &= \lambda \sum_{i\in F} z_i x_i \exp\left(-\beta^T z_i\right) - \lambda \sum_{i\in F} \frac{z_i x_i \exp\left(-\beta^T z_i\right)}{1+\left(1+\lambda x_i \exp\left(-\beta^T z_i\right)\right)} + 2\lambda \sum_{i\in F} \frac{z_i x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} \\ &\quad +\lambda \sum_{i\in C} \frac{z_i x_i \exp\left(-\beta^T z_i\right)}{1+\lambda x_i \exp\left(-\beta^T z_i\right)} + \lambda \sum_{i\in C} z_i x_i \exp\left(-\beta^T z_i\right) - \sum_{i\in F} z_i \end{aligned}$$

where r is the number of failures.

8.1.1. Calculation of the matrix $\widehat{W}$ The elements of the estimated matrix $\widehat{W}$ defined by

$$\widehat{W}_l = \sum_{j=1}^{k} \widehat{C}_{lj} \widehat{A}_j^{-1} \widehat{Z}_j,\ l = 1,2,3\ ;\ j = 1,2,...,k.$$

are obtained as follow

$$\begin{aligned}
\widehat{C}_{1j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{1}{\lambda}-\frac{U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}\right\},\\
\widehat{C}_{2j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{\lambda U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}-1\right\},\\
\widehat{C}_{3j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{\lambda U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}-z_i\right\},
\end{aligned}$$

where,

$$U(x,\vartheta)=xe^{-\beta^T z}, M(x_i,\vartheta)=1+\lambda xe^{-\beta^T z}, \vartheta=(,\lambda,\beta_0,\beta_1).$$

8.1.2. Information matrix $\widehat{I}$ The components of the information matrix $\widehat{I}\left(\widehat{\vartheta}\right)=\left(\widehat{i}_{ll\prime}\right)_{3\times 3}$ are given as follows

$$\begin{aligned}
\widehat{i}_{11} &= \frac{1}{n}\sum_{i=1}^{n}\delta_i\left\{\frac{1}{\lambda}-\frac{U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}\right\}^2,\\
\widehat{i}_{22} &= \frac{1}{n}\sum_{i=1}^{n}\delta_i\left\{\frac{\lambda U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}-1\right\}^2,\\
\widehat{i}_{33} &= \frac{1}{n}\sum_{i=1}^{n}\delta_i\left\{\frac{\lambda U(x_i,\vartheta)}{M(x_i,\vartheta)\left[1+M(x_i,\vartheta)\right]}-z_i\right\}^2,
\end{aligned}$$

$$\begin{aligned}
\widehat{i}_{12} &= -\frac{1}{n}\sum_{i=1}^{n}\delta\left\{{}_i\frac{\left[M(x_i,\vartheta)-\lambda U(x_i,\vartheta)+M(x_i,\vartheta)^2\right]^2}{\lambda\left[1+M(x_i,\vartheta)\right]^2\times M(x_i,\vartheta)^2}\right\},\\
\widehat{i}_{13} &= -\frac{1}{n}\sum_{i=1}^{n}\delta\left\{{}_i\frac{z_i\left[M(x_i,\vartheta)-\lambda U(x_i,\vartheta)+M(x_i,\vartheta)^2\right]^2}{\lambda\left[1+M(x_i,\vartheta)\right]^2\times M(x_i,\vartheta)^2}\right\},
\end{aligned}$$

$$\widehat{i}_{23}=\frac{1}{n}\sum_{i=1}^{n}\delta_i\left\{\frac{z_i\left[M(x_i,\vartheta)-\lambda U(x_i,\vartheta)+M(x_i,\vartheta)^2\right]^2}{\left[1+M(x_i,\vartheta)\right]^2\times M(x_i,\vartheta)^2}\right\}.$$

