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Investigation of super lehmer-3 mean labeling in theta related graphs

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Abstract In graph theory, labeling of vertices and edges according to specific rules offers insight into the structural properties of graphs and their potential applications. This study focuses on a particular type of graph labeling known as Super Lehmer-3 Mean Labeling. A graph is said to admit such a labeling if there exists an injective vertex labeling such that the corresponding edge labels, derived from the Lehmer-3 mean of their incident vertex labels, are also distinct and satisfy certain arithmetic conditions. Graphs that admit such a labeling are termed Super Lehmer-3 Mean Graphs. In this work, we investigate the existence of super Lehmer-3 mean labeling in various families of theta-related graphs. Specifically, we analyze standard theta graphs and their structural variations obtained by subdivisions and attachments of pendant vertices. The results confirm that these modified forms of theta graphs also admit super Lehmer-3 mean labeling. Additionally, we extend our exploration to harmonic mean labeling in selected theta graphs, highlighting the conditions under which this alternative labeling scheme is valid. The findings presented in this paper contribute to the growing body of knowledge in graph labeling theory, offering new characterizations and constructions of labeled graph families. These results may have implications for theoretical studies and applications where such labeling structures are relevant.

Keywords Graph, Lehmer-3 Mean Labeling, Super Lehmer-3 Mean labeling, Harmonic Mean Labeling, Corona Product of G, Subdivision of G, Theta Graph

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1. Introduction

"Graph Theory" is a very important branch of arithmetic, Euler (1707-1782) is understood because the father of Graph Theory likewise as Topology. Graph Theory came into existence throughout the primary half the 18^{th} century. Graph Theory failed to begin to become associate organized branch of mathematics till the last half of the 19^{th} century and, there wasn't even a book on the topic till the primary half the 20^{th} century. Graph Theory is a branch of mathematics concerned with networks of points connected by lines. The subject of graph theory had its beginning in recreational with problems but it has grown into a significant area of mathematical research, application in chemistry, operations research, social science, application in biology and computer science.

Labeled graphs provide mathematical models for a broad range of applications. The qualitative labeling of graph elements have been used in diverse fields such as conflict resolutions in social psychology, energy crises etc. Quantitative labeling of graph elements have been used in missile guidance codes, radar location codes, coding theory, X-ray crystallography, astronomy, circuit design, communication network etc. Today graph theory is one of the flourishing branches of modern algebra with wide applications to combinatorial problems and to classical algebraic problems.

Graph labeling is one of the fascinating areas of graph theory with wide range of applications. A graph labeling is an assignment of integers to the vertices or the edges or both, subject to certain conditions. If the domain is the

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set of vertices we speak about the vertex labeling. If the domain is the set of edges, then the labeling is called edge labeling. Most graph labeling methods trace their origin to the one introduced by Rosa in 1967. Over the past three decades more than 600 papers have been published in graph labeling.

For all other standard terminology and notations, we follow Harary[4]. Graph labeling is used in many areas of sciences and technology. A lot of graph labeling techniques are discussed in [3]. The concept and notation of mean labeling was first introduced by S. Somasundaram and R. Ponraj[10]. S. Somasundaram, S.S.Sandya and T. Pavithra[22] introduced the concept of Lehmer-3 mean labeling of graphs and studied their behavior in [17, 18, 19] and also proved that some Super Lehmer-3 mean labeling and $k-$Super Lehmer-3 mean labeling graphs [20, 21, 23, 24]. Paramaguru. N and Manirathnam. D[9] proved that some cycle related Lehmer-3 Mean Graphs. S. Somasundaram, R. Ponraj and S.S. Sandhya introduced the concept of Harmonic Mean labeling of graphs in [14] and studied their behavior in [15, 12, 13, 11, 16]. Deepika. R[2] investigated super Lehmer-3 mean labeling of path union related graphs. Meena. S and Mani. R[7] investigated this labeling for some cycle related graphs and they also find some new Lehmer-3 mean graphs in [5, 6]. Meena. S and R. Mani[8] proved root square mean labeling of theta related graphs. Prime cordial labeling for theta graph has been proved by Sugumaran. A and Vishnu Prakash. P[25]. In this paper, we examine about the super Lehmer-3 mean labeling of theta related graphs and some new examples are presented and verified. We now give the definitions which are necessary for the present investigation.

Let $h : V(G) \to \{1, 2, \ldots, p+q\}$ be an injective function. The induced edge labeling $h^*(e = uv)$ is defined by $h^*(e) = \lceil \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rceil$ or $\lfloor \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rfloor$, then h is called super Lehmer-3 mean labeling, if $\{h(V(G))\} \cup \{h(e)/e \in E(G)\} = \{1, 2, \ldots, p+q\}$. A graph which admits Super Lehmer-3 mean labeling is called Super Lehmer-3 Mean graph. In this paper, we prove that theta related graphs such as θ_τ, $\theta_\tau \bigodot \overline{K_1}$, $\theta_\tau \bigodot \overline{K_2}$, $\theta_\tau \bigodot \overline{K_3}$, $\theta_\tau \bigodot K_2$, $S(\theta_\tau)$, $(S(\theta_\tau)) \bigodot \overline{K_1}$, $(S(\theta_\tau)) \bigodot \overline{K_2}$, $(S(\theta_\tau)) \bigodot \overline{K_3}$, $(S(\theta_\tau)) \bigodot K_2$. Subdividing all the path vertices of theta graph such as $\theta_\tau \bigodot \overline{K_1}$, $\theta_\tau \bigodot \overline{K_2}$, $\theta_\tau \bigodot \overline{K_3}$ and $\theta_\tau \bigodot K_2$ are all super Lehmer-3 mean graphs. Then some special results for harmonic mean labeling of theta graphs also are investigates in this paper.

Definition 1.1
A graph G with p vertices and q edges is called Lehmer-3 mean graph if it is possible to label the vertices $x \in V$ with distinct labels $h(x)$ from $1, 2, \ldots, q+1$ in such a way that when each edge $e = uv$ is labeled with $h(e = uv) = \lceil \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rceil$ or $\lfloor \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rfloor$, then the resulting edge labels are distinct. In this case, h is called a **Lehmer-3 mean labeling** of G.

Definition 1.2
Let $h : V(G) \to \{1, 2, \ldots, p+q\}$ be an injective function. The induced edge labeling $h^*(e = uv)$ is defined by $h^*(e) = \lceil \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rceil$ or $\lfloor \frac{h(u)^3+h(v)^3}{h(u)^2+h(v)^2} \rfloor$, then h is called super lehmer-3 mean labeling, if $\{h(V(G))\} \cup \{h(e)/e \in E(G)\} = \{1, 2, \ldots, p+q\}$. A graph which admits Super lehmer-3 mean labeling is called **Super Lehmer-3 Mean graph**.

Example 1.2.1
1. **Path:**
   Number of Vertices $p = 6$
   Number of Edges $q = 5$
2. **Cycle:**
   Number of Vertices $p = 9$
   Number of Edges $q = 9$

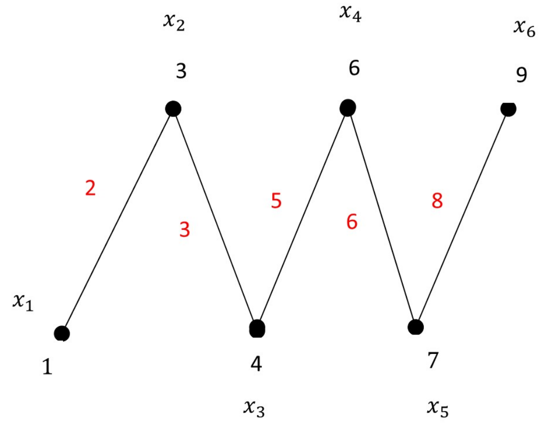

Figure 1. Super Lehmer-3 mean labeling of Path P_6

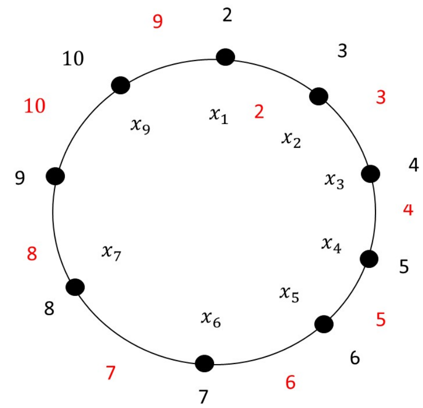

Figure 2. Super Lehmer-3 mean labeling of Cycle C_9

Definition 1.3
A graph G with p vertices and q edges is called a **Harmonic mean graph** if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \ldots, q+1$ in such a way that when each edge $e = uv$ is labeled with $f(e = uv) = \lceil \frac{2f(u)f(v)}{f(u)+f(v)} \rceil$ or $\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ then the edge labels are distinct. In this case f is called a **Harmonic mean labeling** of G.

