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Study of Quantile-based Interval Inaccuracy Measure

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Abstract There are many models for which quantile function are available in tractable form, though their distribution functions are not available in explicit form. Recently researchers have great interest to study the quantile-based Interval entropy measures. In the present paper, we introduce quantile-based inaccuracy measure for doubly truncated random variable and study its properties. We also discuss some characterization results of this proposed measure. Further we propose and study the quantile version of Kullback -Leibler divergence measure for doubly truncated random variable. Finally, We discuss some results of proposed Kullback -Leibler divergence measure.

Keywords Interval entropy, Interval inaccuracy measure, Quantile entropy, Quantile divergence measure, Characterization result.

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1. Introduction

The measure of uncertainty associated with a probability distribution is known as entropy, which was introduced by Shannon [26]. In a sample space with corresponding probabilities $p_1, p_2..., p_n$, there are n mutually exclusive and exhaustive events $E_1, ...E_n$. The entropy for these events is $H(p) = -\sum_{i=1}^{n} p_i \log p_i$, where $p = (p_1, ...p_n)$. Kerridge [8] introduced a generalization of entropy was inaccuracy measure $I(p, q) = -\sum_{i=1}^{n} p_i \log q_i$ of two probability distributions where q_i is the probability that an experimenter assigned to the event and p_i represents the genuine probability of the ith event, then $p = (p_1, p_2...p_n)$ and $q = (q_1, q_2...q_n)$. Nath [23] further expanded this measure to include continuous random variables. The inaccuracy measure between two random variable X and Y is

$$I(X, Y) = -\int_0^\infty f(x) \log g(x) dx, \quad (1)$$

where $f(x)$ is the true probability density function of a random variable and $g(x)$ is an assigned, assumed, or approximated density function. Data for used objects is frequently reduced in reliability and life testing, making it unsuitable for measuring inaccuracy (1). Nair and Gupta [21] therefore expanded (1) for the residual inaccuracy for a pair of random variables. Inaccuracy measures for two past lifetime random variables were presented by Kumar et al. [15]. Chanchal kundu et al. [17] introduce the interval inaccuracy measure for doubly truncated random variables $(X|t_1 \leq X \leq t_2)$ and $(Y|t_1 \leq Y \leq t_2)$ are considered, where $G(u) < G(v)$ and $(t_1, t_2) \in D = (u, v) \in \Re^2 : F(u) < F(v)$ are absolutely continuous. Next, in the interval (t_1, t_2), the interval inaccuracy measure of X and Y is defined as

$$I_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_2)} \log \frac{g(x)}{G(t_2) - G(t_1)} dx. \quad (2)$$

For information and results we studied Di Crescenzo et al.[3], Taneja et al. [29], Sunoj et al. [27], Misagh and Yari [19].

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Kullback–Leibler divergence is a fundamental concept in information theory, statistics, reliability analysis, and related domains to quantify the disrimination between two distributions. Kullback and Leibler [10] studied and proposed Kullback-Leibler divergence between two random variable X and Y as

$$I_{X,Y} = \int_0^{\infty} \left(\log \frac{f(x)}{g(x)} \right) f(x) dx. \tag{3}$$

In [4] Ebrahimi and Kirmani proposed and studied the residual Kullback-Leibler discrimination. Di Crescenzo and Longobardi [2] gives Kullback-Leibler distance for two past lifetime. Yari et al. [18] gives the distance between random lifetime X and Y at the interval (t_1, t_2) is the Kullback-Leibler discrimination measure as

$$KL_{X,Y}(t_1, t_2) = \int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \left(\frac{\frac{f(x)}{F(t_2)-F(t_1)}}{\frac{g(x)}{G(t_2)-G(t_1)}} \right) dx. \tag{4}$$

The distribution function forms the basis for both theoretical studies and practical implementations involving these information measures, yet they might not be appropriate in circumstances where the distribution is not analytically tractable. Quantile function-based approaches have been demonstrated to be an effective and comparable substitute for distribution in the modelling and analysis of statistical data. [5] Gilchrist and Nair et al. [20]. The quantile function of a random variable X is given as

$$Q_1(u) = F^{-1}(u) = inf(x|F(x) \geq u), \;\; 0 \leq u \geq 1. \tag{5}$$

Recent attention has been generated by the research of information measures based on quantile function. Sunoj et al. [6] studied and introduced quantile-based inaccuracy measures, including quantile-based past and residual inaccuracy. For two non-negative, absolutely continuous random variables X and Y with respective quantile functions Q_1 and Q_2 the quantile-based inaccuracy measure is defined as

$$I_Q(X, Y) = -\int_0^1 \log g(Q_1(u)) du. \tag{6}$$

And the quantile-based past inaccuracy measure and quantile-based residual inaccuracy measure is given as

$$I_Q(X, Y, u) = -\int_u^1 \frac{f(Q_1(p))}{\bar{F}(Q_1(u))} \log \frac{g(Q_1(p))}{\bar{G}(Q_1(u))} d(Q_1(p)), \tag{7}$$

$$\bar{I}_Q(X, Y; v) = -\int_0^v \frac{f(Q_1(p))}{F(Q_1(v))} \log \frac{g(Q_1(p))}{G(Q_1(v))} d(Q_1(p)). \tag{8}$$

Sankaran et al.[25] studied and proposed quantile based Kullback Leibler measure, quantile based residual and quantile based past Kullback Leibler divergence measure defined as

$$KL_{X,Y}(Q) = \int_0^1 \log \left(\frac{f(Q_1)(u)}{g(Q_1(u))} \right) f(Q_1(u)) dQ_1(u), \tag{9}$$

$$KL_{X,Y}(Q_1(u)) = -\frac{1}{1-u} \int_u^1 \log((q_1(p))g(Q_1(p)))dp + \log \left(\frac{1 - G(Q_1(u))}{1-u} \right), \tag{10}$$

$$\bar{KL}_{X,Y}(Q_1(v)) = -\frac{1}{v} \int_0^v \log((q_1(p))g(Q_1(p)))dp + \log \left(\frac{G(Q_1(u))}{u} \right). \tag{11}$$

Quantile-based information measures have been studied by many researchers, Baratpour et al. [1], Kumar et al. [12], Kumar et al. [13], Kittaneh [9], Kayal et al. [7].

Many researchers have indicated that the quantile-based approach serves as an alternative to the conventional distribution function approach. In their study, Sankaran et al. [25] developed a quantile-based version of the Kullback-Leibler divergence and investigated its characteristics, with a particular focus on its application to lifetime data analysis. Sunoj et al. [6] proposed a quantile version of inaccuracy measure also discuss many properties of quantile-based inaccuracy measure. However a corresponding study of quantile inaccuracy for doubly truncated random variable and quantile-based Kullback Leibler divergence measure for doubly truncated random variable has not been covered in previous research. This motivates the use of quantile-based approaches, particularly for defining measures like entropy and divergence. Recent research has highlighted the value of quantile-based entropy measures, but there has been limited exploration of inaccuracy and divergence measures in the context of truncated data. This paper aims to fill this gap by developing and studying quantile-based inaccuracy and Kullback–Leibler divergence measures for doubly truncated random variables, offering new tools for statistical inference in such constrained data environments. More results and applications are refer to Kumar and Dangi [16], Zamani and Madadi [31], Wang and Kang [30], Tehlan and Kumar [28], Kumar et al.[14].
In the following paper, We outline our approach as follows. In Section 2, we propose and study quantile- based inaccuracy for doubly truncated random variable and some of its important properties. Some characterization result also discuss. In Section 3, we study quantile-based Kullback-Leibler divergence for doubly truncated random variable. While explicit forms of their distribution functions are lacking, many models have quantile functions that are available in a feasible form also discuss.

