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A Novel Accelerated Failure Time Model with Risk Analysis under Actuarial Data, Censored and Uncensored Application

Mohamed Ibrahim[1,*], Hafida Goual[2], Meribout Kaouter Khaoula[2], Abdullah H. Al-Nefaie[1], Ahmad M. AboAlkhair[1] and Haitham M. Yousof[3]

[1] *Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia.*
[2] *Laboratory of probabilities and statistics LaPS, Badji Mokhtar-Annaba University, 12, P.O. Box, 23000 Annaba Algeria.*
[3] *Department of Statistics, Mathematics and Insurance, Faculty of Commerce, Benha University, Egypt*

Abstract This paper proposes a novel Accelerated Failure Time (AFT) model based on the Weighted Topp-Leone exponential (WTLE) distribution, designed for robust survival analysis under censored and uncensored actuarial and biomedical data. The AFT-WTLE model introduces flexible hazard rate shapes, validated through goodness-of-fit tests and real-world applications, including electric insulating fluid failure times and body fat percentage datasets. Parameter estimation employs maximum likelihood (MLE), Cramér-von Mises (CVM), Anderson-Darling (ADE), and their modified variants (RTADE, AD2LE), with simulation studies demonstrating RTADE's superior accuracy in bias and root mean squared error (RMSE) for small-to-moderate samples. The model's risk assessment capabilities are highlighted via Value-at-Risk (VaR), Tail VaR (TVaR), and tail mean-variance metrics, revealing RTADE and ADE as optimal for capturing extreme tail risks. A modified Nikulin-Rao-Robson (NRR) chi-square test confirms the AFT-WTLE's validity for censored data, with empirical rejection levels aligning closely with theoretical thresholds. Applications to motor failure data and Johnson's body fat dataset illustrate its practical utility in actuarial, healthcare, and engineering domains. Computational efficiency is achieved via the BB algorithm for parameter optimization. Simulation results emphasize improved estimation consistency with increasing sample sizes, particularly for RTADE in high-quantile risk metrics. This work bridges gaps in survival modeling by integrating flexible baseline hazards with advanced risk quantification tools, offering a versatile framework for analyzing complex survival data across disciplines.

Keywords Accelerated Failure Time; Censored Data; Risk Analysis; Goodness-of-Fit Tests; Value-at-Risk; Barzilai-Borwein Optimization; Biomedical Data; Reliability Engineering.

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

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1. Introduction

Parametric models are often sought after for analyzing survival data because they provide insights into the characteristics of failure times and risk functions. However, when failure rates or events like product failures, patient deaths, or disease remission have multiple causes, simple parametric models fall short in capturing the influence of each cause. To address this limitation, accelerated failure time (AFT) models were introduced in statistical literature. In AFT models, explanatory variables (such as temperature, pressure, or medication dosage) represented by covariates directly impact key model functions like failure rates and survival probabilities. Unlike proportional hazards models, which often rely on Cox's semi-parametric approach, AFT models are fully parametric. Additionally, regression parameter estimates from AFT models are robust to omitted covariates and

*Correspondence to: Mohamed Ibrahim (Email: miahmed@kfu.edu.sa). Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia.

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less sensitive to the choice of probability distribution. By adjusting covariate values, engineers and practitioners can manipulate outcomes, making AFT models widely applicable in reliability studies and survival analysis. The primary goal of AFT models is to assess how stress factors (covariates) affect the lifespan of items.

Various baseline distributions form the foundation for different AFT models, including exponential, Weibull, log-logistic, and log-normal models. More advanced models, such as the generalized inverse Weibull AFT model, have also been developed. Statistical tests, such as chi-squared goodness-of-fit tests, have been proposed for evaluating regression models like AFT, proportional hazards, and frailty models.

Among AFT models, the log-logistic distribution is particularly popular due to its ability to exhibit non-monotonic hazard functions, those that initially rise and later declinem unlike the monotonic behavior of the Weibull distribution. Although it has heavier tails, the log-logistic distribution resembles the log-normal distribution in shape. Its straightforward closed-form cumulative distribution function (CDF) makes it computationally advantageous for fitting censored data. The survival function, derived as the complement of the CDF, is essential for handling censored observations. Notably, the Weibull distribution (which includes the exponential distribution as a special case) is unique among distributions because it can be parameterized as either an AFT or a proportional hazards model. However, the monotonicity of the Weibull hazard function may limit its biological applications.

Other distributions suitable for AFT models include the log-normal, gamma, and inverse Gaussian distributions, though they are less commonly used than the log-logistic distribution, partly due to their lack of closed-form CDFs. The Weibull, log-normal, and gamma distributions are specific cases of the generalized gamma distribution, a three-parameter model. To evaluate the performance of estimators, various estimation methods are employed, including maximum likelihood, Cramer-von-Mises, Anderson-Darling (right-tail and left-tail variants), and L-moments. Simulation studies compare these methods across different sample sizes and parameter values, assessing bias, root mean-standard errors, mean absolute differences (MADv), and maximum absolute differences (MaxADv). Based on these evaluations, a new WTLE-AFT model is proposed as a parametric accelerated life model when the baseline survival function belongs to the WTLE model. This model is applicable in reliability modeling and lifetime testing across fields such as electrical insulation, medicine, and lifetime studies. Using the Barzilai-Borwein (BZB) algorithm, the average simulated values of maximum likelihood estimators (MLEs) and their mean squared errors are reported under varying sample sizes. The WTLE-AFT model is validated using a modified chi-square test for both complete and right-censored data scenarios. The theoretical framework of Nikulin-Rao-Robson (NRR) statistics is applied to assess the model's viability. Recent enhancements to the NRR test statistic have improved its utility in validation procedures. For the WTLE-AFT model, the modified NRR test statistic is evaluated at empirical and theoretical levels using maximum likelihood estimation. To further assess the effectiveness of the NRR test statistic, three real-world datasets are analyzed. Following Popović et al. (2016), Lak et al. (2025) presented the cumulative distribution function (CDF) of weighted Topp-Leone family of distribution which given by

$$F_{\gamma,\underline{\xi}}(X) = \gamma / \left\{\gamma - \log\left[1 - \bar{G}_{\underline{\xi}}(X)^2\right]\right\}|_{\gamma>0}, \tag{1}$$

where $\bar{G}_{\underline{\xi}}(X) = 1 - G_{\underline{\xi}}(X)$ refers to the survival function of any baseline model. Hence, $G_{\underline{\xi}}(X) = G_{\lambda}(X)$ refers to the CDF of the exponential baseline model. Then, the CDF and the probability density function (PDF) of WTLE is given by

$$F_{\gamma,\lambda}(X) = \gamma / \{\gamma - \log[1 - \exp(-2\lambda x)]\}|_{\gamma,\lambda>0}, \tag{2}$$

$$f_{\gamma,\lambda}(X) = 2\gamma\lambda \frac{\exp(-2\lambda x)}{[1 - \exp(-\lambda x)][1 + \exp(-\lambda x)]\{\gamma - \log[1 - \exp(-2\lambda x)]\}^2}, \tag{3}$$

Using wolframalpha, for any $0 < \frac{\varsigma_2}{\varsigma_3} < 1$, we can write

$$\frac{\varsigma_1}{\varsigma_1 - \log\left(1 - \frac{\varsigma_2}{\varsigma_3}\right)} = \sum_{\varsigma_4=0}^{\infty} a_{\varsigma_4} \left(\frac{\varsigma_2}{\varsigma_3}\right)^{\varsigma_4}, \tag{4}$$

where

$$
\begin{aligned}
a_0 &= 1, \\
a_1 &= -\frac{1}{\varsigma_1}, \\
a_2 &= -\frac{1}{2\varsigma_1^2}(\varsigma_1 - 2), \\
a_3 &= -\frac{1}{3\,\varsigma_1^3}(\varsigma_1^2 - 3\varsigma_1 + 3), \\
a_4 &= -\frac{1}{12\varsigma_1^4}\left(3\varsigma_1^3 + 11\varsigma_1^2 - 18\varsigma_1 + 12\right), \\
a_5 &= \frac{1}{60\varsigma_1^5}\left(12\varsigma_1^4 - 50\varsigma_1^3 + 105\varsigma_1^2 + 105\varsigma_1^2 - 120\varsigma_1 + 60\right), \cdots .
\end{aligned}
$$

Then, we can obtain an expansion for CDF of WTLE as follows

$$F_{\gamma,\lambda}(x) = \sum_{k=0}^{\infty} a_k \, [\exp(-\lambda x)]^{2k},$$

$$F_{\gamma,\lambda}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{2k} a_k \,(-1)^j \binom{2k}{j} \, [1 - \exp(-\lambda x)]^j$$

which can be simplified by

$$F_{\gamma,\lambda}(x) = \sum_{j=0}^{\infty} b_j \, [1 - \exp(-\lambda x)]^j, \tag{6}$$

where $b_j = \sum_{k=[j/2]}^{\infty} a_k \,(-1)^j \binom{2k}{j}$. The PDF of X follows by differentiating (6) as

$$f_{\gamma,\lambda}(x) = \sum_{j=0}^{\infty} b_{j+1}\, \lambda \exp(-\lambda x)\, [1 - \exp(-\lambda x)]^j, \tag{7}$$

Equation (7) reveals that the WTLE density function is a linear combination of generalized-exponential (EE) densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the EE distribution.

2. Simulations for assessing estimation methods

In this paper we will consider the maximum likelihood estimation (MLE) method, the Cramér-von-Mises estimation (CVME), the Anderson Darling estimation (ADE), the Tail-Anderson Darling estimation (RTADE) and the left Tail-Anderson Darling estimation (LTADE) for estimating the model parameters. Also, the same mehtods will be cosidered in risk anlaysis. To systematically evaluate and compare the effectiveness of various parameter estimation techniques, a detailed numerical simulation study is undertaken. The analysis is based on data generated from the WTLE distribution, with $N = 1000$ independent simulation replications to ensure statistical reliability. Within each replication, synthetic data sets are produced for multiple sample sizes, $n = 15, 30, 50,$ and 100, to explore how estimation performance evolves with increasing data availability. To achieve a robust comparison, multiple evaluation criteria are employed. These include bias, which quantifies the average deviation of an estimator from the true parameter value, and the root mean squared error (RMSE), which encapsulates both bias and variance components. In addition, the mean absolute deviation in distribution (M-AD) is used to measure the average discrepancy between the estimated and actual cumulative distribution functions, while the maximum absolute deviation (Max-AD) identifies the largest such discrepancy across the domain. Together, these metrics provide a multidimensional perspective on estimator performance, capturing both point estimation accuracy and overall distributional fit.

Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study where:

1-BIAS$(\gamma) = \frac{1}{B}\sum\limits_{i=1}^{B}\left(\widehat{\gamma}_i - \gamma\right)$,BIAS$(\lambda) = \frac{1}{B}\sum\limits_{i=1}^{B}\left(\widehat{\lambda}_i - \lambda\right)$,

2-RMSE$(\gamma) = \sqrt{\frac{1}{B}\sum\limits_{i=1}^{B}\left(\widehat{\gamma}_i - \gamma\right)^2}$,RMSE$(\lambda) = \sqrt{\frac{1}{B}\sum\limits_{i=1}^{B}\left(\widehat{\lambda}_i - \lambda\right)^2}$,

3-The M-AD $\left(D_{(\text{abs})}\right)$:D$_{(\text{abs})} = \frac{1}{nB}\sum\limits_{i=1}^{B}\sum\limits_{j=1}^{n}|F_{(\gamma,\lambda)}(x_{ij}) - F_{(\widehat{\gamma},\widehat{\lambda})}(t_{ij})|$ and

4-The Max-AD $\left(D_{(\max)}\right)$:D$_{(\max)} = \frac{1}{B}\sum\limits_{i=1}^{B}\max_j|F_{(\gamma,\lambda)}(x_{ij}) - F_{(\widehat{\gamma},\widehat{\lambda})}(w_{ij})|$.

Table 1 reports the simulation outcomes for estimating parameters λ = 0.4 and γ = 0.2 using five methods across different sample sizes. As expected, all estimators improve in accuracy as sample size increases. The RTADE consistently achieves the lowest bias and RMSE, particularly at small sample sizes (n = 15, 30), highlighting its robustness. MLE, while asymptotically efficient, performs relatively poorly for small n, with higher bias and distributional deviation (Dabs, Dmax). CVM and ADE offer strong mid-range performance, with ADE slightly better in estimating λ. LEADE is generally less precise, especially in smaller samples, but remains competitive as n grows. In terms of distributional fit, RTADE dominates with the smallest Dabs and Dmax values throughout. This suggests its superior capability in capturing both parameter values and distributional shape. Overall, RTADE stands out as the most reliable estimator under the simulated conditions.

Table 2 presents the simulation results for estimating parameters λ = 0.5 and γ = 0.05 across increasing sample sizes. The RTADE method again shows superior performance, particularly evident in the lowest Dabs and Dmax values, indicating its exceptional distributional accuracy. At small sample size (n = 15), ADE also performs well with low bias and RMSE, especially for λ, while LEADE exhibits the poorest precision across all metrics. As the sample size increases, all methods improve markedly; however, MLE becomes more competitive and nearly matches ADE and RTADE in RMSE and bias by $n = 100$. CVM shows moderate performance throughout but lags behind ADE and RTADE in precision. The results emphasize that RTADE consistently delivers the most reliable estimates, with ADE following closely, particularly in small to mid-sized samples. LEADE, though useful in some cases, suffers from higher variability and bias, especially in λ estimation.

Table 3 displays simulation outcomes for parameter values λ = 0.1 and γ = 0.1, focusing on how estimation methods perform with modest parameter magnitudes. At small sample size (n = 15), ADE and RTADE demonstrate superior estimation precision, especially in terms of RMSE and distributional deviations (Dabs and Dmax), outperforming MLE, which shows relatively higher bias in γ. As the sample size increases, all methods improve, but ADE and RTADE continue to dominate, particularly at $n = 100$, where ADE achieves almost negligible Dabs and Dmax values, signaling excellent distributional fidelity. CVM remains moderate, showing lower bias than MLE but slightly higher RMSE than ADE. Interestingly, LEADE consistently performs the weakest in distributional metrics and bias, especially at small n, though it gradually improves. Overall, ADE exhibits the most balanced accuracy, followed closely by RTADE, making them the preferred estimators for this parameter setting.

Table 1: Simulation results for parameter $\lambda = 0.4$ & $\gamma = 0.2$

	n	BIAS γ	BIAS λ	RMSE γ	RMSE λ	Dabs	Dmax
MLE	15	0.03640	0.01460	0.01650	0.00642	0.03918	0.05850
CVM		0.02303	0.01845	0.01161	0.01024	0.03250	0.0493
ADE		0.02116	0.00910	0.01157	0.00728	0.02401	0.03582
RTADE		0.01456	0.00438	0.01116	0.00641	0.01526	0.02269
LEADE		0.02265	0.02071	0.01327	0.01711	0.03413	0.05171
MLE	30	0.01941	0.00830	0.00702	0.00298	0.02199	0.03311
CVM		0.01006	0.00833	0.00476	0.00417	0.01494	0.02264
ADE		0.00701	0.00162	0.00470	0.00326	0.00706	0.01055
RTADE		0.00873	0.00313	0.00505	0.00300	0.00963	0.01448
LEADE		0.01195	0.01077	0.00507	0.00556	0.01837	0.02790
MLE	50	0.01179	0.00517	0.00392	0.00174	0.01378	0.02063
CVM		0.00398	0.00275	0.00269	0.00234	0.00553	0.00838
ADE		0.00564	0.00176	0.00268	0.00195	0.00605	0.00910
RTADE		0.00535	0.00197	0.00276	0.00169	0.00600	0.00901
LEADE		0.00552	0.00436	0.00285	0.00309	0.00808	0.01229
MLE	100	0.00636	0.00321	0.00151	0.00073	0.00782	0.01180
CVM		0.00235	0.00167	0.00125	0.00107	0.00331	0.00502
ADE		0.00236	0.00107	0.00127	0.00093	0.00284	0.00426
RTADE		0.00122	-0.00007	0.00131	0.00081	0.00096	0.00145
LEADE		0.00319	0.00252	0.00134	0.00141	0.00469	0.00714

Table 2: Simulation results for parameter $\lambda = 0.5$ & $\gamma = 0.05$

	n	BIAS γ	BIAS λ	RMSE γ	RMSE λ	Dabs	Dmax
MLE	15	0.00542	0.00858	0.00072	0.00507	0.02583	0.03877
CVM		0.00557	0.00839	0.00083	0.00613	0.02596	0.03912
ADE		0.00436	0.00376	0.00065	0.00446	0.01797	0.02656
RTADE		0.00341	-0.00029	0.00076	0.00445	0.01064	0.01594
LEADE		0.00755	0.01631	0.00099	0.01322	0.03951	0.05982
MLE	30	0.00243	0.00389	0.00029	0.00243	0.01199	0.01796
CVM		0.00282	0.00460	0.00035	0.00274	0.01388	0.02092
ADE		0.00168	0.00020	0.00029	0.00234	0.00571	0.00855
RTADE		0.00161	-0.00014	0.00030	0.00206	0.00513	0.00769
LEADE		0.00380	0.00734	0.00040	0.00374	0.01973	0.02980
MLE	50	0.00085	0.00091	0.00016	0.00147	0.00378	0.00567
CVM		0.00168	0.00264	0.00018	0.00149	0.00824	0.01245
ADE		0.00111	0.00016	0.00015	0.00127	0.00383	0.00574
RTADE		0.00134	0.00114	0.00018	0.00124	0.00562	0.00842
LEADE		0.00236	0.00451	0.0002	0.00195	0.01231	0.01865
MLE	100	0.00050	0.00070	0.00007	0.00068	0.00238	0.00359
CVM		0.00096	0.00187	0.00008	0.00072	0.00511	0.00772
ADE		0.00061	0.00040	0.00008	0.00068	0.00243	0.00365
RTADE		0.00018	-0.00074	0.00008	0.00055	0.00024	0.00046
LEADE		0.00109	0.00240	0.00009	0.00092	0.00611	0.00924

Table 3: Simulation results for parameter $\lambda = 0.1$ & $\gamma = 0.1$

	n	BIAS γ	BIAS λ	RMSE γ	RMSE λ	Dabs	Dmax
MLE	15	0.01640	0.00189	0.00499	0.00025	0.03272	0.04912
CVM		0.00964	0.00220	0.00325	0.00037	0.02439	0.03660
ADE		0.01076	0.00140	0.00267	0.00027	0.02267	0.03401
RTADE		0.00880	0.00088	0.00309	0.00027	0.01765	0.02632
LEADE		0.01405	0.00508	0.00440	0.00136	0.04230	0.06406
MLE	30	0.00832	0.00119	0.00188	0.00011	0.01821	0.02736
CVM		0.00383	0.00078	0.00114	0.00015	0.00951	0.01434
ADE		0.00684	0.00139	0.00135	0.00015	0.01683	0.02527
RTADE		0.00285	0.00014	0.00113	0.00011	0.00530	0.00789
LEADE		0.00549	0.00130	0.00125	0.00019	0.01431	0.02163
MLE	50	0.00456	0.00063	0.00094	0.00006	0.01002	0.01510
CVM		0.00329	0.00078	0.00070	0.00009	0.00866	0.01309
ADE		0.00229	0.00018	0.00071	0.00008	0.00452	0.00676
RTADE		0.00404	0.00076	0.00075	0.00007	0.00979	0.01474
LEADE		0.00436	0.00115	0.00078	0.00012	0.01192	0.01805
MLE	100	0.00283	0.00061	0.00041	0.00003	0.00719	0.01085
CVM		0.00161	0.00037	0.00033	0.00004	0.00421	0.00637
ADE		0.00044	-0.00013	0.00031	0.00004	0.00023	0.00044
RTADE		0.00111	0.00015	0.00033	0.00003	0.00249	0.00373
LEADE		0.00211	0.00058	0.00037	0.00006	0.00592	0.00898

Based on the results in Tables 1, 2, and 3, we note that the ADE (Anderson-Darling Estimation) and RTADE (Right Tail-ADE) consistently offer superior performance across different parameter settings and sample sizes, with lower bias, RMSE, and distributional errors (Dabs, Dmax) compared to other methods. The MLE, while traditional and reliable for larger samples, tends to show higher bias and distributional deviation, especially at smaller sample sizes. The CVM performs moderately well, generally better than MLE but not as robust as ADE or RTADE in tail-sensitive metrics. The LEADE appears to be the least stable method, often producing the highest Dabs and Dmax values, especially when the left tail is not dominant in the data structure. As sample size increases ($n = 100$), all methods improve significantly, but the advantage of ADE and RTADE becomes more pronounced, suggesting better scalability and robustness. The RTADE particularly excels when precision in tail behavior is critical, offering the lowest Dmax in many scenarios. For smaller values of γ or λ, such as in Table 2 (γ = 0.05), ADE and RTADE are notably resilient and maintain estimation quality. In symmetric or balanced parameter settings (e.g., λ = γ = 0.1 in Table 3), ADE demonstrates near-optimal performance in both parameter and distributional estimates. The results emphasize the importance of considering tail sensitivity and distributional behavior in estimator selection, not just point accuracy. The ADE and RTADE emerge as the most reliable and efficient estimation techniques across a variety of settings and should be prioritized in practical applications.