Definition 1.4
A walk in which $x_1 x_2 \ldots x_\tau$ are distinct is called a **path**. A path on τ vertices is denoted by P_τ.

Definition 1.5
A closed path is called a **cycle**. A cycle on $n-$vertices is denoted by C_τ.

Definition 1.6
The **Corona of two graphs** G_1 and G_2 is the graph $G = G_1 \bigodot G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Definition 1.7
A **subdivision graph** $S(G)$ of a graph G is a graph obtained from G by subdividing all edges of G exactly once.

Example 1.7.1

Path:

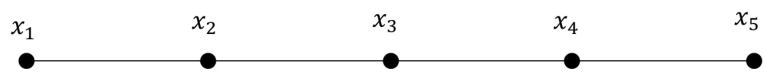

Figure 3. G

Subdivision of Path:

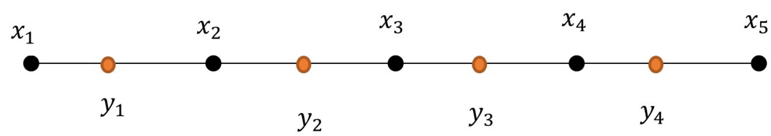

Figure 4. $S(G)$

Definition 1.8
A **theta graph** is a block with two non-adjacent vertices of degree 3 and all other vertices of degree 2 is called a theta graph and it is denoted by θ_γ.

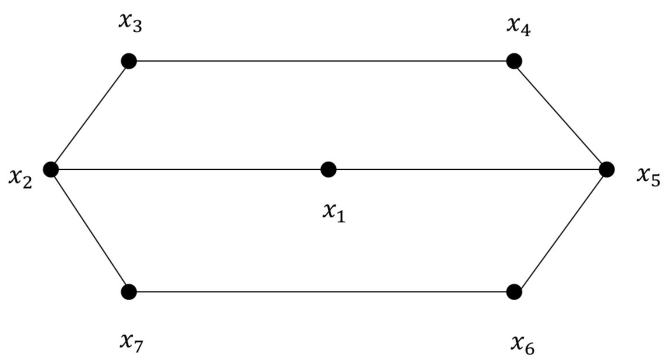

Figure 5. Theta Graph θ_τ

Notation

Here $\lceil h \rceil$ is the least integer greater than or equal to h and $\lfloor h \rfloor$ is the greatest integer less than or equal to h.

2. Main Results

In this paper, we investigate the super lehmer-3 mean labeling of theta related graphs.

Theorem 2.1
θ_τ is a super lehmer-3 mean graph.

Proof
Let $G = \theta_\tau$.
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.
Let $V(G) = \{x_1 x_2 \ldots x_\tau\}$
$E(G) = \{x_k x_{k+1}/1 \leq k \leq \tau - 1\} \cup \{x_2 x_\tau, x_1 x_{\tau-2}\}$.
Define a mapping $h : V(G) \to \{1, 2, \ldots, 2\tau - 1\}$ by

$$h(x_k) = 2k - 1 \text{ for } 1 \leq k \leq \tau.$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 2k \text{ for } 1 \leq k \leq \tau - 1 \\ h(x_1 x_{\tau-2}) &= \tau + 2 \\ h(x_2 x_\tau) &= 2\tau - 1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.1.1
For instance,

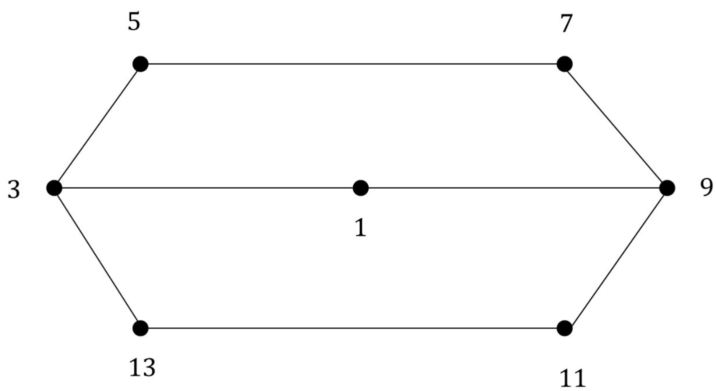

Figure 6. Super lehmer-3 mean labeling of θ_7

Theorem 2.2
$S(\theta_\tau)$ is a super lehmer-3 mean graph.

Proof
Let $G = S(\theta_\tau)$.
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.
Let $y_1 y_2 \ldots y_\tau$ be the vertices of subdivision theta graph $S(G)$.
Let $V(G) = \{x_1 x_2 \ldots x_\tau, y_1 y_2 \ldots y_\tau\}$

$$\begin{aligned} E(G) &= \{x_k b_{k+1}/1 \leq k \leq \tau\} \cup \{y_k x_k/1 \leq k \leq \tau\} \\ &\cup \{y_1 x_{\gamma-2}, x_2 y_{\tau+1}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 4\tau + 1\}$ by

$$\begin{aligned} h(x_k) &= 4k - 1 \text{ for } 1 \leq k \leq \tau \\ h(y_k) &= 4k - 3 \text{ for } 1 \leq k \leq \tau \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k y_{k+1}) &= 4k \text{ for } 1 \leq k \leq \tau \\ h(y_k x_k) &= 4k-2 \text{ for } 1 \leq k \leq \tau \\ h(y_1 x_{\gamma-2}) &= 3\tau - 2 \\ h(x_2 y_{\tau+1}) &= 4\tau - 1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.2.1
For instance,

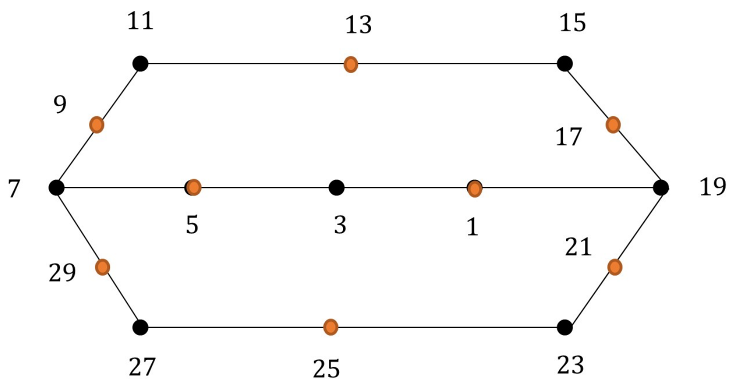

Figure 7. Super lehmer-3 mean labeling of $(S(\theta_7))$

Theorem 2.3
$\theta_\tau \bigodot \bar{K}_1$ is a super lehmer-3 mean graph.