2. Quantile-based Interval Inaccuracy

Let X and Y be two random variables with quantile functions $Q_1(u)$ and $Q_2(u)$, and distribution functions F and G, respectively. For a continuous distribution function F, it holds that $F(Q_1(u)) = u$. Differentiating both sides with respect to u, we get

$$q_1(u) \cdot f(Q_1(u)) = 1. \tag{12}$$

Here, $f(Q_1(u))$ is referred to as the density quantile function, and $q_1(u) = Q_1'(u)$ is the quantile density function corresponding to the quantile function $Q_1(u)$. Next, define $Q_3(u) = Q_2^{-1}(Q_1(u))$, which represents the relative inverse quantile function between the distributions G and F, the quantile function of $F(G^{-1}(\cdot))$. Differentiating $Q_3(u)$, we obtain $q_3(u) = Q_3'(u) = g(Q_1(u)) \cdot q_1(u)$ where g is the density function of G, evaluated at $Q_1(u)$. This setup forms the basis for defining a quantile-based version of the interval inaccuracy measure, which compares the two distributions using their quantile and density functions.

Definition 2.1
Consider two absolutely continuous, non-negative random variables, X and Y, with *QFs* Q_1 and Q_2, respectively. Then the quantile-based interval inaccuracy measure is defined as

$$\begin{aligned} I_Q(X, Y; u_1, u_2) &= -\int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{g(Q_1(u))}{G(Q_1(u_2)) - G(Q_1(u_1))} q_1(u) du. \\ &= -\int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{g(Q_1(u))}{(Q_3(u_2)) - (Q_3(u_1))} q_1(u) du. \\ &= \log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log \frac{q_3(u)}{q_1(u)} du. \end{aligned} \tag{13}$$

Let T be a non-negative continuous random variable representing the time until an event occurs. Let $f(t)$ be the probability density function, $F(t)$ be the cumulative distribution function $\bar{F}(t) = 1 - F(t)$ be the survival function. The hazard rate at time t is defined as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\bar{F}(t)}.$$

This function describes the instantaneous probability of failure at time t, given survival up to that point. If the quantile function $Q(u)$ is known, the hazard rate can also be expressed in terms of quantile function as

$$h(Q(u)) = \frac{1}{q(u)(1-u)}$$

where $Q(u)$ is the quantile function $q(u) = Q'(u)$ is the quantile density function. Now we define hazard quantile and reversed hazard quantile functions at time of interval u_1 to u_2 of random variable X are defined as $H_1(u_1, u_2) = [q_1(u)(u_2 - u_1)]^{-1}$ and $\bar{H}_1(u_1, u_2) = [q_1(u)(u_1 - u_2)]^{-1}$. These are equivalent to the hazard and reverse hazard rate function in terms of the *QF*. Denote $H_3(u_1, u_2)$ and $\bar{H}_3(u_1, u_2)$ the hazard quantile and reversed hazard quantile function corresponding to the quantile function Q_3 . In parallel to these hazard quantile function Q_3 we defined $H_3(u_1, u_2) = [q_3(u)(u_2 - u_1)]^{-1}$ and $\bar{H}_3(u_1, u_2) = [q_3(u)(u_1 - u_2)]^{-1}$. The quantile-based inaccuracy measures (13) has been expressed in terms of hazard quantile and reversed hazard quantile functions.

$$\begin{aligned} I_Q(X, Y; u_1, u_2) &= \log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log \frac{H_1(u_1, u_2)}{H_3(u_1, u_2)} du, \\ &= \log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log \frac{\bar{H}_1(u_1, u_2)}{\bar{H}_3(u_1, u_2)} du. \end{aligned}$$

The quantile version of proportional hazard model is $Q_y(u) = Q_x(1 - (1-u)^{\frac{1}{\theta}})$ and the quantile version of reverse hazard model is $Q_x(u) = Q_y(u^\theta)$ respectively. The quantile-based inaccuracy measures (13) has been expressed in terms of proportional hazard rate model *(PHRM)* , given as follows

$$\begin{aligned} I_Q(X, Y; u_1, u_2) = \theta - \log\theta + \log[\bar{F}^\theta(Q_1(u_1) - \bar{F}^\theta(Q_1(u_2))] + \\ \frac{\theta[\bar{F}(Q_1(u_2)) \log \bar{F}(Q_1(u_2)) - \bar{F}(Q_1(u_1)) \log \bar{F}(Q_1(u_1))]}{\bar{F}(Q_1(u_1)) - \bar{F}(Q_1(u_2))} \\ - \frac{1}{\bar{F}(Q_1(u_1)) - \bar{F}(Q_1(u_2))} \int_{u_1}^{u_2} f(Q_1(u)) \log \lambda_F(u) du. \end{aligned}$$

The quantile-based inaccuracy measures (13) has been expressed in terms of proportional reverse hazard rate model *(QPRHRM)*, given as follows

$$\begin{aligned} `I_Q(X, Y; u_1, u_2) = \theta - \log\theta + \log[F^\theta(u_2) - F^\theta(u_1)] - \frac{\theta[F(Q_1(u_2)) \log F(Q_1(u_2)) - F(Q_1(u_1)) \log F(Q_1(u_1))]}{F(Q_1(u_2)) - F(Q_1(u_1))} \\ - \frac{1}{F(Q_1(u_2)) - F(Q_1(u_1))} \int_{u_1}^{u_2} f(Q_1(u)) \log \phi_F(u) du. \end{aligned}$$

2.1. Properties of Inaccuracy Measures

In general, the measure of interval inaccuracy based on quantiles, as detailed in (12), does not maintain monotonicity with respect to u_1, u_2. The next theorem presents the upper (lower) bounds for $I_Q(X, Y; u_1, u_2)$ based on monotone properties of it. Additionally helpful are bounds in cases when computing the real measure is challenging.

Theorem 2.1
The quantile-based interval inaccuracy measure for $I_Q(X, Y; u_1, u_2)$ is increasing (decreasing) in u_1 and u_2 if one is fixed other is run if and only if

$$I_Q(X, Y; u_1, u_2) \geq (\leq) \frac{(u_2 - u_1) q_3(u_1)}{Q_3(u_2) - Q_3(u_1)} - \log(\frac{q_3(u_1)}{q_1(u_1)(Q_3(u_2) - Q_3(u_1))}).$$

Proof
From (13), we have

$$I_Q(X,Y;u_1,u_2) = \log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log\frac{q_3(u)}{q_1(u)}du.$$

This can be rewritte as

$$(u_2-u_1)I_Q(X,Y;u_1,u_2) = (u_2-u_1)\log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2-u_1}\int \log\frac{q_3(u)}{q_1(u)}du. \quad (14)$$

Taking the derivative of both sides of (14) with respect to u_1 the result as

$$I_Q^{'}(u_1,u_2) = \frac{1}{(u_2-u_1)}[I_Q(X,Y;u_1,u_2) - \frac{(u_2-u_1)q_3(u_1)}{Q_3(u_2)-Q_3(u_1)} + \log(\frac{H_1(u_1,u_2)q_3(u_1)(u_2-u_1)}{Q_3(u_2)-Q_3(u_1)})], \quad (15)$$

Taking the derivative of both sides of (14) with respect to u_2 the result as

$$I_Q^{'}(u_1,u_2) = \frac{1}{(u_2-u_1)}[I_Q(X,Y;u_1,u_2) - \frac{(u_2-u_1)q_3(u_2)}{Q_3(u_2)-Q_3(u_1)} + \log(\frac{H_1(u_1,u_2)q_3(u_2)(u_2-u_1)}{Q_3(u_2)-Q_3(u_1)})], \quad (16)$$

where $H_1(u_1,u_2)$ and $H_2(u_1,u_2)$is hazard quantile function of X. Therefore, given the assumption, the desired result follows immediately. The reverse result can be obtained by reversing the process and hence is omitted. □

We provide the idea of quantile-based interval inaccuracy ratio in this section.