8.2. Simulation of the censored MLEs of the parameters for the AFT-QBHE distribution

In this section, we conduct a comprehensive simulation study to evaluate the performance of the maximum likelihood estimators (MLEs) for the parameters of the Accelerated Failure Time model based on the quasi Burr-Hatke exponential (AFT-QBHE) distribution. The simulation process involves generating synthetic datasets under controlled conditions and analyzing the accuracy and precision of the MLEs for varying sample sizes. To begin, we assume that the AFT-QBHE distribution is the underlying model for the data generation process. The simulation is repeated N = 10,000 times to ensure robustness and reliability of the results. The true parameter values used in the simulation are set as λ = 0.58 , representing the scale parameter of the distribution, β_0 = 0.53 , corresponding to the intercept term in the regression model, β_1 = 0.38 , representing the coefficient of the explanatory variable. For each replication, we generate synthetic datasets with six different sample sizes: $n = 15, n = 30, n = 50, n = 150, n = 300$, and $n = 500$. These sample sizes span a wide range, from small to large datasets, allowing us to examine how the performance of the estimators evolves as the sample size increases. The primary objective of the simulation study is to compute the mean simulated MLEs for the parameters λ, β_0, and β_1 , along with their corresponding MSEs. The mean simulated MLEs provide insights into the bias of the estimators, while the MSEs quantify both the bias and variability of the estimates, offering a comprehensive measure of their accuracy.

Table 5: The censored MLEs $\left(\widehat{\lambda}, \widehat{\beta}_0, \widehat{\beta}_1\right)$ of AFT-QBHE's parameters and their mean squared errors.

$N = 10,000$	$n = 15$	$n = 30$	$n = 50$	$n = 150$	$n = 300$	$n = 500$
$\widehat{\lambda}$	0.6097	0.5987	0.5968	0.5845	0.5823	0.5821
SME	2.5016×10^{-3}	1.7337×10^{-3}	1.1190×10^{-3}	4.4723×10^{-4}	2.6714×10^{-4}	1.2759×10^{-4}
$\widehat{\beta}_0$	0.5497	0.5420	0.5400	0.5371	0.5352	0.5321
SME	2.0320×10^{-3}	1.5970×10^{-3}	1.3457×10^{-3}	9.3368×10^{-4}	8.2201×10^{-4}	2.2105×10^{-4}
$\widehat{\beta}_1$	0.3897	0.3887	0.3867	0.3822	0.3801	0.3797
SME	1.8854×10^{-3}	9.6180×10^{-4}	5.3510×10^{-4}	3.3594×10^{-4}	2.9358×10^{-4}	2.4539×10^{-4}

These findings underscore the reliability of the MLEs for the AFT-QBHE model across different sample sizes and parameter settings. The simulation study not only validates the theoretical properties of the estimators but also provides practical guidance on their performance in real-world applications. The results demonstrate that the proposed AFT-QBHE model is well-suited for analyzing survival data, particularly when the sample size is sufficiently large to ensure accurate and stable parameter estimates.

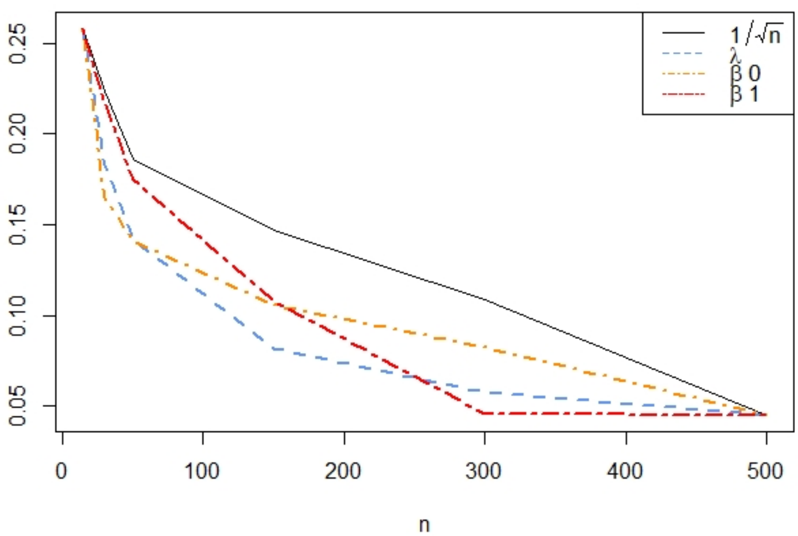

Figure 3. The $\sqrt{n}$-convergence of the censored MLEs $\widehat{\lambda}$, $\widehat{\beta_0}$ and $\widehat{\beta_1}$ of the $QBHE$ distribution.