Proof
Let $G = \theta_\tau \bigodot \bar{K}_1$.
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.
Let $y_1 y_2 \ldots y_\tau$ be the pendant vertices attached at $x_1 x_2 \ldots x_\tau$ respectively.
Let $V(G) = \{x_1 x_2 \ldots x_\tau, y_1 y_2 \ldots y_\tau\}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \leq k \leq \tau - 1\} \cup \{x_k y_k/1 \leq k \leq \tau\} \\ &\cup \{x_2 x_\tau, x_1 x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \rightarrow \{1, 2, \ldots, 4\tau - 1\}$ by

$$\begin{aligned} h(x_k) &= 4k-3 \text{ for } 1 \leq k \leq \tau \\ h(y_k) &= 4k-1 \text{ for } 1 \leq k \leq \tau \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 4k \text{ for } 1 \leq k \leq \tau - 1 \\ h(x_k y_k) &= 4k-2 \text{ for } 1 \leq k \leq \tau \\ h(x_1 x_{\tau-2}) &= 2\tau + 3 \\ h(x_2 x_\tau) &= 3\tau - 4 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.3.1
For instance,

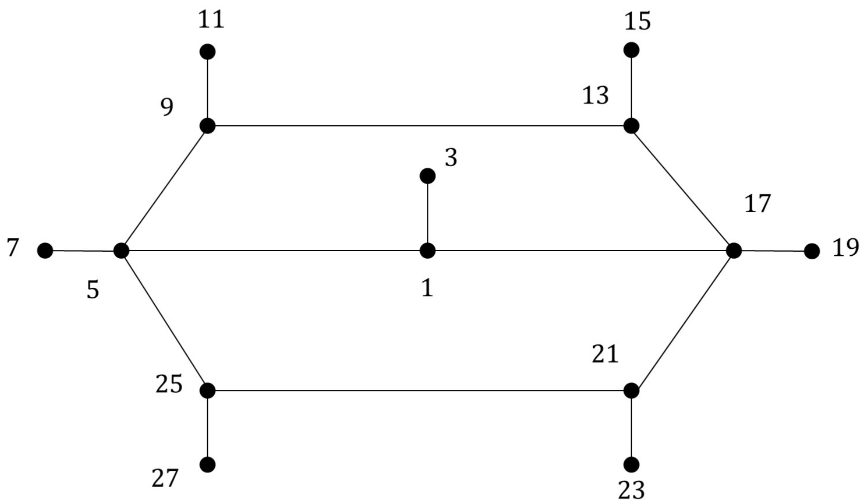

Figure 8. Super lehmer-3 mean labeling of $\theta_7 \bigodot \bar{K_1}$

Theorem 2.4
$(S(\theta_\tau)) \bigodot \bar{K_1}$ is a super lehmer-3 mean graph.

Proof
Let $G = (S(\theta_\tau)) \bigodot \bar{K_1}$.
Let $x_1 x_2 \ldots x_{2\tau+1}$ be the vertices of subdivision theta graph $S(G)$.
Let $y_1 y_2 \ldots y_{2\tau+1}$ be the pendant vertices attached at $x_1 x_2 \ldots x_{2\tau+1}$ respectively.
Let $V(G) = \{x_1 x_2 \ldots x_{2\tau+1}, y_1 y_2 \ldots y_{2\tau+1}\}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \le k \le 2\tau\} \cup \{x_k y_k/1 \le k \le 2\tau+1\} \\ &\cup \{x_1 x_{\tau+3}, x_4 x_{2\tau+1}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 8\tau+3\}$ by

$$\begin{aligned} h(x_k) &= 4k-3 \text{ for } 1 \le k \le 2\tau+1 \\ h(y_k) &= 4k-1 \text{ for } 1 \le k \le 2\tau+1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 4k \text{ for } 1 \le k \le 2\tau \\ h(x_k y_k) &= 4k-2 \text{ for } 1 \le k \le 2\tau+1 \\ h(x_1 x_{\tau+3}) &= 5\tau+3 \\ h(x_4 x_{2\tau+1}) &= 8\tau-1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.4.1
For instance,

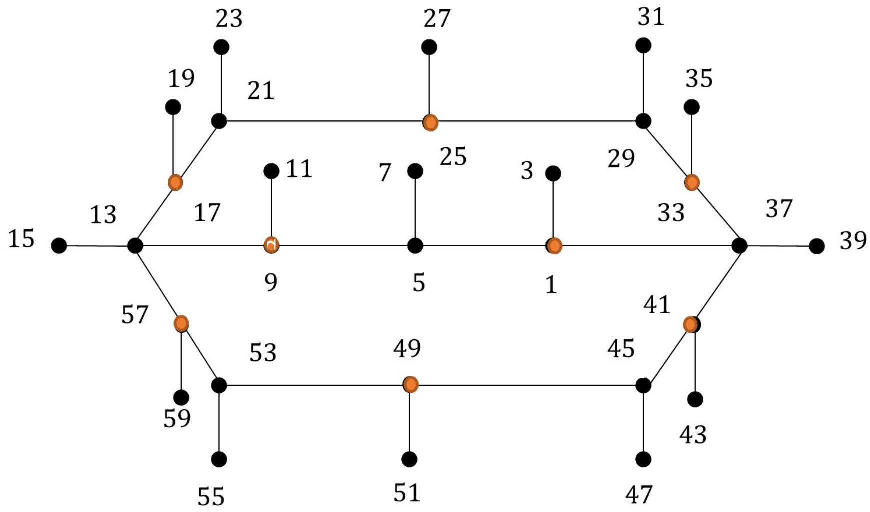

Figure 9. Super lehmer-3 mean labeling of $(S(\theta_7)) \bigodot \bar{K_1}$

Theorem 2.5
$\theta_\tau \bigodot \bar{K_2}$ is a super lehmer-3 mean graph.

Proof
Let $G = \theta_\tau \bigodot \bar{K_2}$.
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.
Let $y_1 y_2 \ldots y_\tau$ and $z_1 z_2 \ldots z_\tau$ be the pendant vertices attached at $x_1 x_2 \ldots x_\tau$ respectively.
Let $V(G) = \{x_1 x_2 \ldots x_\tau, y_1 y_2 \ldots y_\tau, z_1 z_2 \ldots z_\tau\}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \leq k \leq \tau - 1\} \cup \{x_k y_k/1 \leq k \leq \tau\} \\ &\quad \cup \{x_k z_k/1 \leq k \leq \tau\} \cup \{x_2 x_\tau, x_1 x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 6\tau - 1\}$ by

$$\begin{aligned} h(x_k) &= 6k - 5 \text{ for } 1 \leq k \leq \tau \\ h(y_k) &= 6k - 3 \text{ for } 1 \leq k \leq \tau \\ h(z_k) &= 6k - 1 \text{ for } 1 \leq k \leq \tau \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 6k \text{ for } 1 \leq k \leq \tau - 4 \\ h(x_k x_{k+1}) &= 6k - 1 \text{ for } \tau - 3 \leq k \leq \tau - 1 \\ h(x_k y_k) &= 6k - 4 \text{ for } 1 \leq k \leq \tau \\ h(x_k z_k) &= 6k - 2 \text{ for } 1 \leq k \leq \tau \\ h(x_1 x_{\tau-2}) &= 3\tau + 4 \\ h(x_2 x_\tau) &= 5\tau + 1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.5.1
For instance,

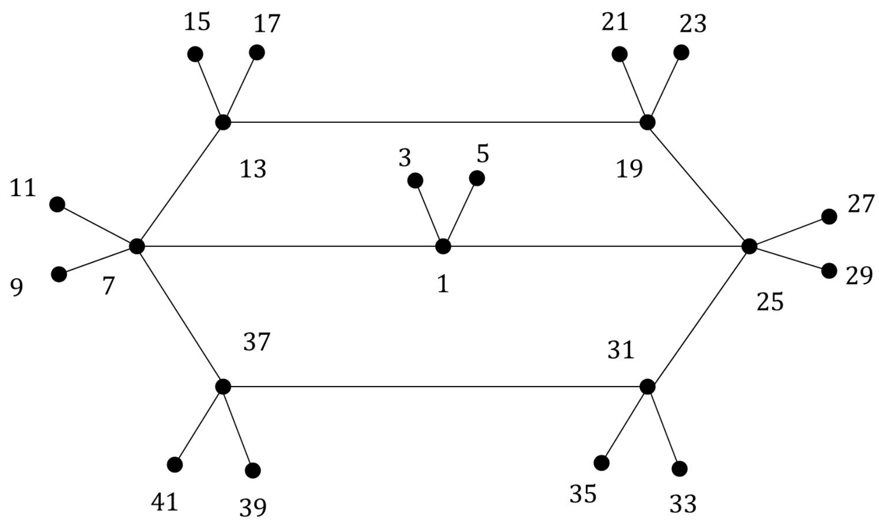

Figure 10. Super lehmer-3 mean labeling of $\theta_7 \bigodot \bar{K}_2$

Theorem 2.6
$(S(\theta_\tau)) \bigodot \bar{K}_2$ is a super lehmer-3 mean graph.