Proposition 2.1
For X and Y it is given by

$$I_QR(X,Y;u_1,u_2) = \frac{I_Q(X,Y;u_1,u_2)}{H(Q;u_1,u_2)} \quad (17)$$

where $H(Q;u_1,u_2)$ is the quntile Shannon interval entropy, $I_QR(X,Y;u_1,u_2)$ provide dimensional measure of closeness between X and Y if $Q_1 = Q_2$ then $I_QR(X,Y;u_1,u_2) = 1$. Further analog the quantile-based interval inaccuracy measure is not symmetric that is $I_QR(X,Y;u_1,u_2) \neq I_QR(X,Y;u_2,u_1)$.
Next we find the decomposition of quantile-based inaccuracy in the terms or past quantile inaccuracy, residual quantile inaccuracy and $I_Q(X,Y;u_1,u_2)$.

Proposition 2.2
For arbitrary lifespan of (X,Y) the function $I_Q(X,Y)$ can be expressed as.

Proof

$$\begin{aligned} I_Q(X,Y) &= F(Q_1(u_1))\bar{I}_Q(X,Y;u_1) + [F(Q_1(u_2)) - F(Q_2(u_1))]I_Q(X,Y;u_1,u_2) + \bar{F} \\ &(Q_1(u_2))I_Q(X,Y;u_2) - [\log G(Q_1(u_1)) + \log\bar{G}(Q_1(u_2)) + \log G(Q_1(u_2)) - G(Q_1(u_1))]. \end{aligned}$$

Four parts make up the quantile inaccuracy measure. i) the random variable's inaccuracy measure truncated above u_1. ii) the inaccuracy measure in the range (u_1,u_2), assuming that the item failed before to u_2 but subsequent to u_1. iii) the measure of inaccuracy for random variables truncated below u_2; and iv) the measure of inaccuracy determines if the item has failed before u_1, in between u_1 and u_2, or after u_2. When $u_1 = u_2 = u$ then above can be expressed as

$$I_Q(X,Y) = F(Q_1(u))\bar{I}_Q(X,Y) + \bar{F}(Q_1(u))I_Q(X,Y) - \log G(Q_1(u)) - \log\bar{G}(Q_1(u)).$$

□

The following section presents the bounds of the quantile-based interval inaccuracy.
On differentiating (13) with respect to u_1 and u_2, we get

$$\frac{\partial I_Q(X,Y;u_1,u_2)}{\partial u_1} = H_1^X(u_1,u_2)[I_Q(X,Y;u_1,u_2) + \log H_1^Y(u_1,u_2)] + H_1^Y(u_1,u_2), \tag{18}$$

and

$$\frac{\partial I_Q(X,Y;u_1,u_2)}{\partial u_2} = H_2^X(u_1,u_2)[I_Q(X,Y;u_1,u_2) + \log H_2^Y(u_1,u_2)] + H_2^Y(u_1,u_2). \tag{19}$$

When $I_Q(X,Y;u_1,u_2)$ is increasing in each of u_1 and u_2 while keeping the other fixed, then equation (18) and (19) together imply that

$$\frac{H_1^Y(u_1,u_2)}{H_1^X(u_1,u_2)} - \log H_1^Y(u_1,u_2) \le I_Q(X,Y;u_1,u_2) \le \frac{H_2^Y(u_1,u_2)}{H_2^X(u_1,u_2)} - \log H_2^Y(u_1,u_2). \tag{20}$$

The follwing preposition provides the bounds for the quantile interval inaccuracy measure. The proof is derived from (13) and is therefore omitted.

Proposition 2.3
If $g(Q(u))$ is decreasing in u, then

$$-\log H_1^Y(u_1,u_2) \le -I_Q(X,Y;u_1,u_2) \le -\log H_2^Y(u_1,u_2). \tag{21}$$

For increasing $g(Q(u))$ the above inequalities are reversed.
In the next two theorems, we gives upper and lower bounds for the quantile interval inaccuracy, based on the monotonic behavior of the quantile generalized failure rate function for Y note that $\frac{\partial H_1^Y(u_1,u_2)}{\partial u_1} = H_1^Y(u_1,u_2)(\frac{g'(Q_1(u_1))}{g(Q_1(u))} + H_1^Y(u_1,u_2))$ and $\frac{\partial H_1^Y(u_1,u_2)}{\partial u_2} = -H_1^Y(u_1,u_2)H_2^Y(u_1,u_2))$.

Theorem 2.2
For fixed u_2,
(i) if $H_1^Y(u_1,u_2)$ is decreasing in u_1 then $I_Q(X,Y;u_1,u_2) \ge -\log H_1^Y(u_1,u_2)$;
(ii) if $H_1^Y(u_1,u_2)$ is increasing in u_1, then

$$I_Q(X,Y;u_1,u_2) \le -\log H_1^Y(u_1,u_2) - \int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{G(Q_1(u_2)) - G(Q_1(x))}{G(Q_1(u_2)) - G(Q_1(u_1))} d(Q_1(u)). \tag{22}$$

Proof
Observe that (13) can be rewritten as

$$I_Q(X,Y;u_1,u_2) = -\int_{u_1}^{u_2} \frac{f(Q_1(u)\log H_1^Y(u_1,u_2)d(Q_1(u_1))}{F(Q_1(u_2)) - F(Q_1(u_1))} - \int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{G(Q_1(u_2)) - G(Q_1(x))}{G(Q_1(u_2)) - G(Q_1(u_1))} d(Q_1(u)). \tag{23}$$

For $1 < x$, $\log \frac{G(Q_1(u_2))-G(Q_1(x))}{G(Q_1(u_2))-G(Q_1(u_1))} \le 0$, and $H_1^Y(x,u_2) \le \log H_1^Y(u_1,u_2)$ if $H_1^Y(u_1,u_2)$ is decreasing in u_1 hence from (23) we obtain,

$$\begin{aligned} I_Q(X,Y;u_1,u_2) &\ge -\int_{u_1}^{u_2} \frac{f(Q_1(u)}{F(Q_1(u_2)) - F(Q_1(u_1))} \log H_1^Y(x,u_2)d(Q_1(u_1)) \\ &\ge -\log H_1^Y(u_1,u_2). \end{aligned}$$

(ii) The second section proceeds naturally by utilising the knowledge that $\log H_1^Y(x, u_2) \geq \log H_1^Y(u_1, u_2)$.
Let us suppose that, for a fixed u_2, we aim to characterize the distributions that achieve the upper and lower bounds of the quantile-based interval inaccuracy measure, as stated in the preceding theorem. $I_Q(X, Y; u_1, u_2) = \log H_1^Y(u_1, u_2)$ and $H_1^Y(u_1, u_2)$ is decreasing in u_1. Then by differentiating with respect u_1 and using (18), we get a result $\frac{g'(Q_1(u_1))}{g(Q(u_1))} = 0$, which in turn gives via $\frac{\partial}{\partial u_1} H_1^Y(u_1, u_2) = (H_1^Y(u_1, u_2))^2$ that $H_1^Y(u_1, u_2)$ cannot be decreasing, constant, or zero. Consequently, the inequality in part (i) of the preceding theorem must be strict. (ii) Next, we will assume the inequality in the second part of the theorem is established i.e.,

$$I_Q(X, Y; u_1, u_2) = -\log H_1^Y(u_1, u_2) - \int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{G(Q_1(u_2)) - G(Q_1(x))}{G(Q_1(u_2)) - G(Q_1(u_1))} d(Q_1(u))$$

and $H_1^Y(u_1, u_2)$ is increasing in u_1. Differentiating with respect to u_1 and applying (18), we obtain, $H_1^Y(u_1, u_2) + \frac{g'(Q_1(u_1))}{g(Q(u_1))} = 0$, which gives that $H_1^Y(u_1, u_2) = k(Q(u_2))$ which is either a function of u_2 only or a constant. While the proof for the next theorem is similar to the previous one, we provide an overview of the proof here for completeness. □