3. Risk analysis under artificial data

In this Section, we will check the above-mentioned estimation methods in risk analysis. The quantile levels (70%, 80%, 90%) are considered for all risk indicators (VaRq(X), TVaRq(X), TVq(X), TMVq(X) and ELq(X)). Table 4 presents key risk indicators (KRIs) under artificial data for $n = 15$, highlighting how different estimation methods impact tail-based financial measures. Across all quantile levels (70%, 80%, 90%), RTADE consistently produces the highest values for VaR and tail risk measures like TVaR and TMVq, suggesting it captures extreme risk better than others. ADE also shows strong performance, especially in terms of TVaRq and TMVq, indicating robustness in heavy-tailed scenarios. MLE underestimates most risk metrics, especially at higher quantiles, possibly overlooking tail events. CVM and LEADE yield similar profiles, slightly outperforming MLE but falling short of ADE and RTADE. Notably, ELq(X) decreases as the quantile increases, as expected, but RTADE maintains the highest expected losses, reflecting its conservative bias. The results confirm that tail-adaptive estimators (ADE, RTADE) offer superior risk sensitivity, even in small samples. For critical risk management, RTADE appears to be the most prudent method. Table 5 provides KRIs under artificial data for $n = 30$, revealing consistent trends across methods, especially at higher quantiles. ADE and RTADE continue to dominate in capturing higher tail risks, with both showing elevated values for TVaRq, TMVq, and ELq across all quantile levels. Notably, ADE reaches the highest values, especially at the 90% level, implying it better reflects extreme-event exposure. MLE, though slightly improved from the $n = 15$ case, still yields lower estimates, signaling potential underestimation of tail risk. CVM offers slight enhancements over MLE but remains more conservative than RTADE and ADE. LEADE balances between bias reduction and tail sensitivity, performing better than MLE but not matching the tail responsiveness of RTADE. Overall, with the larger sample size, differences narrow slightly, but tail-adaptive methods remain preferable for robust risk estimation. This confirms their advantage even as data availability increases.

Table 4: KRIs under artificial data for n=15.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.23640	0.41460					
70%			2.82174	4.22178	1.68434	5.06395	1.40004
80%			3.44665	4.77318	1.59822	5.57229	1.32653
90%			4.40497	5.66744	1.52219	6.42853	1.26247
CVM	0.22303	0.41845					
70%			2.86194	4.25094	1.65556	5.07872	1.38901
80%			3.48248	4.79784	1.57012	5.58290	1.31536
90%			4.43306	5.68435	1.49479	6.43174	1.25129
ADE	0.22116	0.40910					
70%			2.93719	4.35822	1.73246	5.22445	1.42104
80%			3.57212	4.91771	1.64293	5.73918	1.34559
90%			4.54459	5.82456	1.56402	6.60657	1.27997
RTADE	0.21456	0.40438					
70%			3.00717	4.44575	1.77423	5.33286	1.43857
80%			3.65022	5.01205	1.68213	5.85312	1.36183
90%			4.63460	5.92973	1.60100	6.73023	1.29513
LEADE	0.22265	0.42071					
70%			2.84850	4.23011	1.63792	5.04908	1.38162
80%			3.46575	4.77410	1.55337	5.55078	1.30834
90%			4.41127	5.65587	1.47883	6.39528	1.24460

Table 6 highlights the KRIs under artificial data for $n = 50$, showing further convergence in risk measure estimates as sample size grows. ADE and RTADE exhibit nearly identical outputs, indicating stability and precision in estimating tail risks, especially for higher quantiles. Both consistently report the highest TVaRq, TMVq, and ELq, confirming their responsiveness to extreme values. MLE shows continued underestimation compared to the others, particularly at the 90% level, despite moderate improvement from smaller nn. CVM and LEADE provide middle-ground estimates, with LEADE slightly outperforming CVM in most metrics. The near overlap between ADE and RTADE suggests RTADE's added robustness does not compromise accuracy, reinforcing its appeal. As nn increases, the advantage of tail-adaptive estimators remains evident, though all methods begin to align more closely, reflecting enhanced reliability with more data. Table 7 displays the KRIs under artificial data for $n = 100$, confirming the strong performance of tail-adaptive methods in large samples. RTADE consistently leads across all quantiles, producing the highest values for TVaRq, TMVq, and ELq, emphasizing its superior sensitivity to tail risks. ADE follows closely, with almost indistinguishable results, validating its precision. MLE, while improved from smaller nn, remains the lowest across all metrics, indicating a more conservative estimate of extreme outcomes. CVM and LEADE deliver intermediate values, with LEADE showing slightly better balance in terms of efficiency and risk awareness. As nn increases to 100, differences across methods narrow, but adaptive estimators still retain an edge, particularly in capturing nuanced tail behaviors. This underscores their robustness and reliability in high-dimensional, risk-sensitive environments, making them strong candidates for operational use.

Table 5: KRIs under artificial data for n=30.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.21941	0.40830					
70%			2.95218	4.37623	1.73949	5.24597	1.42405
80%			3.58853	4.93688	1.64949	5.76163	1.34835
90%			4.56304	5.84556	1.57018	6.63065	1.28252
CVM	0.21006	0.40833					
70%			3.00294	4.42825	1.74086	5.29868	1.42532
80%			3.64025	4.98929	1.65021	5.81439	1.34904
90%			4.61549	5.89826	1.57040	6.68346	1.28277
ADE	0.20701	0.40162					
70%			3.07046	4.52002	1.79999	5.42002	1.44956
80%			3.71874	5.09055	1.70607	5.94359	1.37181
90%			4.71053	6.01483	1.62341	6.82653	1.30430
RTADE	0.20873	0.40313					
70%			3.04912	4.49300	1.78624	5.38612	1.44388
80%			3.69479	5.06132	1.69315	5.90790	1.36653
90%			4.68272	5.98207	1.61119	6.78766	1.29935
LEADE	0.21195	0.41077					
70%			2.97464	4.39121	1.71990	5.25116	1.41657
80%			3.60796	4.94882	1.63046	5.76405	1.34086
90%			4.57725	5.85233	1.55170	6.62817	1.27508

Table 6: KRIs under artificial data for n=50.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.21179	0.40517					
70%			3.01663	4.45280	1.76778	5.33669	1.43617
80%			3.65872	5.01813	1.67585	5.85605	1.35940
90%			4.64142	5.93412	1.59488	6.73156	1.29270
CVM	0.20398	0.40275					
70%			3.07936	4.52529	1.79043	5.42051	1.44594
80%			3.72615	5.09437	1.69682	5.94278	1.36821
90%			4.71541	6.01616	1.61445	6.82338	1.30075
ADE	0.20564	0.40176					
70%			3.07729	4.52654	1.79898	5.42603	1.44926
80%			3.72549	5.09694	1.70503	5.94945	1.37145
90%			4.71705	6.02094	1.62234	6.83211	1.30389
RTADE	0.20535	0.40197					
70%			3.07739	4.52594	1.79718	5.42454	1.44855
80%			3.72530	5.09606	1.70330	5.94771	1.37077
90%			4.71637	6.0196	1.62068	6.82995	1.30323
LEADE	0.20552	0.40436					
70%			3.05818	4.49813	1.77593	5.38609	1.43995
80%			3.70223	5.06486	1.68318	5.90645	1.36264
90%			4.68742	5.98293	1.60154	6.78370	1.29551

Table 7: KRIs under artificial data for n=100.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.20636	0.40321					
70%			3.06206	4.50599	1.78592	5.39895	1.44393
80%			3.70786	5.07431	1.69269	5.92065	1.36645
90%			4.69578	5.99495	1.61064	6.80027	1.29917
CVM	0.20235	0.40167					
70%			3.09723	4.54731	1.80039	5.44750	1.45008
80%			3.74595	5.11799	1.70615	5.97106	1.37204
90%			4.73802	6.04233	1.62324	6.85395	1.30432
ADE	0.20236	0.40107					
70%			3.10176	4.55399	1.80575	5.45687	1.45223
80%			3.75144	5.12552	1.71123	5.98114	1.37408
90%			4.74499	6.05124	1.62807	6.86528	1.30626
RTADE	0.20122	0.39993					
70%			3.11731	4.57384	1.81622	5.48195	1.45652
80%			3.76896	5.14704	1.72108	6.00758	1.37808
90%			4.76543	6.07543	1.63739	6.89413	1.31001
LEADE	0.20319	0.40252					
70%			3.08574	4.53264	1.79267	5.42898	1.44690
80%			3.73300	5.10208	1.69889	5.95153	1.36908
90%			4.72291	6.02445	1.61637	6.83263	1.30154

Based on Tables 4 through 7, we observe that as the sample size nn increases, all methods demonstrate improved stability and accuracy in estimating key risk indicators (KRIs). RTADE consistently yields the highest values for tail-focused measures such as TVaRq(X), TMVq(X), and ELq(X), indicating superior sensitivity to extreme events. ADE closely follows, confirming its reliability, especially in larger samples. In contrast, MLE tends to underestimate tail risk, producing the lowest KRI values across all nn, while CVM and LEADE offer intermediate performance, with LEADE showing moderate gains in some quantiles. The differences between methods diminish with larger nn, but adaptive techniques maintain an edge in capturing risk dynamics. Moreover, the parameters $\gamma\gamma$ and $\lambda\lambda$ converge more clearly with increasing nn, particularly under adaptive methods, suggesting enhanced estimator robustness and efficiency in modeling tail risk.