Proof
Let $G = (S(\theta_\tau)) \bigodot \bar{K}_2$.
Let $x_1 x_2 \dots x_{2\tau+1}$ be the vertices of subdivision theta graph $S(G)$.
Let $y_1 y_2 \dots y_{2\tau+1}$ and $z_1 z_2 \dots z_{2\tau+1}$ be the pendant vertices attached at $x_1 x_2 \dots x_{2\tau+1}$ respectively.
Let $V(G) = \{x_1 x_2 \dots x_{2\tau+1}, y_1 y_2 \dots y_{2\tau+1}, z_1 z_2 \dots z_{2\tau+1}\}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \leq k \leq 2\tau\} \cup \{x_k y_k/1 \leq k \leq 2\tau+1\} \\ &\cup \{x_k z_k/1 \leq k \leq 2\tau+1\} \cup \{x_1 x_{\tau+3}, x_4 x_{2\tau+1}\} \end{aligned}$$

Define a mapping $h : V(G) \rightarrow \{1, 2, \dots, 13\tau - 2\}$ by

$$\begin{aligned} h(x_k) &= 6k-5 \text{ for } 1 \leq k \leq 2\tau+1 \\ h(y_k) &= 6k-3 \text{ for } 1 \leq k \leq 2\tau+1 \\ h(z_k) &= 6k-1 \text{ for } 1 \leq k \leq 2\tau+1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 6k \text{ for } 1 \leq k \leq \tau-4 \\ h(x_k x_{k+1}) &= 6k-1 \text{ for } \tau-3 \leq k \leq 2\tau \\ h(x_k y_k) &= 6k-4 \text{ for } 1 \leq k \leq 2\tau+1 \\ h(x_k z_k) &= 6k-2 \text{ for } 1 \leq k \leq 2\tau+1 \\ h(x_1 x_{\tau+3}) &= 8\tau-2 \\ h(x_4 x_{2\tau+1}) &= 12\tau-3 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.6.1
For instance,

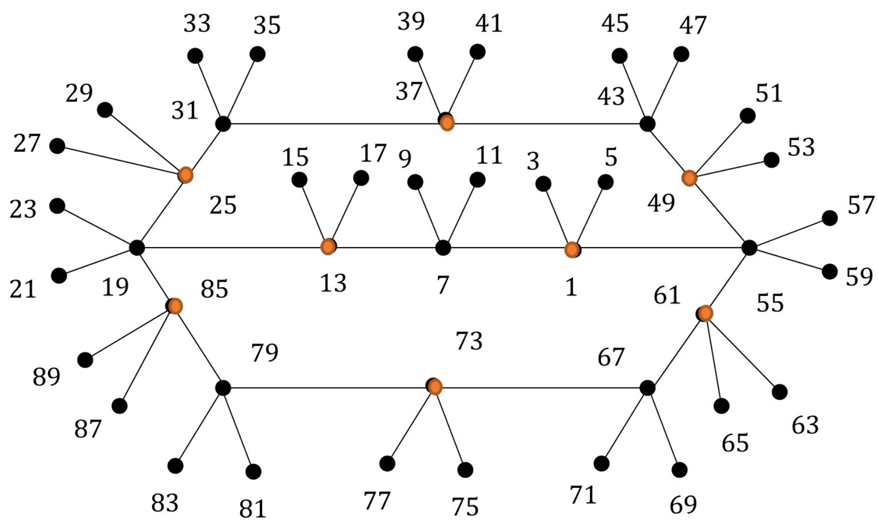

Figure 11. Super lehmer-3 mean labeling of $(S(\theta_7)) \odot \bar{K_2}$

Theorem 2.7
$\theta_\tau \odot \bar{K_3}$ is a super lehmer-3 mean graph.

Proof
Let $G = \theta_\tau \odot \bar{K_3}$.
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.
Let $y_1 y_2 \ldots y_\tau, y'_1 y'_2 \ldots y'_\tau$ and $z_1 z_2 \ldots z_\tau$ be the pendant vertices attached at $x_1 x_2 \ldots x_\tau$ respectively.
Let $V(G) = \{x_1 x_2 \ldots x_\tau, y_1 y_2 \ldots y_\tau, y'_1 y'_2 \ldots y'_\tau, z_1 z_2 \ldots z_\tau\}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \leq k \leq \tau - 1\} \cup \{x_k y_k/1 \leq k \leq \tau\} \\ &\cup \{x_k y'_k/1 \leq k \leq \tau\} \cup \{x_k z_k/1 \leq k \leq \tau\} \cup \{x_2 x_\tau, x_1 x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 8\tau - 1\}$ by

$$\begin{aligned} h(x_k) &= 8k - 7 \text{ for } 1 \leq k \leq \tau \\ h(y_k) &= 8k - 5 \text{ for } 1 \leq k \leq \tau \\ h(y'_k) &= 8k - 3 \text{ for } 1 \leq k \leq \tau \\ h(z_k) &= 8k - 1 \text{ for } 1 \leq k \leq \tau \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_k x_{k+1}) &= 8k \text{ for } 1 \leq k \leq \tau - 5 \\ h(x_k x_{k+1}) &= 8k - 1 \text{ for } \tau - 4 \leq k \leq \tau - 3 \\ h(x_k x_{k+1}) &= 8k - 2 \text{ for } \tau - 2 \leq k \leq \tau - 1 \\ h(x_k y_k) &= 8k - 6 \text{ for } 1 \leq k \leq \tau \\ h(x_k y'_k) &= 8k - 4 \text{ for } 1 \leq k \leq \tau \\ h(x_k z_k) &= 8k - 2 \text{ for } 1 \leq k \leq \tau - 5 \\ h(x_k z_k) &= 8k - 3 \text{ for } \tau - 4 \leq k \leq \tau \\ h(x_1 x_{\tau-2}) &= 5\tau - 2 \\ h(x_2 x_\tau) &= 7\tau - 2 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.7.1
For instance,

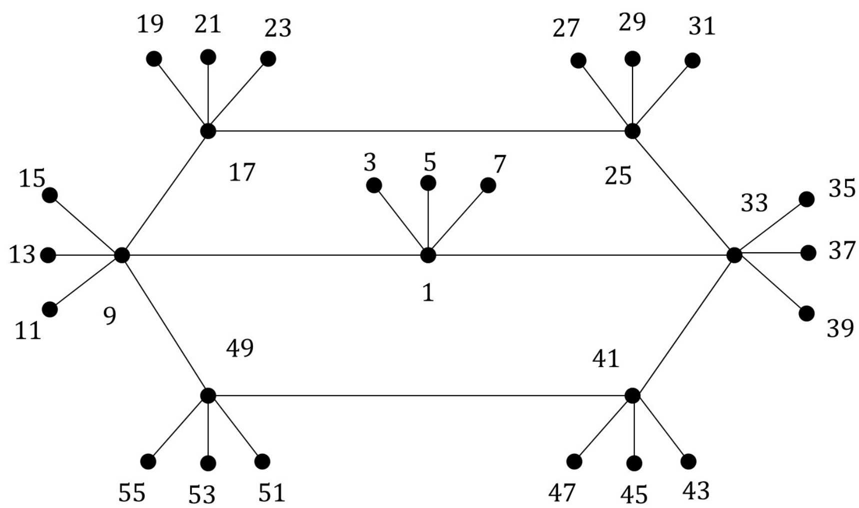

Figure 12. Super lehmer-3 mean labeling of $\theta_7 \bigodot \bar{K_3}$

Theorem 2.8
$(S(\theta_\tau)) \bigodot \bar{K_3}$ is a super lehmer-3 mean graph.