8.3. Simulated distribution of K_n^2 statistic for the right-censored AFT-QBHE distribution

We compute 5000 simulations of samples data (sample sizes : $n = 15$, $n = 30$, $n = 50$, $n = 150$, $n = 300$ and $n = 500$) from AFT-QBHE distribution, after calculating the values of criteria statistics K_n^2, we count the number of rejection's cases of the null hypothesis H_0, when $K_n^2 > \chi^2_\alpha(k)$, with different significance level α $(\alpha = 1\%, 5\%, 1\%)$. The results of simulated levels of K_n^2 against their theoretical values are shown in the following Table 6.

Table 6: Empirical levels K_n^2 and corresponding theoretical levels.

$N = 5000$	$n = 15$	$n = 30$	$n = 50$	$n = 150$	$n = 300$	$n = 500$
$\alpha = 1\%$	0.016	0.015	0.015	0.013	0.010	0.010
$\alpha = 5\%$	0.054		0.053	0.053	0.052	0.050
$\alpha = 10\%$	0.16	0.14	0.14	0.13	0.099	0.109

As can be seen, the calculated empirical level K_n^2 values are extremely similar to the equivalent theoretical level value. Consequently, we draw the conclusion that the suggested test is excellent for the AFT-QBHE distribution. in Figure 4 we plot, the histograms of simulated K_n^2, compared with the chi-square distribution with k degrees of freedom Figure 4 shows the statistical distribution of K_n^2 for different parameters values estimated above and various r grouping intervals is chi-square with k degrees of freedom. As a result, the NRR K_n^2 statistic's limit distribution is chi-square.

8.4. Applications to real censored data

We take into account the following real data sets and confirm the presumption that their distribution is consistent with the AFT-QBHE model in order to demonstrate the applicability of the proposed modified chi-square GoF test.

8.4.1. Censored motor data These reliability datasets, accessible in the survival package of R software, record the time to failure (or breakdown) of motor insulation systems under varying temperature conditions. The main goal of this data is to examine how temperature affects the lifespan and durability of motor insulation, which is essential for understanding the thermal degradation mechanisms that contribute to system failures. Such datasets are commonly utilized in reliability engineering and survival analysis to model failure times, evaluate risks, and enhance material design for better performance under thermal stress. Below, Table 6 provides a summary of the motor dataset.

Table 6: The breakdown of motor data set.

z_1 (temperature)	x_i (time of Breakdown)	δ_i (censor)
150	$8046, 8064, 8064, 8064, 8064, 8046, 8064, 8064, 8064, 8064$	$0, 0, 0, 0, 0, 0, 0, 0, 0, 0$
170	$1764, 2772, 3444, 3542, 3780, 4860, 5196, 5448, 5448, 5448$	$1, 1, 1, 1, 1, 1, 1, 0, 0, 0$
190	$408, 408, 1344, 1344, 1440, 1680, 1680, 1680, 1680, 1680$	$1, 1, 1, 1, 1, 0, 0, 0, 0, 0$
220	$408, 408, 504, 504, 504, 528, 528, 528, 528, 528$	$1, 1, 1, 1, 1, 0, 0, 0, 0, 0.$

In case of $\varphi(z) = z$ log-linear assumption: Using R statistical software (the BB package) we find the values of the censored MLEs of AFT-QBHE distribution parameters :

$$\widehat{\lambda} = 0.25491, \widehat{\beta}_0 = 2.61542, \widehat{\beta}_1 = 4.91572,$$

we choose $r = 15$ intervals and the estimated FIM can be expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.002154 & 0.95782 & -2.31574 \\ & 0.95354 & -1.300024 \\ & & 0.632591 \end{pmatrix},$$

and then the modified NRR statistic : $K^2 = 20.930154$. For the critical value : $\alpha = 0.01$, we find $K^2 > \chi^2_{0.01}(14) = 29.1412$.