4. Risk analysis under insurance claims data

Historical insurance data is frequently organized in a triangular format to illustrate how claims evolve over time for each corresponding underwriting or accident period. The ”origin period” typically represents the year a policy was issued, the year a loss occurred, or another defined timeframe (e.g., quarterly or monthly intervals). The term ”claim age” or ”development lag” refers to the time elapsed since the origin period, tracking how claims progress over subsequent periods. Individual policy data is often grouped into homogeneous categories, such as business lines, risk types, or organizational divisions. In this study, we analyze a real-world example using a claims payment triangle from a U.K. Motor Non-Comprehensive insurance portfolio, with origin years spanning 2007 to 2013 (see Mohamed et al. (2024) and Mohamed et al. (2024)). The dataset is structured conventionally, with columns specifying the origin year (2007–2013), the development year, and the incremental claim payments recorded for each period. These data are recently analyzed by Mohamed et al. (2024), Alizadeh et al. (2025) and Yousof et al. (2025).

Table 8: KRIs under insurance claims data.

Method	$\widehat{\beta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.07986	0.00049					
70%			3455.5	4657.039	1220190.9	614752.5	1201.5
80%			3997.4	5128.610	1150999.9	580628.6	1131.2
90%			4817.9	5888.959	1090844.4	551311.2	1071.0
CVM	0.9932	0.00009					
70%			5755.0	11596.6	31810269.6	15916731.5	5841.6
80%			8228.2	13936.7	31034728.3	15531300.9	5708.4
90%			12270.0	17840.3	30263790.3	15149735.5	5570.2
ADE	0.7948	0.00012					
70%			5164.7	9725.4	18976568.1	9498009.4	4560.7
80%			7123.4	11544.2	18381531.2	9202309.9	4420.8
90%			10270.4	14555.5	17828545.9	8928828.5	4285.1
RTADE	0.47379	0.00018					
70%			4720.9	7879.2	8791578.1	4403668.3	3158.3
80%			6108.2	9129.8	8416152.3	4217205.9	3021.7
90%			8277.6	11175.9	8075702.8	4049027.4	2898.3
AD2LE	1.13316	0.00008					
70%			6096.8	12863.2	43335940.3	21680833.4	6766.4
80%			8932.2	15582.2	42491859.7	21261512.1	6650.0
90%			13622.6	20141.7	41615264.6	20827774.0	6519.1

Table 8 presents a comparative assessment of KRIs derived from insurance claims data using five estimation methods like the MLE, CVM, ADE, RTADE, and AD2LE. Each method is evaluated based on two model parameters (β and λ) and a suite of risk measures, including VaRq(X),TVaRq(X),TVq(X),TMVq(X) and

ELq(X) at the 70%, 80%, and 90% levels. Notably, the CVM and AD2LE methods produce significantly higher values across TVaRq, TVq, and TMVq, indicating a heightened sensitivity to extreme tail risks, whereas MLE yields the most conservative estimates with the lowest figures across all metrics, especially in ELq, suggesting minimal tail heaviness. ADE and RTADE offer intermediate risk profiles, with RTADE producing the most compact and stable set of risk indicators, particularly suitable for balanced risk management. The progression across quantiles reveals an expected increase in tail risk and volatility, but a decreasing trend in ELq for all methods—highlighting the shift of mass in heavy-tailed distributions beyond the quantile threshold. Overall, the table underscores the impact of method selection on perceived risk: while MLE may appeal in stable environments, methods like AD2LE and CVM are more aligned with conservative strategies focused on capturing worst-case loss scenarios.

Based on Table 8, we note that the choice of estimation method leads to starkly different assessments of risk exposure under insurance claims data, with numerical results underscoring these contrasts. The Maximum Likelihood Estimation (MLE) method consistently produces the most conservative risk estimates. For example, at the 90% quantile, MLE reports a TVq(X) of 1,090,844.47 and ELq(X) of only 1,071.04, significantly lower than other methods. In contrast, the AD2LE method, generates extremely high tail risk measures: at the same 90% level, it yields a TVq(X) of 41,615,264.69 and ELq(X) of 6,519.12, revealing its focus on capturing extreme events. Similarly, CVM estimates a TVq(X) of 30,263,790.29 and ELq(X) of 5,570.25 at 90%, again emphasizing substantial tail risk sensitivity. The ADE and RTADE methods offer intermediate profiles. For instance, at the 90% level, ADE gives TVq(X) = 17,828,545.94 and ELq(X) = 4,285.15, while RTADE provides more moderate values, TVq(X) = 8,075,702.86 and ELq(X) = 2,898.32, making it a potential compromise between efficiency and risk sensitivity. Notably, across all methods, VaRq(X), TVaRq(X), and TVq(X) increase with the quantile level, which is expected as we move deeper into the tail. Yet, ELq(X) decreases across quantiles in every method: for example, MLE's ELq drops from 1,201.52 at 70% to 1,071.04 at 90%, and similarly, AD2LE's ELq decreases from 6,766.45 to 6,519.12. This pattern suggests that while higher quantiles reflect greater potential losses, the expected loss within those quantiles becomes more concentrated and less dispersed, particularly under light-tailed assumptions. Numerical comparisons reinforce that MLE offers the lowest risk estimates, suitable for stable conditions, while AD2LE and CVM aggressively account for tail risk, ideal for stress-testing and conservative scenarios. ADE and RTADE occupy the middle ground, providing flexible alternatives depending on the insurer's risk appetite.

5. The WTLE-AFT model

A comprehensive review of the recent literature reveals a growing interest in extending and validating statistical models for applications in actuarial science, survival analysis, reliability engineering, and risk assessment. Several studies have focused on developing new probability distributions that offer greater flexibility in modeling real-world data with varying shapes and tail behaviors. For instance, Zamani et al. (2022) explored the Extended Exponentiated Chen distribution, analyzing its mathematical properties and demonstrating its effectiveness in fitting real-life datasets. Similarly, Dey et al. (2017) provided detailed insights into the Exponentiated Chen distribution, emphasizing estimation methods and practical applications. In another direction, Ibrahim et al. (2025a) proposed a Reciprocal Weibull model for medical and reliability data, incorporating sequential sampling plans and truncated life testing to enhance model validation. The use of modified goodness-of-fit tests, particularly the Nikulin–Rao–Robson statistic, has also gained attention for validating parametric models under censored and uncensored data scenarios, as demonstrated by Goual and Yousof (2019, 2020), Yousof et al. (2023c), and Salem et al. (2023). These works highlight the importance of robust validation techniques in improving the reliability of statistical models used in financial risk, insurance claims, and biomedical research. Other contributions include Mansour et al. (2020d), who introduced a new log-logistic lifetime model with copula-based dependence structures and applied it to real datasets using various estimation methods. Meanwhile, Alizadeh et al. (2024) developed an Extended Gompertz model for assessing extreme stress data, while Ramaki et al. (2025) studied a Weighted Flexible Weibull model for extreme event analysis. Risk indicators such as Value-at-Risk (VaR), Tail-Value-at-Risk (TVaR), tail variance, and mean excess loss have been widely utilized in actuarial modeling, as seen in Yousof et al. (2023d), Mohamed et al. (2024), and Ibrahim et al. (2025d), where different estimation techniques—such as maximum likelihood, least squares, and Cramer–von Mises—were compared for their performance in capturing tail behavior and quantifying uncertainty. Additionally, works like Abonongo et al. (2025) and Shehata et al. (2024) applied accelerated failure time models and Bayesian inference techniques to analyze right-censored data from clinical trials and reliability studies. Furthermore, Yousof and his collaborators have made significant contributions in proposing new models tailored for specific domains, including the Double Burr Type XII model (Ibrahim et al., 2022), the Lomax inverse Weibull model (Goual et al., 2020), and the Topp-Leone-Lomax model (Yadav et al., 2020), all of which were validated using advanced goodness-of-fit procedures. The Bagdonavičius–Nikulin family of tests has been frequently employed in these validations, especially in the presence of censoring, as noted by Yousof et al. (2021b, 2022b, 2023a), Mansour et al. (2020f), and Bagdonavicius & Nikulin (2011a,b). These studies collectively underscore the evolving landscape of statistical modeling, where novel distributions are being rigorously tested and refined to better capture the complexities of real-world phenomena across diverse fields such as finance, medicine, engineering, and environmental sciences. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)).

In this section, we propose a new accelerated failure time model. For this, we suppose that n independent failure time variables are observed and we consider that the hypothesis H_0 stating that the survival function given the vector of explanatory variables $z(X) = (z_0(X), z_1(X), ..., z_m(X)), \quad z_0(X) = 1$ (covariates such as temperature, stress,...etc) has the form

$$S(x|z) = S_0\left(\int_0^{\varsigma} e^{-\beta^T z(u)} du; \zeta\right),$$

where $\beta = (\beta_0, \beta_1, ..., \beta_m)^T$ is a vector of unknown regression parameters, the function S_0 is a specified functional of time and does not depend on z_i. If explanatory variables are constant over time, the parametric accelerated failure time (AFT) model has the form

$$S(x|z) = S_0\left[\exp\left(-\beta^T z\right) t; \zeta\right].$$

Consider the WTLE distribution as baseline distribution where

$$H_0 = F(t) = F_{\text{AFT}}(x, \lambda, \beta) = F_{AFT-WTLE}.$$

So, the CDF of the AFT model can be expressed as

$$F_{AFT-WTLE}(x; \gamma, \lambda) = \frac{\gamma}{\gamma - \log\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]}, x > 0; \gamma, \lambda > 0,$$

and then, the PDF of the AFT model can be re-expressed as

$$\begin{aligned} f_{AFT-WTLE}(x; \gamma, \lambda) &= \frac{2\lambda\gamma \exp\left(-2\lambda x e^{-\beta^T z}\right)}{\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]\left[1 + \exp\left(-\lambda x e^{-\beta^T z}\right)\right]\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]\right\}} \\ &= \frac{2\lambda\gamma \exp\left(\lambda x e^{-\beta^T z}\right)}{\left[\exp\left(\lambda x e^{-\beta^T z}\right) - 1\right]\left[1 + \exp\left(\lambda x e^{-\beta^T z}\right)\right]^2\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]\right\}} \end{aligned}$$

Analogously, the corresponding survival function (SF), HRF and cumulative HRF of the AFT model are given by

$$S_{AFT-WTLE} = S_0\left(x e^{-\beta^T z}\right) = 1 - \frac{\gamma}{\gamma - \log\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]},$$

and

$$h_{AFT-WTLE} = -\frac{2\lambda\gamma \exp\left(\lambda x e^{-\beta^T z}\right)}{\left[\exp\left(\lambda x e^{-\beta^T z}\right) - 1\right]\left[1 + \exp\left(\lambda x e^{-\beta^T z}\right)\right]^2 \log\left[1 - \exp\left(-2\lambda x e^{-\beta^T z}\right)\right]}.$$

6. The MLE for the WTLE-AFT model

In this section, we apply the maximum likelihood method to estimate the parameters of the AFT for the WTLE distribution. We give a detailed description of the method as well as the score functions and the elements of the FIM.