Proof
Let $G = (S(\theta_\tau)) \bigodot \bar{K_3}$.

Let $x_1 x_2 \ldots x_{2\tau+1}$ be the vertices of subdivision theta graph $S(G)$.

Let $y_1 y_2 \ldots y_{2\tau+1}$, $y_1' y_2' \ldots y_{2\tau+1}'$ and $z_1 z_2 \ldots z_{2\tau+1}$ be the pendant vertices attached at $x_1 x_2 \ldots x_{2\tau+1}$ respectively.

Let $V(G) = \{x_1 x_2 \ldots x_{2\tau+1}, y_1 y_2 \ldots y_{2\tau+1}, y_1' y_2' \ldots y_{2\tau+1}', z_1 z_2 \ldots z_{2\tau+1}$

$$\begin{aligned} E(G) &= \{x_k x_{k+1}/1 \le k \le 2\tau\} \cup \{x_k y_k/1 \le k \le 2\tau+1\} \\ &\cup \{x_k y_k'/1 \le k \le 2\tau+1\} \cup \{x_k z_k/1 \le k \le 2\tau+1\} \cup \{x_1 x_{\tau+3}, x_4 x_{2\tau+1}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 17\tau\}$ by

$$\begin{aligned} h(x_k) &= 8k - 7 \text{ for } 1 \le k \le 2\tau+1 \\ h(y_k) &= 8k - 5 \text{ for } 1 \le k \le 2\tau+1 \\ h(y_k') &= 8k - 3 \text{ for } 1 \le k \le 2\tau+1 \\ h(z_k) &= 8k - 1 \text{ for } 1 \le k \le 2\tau+1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned}
h(x_k x_{k+1}) &= 8k \text{ for } 1 \le k \le \tau - 5 \\
h(x_k x_{k+1}) &= 8k - 2 \text{ for } \tau - 4 \le k \le 2\tau \\
h(x_k y_k) &= 8k - 6 \text{ for } 1 \le k \le 2\tau + 1 \\
h(x_k y_k') &= 8k - 4 \text{ for } 1 \le k \le 2\tau - 1 \\
h(x_k y_k') &= 8k - 5 \text{ for } 2\tau \le k \le 2\tau + 1 \\
h(x_k z_k) &= 8k - 2 \text{ for } 1 \le k \le \tau - 5 \\
h(x_k z_k) &= 8k - 3 \text{ for } \tau - 4 \le k \le 2\tau + 1 \\
h(x_1 x_{\tau+3}) &= 10\tau + 2 \\
h(x_4 x_{2\tau+1}) &= 15\tau + 3
\end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.8.1
For instance,

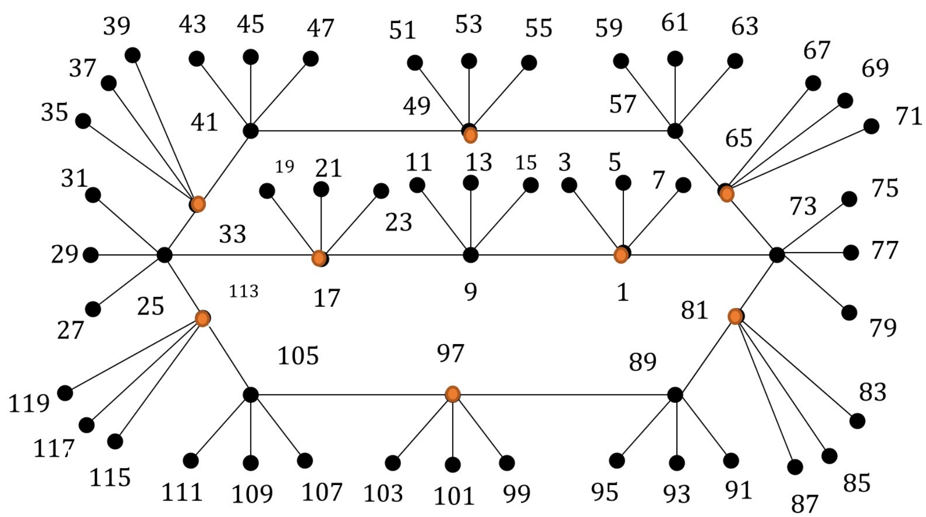

Figure 13. Super lehmer-3 mean labeling of $(S(\theta_7)) \bigodot \bar{K_3}$

Theorem 2.9
$\theta_\tau \bigodot K_2$ is a super lehmer-3 mean graph.

Proof
Let $G = \theta_\tau \bigodot K_2$.
Let $x_1 x_2 \dots x_\tau$ be the vertices of theta in G.
Let $y_1 y_2 \dots y_\tau$ and $z_1 z_2 \dots z_\tau$ be the adjacent vertices attached at $x_1 x_2 \dots x_\tau$ respectively.
Let $V(G) = \{x_1 x_2 \dots x_\tau, y_1 y_2 \dots y_\tau, z_1 z_2 \dots z_\tau\}$

$$\begin{aligned}
E(G) &= \{x_k x_{k+1}/1 \le k \le \tau - 1\} \cup \{x_k y_k/1 \le k \le \tau\} \\
&\quad \cup \{x_k z_k/1 \le k \le \tau\} \cup \{y_k z_k/1 \le k \le \tau\} \\
&\quad \cup \{x_2 x_\tau, x_1 x_{\tau-2}\}
\end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 7\tau - 2\}$ by

$$\begin{aligned}
h(x_k) &= 6k - 5 \text{ for } 1 \leq k \leq \tau - 3 \\
h(x_k) &= 8k - 13 \text{ for } \tau - 2 \leq k \leq \tau \\
h(y_k) &= 6k - 3 \text{ for } 1 \leq k \leq \tau - 3 \\
h(y_k) &= 8k - 11 \text{ for } \tau - 2 \leq k \leq \tau \\
h(z_k) &= 6k - 1 \text{ for } 1 \leq k \leq \tau - 3 \\
h(z_k) &= 8k - 9 \text{ for } \tau - 2 \leq k \leq \tau
\end{aligned}$$

Then the edge labels are

$$\begin{aligned}
h(x_k x_{k+1}) &= 6k \text{ for } 1 \leq k \leq \tau - 3 \\
h(x_k x_{k+1}) &= 8k - 8 \text{ for } \tau - 2 \leq k \leq \tau - 1 \\
h(x_k y_k) &= 6k - 4 \text{ for } 1 \leq k \leq \tau - 3 \\
h(x_k y_k) &= 8k - 12 \text{ for } \tau - 2 \leq k \leq \tau \\
h(y_k z_k) &= 6k - 1 \text{ for } 1 \leq k \leq \tau - 3 \\
h(y_k z_k) &= 8k - 9 \text{ for } \tau - 2 \leq k \leq \tau \\
h(x_k z_k) &= 6k - 2 \text{ for } 1 \leq k \leq \tau - 3 \\
h(x_k z_k) &= 8k - 10 \text{ for } \tau - 2 \leq k \leq \tau \\
h(x_1 x_{\tau-2}) &= 4\tau - 2 \\
h(x_2 x_\tau) &= 6\tau
\end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.9.1
For instance,

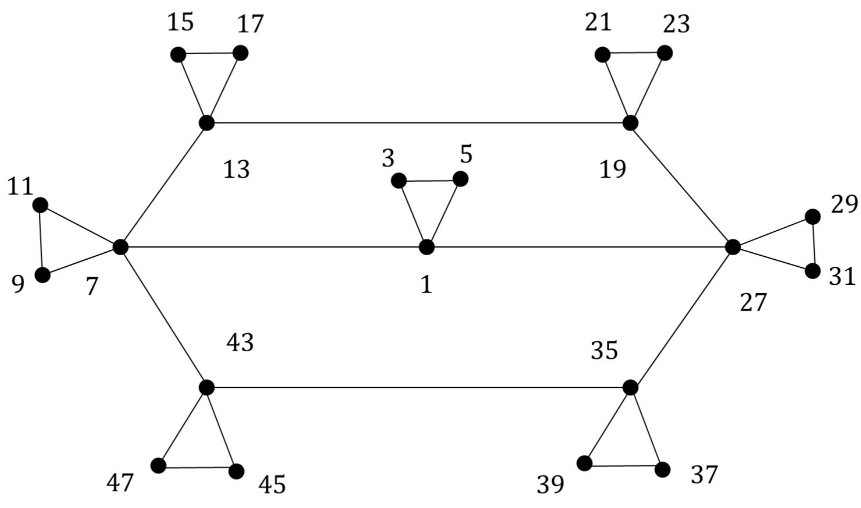

Figure 14. Super lehmer-3 mean labeling of $\theta_7 \bigodot K_2$

Theorem 2.10
$(S(\theta_\tau)) \bigodot K_2$ is a super lehmer-3 mean graph.