Theorem 2.3
For fixed u_1,
(i) if $H_2^Y(u_1, u_2)$ is increasing in u_2 then $I_Q(X, Y; u_1, u_2) \geq -\log H_2^Y(u_1, u_2)$;
(ii) if $H_2^Y(u_1, u_2)$ is decreasing in u_2, implies that

$$I_Q(X, Y; u_1, u_2) \leq -\log H_2^Y(u_1, u_2) - \int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{G(Q_1(x)) - G(Q_1(u_1))}{G(Q_1(u_2)) - G(Q_1(u_1))} d(Q_1(u)). \tag{24}$$

Proof
Observe that (13) can be expressed as

$$I_Q(X, Y; u_1, u_2) = -\int_{u_1}^{u_2} \frac{f(Q_1(u) \log H_2^Y(u_1, x) d(Q_1(u_1))}{F(Q_1(u_2)) - F(Q_1(u_1))} - \int_{u_1}^{u_2} \frac{f(Q_1(u))}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \frac{G(Q_1(x)) - G(Q_1(u_1))}{G(Q_1(u_2)) - G(Q_1(u_1))} d(Q_1(u)). \tag{25}$$

The remaining proof follows from (25) by applying the fact that, for $x < u_2$, $\log \frac{G(Q_1(x)) - G(Q_1(u_1))}{G(Q_1(u_2)) - G(Q_1(u_1))} \leq 0$, and $H_2^Y(u_1, x) \leq \log H_2^Y(u_1, u_2)$ if $H_2^Y(u_1, u_2)$ is increasing (decreasing) in u_2. By using the aforementioned analogy, we can demonstrate that the equality in the theorem's first section will not hold. The equality will apply to the second section if $H_{u_1}^{u_2} = k$. □

2.2. *Characterization Theorem*

In this section, we analyze a class of continuous lifetime distributions and establish characterization theorems using the quantile generalized failure rate function, the quantile geometric vitality function, and the quantile inaccuracy measure, under the frameworks of the quantile proportional hazard rate model and the quantile proportional reverse hazard rate model.
The general characterization problem involves determining the conditions under which the quantile interval inaccuracy measure uniquely determines the underlying distribution function. In what follows, we present and discuss a corresponding characterization result.

Theorem 2.4
For two absolutely continuous non-negative random variable X and Y, if $I_Q(X, Y; u_1, u_2)$ increasing with u_1, (when u_2 is fixed) and decreases with u_2 (when u_1 is fixed), and if $H_i^Y(u_1, u_2) = \theta H_i^X(u_1, u_2)$; for $\theta > 0$; and $i = 1, 2$ respectively. Then $I_Q(X, Y; u_1, u_2)$ uniquely determines the distribution function.

Proof
Using (18) and (19), we derive

$$\frac{\partial}{\partial u_1} I_Q(X,Y;u_1,u_2) = H_1^X(u_1,u_2)[I_Q(X,Y;u_1,u_2) + \log\theta - \theta + \log H_1^X(u_1,u_2)] \tag{26}$$

and,

$$\frac{\partial}{\partial u_2} I_Q(X,Y;u_1,u_2) = -H_2^X(u_1,u_2)[I_Q(X,Y;u_1,u_2) + \log\theta - \theta + \log H_2^X(u_1,u_2)]. \tag{27}$$

Then for any fixed u_1 and arbitrary u_2, $H_1^X(u_1,u_2)$ is a positive solution of the equation $\eta(x_{u_2}) = 0$, where

$$\eta(x_{u_2}) = x_{u_2}[I_Q(X,Y;u_1,u_2) + \theta - \theta + \log\eta(x_{u_2})] - \frac{\partial}{\partial u_1} I_Q(X,Y;u_1,u_2). \tag{28}$$

Similarly, for fixed u_2 and arbitrary u_1, H_2^X is a positive solution of the equation $\zeta(y_{t_1}) = 0$, where

$$\zeta(y_{u_1}) = y_{u_1}[I_Q(X,Y;u_1,u_2) + \log\theta - \theta\log y_{u_1}] + \frac{\partial}{\partial u_2} I_Q(X,Y;u_1,u_2). \tag{29}$$

Taking the derivatives of $\eta(x_{u_2})$ with respect to x_{u_2} and $\zeta(y_{u_1})$ with respect to y_{u_1}, we get

$$\frac{\partial\eta(x_{u_2})}{\partial x_{u_2}} = I_Q(X,Y;u_1,u_2) + \log\theta - \theta + 1 + \log\eta(x_{u_2}). \tag{30}$$

and

$$\frac{\partial\zeta(y_{u_1})}{\partial y_{u_1}} = I_Q(X,Y;u_1,u_2) + \log\theta - \theta + 1 + \log\eta(y_{u_1}). \tag{31}$$

Further second order derivative are $\frac{\partial^2\eta(x_{u_2})}{\partial x_{u_2}^2} = \frac{1}{x_{u_2}} > 0$ and $\frac{\partial^2\zeta(y_{u_1})}{\partial y_{u_1}^2} = \frac{1}{y_{u_1}} > 0$. So, both of $\eta(x_{u_2})$ and $\zeta(y_{u_1}$ minimized at $x_{u_2} = \exp[\theta - \log\theta - 1 - I_Q(X,Y;u_1,u_2)] = y_{u_1}$ respectively. Here $\eta(0) = -\frac{\partial}{\partial u_1} I_Q(X,Y;u_1,u_2) <$ 0; Since we assume that $I_Q(X,Y;u_1,u_2)$ is increasing in u_1 and also when $(x_{u_2} \to \infty)$, $\eta(x_{u_2}) \to \infty$. Similarly $\zeta(0) = \frac{\partial}{\partial u_2} I_Q(X,Y;u_1,u_2) < 0$, and $\zeta(y_{u_1}) \to \infty$ as $y_{u_1} \to \infty$. Therefore, both the equation $\eta(x_{u_2}) = 0, \zeta(x_{u_2} =$ 0 have unique positive solution $H_1^X(u_1,u_2)$ and $H_2^X(u_1,u_2)$, respectively. Thus, the proof is completed by utilizing the result that the *QGFR* function uniquely characterizes the distribution function, as demonstrated by Navarro and Ruiz [24]. □

Example 2.1
Suppose X and Y be follow Pareto II distribution with their quantile function respectively, $Q_1(u) = b((1-u)^{\frac{-1}{p_1}} - 1); p_1 > 0\ , b > 0$ and $Q_2(u) = b((1-u)^{\frac{-1}{p_1}} - 1); p_1 > 0\ , b > 0$. Then $Q_3(u) = 1 - (1-u)^{\frac{p_2}{p_1}}$ and $q_3(u) = \frac{p_2}{p_1}(1-u)^{\frac{p_2}{p_1}-1}$. Then from (13) we have

$$I_Q(X,Y;u_1,u_2) = \log(Q_3(u_2) - Q_3(u_1)) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \log\frac{q_3(u)}{q_1(u)} du.$$

$$I_Q(X,Y;u_1,u_2) = \log((1-u_1)^{\frac{p_2}{p_1}} - (1-u_2)^{\frac{p_2}{p_1}}) - \log(\frac{1}{p_2}) - \frac{1}{u_2 - u_1}(\frac{p_2+1}{p_1})((u_2\log(1-u_2) \\ - u_2 + \log(1-u_2)) - (u_1\log(1-u_1) + u_1 - \log(1-u_1))).$$

$$I_Q(X,Y;u_2,u_1) = \log((1-u_1)^{\frac{p_1}{p_2}} - (1-u_2)^{\frac{p_1}{p_2}}) - \log(\frac{1}{p_1}) - \frac{1}{u_2 - u_1}(\frac{p_1+1}{p_2}) \\ ((u_2\log(1-u_2) - u_2 + \log(1-u_2)) - (u_1\log(1-u_1) + u_1 - \log(1-u_1))).$$