9. Conclusions and future points

In this study, we introduced a novel exponential accelerated failure time (AFT) model based on the quasi Burr-Hatke exponential (QBHE) distribution. The model's fundamental properties were explored, and its validity was assessed using various estimation techniques and GoF tests. Simulation studies demonstrated that maximum likelihood estimators performed well across different sample sizes, with bias and RMSE decreasing as sample size increased. The modified chi-square test based on Nikulin-Rao-Robson statistics proved effective for validating the model in both complete and censored data scenarios. Applications to real-world datasets, including electric insulating fluid, body fat measurements, and motor insulation data, confirmed the model's practical utility. Results showed that our proposed AFT-QBHE model provided an appropriate fit for these diverse datasets. Furthermore, simulation studies indicated that the NRR statistic's limiting distribution followed a chi-square distribution, supporting the robustness of our approach. The model's performance was consistent across various parameter values and grouping intervals, demonstrating its flexibility and reliability. Comparisons between empirical and theoretical levels of the test statistic showed close agreement, confirming the suitability of the proposed methodology. This research contributes to the field of survival analysis by offering a new parametric AFT model that can be applied to both complete and right-censored data. The QBHE-AFT model shows particular promise in reliability studies and lifetime data analysis across multiple disciplines. Future research could explore extensions of this model to accommodate more complex censoring mechanisms or incorporate additional covariates. The successful application of BB algorithm for parameter estimation suggests potential for further optimization in computational methods. Overall, this study provides a comprehensive framework for analyzing survival data using the proposed QBHE-AFT model, contributing valuable tools for researchers in reliability engineering, medical research, and other fields dealing with time-to-event data.

While this study presents a novel QBHE-AFT model with valuable applications, it has certain limitations. The model's performance is evaluated primarily through simulation studies and a limited number of real-world datasets, which may not fully represent the complexity of various practical scenarios. The research focuses on right-censored data and does not extensively explore other types of censoring mechanisms, potentially limiting its applicability in more complex survival analysis contexts. Additionally, the study assumes specific parametric forms for the baseline distribution, which might not always be suitable for all types of lifetime data. Finally, while multiple estimation methods are considered, the computational efficiency and robustness of these methods under very large datasets or high-dimensional covariates are not thoroughly investigated.

Based on the cited literature, the QBHE distribution holds significant promise for future statistical and applied modeling. Inspired by Ahmed et al. (2022), who developed a novel G-family for sampling plans, QBHE could be tailored for truncated or censored quality control schemes. In line with the flexible models in Alizadeh et al. (2024), the QBHE can be utilized in threshold-based risk analysis under extreme environmental stress. Its structure could be extended for actuarial applications, following methodologies from Alizadeh et al. (2025), Alizadeh et al. (2023), and Hamedani et al. (2023), to model skewed and heavy-tailed claims or reinsurance data. Building upon entropy and order-statistic-based analyses in Elbatal et al. (2024) and Hashempour et al. (2024a), QBHE can contribute to entropy maximization and P-mean evaluations in revenue and loss modeling. Moreover, incorporating copula-based dependence structures as shown in Hamed et al. (2022) and Khedr et al. (2023), the QBHE can be integrated into multivariate risk frameworks. The Bayesian and non-Bayesian frameworks by Ibrahim et al. (2023), as well as discrete modeling approaches in Yousof et al. (2024a), indicate the potential of QBHE extensions for handling overdispersed, skewed, or zero-inflated discrete data. Its use in extreme value modeling, particularly in finance and insurance, is suggested by models in Yousof et al. (2023b), Salem et al. (2023), and Hashempour et al. (2024b). For survival and frailty analysis under heterogeneity, as examined by Loubna et al. (2024), QBHE can offer alternative formulations with improved tail behavior. Given the success of existing models in Value-at-Risk (VaR) and stress testing—see Korkmaz et al. (2018), Aljadani et al. (2024), and Yousof et al. (2024b), the QBHE could serve as a novel base for estimating risk measures under financial extremes. Additionally, following Das et al. (2025), Ibrahim et al. (2025), and Yousof et al. (2023b), the distribution may support economic peak modeling, automobile claims forecasting, and heavy-tailed value estimation in real estate and transportation sectors. Furthermore, applications involving bimodal and symmetric insurance data, as developed in Yousof et al. (2023c, 2023d and 2023e), and precipitation risk modeling from Hashempour et al. (2024b), affirm QBHE's relevance across climate and disaster risk domains. Finally, the statistical validation strategies used across studies such as Teghri et al. (2024), Rasekhi et al. (2022), and Shrahili et al. (2021) pave the way for QBHE to undergo rigorous goodness-of-fit, simulation-based assessment, and risk-based policy design. For more related works in risk analysis and other related works for future points see Alizadeh et al. (2025), Das et al. (2025), Ibrahim et al. (2025a,b,c,d), Ramaki et al. (2025), Yousof et al. (2025), Taghipour et al. (2025), AlKhayyat et al. (2025) and Abonongoet al. (2025).

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