Proof
Let $G = (S(\theta_\tau)) \bigodot K_2$.
Let $x_1x_2\ldots x_{2\tau+1}$ be the vertices of subdivision theta graph $S(G)$.
Let $y_1y_2\ldots y_{2\tau+1}$ and $z_1z_2\ldots z_{2\tau+1}$ be the adjacent vertices attached at $x_1x_2\ldots x_{2\tau+1}$ respectively.
Let $V(G) = \{x_1x_2\ldots x_{2\tau+1}, y_1y_2\ldots y_{2\tau+1}, z_1z_2\ldots z_{2\tau+1}\}$

$$\begin{aligned} E(G) &= \{x_kx_{k+1}/1 \le k \le 2\tau\} \cup \{x_ky_k/1 \le k \le 2\tau+1\} \\ &\cup \{y_kz_k/1 \le k \le 2\tau+1\} \cup \{x_kz_k/1 \le k \le 2\tau+1\} \\ &\cup \{x_1x_{\tau+3}, x_4x_{2\tau+1}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 16\tau - 1\}$ by

$$\begin{aligned} h(x_k) &= 6k - 5 \text{ for } 1 \le k \le \tau - 3 \\ h(x_k) &= 8k - 13 \text{ for } \tau - 2 \le k \le 2\tau + 1 \\ h(y_k) &= 6k - 4 \text{ for } 1 \le k \le \tau - 3 \\ h(y_k) &= 8k - 11 \text{ for } \tau - 2 \le k \le 2\tau + 1 \\ h(z_k) &= 6k - 1 \text{ for } 1 \le k \le \tau - 3 \\ h(z_k) &= 8k - 9 \text{ for } \tau - 2 \le k \le 2\tau + 1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(x_kx_{k+1}) &= 6k \text{ for } 1 \le k \le \tau - 3 \\ h(x_kx_{k+1}) &= 8k - 8 \text{ for } \tau - 2 \le k \le 2\tau \\ h(x_ky_k) &= 6k - 4 \text{ for } 1 \le k \le \tau - 3 \\ h(x_ky_k) &= 8k - 12 \text{ for } \tau - 2 \le k \le 2\tau + 1 \\ h(y_kz_k) &= 6k - 1 \text{ for } 1 \le k \le \tau - 3 \\ h(y_kz_k) &= 8k - 9 \text{ for } \tau - 2 \le k \le 2\tau + 1 \\ h(x_kz_k) &= 6k - 2 \text{ for } 1 \le k \le \tau - 3 \\ h(x_kz_k) &= 8k - 10 \text{ for } \tau - 2 \le k \le 2\tau + 1 \\ h(x_1x_{\tau+3}) &= 10\tau - 4 \\ h(x_4x_{2\tau+1}) &= 15\tau \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.10.1
For instance,

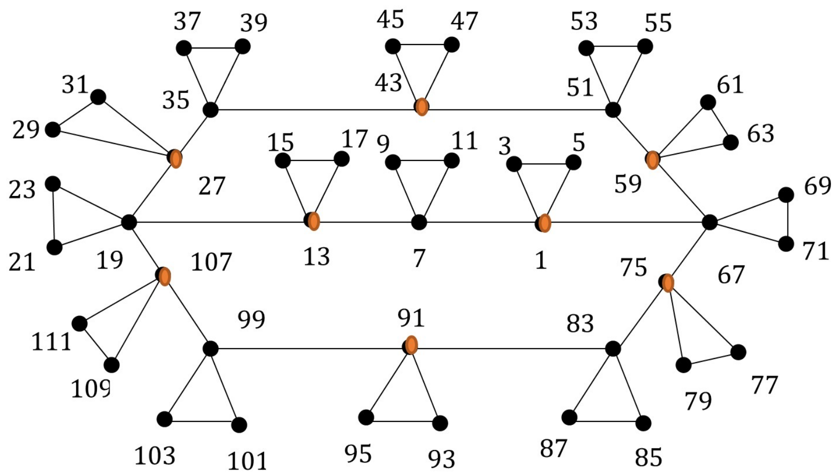

Figure 15. Super lehmer-3 mean labeling of $(S(\theta_7)) \odot K_2$

Theorem 2.11
The graph G obtained by subdividing all the path vertices of theta graph $\theta_\tau \odot \bar{K}_1$ is a super lehmer - 3 mean graph.

Proof
Let $x_1 x_2 \ldots x_\tau$ be the vertices of theta in G.

Let $y_1 y_2 \ldots y_\tau$ be the pendant vertices attached at $x_1 x_2 \ldots x_\tau$ respectively.

Let $z_1 z_2 \ldots z_{\tau+1}$ be the vertices obtained by subdividing all the path edges $x_k x_{k+1}$, $x_1 x_{\tau-2}$ and $x_2 x_\tau$ respectively.

Let $V(G) = \{x_1 x_2 \ldots x_\tau, y_1 y_2 \ldots y_\tau, z_1 z_2 \ldots z_{\tau+1}\}$

$$\begin{aligned} E(G) &= \{z_k x_k / 1 \leq k \leq \tau\} \cup \{x_k z_{k+1} / 1 \leq k \leq \tau\} \\ &\quad \cup \{x_k y_k / 1 \leq k \leq \tau\} \cup \{x_2 z_{\tau+1}, x_1 x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 6\tau + 1\}$ by

$$\begin{aligned} h(x_k) &= 6k - 3 \text{ for } 1 \leq k \leq \tau \\ h(y_k) &= 6k - 1 \text{ for } 1 \leq k \leq \tau \\ h(z_k) &= 6k - 5 \text{ for } 1 \leq k \leq \tau + 1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(z_k x_k) &= 6k - 4 \text{ for } 1 \leq k \leq \tau \\ h(x_k z_{k+1}) &= 6k \text{ for } 1 \leq k \leq \tau \\ h(x_k y_k) &= 6k - 2 \text{ for } 1 \leq k \leq \tau \\ h(z_1 x_{\tau-2}) &= 4\tau - 1 \\ h(x_2 z_{\tau+1}) &= 6\tau - 1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.11.1
For instance,

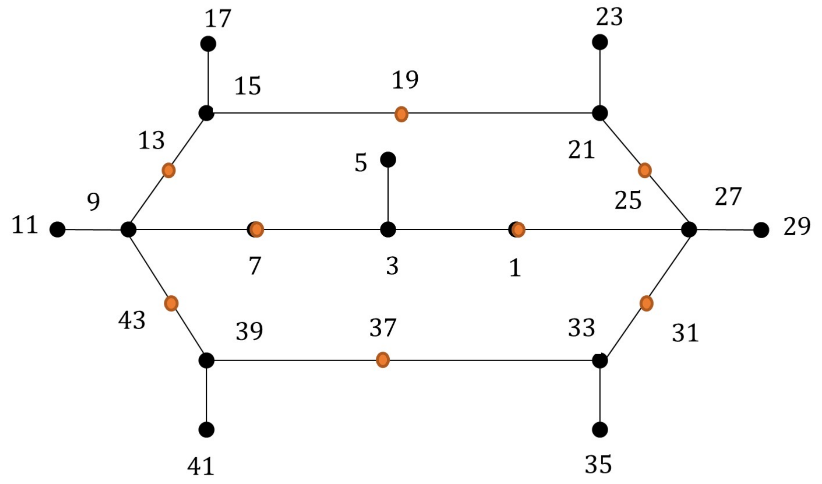

Figure 16. A Super lehmer-3 mean labeling of Subdividing all the path vertices of theta graph $\theta_7 \bigodot \bar{K}_1$

Theorem 2.12
The graph G obtained by subdividing all the path vertices of theta graph $\theta_\tau \bigodot \bar{K}_2$ is a super lehmer - 3 mean graph.