This clear shows that the quantile-based interval inaccuracy measure is not symmetric (see Figure 1), that is in general, $I_Q(X,Y;u_1,u_2) \neq I_Q(Y,X;u_2,u_1)$.
In figure 1, Plotting the interval inaccuracy measures acquired in example (2.1) with regard to u_1 is done assuming $u_2 = 0.8$. The plot of $I_Q(X,Y;u_1,u_2), I_Q(Y,X;u_2,u_1)$ is shown by the curve line with the bold (thick). From the figure, we notice that the case considered in example (2.1), inaccuracy measure cross at the point when $u_1 = u_2 = 0.8$. We can see the measure $I_Q(X,Y;u_1,u_2)$ represent the bold line which is not monotonic in the interval of $(0,1)$ and $I_Q(X,Y;u_1,u_2) \geq I_Q(Y,X;u_2,u_1)$ at the interval of $(0, 0.8)$.

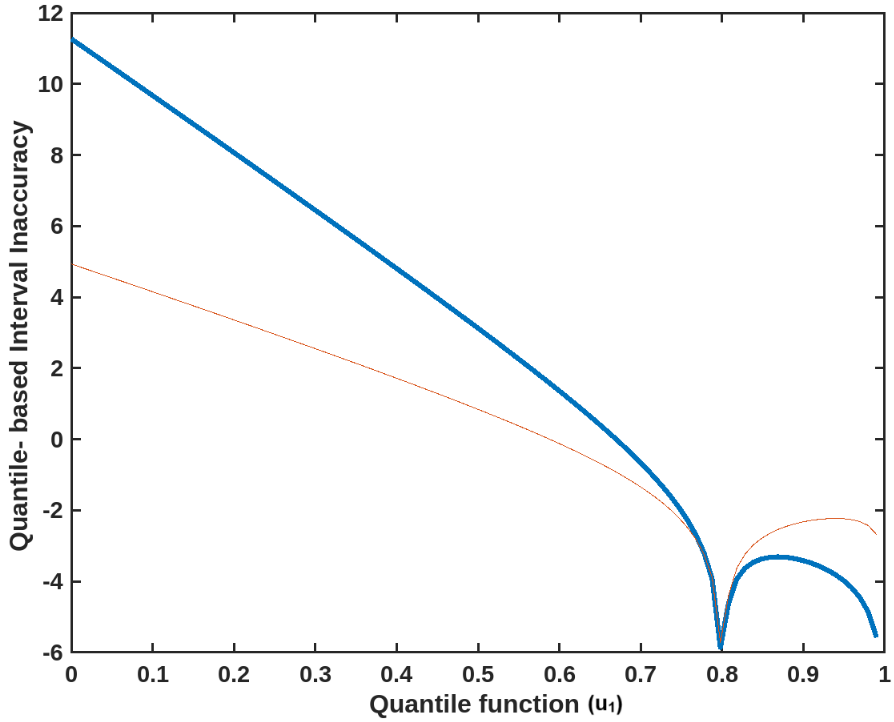

Figure 1. Quantile- based Interval Inaccuracy vs Quantile function

An example is provided below to illustrate the effectiveness of the quantile-based interval Inaccuracy measure in equation (13).

Example 2.2
X and Y be two non-negative exponential distribution with their quantile functions respectively $Q_1(u) = -\frac{1}{\lambda_1}\log(1-u)\ ;\lambda_1 > 0$ and $Q_2(u) = -\frac{1}{\lambda_2}\log(1-u)\ ;\lambda_2 > 0$, Then $Q_3(u) = Q_2^{-1}(Q_1(u)) = 1-(1-u)^{\frac{\lambda_2}{\lambda_1}}$ and $q_3(u) = \frac{\lambda_2}{\lambda_1}(1-u)^{\frac{\lambda_2}{\lambda_1}} - 1$. Thus from (13) we get

$$I_Q(X,Y;u_1,u_2) = \log\left((1-u_1)^{\frac{\lambda_2}{\lambda_1}} - (1-u_2)^{\frac{\lambda_2}{\lambda_1}}\right) - \log\left(\frac{\lambda_2}{\lambda_1}\right) - \left(\frac{1}{u_2-u_1}\right)\left(\frac{\lambda_2}{\lambda_1}-1\right)((u_2\log(1-u_2) - u_2 + \log(1-u_2)) - (u_1\log(1-u_1) + u_1 - \log(1-u_1))) + \log\left(\frac{1}{\lambda_1}\right) + \frac{1}{u_2-u_1}(\log(\frac{1}{1-u_2})u_2 - u_2 + \log(1-u_2) - \log(\frac{1}{1-u_1})u_1 + u_1 - \log(1-u_1)).$$

In figure 2, assuming different value of u_2, $\lambda_1 = 2, \lambda_2 = 1.2$ then the quantile-based interval inaccuracy measures attained in example (2.2) are mapped according to u_1. In figure 3, assuming different value of u_1, $\lambda_1 = 2, \lambda_2 = 1.2$ then the quantile-based interval inaccuracy measures attained in example (2.2) are mapped according to u_2.