6.1. The MLE derivations

Let $x_1, \ldots, x_n$ be a RS from the AFT for the WTLE model with parameters λ, γ and β. Let $\underline{\mathbf{V}} = (\lambda, \gamma, \beta_0, \beta_1)^\intercal$ be the 4×1 parameter vector. For determining the MLE of $\underline{\mathbf{V}}$, we have the log-likelihood function

$$\begin{aligned} \ell &= \ell\left(x; \underline{\mathbf{V}}\right) = n \log\left(2\gamma\lambda\right) - 2\lambda \sum_{i=1}^{n} x_i e^{-\beta^T z_i} - \sum_{i=1}^{n} \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right] \\ &\quad - \sum_{i=1}^{n} \log\left[1 + \exp\left(-\lambda x_i e^{-\beta^T z}\right)\right] - \sum_{i=1}^{n} \log\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\} \end{aligned}$$

The score vector $\mathbf{I}_{(\underline{\mathbf{V}})} = \frac{\partial \ell}{\partial \underline{\mathbf{V}}} = \left(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \gamma}, \frac{\partial \ell}{\partial \beta_0}, \frac{\partial \ell}{\partial \beta_1}\right)^\intercal$ is given by

$$\begin{aligned} \frac{\partial \ell\left(x_i; \underline{\mathbf{V}}\right)}{\partial \gamma} &= \frac{n}{\gamma} - \sum_{i=1}^{n} \frac{2x_i e^{-\beta^T z}}{\gamma - \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]}, \\ \frac{\partial \ell\left(x_i; \underline{\mathbf{V}}\right)}{\partial \lambda} &= \frac{n}{\lambda} - 2\sum_{i=1}^{n} x_i e^{-\beta^T z_i} + \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z}}{1 + \exp\left(\lambda x_i e^{-\beta^T z}\right)} - \sum_{i=1}^{n} \frac{2x_i e^{-\beta^T z}}{\exp\left(2\lambda x_i e^{-\beta^T z}\right) - 1} \\ &\quad + \sum_{i=1}^{n} \frac{2x_i e^{-\beta^T z}}{\left[\exp\left(2\lambda x_i e^{-\beta^T z}\right) - 1\right]\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\}}, \\ \frac{\partial \ell\left(x_i; \underline{\mathbf{V}}\right)}{\partial \beta_0} &= 2\lambda \sum_{i=1}^{n} x_i e^{-\beta^T z_i} + 2\lambda \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-2\lambda x_i e^{-\beta^T z}\right)}{1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)} - \lambda \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-\lambda x_i e^{-\beta^T z}\right)}{1 + \exp\left(-\lambda x_i e^{-\beta^T z}\right)} \\ &\quad - 2\lambda \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-2\lambda x_i e^{-\beta^T z}\right)}{\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\}}, \\ \frac{\partial \ell\left(x_i; \underline{\mathbf{V}}\right)}{\partial \beta_l} &= 2\lambda z_l \sum_{i=1}^{n} x_i e^{-\beta^T z_i} - \lambda z_l \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-\lambda x_i e^{-\beta^T z}\right)}{1 + \exp\left(-\lambda x_i e^{-\beta^T z}\right)} + 2\lambda z_l \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-2\lambda x_i e^{-\beta^T z}\right)}{1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)} \\ &\quad - 2\lambda z_l \sum_{i=1}^{n} \frac{x_i e^{-\beta^T z} \exp\left(-2\lambda x_i e^{-\beta^T z}\right)}{\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\left\{\gamma - \log\left[1 - \exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\}}. \end{aligned}$$

Setting the nonlinear system of equations $\mathbf{I}_{(\lambda)} = 0, \mathbf{I}_{(\gamma)} = 0, \mathbf{I}_{(\beta_0)} = 0$ and $\mathbf{I}_{(\beta_1)} = 0$ and solving them simultaneously yields the MLE $\widehat{\underline{\mathbf{V}}} = (\widehat{\lambda}, \widehat{\gamma}, \widehat{\beta}_0, \widehat{\beta}_1)^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. Since, we can not find the explicit formulas for the MLEs of the parameters, we use numerical methods such as the Newton Raphson method, the Monte Carlo method, the BB algorithm or others.

6.2. Assessing the WTLE-AFT model via a simulation study

We carry out an important study by simulation using the R programming software. In the following, we present the results obtained by means of numerical method (the method of Newton Raphson). Suppose that the AFT for the

WTLE distribution is considered. The data is iterated $N = 5500$ times, with $\lambda = 1.5, \gamma = 2.5, \beta_0 = 2.90, \beta_1 = 0.45$ as values of the parameters. Using BB algorithm (see Ravi (2009)) in R software for calculating the averages of the simulated values of the MLEs $\widehat{\lambda}, \widehat{\gamma}, \widehat{\beta}_0, \widehat{\beta}_1$ parameters and their mean squared errors (MSE), sample sizes are $n = 15,\ n = 25,\ n = 45,\ n = 125,\ n = 350$ and $n = 600$. Table 9 lists the square mean errors for the parameters' MLEs (MSE).

Table 9: MLEs $\left(\widehat{\lambda}, \widehat{\gamma}, \widehat{\beta}_0, \widehat{\beta}_1\right)$ of the parameters and their mean squared errors.

$N = 6000$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\widehat{\lambda}$	1.55449	1.54081	1.52993	1.52784	1.51009	1.50487
MSE	1.7006×10^{-2}	4.2651×10^{-3}	3.625×10^{-3}	2.1935×10^{-3}	5.3625×10^{-4}	4.9527×10^{-4}
$\widehat{\gamma}$	2.55412	2.55401	2.54161	2.53113	2.51418	2.50464
MSE	0.054212	0.048625	0.043218	0.036258	0.021473	0.016895
$\widehat{\beta}_0$	2.92091	1.91660	1.91115	1.90384	1.90217	1.90062
MSE	5.0325×10^{-2}	4.9586×10^{-2}	3.3625×10^{-3}	1.0368×10^{-3}	8.2658×10^{-4}	6.3544×10^{-4}
$\widehat{\beta}_1$	0.46301	0.46001	0.45531	0.45182	0.45081	0.45042
MSE	0.053254	0.049998	7.0325×10^{-2}	4.9538×10^{-2}	2.3265×10^{-3}	1.9502×10^{-3}

The results obtained from the proposed methods are compelling and statistically meaningful, as demonstrated in the accompanying table. The performance of the models is not only evaluated through goodness-of-fit measures but also supported by precise parameter estimation, with relatively low standard errors even under small sample conditions.

7. Validation of the WTLE-AFT model

Any traditional test, such as Pearson's chi-square, Kolmogorov-Smirnov statistic, Anderson Darling statistic, and other statistics, can be used to validate the selection of model employed in analysis in the case of a well-defined distribution. However, when the parameters are unknown and must be estimated from the sample, the classical tests are no longer appropriate, and the test statistical distributions rely on the model put forth and the estimation technique utilised. In case of complete data, various techniques are used to verify the adequacy of mathematical models to data from observation. The most ommon tests are those based on Pearson's Chi-square statistics. Nevertheless, these can not be applied in all situations, especially when the data is censored or when the the parameters of the model are unknown. Nikulin (1973) and Rao and Robson (1974) each independently presented a statistic for the whole data that is now known as the NRR statistic. At the limit, this statistic, which is based on the MLEs on the initial data, likewise exhibits a Chi-square distribution. For more details on the construction of these statistics, we can see Voinov et al. (2013) and Goual et al. (2019).These methods were used to adapt observations to the distribution of Lomax inverse Weibull (Goual et al., 2020), the Burr XII inverse Rayleigh model (Goual et al., 2019), and the Lindley exponentiated model (Goual et al., 2019). In this section, we build a modified chi-square type test based on the NRR test statistic for the WTLE model. Other works could cover a broad range of topics including new probability models (Al-babtain et al., 2020), copula-based extensions (Alizadeh et al., 2018; Mansour et al., 2020c), and regression modeling approaches (Afify et al., 2018; Yousof et al., 2019). Several studies have introduced generalized distributions such as the transmuted Weibull-G and Lindley-Weibull families, emphasizing their applicability to real-world data (Cordeiro et al., 2018; Alizadeh et al., 2018). Risk measures like Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) are explored in depth, especially within actuarial science and financial risk modeling (Mohamed et al., 2024; Yousof et al., 2023d). Theoretical developments include Bayesian and classical inference methods under censored data scenarios (Elgohari & Yousof, 2020; Rasekhi et al., 2020), while practical applications span insurance, medicine, and engineering reliability (Yousof et al., 2023; Abonongo et al., 2025). These works collectively contribute to the advancement of statistical theory and its application in modeling complex datasets using novel distributions and rigorous validation techniques (Mustafa et al., 2018; Salah et al., 2020; Goual et al., 2019).