Proof
Let $x_1x_2\dots x_\tau$ be the vertices of theta in G.

Let $y_1y_2\dots y_\tau$ and $y'_1y'_2\dots y'_\tau$ be the pendant vertices attached at $x_1x_2\dots x_\tau$ respectively.

Let $z_1z_2\dots z_{\tau+1}$ be the vertices obtained by subdividing all the path edges x_kx_{k+1}, $x_1x_{\tau-2}$ and x_2x_τ respectively.

Let $V(G) = \{x_1x_2\dots x_\tau, y_1y_2\dots y_\tau, y'_1y'_2\dots y'_\tau, z_1z_2\dots z_{\tau+1}\}$

$$\begin{aligned} E(G) &= \{z_kx_k/1 \le k \le \tau\} \cup \{x_kz_{k+1}/1 \le k \le \tau\} \\ &\cup\{x_ky_k/1 \le k \le \tau\} \cup \{x_ky'_k/1 \le k \le \tau\} \\ &\cup\{x_2z_{\tau+1}, x_1x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \rightarrow \{1, 2, \dots, 8\tau + 1\}$ by

$$\begin{aligned} h(x_k) &= 8k-5 \text{ for } 1 \le k \le \tau \\ h(y_k) &= 8k-3 \text{ for } 1 \le k \le \tau \\ h(y'_k) &= 8k-1 \text{ for } 1 \le k \le \tau \\ h(z_k) &= 8k-7 \text{ for } 1 \le k \le \tau+1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(z_kx_k) &= 8k-6 \text{ for } 1 \le k \le \tau \\ h(x_kz_{k+1}) &= 8k \text{ for } 1 \le k \le \tau-5 \\ h(x_kz_{k+1}) &= 8k-7 \text{ for } \tau-4 \le k \le \tau \\ h(x_ky_k) &= 8k-4 \text{ for } 1 \le k \le \tau \\ h(x_ky'_k) &= 8k-2 \text{ for } 1 \le k \le \tau \\ h(z_1x_{\tau-2}) &= 5\tau \\ h(x_2z_{\tau+1}) &= 8\tau \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.12.1
For instance,

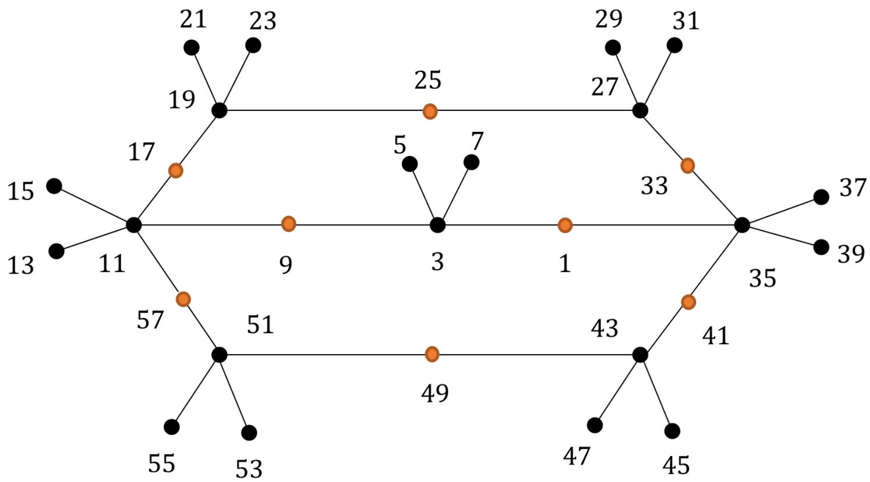

Figure 17. A Super lehmer-3 mean labeling of Subdividing all the path vertices of theta graph $\theta_7 \bigodot \bar{K}_2$

Theorem 2.13
The graph G obtained by subdividing all the path vertices of theta graph $\theta_\tau \bigodot \bar{K}_3$ is a super lehmer - 3 mean graph.

Proof
Let $x_1x_2\ldots x_\tau$ be the vertices of theta in G.
Let $y_1y_2\ldots y_\tau$, $y_1'y_2'\ldots y_\tau'$ and $y_1''y_2''\ldots y_\tau''$ be the pendant vertices attached at $x_1x_2\ldots x_\tau$ respectively.
Let $z_1z_2\ldots z_{\tau+1}$ be the vertices obtained by subdividing all the path edges x_kx_{k+1}, $x_1x_{\tau-2}$ and x_2x_τ respectively.
Let $V(G) = \{x_1x_2\ldots x_\tau, y_1y_2\ldots y_\tau, y_1'y_2'\ldots y_\tau', y_1''y_2''\ldots y_\tau'', z_1z_2\ldots z_{\tau+1}\}$

$$\begin{aligned} E(G) &= \{z_kx_k/1 \le k \le \tau\} \cup \{x_kz_{k+1}/1 \le k \le \tau\} \\ &\cup\{x_ky_k/1 \le k \le \tau\} \cup \{x_ky_k'/1 \le k \le \tau\} \\ &\cup\{x_ky_k''/1 \le k \le \tau\} \cup \{x_2z_{\tau+1}, x_1x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 10\tau + 1\}$ by

$$\begin{aligned} h(x_k) &= 10k - 7 \text{ for } 1 \le k \le \tau \\ h(y_k) &= 10k - 5 \text{ for } 1 \le k \le \tau \\ h(y_k') &= 10k - 3 \text{ for } 1 \le k \le \tau \\ h(y_k'') &= 10k - 1 \text{ for } 1 \le k \le \tau \\ h(z_k) &= 10k - 9 \text{ for } 1 \le k \le \tau + 1 \end{aligned}$$

Then the edge labels are

$$\begin{aligned} h(z_k x_k) &= 10k-8 \text{ for } 1 \leq k \leq \tau \\ h(x_1 z_2) &= \tau+3 \\ h(x_k z_{k+1}) &= 10k-2 \text{ for } \tau-5 \leq k \leq \tau \\ h(x_k y_k) &= 10k-6 \text{ for } 1 \leq k \leq \tau \\ h(x_k y'_k) &= 10k-4 \text{ for } 1 \leq k \leq \tau \\ h(x_1 y''_1) &= \tau+1 \\ h(x_k y''_k) &= 10k-3 \text{ for } 2 \leq k \leq \tau \\ h(z_1 x_{\tau-2}) &= 6\tau+1 \\ h(x_2 z_{\tau+1}) &= 10\tau-1 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.13.1
For instance,

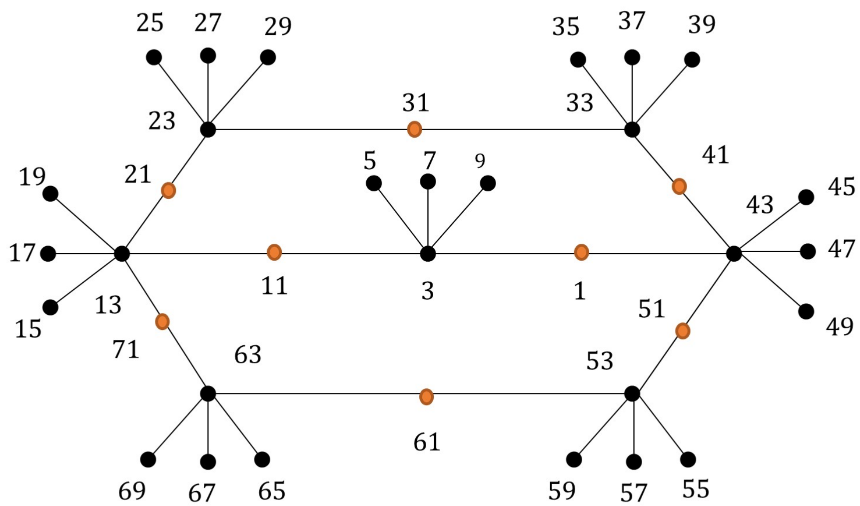

Figure 18. A Super lehmer-3 mean labeling of Subdividing all the path vertices of theta graph $\theta_7 \bigodot \bar{K_3}$

Theorem 2.14
The graph G obtained by subdividing all the path vertices of theta graph $\theta_\tau \bigodot K_2$ is a super lehmer - 3 mean graph.