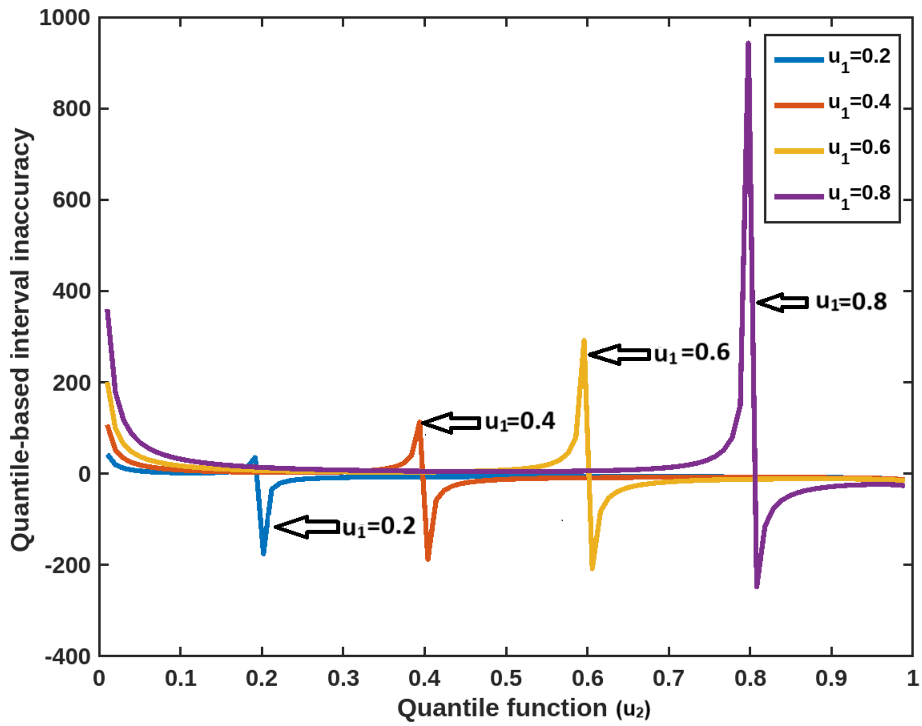

Figure 2. Quantile-based Interval Inaccuracy vs Quantile function

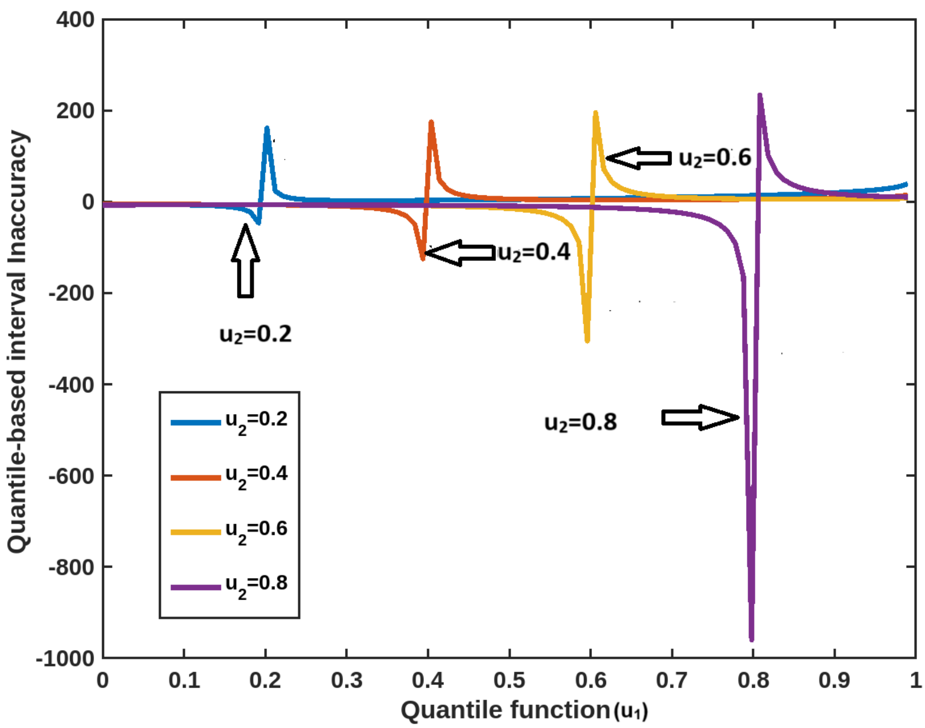

Figure 3. Quantile- based Interval Inaccuracy vs Quantile function

Example 2.3
Assume that there are two distribution function with quantile function $Q_1(u) = c_1 u^{\lambda_1}(1-u)^{-\lambda_2}\ \ ; c_1, \lambda_2 > 0$ and $Q_2(u) = c_2 u^{\frac{1}{\lambda_3}}\ \ ; c_2, \lambda_3 > 0$. Here $Q_1(u)$ and $Q_2(u)$ are the quantile function of Davis and Power distribution respectively $Q_3(u) = (\frac{c_1}{c_2})^{\lambda_3} u^{\lambda_1\lambda_3}(1-u)^{-\lambda_2\lambda_3}$. Then quantile-based interval inaccuracy (13) is given as

$$\begin{aligned} I_Q(X,Y;u_1,u_2) &= \log((\frac{c_1}{c_2})^{\lambda_3} u_2^{\lambda_3}(1-u_2)^{-\lambda_3} - (\frac{c_1}{c_2})^{\lambda_3}(u_1)^{\lambda_3}(1-u_1)^{-\lambda_3}) - \log(\lambda_3) + \log c_1 \\ &- \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log((\frac{c_1}{c_2})^{\lambda_3} u^{\lambda_3}(1-u)^{-\lambda_3}(\frac{1}{u}+\frac{1}{1-u})du \\ &+ \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log((1-u)+u(1-u)^{-2})du. \end{aligned}$$

Example 2.4
Assumed the QF of Govindarajulu and Inverted reciprocal exponential distribution as $Q_1(u) = 2u - u^2$ and $Q_2(u) = \frac{-\lambda}{\log u}, ; \lambda > 0$. Then $Q_3(u)$ and $q_3(u)$ can be obtained as $Q_3(u) = \exp(\frac{-\lambda}{2u-u^2})$, $q_3(u) = (\frac{2\lambda(1-u)}{(2u-u^2)^2})(\exp -(\frac{\lambda}{2u-u^2}))$. Substituting these values in (13) we obtain

$$\begin{aligned} I_Q(X,Y;u_1,u_2) &= \log(\exp(-\frac{\lambda}{2u_2-u^2}) - \exp(-\frac{\lambda}{2u_1-u^1})) - \log(2\lambda) \\ &- \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(\frac{(1-u)}{(2u-u^2)^2}\exp(-\frac{\lambda}{2u-u^2})du \\ &+ \log 2 + \frac{1}{u_2-u_1}(u_2\log(1-u_2) - u_2 + \log(1-u_2) - u_1\log(1-u_1) + u_1 - \log(1-u_1)). \end{aligned}$$

Example 2.5
Suppose X and Y be follow Pareto II distribution with quantile function respectively, $Q_1(u) = b((1-u)^{\frac{-1}{p_1}} - 1)$; $p_1 > 0\ , b > 0$ and $Q_2(u) = b((1-u)^{\frac{-1}{p_1}} - 1); p_1 > 0\ , b > 0$. Then $Q_3(u) = 1-(1-u)^{\frac{p_2}{p_1}}$ and $q_3(u) = \frac{p_2}{p_1}(1-u)^{\frac{p_2}{p_1}-1}$. Then from (13) we get

$$\begin{aligned} I_Q(X,Y;u_1,u_2) &= \log[(1-u_1)^{\frac{p_2}{p_1}} - (1-u_2)^{\frac{p_2}{p_1}}] - \log(\frac{1}{p_2}) - \frac{1}{u_2-u_1}(\frac{p_2+1}{p_1})((u_2\log(1-u_2) - u_2 \\ &+ \log(1-u_2)) - (u_1\log(1-u_1) + u_1 - \log(1-u_1))). \end{aligned}$$

Example 2.6
Let X and Y where X follow generalized lambda distribution and Y follow the uniform distribution with their quantile functions $Q_1(u) = \lambda_1 + \frac{1}{\lambda_2}(u^{\lambda_3} - (1-u)^{\lambda_4})$ and $Q_2(u) = \frac{2u}{\lambda_2} + (\lambda_1 - \frac{1}{\lambda_2})$, where $\lambda_i > 0;\ \ i = 1,2,3,4$, respectively with support $(\lambda_1 + \frac{1}{\lambda_2}, \lambda_1 = \frac{1}{\lambda_2})$. Simple calculation lead to $Q_3 = \frac{1}{2}(u^{\lambda_3} - (1-u)^{\lambda_4} + 1)$ and $q_3(u) = \frac{1}{2}(\lambda_3 u^{\lambda_3 - 1} + \lambda_4(1-u)^{\lambda_4-1})$. Then from (13) we get

$$\begin{aligned} I_Q(X,Y;u_1,u_2) &= \log(\frac{1}{2}(u_2^{\lambda_3} - (1-u_2)^{\lambda_4} + 1) - \frac{1}{2}(u_1{}^{\lambda_3} - (1-u_1)^{\lambda_4} + 1)) - \frac{1}{(u_2-u_1)} \\ &\int_{u_1}^{u_2} \log(\frac{1}{2}(\lambda_3 u^{\lambda_3-1} + \lambda_4(1-u)^{\lambda_4-1}))du + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(\frac{1}{\lambda_2}(\lambda_3 u^{\lambda_3-1} + \lambda_4(1-u)^{\lambda_4-1})). \end{aligned}$$

Example 2.7
Consider Van- Staden loots and uniform distribution with their quantile function where $Q_1(u) = \lambda_1 + \lambda_2((\frac{1-\lambda_3}{\lambda_4})(u^{\lambda_4} - 1) - (\frac{\lambda_3}{\lambda_4})((1-u)^{\lambda_4} - 1))$ and $Q_2(u) = u$, respectively where $\lambda_i > 0$ for $i = 1,2,3,4$. Here $Q_3(u) = \lambda_1 + \lambda_2((\frac{1-\lambda_3}{\lambda_4})(u^{\lambda_4} - 1) - (\frac{\lambda_3}{\lambda_4})((1-u)^{\lambda_4} - 1))$ and $q_3(u) = \lambda_2((1-\lambda_3)u^{\lambda_4-1} + \lambda_3(1-u)^{\lambda_4-1})$.