7.1. The NRR statistic test for the WTLE-AFT model

To test the hypothesis H_0 according to which $T_1, T_2, \cdots, T_n$, an n-sample comes from a parametric family $F_{\underline{\mathbf{V}}}(t)$

$$H_0 : \Pr\{T_i \leq t\} = F_{\underline{\mathbf{V}}}(t), \quad t \in \mathbb{R},$$

where $\underline{\mathbf{V}} = (\underline{\mathbf{V}}_1, \underline{\mathbf{V}}_2, \cdots, \underline{\mathbf{V}}_s)^T$ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson (1974) proposed K^2 the NRR statistic defined as below. Observations $T_1, T_2, \cdots, T_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_r$ mutually disjoint $\mathbf{I}_j =]a_j\text{-}1; a_j]$; where $j = \overline{1; r}$. The limits a_j of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\underline{\mathbf{V}}) = p_j(\underline{\mathbf{V}}; a_{j-1}, a_j) = \int_{a_{j-1}}^{a_j} f_{\underline{\mathbf{V}}}(t)dt|_{(\ j=1,2,\cdots,r)},$$

so

$$a_j = F^{-1}\left(\frac{j}{r}\right)|_{(j=1,\cdots,r-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_r)^T$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{t_i \in \mathbf{I}_j\}}\ |_{(j=1,\ldots,r)}.$$

The NRR statistic is given by

$$K^2(\widehat{\underline{\mathbf{V}}}_n) = X_n^2(\widehat{\underline{\mathbf{V}}}_n) + \frac{1}{n}\mathbf{L}^T(\widehat{\underline{\mathbf{V}}}_n)(\mathbf{I}(\widehat{\underline{\mathbf{V}}}_n) - \mathbf{J}(\widehat{\underline{\mathbf{V}}}_n))^{-1}\mathbf{L}(\widehat{\underline{\mathbf{V}}}_n),$$

where

$$X_n^2(\underline{\mathbf{V}}) = \left(\frac{\nu_1 - np_1(\underline{\mathbf{V}})}{\sqrt{np_1(\underline{\mathbf{V}})}}, \frac{\nu_2 - np_2(\underline{\mathbf{V}})}{\sqrt{np_2(\underline{\mathbf{V}})}}, \cdots, \frac{\nu_r - np_r(\underline{\mathbf{V}})}{\sqrt{np_r(\underline{\mathbf{V}})}}\right)^T$$

and $\mathbf{J}(\underline{\mathbf{V}})$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\underline{\mathbf{V}}) = B(\underline{\mathbf{V}})^T B(\underline{\mathbf{V}}),$$

with

$$B(\underline{\mathbf{V}}) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial p_i(\underline{\mathbf{V}})}{\partial \mu}\right]_{r\times s}|_{(i=1,2,\cdots,r \text{ and } k=1,\cdots,s)},$$

then

$$\mathbf{L}(\underline{\mathbf{V}}) = (\mathbf{L}_1(\underline{\mathbf{V}}), ..., \mathbf{L}_s(\underline{\mathbf{V}}))^T \text{ with } \mathbf{L}_k(\underline{\mathbf{V}}) = \sum_{i=1}^{r}\frac{\nu_i}{p_i}\frac{\partial}{\partial \underline{\mathbf{V}}_k}p_i(\underline{\mathbf{V}}),$$

where $\mathbf{I}_n(\widehat{\underline{\mathbf{V}}_n})$ represents the estimated FIM and $\widehat{\underline{\mathbf{V}}_n}$ is the maximum likelihood estimator of the parameter vector. The K^2 statistic follows a distribution of chi-square χ^2_{r-1} with $(r-1)$ degrees of freedom.

7.2. Simulation studies under the NRR statistic K^2

Consider a sample $T_{1:n}$ where $T = T_{1:n} = (T_1, T_2, \cdots, T_n)^T$. If these data are distributed in accordance with the WTLE-AFT model, then $P\{T_{1:n} \leq t\} = F_{\underline{\mathbf{V}}}(t)$; with unknown parameters $\underline{\mathbf{V}} = (\lambda, \beta_0, \beta_1)^T$, by fitting the NRR statistic created in the preceding section, a chi-square goodness-of-fit test is created. The MLEs $\widehat{\underline{\mathbf{V}}_n}$ of the unknown parameters of the AFT-WTLE model are computed on the initial data. Since, the statistic K^2 not dependent on the parameters, we can therefore use the estimated Fisher information matrix (FIM) $I_n(\widehat{\underline{\mathbf{V}}_n})$. All the components of the statistic K^2, for the AFT-WTLE distribution are provided, therefore K^2 can be deduced easily.

In order to support the results obtained in this work, a numerical simulation is performed. Therefore, in order to test the null hypothesis H_0 of the AFT-WTLE model, we calculated 5500 sample data simulations ($n = 15,\ n = 25,\ n = 45,\ n = 125,\ n = 350$ and $n = 600$) from AFT-WTLE distribution, after calculating the value of the criterion statistic K^2, we count the number of rejected cases of the null hypothesis H_0. When $K^2 > \chi^2\,(k = r - 1)$, the significance is different level $\alpha\,(1\%, 5\%, 10\%)$. The simulation results of the rejected level of K^2 and its theoretical value are shown in Table 10 below.

Table 10: Empirical levels K^2 and corresponding theoretical levels.

$N = 6000$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\alpha = 0.01$	0.0185	0.0164	0.0152	0.0132	0.0119	0.0112
$\alpha = 0.05$	0.0531	0.0519	0.0513	0.0509	0.0504	0.0502
$\alpha = 0.1$	0.1302	0.1213	0.1101	0.1093	0.1031	0.1021

The results show that the computed empirical level closely matches the corresponding theoretical level. Based on this, we conclude that the proposed test is highly appropriate for the AFT-WTLE distribution. This finding supports the claim that the K^2 statistic asymptotically follows a chi-squared distribution with degrees of freedom given by $k = r - 1$.

7.3. Applications to real data

We take into account the following real data sets and confirm the presumption that their distribution is consistent with the AFT-WTLE model in order to demonstrate the applicability of the proposed modified chi-square goodness-of-fit test.

7.3.1. Electric insulating fluid data The failure times of 76 electrical insulating fluids tested at voltages ranging from 26 to 38 kilovolts are provided in Lawless (2003), from which this information was derived. Bagdonavicius and Nikulin (2011) used this data and examined its fit with the exponential and Weibull AFT power-rule models. In this part, we evaluate how well these data fit our suggested AFT-WTLE model. The data are:

Voltage level (z_i)	n_i	Breakdown time x_i
26	3	5.79,1579.52,2323.7
28	5	68.85,426.07,110.29,108.29,1067.6
30	11	17.05,22.66,21.01,175.88,139.07,144.12,
		20.46,43.40,194.90,47.30,7.74
32	15	0.40,82.85,9.88,89.29,215.10,2.75,0.79,
		15.93,3.91,0.27,0.69,100.58,27.80,13.95,53.24
34	19	0.96,4.15,0.19,0.78,8.01,31.75,7.35,6.50,8.27,33.91,
		32.52,3.16,4.85,2.78,4.67,1.31,12.06,36.71,72.89
36	15	1.97,0.59,2.58,1.69,2.71,25.50,0.35,0.99,
		3.99,3.67,2.07,0.96,5.35,2.90,13.77
38	8	0.47,0.73,1.40,0.74,0.39,1.13,0.09,2.38

1- In case of $\varphi(z) = z$ log-linear assumption:
Using R statistical software (the BB package) we find the values of the MLEs of AFT-WTLE distribution parameters :

$$\widehat{\lambda} = 0.3524, \widehat{\gamma} = 0.6258, \widehat{\beta}_0 = 1.358, \widehat{\beta}_1 = -0.10958,$$

we choose $r = 8$ intervals and the estimated FIM can be expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 1.30254 & -1.36201 & 2.01472 & 0.00254 \\ & 0.95257 & -3.00265 & -4.32002 \\ & & 0.95387 & 0.85631 \\ & & & 2.95002 \end{pmatrix},$$

and then the NRR statistic : $K^2 = 17.003519$. For the critical value : $\alpha = 0.01$, we find $K^2 < \chi^2_{0.01}(7) = 18.4753$.

So, we can assume that in this case, electric insulating fluid data of Lawless (2003) correspond appropriately to the $AFT-WTLE$ model.

2- In case of $\varphi(z) = \log(z)$ power-rule assumption:
We find the values of the MLEs of the AFT-WTLE distribution parameters:

$$\widehat{\lambda} = 0.8834, \widehat{\gamma} = 0.13590, \widehat{\beta}_0 = 0.63574, \widehat{\beta}_1 = 0.30214,$$

we take $r = 8$ intervals and the estimated FIM can be:

$$I\left(\widehat{\underline{\mathbf{V}}}\right) = \begin{pmatrix} 0.03254 & 0.32658 & -4.00215 & 0.90247 \\ & 1.02548 & 1.032964 & -3.65891 \\ & & 1.03254 & 0.11511 \\ & & & 0.93578 \end{pmatrix},$$

the NRR statistic is $K^2 = 11.205881$. For the critical values : $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$, we find

$$K^2 < \chi^2_{0.01}(7) = 18.4753$$

$$K^2 < \chi^2_{0.05}(7) = 14.0671$$

$$K^2 < \chi^2_{0.1}(7) = 12.0170$$

respectively.

So, we can assume that electric insulating fluid data of Lawless (2003) correspond appropriately to the $AFT-WTLE$ model in case of power-rule assumption.

3- In case of $\varphi(z) = 1/z$ arrehnius model:

We fit these data by the AFT-WTLE model. Using R statistical software (the BB package) we find the values of the MLEs of the AFT-WTLE distribution parameters :

$$\widehat{\lambda} = 2.03654, \widehat{\gamma} = 0.96895, \widehat{\beta}_0 = 3.00625, \widehat{\beta}_1 = -4.03269,$$

we take $r = 8$ intervals and the estimated FIM expressed as :

$$I\left(\widehat{\underline{\mathbf{V}}}\right) = \begin{pmatrix} 1.02547 & -8.03254 & -5.09573 & 1.00021 \\ & 0.32658 & 0.99996 & 0.95174 \\ & & 2.00514 & -1.03254 \\ & & & 1.03054 \end{pmatrix},$$

the NRR statistic is: $K^2 = 19.3475123$. For the critical value : $\alpha = 0.01$, we find $K^2 > \chi^2_{0.01}(7) = 18.4753$.
In case of arrehnius model, we can assume that electric insulating fluid data of Lawless (2003) do not correspond appropriately to our model.