Proof
Let $x_1 x_2 \dots x_\tau$ be the vertices of theta in G.

Let $y_1 y_2 \dots y_\tau$ and $y'_1 y'_2 \dots y'_\tau$ be the adjacent vertices attached at $x_1 x_2 \dots x_\tau$ respectively.

Let $z_1 z_2 \dots z_{\tau+1}$ be the vertices obtained by subdividing all the path edges $x_k x_{k+1}$, $x_1 x_{\tau-2}$ and $x_2 x_\tau$ respectively.

Let $V(G) = \{x_1 x_2 \dots x_\tau, y_1 y_2 \dots y_\tau, y'_1 y'_2 \dots y'_\tau, z_1 z_2 \dots z_{\tau+1}\}$

$$\begin{aligned} E(G) &= \{z_k x_k / 1 \leq k \leq \tau\} \cup \{x_k z_{k+1} / 1 \leq k \leq \tau\} \\ &\cup \{x_k y_k / 1 \leq k \leq \tau\} \cup \{x_k y'_k / 1 \leq k \leq \tau\} \\ &\cup \{y_k y'_k / 1 \leq k \leq \tau\} \cup \{x_2 z_{\tau+1}, x_1 x_{\tau-2}\} \end{aligned}$$

Define a mapping $h : V(G) \to \{1, 2, \ldots, 8\tau + 2\}$ by

$$\begin{aligned}
h(x_k) &= 8k - 5 \text{ for } 1 \leq k \leq \tau - 4 \\
h(x_k) &= 8k - 4 \text{ for } \tau - 3 \leq k \leq \tau \\
h(y_k) &= 8k - 3 \text{ for } 1 \leq k \leq \tau - 4 \\
h(y_k) &= 8k - 2 \text{ for } \tau - 3 \leq k \leq \tau \\
h(y'_k) &= 8k - 1 \text{ for } 1 \leq k \leq \tau - 4 \\
h(y'_k) &= 8k \text{ for } \tau - 3 \leq k \leq \tau \\
h(z_k) &= 8k - 7 \text{ for } 1 \leq k \leq \tau - 4 \\
h(z_k) &= 8k - 6 \text{ for } \tau - 3 \leq k \leq \tau + 1
\end{aligned}$$

Then the edge labels are

$$\begin{aligned}
h(z_k x_k) &= 8k - 6 \text{ for } 1 \leq k \leq \tau - 4 \\
h(z_k x_k) &= 8k - 5 \text{ for } \tau - 3 \leq k \leq \tau \\
h(x_k z_{k+1}) &= 8k \text{ for } 1 \leq k \leq \tau \\
h(x_k y_k) &= 8k - 4 \text{ for } 1 \leq k \leq \tau - 4 \\
h(x_k y_k) &= 8k - 3 \text{ for } \tau - 3 \leq k \leq \tau \\
h(y_k y'_k) &= 8k - 1 \text{ for } 1 \leq k \leq \tau \\
h(x_k y'_k) &= 8k - 2 \text{ for } 1 \leq k \leq \tau \\
h(z_1 x_{\tau-2}) &= 5\tau + 1 \\
h(x_2 z_{\tau+1}) &= 8\tau + 1
\end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

Example 2.14.1
For instance,

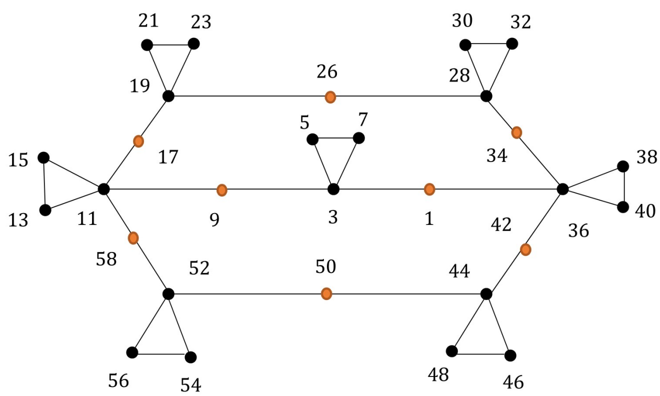

Figure 19. A Super lehmer-3 mean labeling of Subdividing all the path vertices of theta graph $\theta_7 \odot K_2$

3. Harmonic Mean Labeling

In this section, we investigate the harmonic mean labeling of theta graphs.

Theorem 3.1
θ_τ is a harmonic mean graph.

Proof
Let $G = \theta_\tau$.
Let $x_1 x_2 \dots x_\tau$ be the vertices of theta in G.
Let $V(G) = \{x_1 x_2 \dots x_\tau\}$
$E(G) = \{x_k x_{k+1} / 1 \leq k \leq \tau - 1\} \cup \{x_2 x_\tau, x_1 x_{\tau-2}$.

Define a mapping $h : V(G) \to \{1, 2, \dots, \tau + 1\}$ by

$$h(x_k) = k \text{ for } 1 \leq k \leq \tau$$

Then the edge labels are

$$\begin{aligned} h(x_1 x_2) &= 1 \\ h(x_k x_{k+1}) &= k + 1 \text{ for } \tau - 5 \leq k \leq \tau - 4 \\ h(x_k x_{k+1}) &= k + 2 \text{ for } \tau - 3 \leq k \leq \tau - 1 \\ h(x_2 x_\tau) &= \tau - 2 \\ h(x_1 x_{\tau-2}) &= \tau - 5 \end{aligned}$$

Hence h is a super lehmer-3 mean labeling of G. □

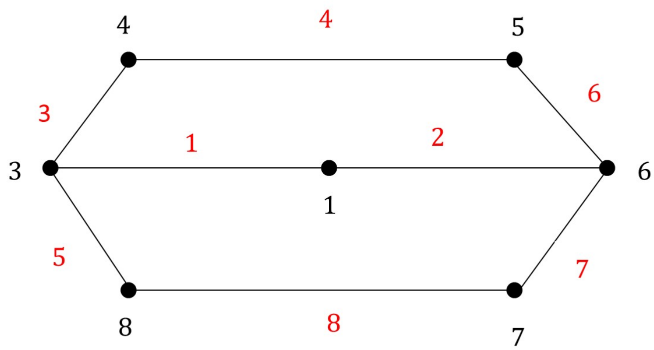

Figure 20. Harmonic mean labeling of θ_7

In this future, we investigate the harmonic mean labeling of theta related graphs like as $\theta_\tau \odot \bar{K}_1$, $\theta_\tau \odot \bar{K}_2$, $\theta_\tau \odot \bar{K}_3$, $\theta_\tau \odot K_2$ and etc...

4. Conclusion

It is very interesting to find whether a graph admits super lehmer-3 mean labeling or not. We present fourteen new results on super lehmer-3 mean labeling of theta related graphs. Then one special result investigated in harmonic mean labeling graph. In future, we investigate the graphs like as triangular snake, quadrilateral snake, dragon graph, and etc... The investigation about similar results for various graphs families is an open area of research.

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