Then from (13) we obtain

$$I_Q(X,Y;u_1,u_2) = \log(\lambda_1+\lambda_2((\frac{1+\lambda_3}{\lambda_4})(u_2^{\lambda_4}-1)-(\frac{\lambda_3}{\lambda_4})((1-u_2)^{\lambda_4}-1)) - (\lambda_1+\lambda_2((\frac{1-\lambda_3}{\lambda_4}) (u_1^{\lambda_4}-1)-(\frac{\lambda_3}{\lambda_4})((1-u_1)^{\lambda_4}-1)))) - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log((1-\lambda_3)u^{\lambda_4-1}+\lambda_3(1-u)^{\lambda_4-1}) + \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log((1-\lambda_3)u^{\lambda_4-1}-\lambda_3(1-u)^{\lambda_4-1}).$$

Example 2.8
Let X and Y two distribution function where X follow Power-Pareto distribution with quantile function then $Q_1(u)=\frac{cu^{\lambda}}{1-u}$, $c,\ \lambda>0$ and Y have quantile function $Q_2(u)=\alpha u^{\beta}$ $\alpha,\beta>0$ where $Q_3(u)=\frac{c^{\frac{1}{\beta}}u^{\frac{\lambda}{\beta}}}{(1-u)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}$ and $q_3(u)=\frac{c^{\frac{1}{\beta}}}{\alpha^{\frac{1}{\beta}}}[\frac{\lambda}{\beta}(u)^{\frac{\lambda}{\beta}-1}(1-u)^{\frac{-1}{\beta}}-\frac{1}{\beta}(u)^{\frac{\lambda}{\beta}}(1-u)^{\frac{1}{\beta}-1}]$. Then from (13) we get

$$I_Q(X,Y;u_1,u_2) = \log\left[\frac{c^{\frac{1}{\beta}}u_2^{\frac{\lambda}{\beta}}}{(1-u_2)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}-\frac{c^{\frac{1}{\beta}}u_1^{\frac{\lambda}{\beta}}}{(1-u_1)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}\right]-\log\left(\frac{c^{\frac{1}{\beta}}}{\alpha^{\frac{1}{\beta}}}\right)-\frac{1}{u_2-u_1} \int_{u_1}^{u_2}\log\left(\frac{\lambda}{\beta}(u)^{\frac{\lambda}{\beta}-1}(1-u)^{\frac{-1}{\beta}}-(1-u)^{\frac{1}{\beta}-1}(\frac{1}{\beta})u^{\frac{\lambda}{\beta}}\right) +\log c+\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\frac{(1-u)\lambda u^{\lambda-1}+u^{\lambda}}{(1-u)^2}du.$$

Example 2.9
Let X follows a generalized lambda distribution function and Y follows power distribution where $Q_1(u)=\frac{1}{\theta_1}+(\frac{1}{\theta_2})(u^{\theta_3}-(1-u)^{\theta_4})$ and $Q_2(u)=u^{\alpha}$, $\theta_1,\theta_2>0$ $Q_3(u)=(\frac{1}{\theta_1}+(\frac{1}{\theta_2})(u^{\theta_3}-(1-u)^{\theta_4}))^{\frac{1}{\alpha}}$ where $q_3(u)=\frac{1}{\theta_2}[\theta_3(u)^{\theta_3-1}+\theta_4(1-u)^{\theta_4-1}]$. From (13) we get

$$I_Q(X,Y;u_1,u_2)=\log\left[\left(\frac{1}{\theta_1}+(\frac{1}{\theta_2})({u_1}^{\theta_3}-(1-u_2)^{\theta_4})\right)^{\frac{1}{\alpha}}-\left(\frac{1}{\theta_1}+(\frac{1}{\theta_2})({u_1}^{\theta_3}-(1-u_1)^{\theta_4})\right)^{\frac{1}{\alpha}}\right].$$

3. Quantile-based Interval Kullback-Leibler Divergence

As an extension of (4) we propose quantile-based Kullback-Leibler divergence measure for doubly truncated random variable

$$KL_{X,Y}(Q;u_1,u_2)=\int_{u_1}^{u_2}\frac{f(Q_1(u)}{F(Q_1(u_2))-F(Q_1(u_1))}\log\left(\frac{\frac{f(Q_1(u))}{F(Q_1(u_2))-F(Q_1(u_1))}}{\frac{g(Q_1(u))}{G(Q_1(u_2))-G(Q_1(u_1))}}\right)d(Q_1(u)). \tag{32}$$

Assuming both systems X and Y have survived up to time u_1 and are observed to fail by time u_2, $KL_{X,Y}(Q;u_1,u_2)$ assesses the discrepancy between their failure time in the interval (u_1,u_2).
$KL_{X,Y}(Q;u_1,u_2)$ satisfies all properties of quantile-based Kullback-Leibler discrimination measure can be rewrite as,

$$KL_{X,Y}(Q;u_1,u_2)=-\int_{u_1}^{u_2}\frac{f(Q_1(u))}{F(Q_1(u_2))-F(Q_1(u_1))}\log\frac{g(Q_1(u))}{G(Q_1(u_2))-G(Q_1(U_1))}d(Q_1(u))-H(Q;u_1,u_2), \tag{33}$$

$$=\log\frac{G(Q_1(u_2))-G(Q_1(u_1))}{u_2-u_1}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\frac{f(Q_1(u))}{g(Q_1(u))}dQ(u). \tag{34}$$

Another way of (32) written as

$$KL_{X,Y}(Q;u_1,u_2) = \log \frac{Q_3(u_2)) - Q_3(u_1)}{u_2 - u_1} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log(q_3(u))du. \tag{35}$$

Now in furthers results we observe some decomposition of quantile Kullbcak-Leibler divergence measure in the terms or quantile-based past Kullbcak-Leibler divergence measure, quantile-based residual Kullbcak-Leibler divergence measure and $KL_{X,Y}(Q;u_1,u_2)$.

Proposition 3.1
Let X and Y represent the non-negative lifetime of two systems. For all $0 \leq u_1 < u_2 < \infty$, the quantile-based Kullback-Leibler discrimination measure is decomposed as follow:

$$\begin{aligned} X,Y(Q) &= F(Q_1(u_1))\bar{K}L_{X,Y}(u_1) + \bar{F}(Q_1(u_2))KL_{X,Y}(u_2) + (F(Q_1(u_2) - F(Q_1(u_1))KL_{X,Y}(Q;u_1,u_2) \\ &+ F(Q_1(u_1)) \log \frac{F(Q_1(u_1))}{G(Q_1(u_1))} + \bar{F}(Q_1(u_2)) \log \frac{\bar{F}(Q_1(u_2))}{\bar{G}(Q_1(u_2))} + F(Q_1(u_2)) - F(Q_1(u_1)) \\ &\log \frac{F(Q_1(u_1)) - F(Q_1(u_1))}{G(Q_1(u_2)) - G(Q_1(u_1))}. \end{aligned}$$

Accept the following analysis: There are four components that make up the quantile-based Kullback discrimination measure between the random lifetimes of systems X and Y: There are four types of discrimination that need to be made: i) between the past lives of two systems at time u_1; ii) between the residual lifetime of X and Y that have both survived up to time u_2; iii) between the lifetimes of both systems in the interval (u_1,u_2); iv) between two random variables to determine whether the systems have failed at u_1, between u_1 and u_2, or after u_2.