7.3.2. Body fat data set The data of Neter et al. (1996) provides information on $(n = 20)$ body fat, triceps skinfold thickness, thigh circumference, and mid-arm circumference for twenty healthy females aged 20 to 34. The data is :

z_{i1} (triceps skinfold measurement)	z_{i2} (thigh circumference)	x_i(body-fat)
$19.5, 24.7, 30.7$	$43.1, 49.8, 51.9$	$11.9, 22.8, 18.7$
$29.8, 19.1, 25.6$	$54.3, 42.2, 53.9$	$20.1, 12.9, 21.7$
$31.4, 27.9, 22.1$	$58.5, 52.1, 49.9$	$27.1, 25.4, 21.3$
$25.5, 31.1, 30.4$	$53.5, 56.6, 56.7$	$19.3, 25.4, 27.2$
$18.7, 19.7, 14.6, 29.5$	$46.5, 44.2, 42.7, 54.4$	$11.7, 17.8, 12.8, 23.9$
$27.7, 30.2, 22.7, 25.2$	$55.3, 58.6, 48.2, 51.0$	$22.6, 25.4, 14.8, 21.1$

For $\varphi(z) = z$ as a log-linear assumption: We fit these data by the AFT-WTLE model. Using R statistical software (the BB package) we find the values of the MLEs of AFT-WTLE distribution parameters :

$$\widehat{\lambda} = 3.08213, \widehat{\gamma} = 0.80064, \widehat{\beta}_0 = 0.63254, \widehat{\beta}_1 = -0.06637, \widehat{\beta}_2 = 0.08475.$$

we take $r = 4$ intervals and the estimated FIM expressed as

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.12354 & -5.12478 & 0.36985 & 0.14256 & 1.02685 \\ & 1.02458 & -3.95126 & -2.51984 & 0.32647 \\ & & 1.92501 & 1.75391 & 0.62847 \\ & & & 0.84457 & 0.74125 \\ & & & & 0.62359 \end{pmatrix},$$

and then the NRR statistic : $K^2 = 6.0032145$. For different critical values : $\alpha = 1\%, \alpha = 5\%$ and $\alpha = 10\%$, we find

$$K^2 < \chi^2_{0.01}(3) = 11.3448, K^2 < \chi^2_{0.05}(3) = 7.8147$$

$$K^2 < \chi^2_{0.1}(3) = 6.2513,$$

respectively.

We can assume that, the body fat dat can be adjusted properly to an AFT-WTLE model, in case of a log-linear assumption.

7.3.3. Johnson's data set Johnson (1996) used a dataset with a response variable (the estimated percentage of body fat) and 13 continuous covariates (age, weight, height and 10 measurements of the body circumference) in $n = 252$ males to illustrate some problems with multiple regression analysis. The aim was to predict percentage body fat from the covariates. These dataset is available on the 'mfp' package in R software.

Variable	Name	Details	Variable	Name	Details
z_1	age	Age (years)	z_8	thigh	Circumference (cm)
z_2	weight	Weight (lb)	z_9	knee	Circumference (cm)
z_3	height	Height (in)	z_{10}	ankle	Circumference (cm)
z_4	neck	Circumference (cm)	z_{11}	bicepes	Circumference (cm)
z_5	chest	Circumference (cm)	z_{12}	forearm	Circumference (cm)
z_6	ab	Circumference (cm)	z_{13}	wrist	Circumference (cm)
z_7	hip	Circumference (cm)	x	pcfat	Body fat (%)

In our case, we used two covariates density (Density determined from underwater weighing gm/cm^3) and age (years). We consider the log linear assumption $(\varphi(z) = z)$ and we fit this data by the AFT-WTLE model. The values of the MLEs parameters :

$$\widehat{\lambda} = 1.03225, \widehat{\gamma} = 0.73195, \widehat{\beta}_0 = -1.0032, \widehat{\beta}_1 = 0.45317, \widehat{\beta}_2 = 0.26346,$$

We take $r = 15$ intervals and the estimated FIM $I\left(\underline{\widehat{\mathbf{V}}}\right)$ expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 2.03147 & -1.03024 & -8.32659 & 0.74623 & -9.30120 \\ & 0.00314 & 0.21485 & 2.00003 & -1.33623 \\ & & 1.32486 & -4.03162 & 3.03014 \\ & & & 1.35748 & 0.96853 \\ & & & & 0.98631 \end{pmatrix},$$

The NRR statistic test: $K^2 = 20.3614902$. For different critical values : $\alpha = 0.01, \alpha = 0.05$ and $\alpha = 0.1$, we find $K^2 < \chi^2_{0.01}(14) = 29.1412, K^2 < \chi^2_{0.05}(14) = 23.6847$ and $K^2 < \chi^2_{0.1}(14) = 21.06414$, respectively.

One can affirm that our proposed AFT-WTLE model with the log-linear assumption $(\varphi(z) = z)$ can be an appropriate distribution of this data.

8. Censored case

8.1. Maximum likelihood estimation

The likelihood function of the right-censored $AFT-WTLE$ is

$$
\begin{aligned}
L\left(x;\underline{\mathbf{V}}\right) &= \prod_{i=1}^{n} f\left(x_i;\underline{\mathbf{V}}\right) = \prod_{i=1}^{n} \lambda^{\delta_i}\left(x_i;\underline{\mathbf{V}}\right) S\left(x_i;\underline{\mathbf{V}}\right), \qquad \delta_i = 1_{\{T_i \leq C_i\}} \\
&= \prod_{i=1}^{n} \left\{ \frac{2\lambda\gamma \exp\left(\lambda x e^{-\beta^T z}\right)}{\left[\exp\left(\lambda x e^{-\beta^T z}\right)-1\right]\left[1+\exp\left(\lambda x e^{-\beta^T z}\right)\right]^2 \log\left[1-\exp\left(-2\lambda x e^{-\beta^T z}\right)\right]} \right\}^{\delta_i} \\
&\quad \times \left\{ 1 - \frac{\gamma}{\gamma - \log\left[1-\exp\left(-2\lambda x e^{-\beta^T z}\right)\right]} \right\}.
\end{aligned}
$$

The log-likelihood function is

$$
\begin{aligned}
\ell\left(x;\underline{\mathbf{V}}\right) &= \sum_{i=1}^{n} \delta_i \log\left(2\gamma\lambda\right) + \lambda \sum_{i=1}^{n} \delta_i x_i e^{-\beta^T z} - \sum_{i=1}^{n} \delta_i \log\left[1-\exp\left(\lambda x_i e^{-\beta^T z}\right)\right] \\
&\quad -2\sum_{i=1}^{n} \delta_i \log\left[1+\exp\left(\lambda x_i e^{-\beta^T z}\right)\right] - \sum_{i=1}^{n} \delta_i \log\left\{\gamma - \log\left[1-\exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\} \\
&\quad + \sum_{i=1}^{n} \log\left[1 - \frac{\gamma}{\gamma - \log\left[1-\exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]}\right]. \\
&= \sum_{i\in F} \log\left(2\gamma\lambda\right) + \lambda \sum_{i\in F} x_i e^{-\beta^T z} - \sum_{i\in F} \log\left[1-\exp\left(\lambda x_i e^{-\beta^T z}\right)\right] - 2\sum_{i\in F} \log\left[1+\exp\left(\lambda x_i e^{-\beta^T z}\right)\right] \\
&\quad + \sum_{i\in F} \log\left[1 - \frac{\gamma}{\gamma - \log\left[1-\exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]}\right] - \sum_{i\in F} \log\left\{\gamma - \log\left[1-\exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]\right\} \\
&\quad + \sum_{i\in C} \log\left[1 - \frac{\gamma}{\gamma - \log\left[1-\exp\left(-2\lambda x_i e^{-\beta^T z}\right)\right]}\right],
\end{aligned}
$$

where F and C denote the sets of uncensored ($\delta_i = 1$) and censored ($\delta_i = 0$) observations, respectively.

The score functions for the parameters λ, γ, β_0 and β_1 are given by

$$
\begin{aligned}
\frac{\partial \ell\left(x_i ; \gamma, \lambda, \beta\right)}{\partial \gamma} &= \frac{r}{\gamma}-\sum_{i \in C} \frac{1}{\gamma-\log \left[1-\exp \left(-2 \lambda x_i e^{-\beta^T z}\right)\right]} \\
\frac{\partial \ell\left(x_i ; \gamma, \lambda, \beta\right)}{\partial \lambda} &= \frac{r}{\lambda}+\sum_{i \in F} L(X)\left[\frac{1}{1-\exp \left(\lambda x_i e^{-\beta^T z}\right)}-\frac{2}{1+\exp \left(\lambda x_i e^{-\beta^T z}\right)}\right] \\
&\quad -2 \sum_{i \in F} \frac{x_i e^{-\beta^T z} \exp \left(-2 \lambda x_i e^{-\beta^T z}\right)}{\left[1-\exp \left(-2 \lambda x_i e^{-\beta^T z}\right)\right] \times M(x)} \\
&\quad -2 \gamma \sum_{i \in C} \frac{x_i e^{-\beta^T z}}{\left[\exp \left(2 \lambda x_i e^{-\beta^T z}\right)-1\right] \times M(x) \times[\gamma-M(x)]} \\
\frac{\partial \ell\left(x_i ; \gamma, \lambda, \beta\right)}{\partial \beta_0} &= -\lambda \sum_{i \in F} x_i e^{-\beta^T z}-\lambda \sum_{i \in F} L(X)\left[\frac{1}{1-\exp \left(\lambda x_i e^{-\beta^T z}\right)}-\frac{2}{1+\exp \left(\lambda x_i e^{-\beta^T z}\right)}\right] \\
&\quad +2 \lambda \sum_{i \in F} \frac{x_i e^{-\beta^T z} \exp \left(-2 \lambda x_i e^{-\beta^T z}\right)}{\left[1-\exp \left(-2 \lambda x_i e^{-\beta^T z}\right)\right] \times M(x)} \\
&\quad -2 \lambda \gamma \sum_{i \in C} \frac{x_i e^{-\beta^T z}}{\left[\exp \left(2 \lambda x_i e^{-\beta^T z}\right)-1\right] \times M(x) \times[\gamma-M(x)]}, \\
\frac{\partial \ell\left(x_i ; \gamma, \lambda, \beta\right)}{\partial \beta_l} &= -\lambda z_l \sum_{i \in F} x_i e^{-\beta^T z}-\lambda z_l \sum_{i \in F} L(X)\left[\frac{1}{1-\exp \left(\lambda x_i e^{-\beta^T z}\right)}-\frac{2}{1+\exp \left(\lambda x_i e^{-\beta^T z}\right)}\right] \\
&\quad +2 \lambda \sum_{i \in F} z_l \frac{x_i e^{-\beta^T z} \exp \left(-2 \lambda x_i e^{-\beta^T z}\right)}{\left[1-\exp \left(-2 \lambda x_i e^{-\beta^T z}\right)\right] \times M(x)} \\
&\quad -2 \lambda \gamma \sum_{i \in C} z_l \frac{x_i e^{-\beta^T z}}{\left[\exp \left(2 \lambda x_i e^{-\beta^T z}\right)-1\right] \times M(x) \times[\gamma-M(x)]}
\end{aligned}
$$

where r is the number of failures.

8.1.1. Calculation of the matrix $\widehat{W}$ The elements of the estimated matrix $\widehat{W}$ defined by

$$
\widehat{W_l}=\sum_{j=1}^k \widehat{C}_{lj} \widehat{A}_j^{-1} \widehat{Z}_j,\ l=1,2,3\ ;\ j=1,2,...,k.
$$

are obtained as follow :

$$\begin{aligned}
\widehat{C}_{1j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\frac{M\left(x_i,\underline{\mathbf{V}}\right)}{1+\gamma M\left(x_i,\underline{\mathbf{V}}\right)},\\
\widehat{C}_{2j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{1}{\lambda}-\frac{L\left(x_i,\underline{\mathbf{V}}\right)}{M\left(x_i,\underline{\mathbf{V}}\right)\left[1+\gamma M\left(x_i,\underline{\mathbf{V}}\right)\right]}\right\},\\
\widehat{C}_{3j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{\lambda L\left(x_i,\underline{\mathbf{V}}\right)}{M\left(x_i,\underline{\mathbf{V}}\right)\left[1+\gamma M\left(x_i,\underline{\mathbf{V}}\right)\right]}-1\right\},\\
\widehat{C}_{4j} &= \frac{1}{n}\sum_{i:X_i\in I_j}^{k}\delta_i\left\{\frac{\lambda L\left(x_i,\underline{\mathbf{V}}\right)}{M\left(x_i,\underline{\mathbf{V}}\right)\left[1+\gamma M\left(x_i,\underline{\mathbf{V}}\right)\right]}-z_i\right\},
\end{aligned}$$

Where,

$$\begin{aligned}
L\left(x,\underline{\mathbf{V}}\right) &= x\exp\left(\lambda x_i e^{-\beta^T z}\right)e^{-\beta^T z},\\
M\left(x_i,\underline{\mathbf{V}}\right) &= \log\left[1-\exp\left(-2\lambda x e^{-\beta^T z}\right)\right],\underline{\mathbf{V}}=\left(\gamma,\lambda,\beta_0,\beta_1\right).
\end{aligned}$$

8.2. Simulation of the censored MLEs of the parameters for the AFT-WTLE distribution

In this section, we conduct a comprehensive simulation study to evaluate the performance of the maximum likelihood estimators (MLEs) for the parameters of the Accelerated Failure Time model based on the quasi Burr-Hatke exponential (AFT-WTLE) distribution. The simulation process involves generating synthetic datasets under controlled conditions and analyzing the accuracy and precision of the MLEs for varying sample sizes. To begin, we assume that the AFT-WTLE distribution is the underlying model for the data generation process. The simulation is repeated $N = 5500$ times to ensure robustness and reliability of the results. The true parameter values used in the simulation are set as $\lambda = 0.45$ and $\gamma = 0.75$, representing the scale and shift parameters of the distribution, $\beta_0 = 1.25$, corresponding to the intercept term in the regression model, $\beta_1 = 0.68$, representing the coefficient of the explanatory variable. For each replication, we generate synthetic datasets with six different sample sizes: $n = 15, n = 25, n = 45, n = 125, n = 350$, and $n = 600$. These sample sizes span a wide range, from small to large datasets, allowing us to examine how the performance of the estimators evolves as the sample size increases. The primary objective of the simulation study is to compute the mean simulated MLEs for the parameters λ, γ, β_0, and β_1 , along with their corresponding MSEs. The mean simulated MLEs provide insights into the bias of the estimators, while the MSEs quantify both the bias and variability of the estimates, offering a comprehensive measure of their accuracy. Table 11 below gives the censored MLEs $\left(\widehat{\lambda},\widehat{\gamma},\widehat{\beta}_0,\widehat{\beta}_1\right)$ of AFT-WTLE's parameters and their mean squared errors.

Table 11: The censored MLEs $\left(\widehat{\lambda},\widehat{\gamma},\widehat{\beta}_0,\widehat{\beta}_1\right)$ of AFT-WTLE's parameters and their mean squared errors.

$N = 5500$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\widehat{\lambda}$	0.46109	0.45522	0.45502	0.45117	0.45016	0.45006
SME	3.625×10^{-2}	2.1547×10^{-2}	1.8524×10^{-2}	4.5247×10^{-3}	2.6772×10^{-3}	2.5124×10^{-3}
$\widehat{\gamma}$	0.75553	0.75411	0.75209	0.75118	0.75012	0.75003
SME	0.05524	0.05501	0.05486	0.05304	0.05271	0.05108
$\widehat{\beta}_0$	1.26345	1.26003	1.25367	1.25308	1.25092	1.25034
SME	0.054328	0.053194	0.048965	0.43781	0.038847	0.021953
$\widehat{\beta}_1$	0.69215	0.69002	0.68423	0.68231	0.68106	0.68041
SME	0.052413	0.051467	0.050001	0.048652	0.036158	0.021548

These findings underscore the reliability of the MLEs for the AFT-WTLE model across different sample sizes and parameter settings. The simulation study not only validates the theoretical properties of the estimators but also provides practical guidance on their performance in real-world applications. The results demonstrate that the proposed AFT-WTLE model is well-suited for analyzing survival data, particularly when the sample size is sufficiently large to ensure accurate and stable parameter estimates.

8.3. Simulated distribution of K_n^2 statistic for the right-censored AFT-WTLE distribution

We compute 6000 simulations of samples data (sample sizes : $n = 15$, $n = 25$, $n = 45$, $n = 125$, $n = 350$ and $n = 600$) from $AFT - WTLE$ distribution, after calculating the values of criteria statistics K_n^2, we count the number of rejection's cases of the null hypothesis H_0, when $K_n^2 > \chi_\alpha^2(k)$, with different significance level α ($\alpha = 1\%, 5\%, 1\%$). The results of simulated levels of K_n^2 against their theoretical values are shown in the following Table 12.

Table 12: Empirical levels K_n^2 and corresponding theoretical levels.

$N = 6000$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\alpha = 1\%$	0.01641	0.01501	0.01435	0.01321	0.01045	0.01003
$\alpha = 5\%$	0.05387	0.52911	0.05198	0.05087	0.05023	0.05014
$\alpha = 10\%$	0.15612	0.14857	0.13012	0.11968	0.10096	0.10046

As can be seen, the calculated empirical level K_n^2 values are extremely similar to the equivalent theoretical level value. Consequently, we draw the conclusion that the suggested test is excellent for the AFT-WTLE distribution.

8.4. Applications to real censored data

We take into account the following real data sets and confirm the presumption that their distribution is consistent with the AFT-WTLE model in order to demonstrate the applicability of the proposed modified chi-square goodness-of-fit test.

8.4.1. Motor data These reliability datasets, accessible in the survival package of R software, record the time to failure (or breakdown) of motor insulation systems under varying temperature conditions. The main goal of this data is to examine how temperature affects the lifespan and durability of motor insulation, which is essential for understanding the thermal degradation mechanisms that contribute to system failures. Such datasets are commonly utilized in reliability engineering and survival analysis to model failure times, evaluate risks, and enhance material design for better performance under thermal stress. Below, Table 13 provides a summary of the motor dataset.

Table 13: The breakdown of motor data set.

z_1 (temperature)	x_i (time of Breakdown)	δ_i (censor)
150	$8046, 8064, 8064, 8064, 8064, 8046, 8064, 8064, 8064, 8064$	$0, 0, 0, 0, 0, 0, 0, 0, 0, 0$
170	$1764, 2772, 3444, 3542, 3780, 4860, 5196, 5448, 5448, 5448$	$1, 1, 1, 1, 1, 1, 1, 0, 0, 0$
190	$408, 408, 1344, 1344, 1440, 1680, 1680, 1680, 1680, 1680$	$1, 1, 1, 1, 1, 0, 0, 0, 0, 0$
220	$408, 408, 504, 504, 504, 528, 528, 528, 528, 528$	$1, 1, 1, 1, 1, 0, 0, 0, 0, 0.$

In case of $\varphi(z) = z$ log-linear assumption and using R statistical software (the BB package) we find the values of the censored MLEs of AFT-WTLE distribution parameters :

$$\widehat{\lambda} = 0.25491, \widehat{\beta}_0 = 2.61542, \widehat{\beta}_1 = 4.91572,$$

we choose $r = 15$ intervals and the estimated FIM can be expressed as :

$$I\left(\underline{\widehat{\mathbf{V}}}\right) = \begin{pmatrix} 0.002154 & 0.95782 & -2.31574 \\ & 0.95354 & -1.300024 \\ & & 0.632591 \end{pmatrix},$$

and then the modified NRR statistic : $K^2 = 20.930154$. For the critical value : $\alpha = 0.01$, we find $K^2 > \chi_{0.01}^2(14) = 29.1412$.

9. Conclusions

This study introduced a novel Accelerated Failure Time (AFT) model based on the Weighted Topp-Leone exponential (WTLE) distribution, offering a robust framework for analyzing censored and uncensored survival data. The proposed AFT-WTLE model demonstrated superior flexibility in capturing hazard rate shapes, validated through rigorous goodness-of-fit tests and empirical applications to real-world datasets, including electric insulating fluid failure times and body fat percentage measurements. Maximum likelihood estimation (MLE), Cramér-von Mises (CVM), and Anderson-Darling (ADE) methods were employed to estimate model parameters, with simulation studies confirming their accuracy and consistency across varying sample sizes. Notably, the RTADE method emerged as the most reliable estimator under simulated conditions, balancing precision and adaptability in risk dynamics. The model's applicability extended to risk analysis, where it effectively quantified Value-at-Risk (VaR), Tail VaR (TVaR), and other risk metrics, highlighting MLE's stability for conservative risk estimates and ADE/CVM's sensitivity to tail risks. For censored data, the modified NRR chi-square test validated the AFT-WTLE's validity, with empirical levels closely aligning with theoretical thresholds. Real-world applications, such as motor failure data and Johnson's body fat dataset, underscored the model's practical utility in actuarial, biomedical, and engineering contexts. Furthermore, the study addressed computational challenges via the BB algorithm, enabling efficient parameter optimization. Future work could explore extensions to handle interval-censored data or incorporate time-varying covariates. The AFT-WTLE model bridges a critical gap in survival analysis by integrating flexible baseline hazards with robust risk assessment tools, offering valuable insights for researchers in reliability engineering, healthcare analytics, and financial risk management.

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