This section investigates the properties of $KL_{X,Y}(Q;u_1,u_2)$ and identifies its similarities with $KL_{X,Y}(Q_1(u_1))$ and $\bar{KL}_{X,Y}(Q_1(u_2))$. The interval distance has lower and upper bounds given by the following assertion. First, we define ordering by likelihood ratio.

Definition 3.1
X is considered larger than Y is likelihood ratio sense $(X \geq Y)$ if the ratio $\frac{f(Q(u))}{g(Q(u))}$ is increasing in u over the combined supports of X and Y.

On differentiating (32) with respect to u_1 and u_2, we get

$$\frac{\partial KL_{X,Y}(Q;u_1,u_2)}{\partial u_1} = H_1^X(u_1,u_2)[\log H_1^Y(u_1,u_2) - \log H_1^X(u_1,u_2) + 1 + KL_{X,Y}(Q;u_1,u_2)] - H_1^Y(u_1,u_2). \tag{36}$$

and

$$\frac{\partial KL_{X,Y}(Q;u_1,u_2)}{\partial u_2} = -H_2^X(u_1,u_2)[\log H_2^Y(u_1,u_2) - \log H_2^X(u_1,u_2) + 1 + KL_{X,Y}(Q;u_1,u_2)] - H_2^Y(u_1,u_2). \tag{37}$$

When $KL_{X,Y}(Q;u_1,u_2)$ is increasing in each of u_1 and u_2 while other fixed then the above together implies,

$$\begin{aligned} \frac{H_1^Y(u_1,u_2)}{H_1^X(u_1,u_2)} + \log H_1^X(u_1,u_2) - \log H_1^Y(u_1,u_2) - 1 &\leq KL_{X,Y}(Q;u_1,u_2) \\ &\leq \frac{H_2^Y(u_1,u_2)}{H_2^X(u_1,u_2)} + \log H_2^X(u_1,u_2) - 1 - \log H_2^Y(u_1,u_2). \end{aligned}$$

Theorem 3.1
Let X and Y are random variables with common support $(0,1)$ then,
(i) $X \geq Y$ implies:

$$\log \frac{H_1^X(u_1,u_2)}{H_1^Y(u_1,u_2)} \leq KL_{X,Y}(Q;u_1,u_2) \leq \log \frac{H_2^X(u_1,u_2)}{H_2^Y(u_1,u_2)} \tag{38}$$

when the $\frac{f(Q_1(u))}{g(Q_1(u))}$ is decreasing in $u>0$ then the above inequality are reserved.
(ii) Decreasing $g(Q(u))$ in $u>0$

$$\log \frac{1}{H_1^Y(u_1,u_2)} \leq KL_{X,Y}(Q;u_1,u_2) + H_X(u_1,u_2) \leq \log \frac{1}{H_2^Y(u_1,u_2)} \tag{39}$$

for increasing $g(Q(u))$ the inequality are reversed.

Proof
Because of increasing $\frac{f(Q_1(u))}{g(Q_1(u))} > 0$, from (32) we have

$$KL_{X,Y}(Q;u_1,u_2) \leq \int_{u_1}^{u_2} \frac{f(Q_1(u)}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \left(\frac{\frac{f(Q_1(u))}{F(Q_1(u_2))-F(Q_1(u_1))}}{\frac{g(Q_1(u))}{G(Q_1(u_2))-G(Q_1(u_1))}} \right) d(Q_1(u)) = \log \frac{H_2^X(u_1,u_2)}{H_2^Y(u_1,u_2)}.$$

and

$$KL_{X,Y}(Q;u_1,u_2) \geq \int_{u_1}^{u_2} \frac{f(Q_1(u)}{F(Q_1(u_2)) - F(Q_1(u_1))} \log \left(\frac{\frac{f(Q_1(u))}{F(Q_1(u_2))-F(Q_1(u_1))}}{\frac{g(Q_1(u))}{G(Q_1(u_2))-G(Q_1(u_1))}} \right) d(Q_1(u)) = \log \frac{H_1^X(u_1,u_2)}{H_1^Y(u_1,u_2)}$$

which gives (38). When $\frac{f(Q_1(u))}{g(Q_1(u))}$ is decreasing, the proof is similar. Furthermore, for all $u_1 < u < u_2$ decreasing $g(Q_1(u))$ in $u>0$ implies $g(Q(u_2)) < g(Q(u)) < g(Q(u_1))$ then we get:

$$KL_{X,Y}(Q;u_1,u_2) \leq -\log H_2^Y(u_1,u_2) - H_X(u_1,u_2),$$

and

$$KL_{X,Y}(Q;u_1,u_2) \geq -\log H_1^Y(u_1,u_2) - H_X(u_1,u_2).$$

so that (39) holds when $g(Q_1(u))$ is increasing then proof is similar. □

In the following theorem, sufficient condition for $KL_{X_1,Y}(Q;u_1,u_2)$ to be smaller then $KL_{X_2,Y}(Q;u_1,u_2)$.

Theorem 3.2
Let X_1, X_2 and Y be non-negative random variables with probability density function f_1, f_2 and g, respectively. If $X_1 \geq Y$, then $KL_{X_1,Y}(Q;u_1,u_2) \leq KL_{X_2,Y}(Q;u_1,u_2)$.

Proof
From (34) we have

$$\begin{aligned}
&KL_{X_1,Y}(Q;u_1,u_2) - KL_{X_2,Y}(Q;u_1,u_2) \\
&= -KL_{X_2,X_1}(Q;u_1,u_2) + \int_{u_1}^{u_2} \left[\frac{1}{F_1(Q_1(u_2)) - F_1(Q_1(u_1))} - \frac{1}{F_2(Q_1(u_2)) - F_2(Q_1(u_1))} \right] \log \frac{f_1(Q_1(u))}{g(Q_1(u))} du \\
&\leq \int_{u_1}^{u_2} \left[\frac{1}{F_1(Q_1(u_2)) - F_1(Q_1(u_1))} - \frac{1}{F_2(Q_1(u_2)) - F_2(Q_1(u_1))} \right] \log \frac{f_1(Q_1(u))}{g(Q_1(u))} du \\
&\leq \frac{f_1(Q_1(u_2))}{g(Q_1(u))} \int_{u_1}^{u_2} \left[\frac{1}{F_1(Q_1(u_2)) - F_1(Q_1(u_1))} - \frac{1}{F_2(Q_1(u_2)) - F_2(Q_1(u_1))} \right] du = 0
\end{aligned}$$

where the first inequality arises because $KL_{X_2,X_1}(Q;u_1,u_2) \geq 0$ and the second inequality follows from the fact that $\frac{f_1(Q_1(u))}{g(Q_1(u))}$ is increasing in $Q_1(u \geq 0)$. □

Proposition 3.2
For a continuous, differentiable and invertible function $\phi(.)$, the expression of quantile based Kullback Leibler divergence measure (32) is defined as

$$KL_{\phi_1(X),\phi_2(Y)}(Q;u_1,u_2) = -\frac{1}{Q_1^{-1}(h(Q_1(u_2)) - Q_1^{-1}(h(Q_1(u_1))}\int_{u_1}^{u_2}\log\left(\frac{\frac{d}{du}(Q_1^{-1}(h(Q_1(u)))}{\frac{d}{du}Q_2^{-1}(h(Q_1(u))}\right)$$
$$\frac{d}{dp}(Q_1^{-1}(h(Q_1(u)))dp + \log\left(\frac{Q_2^{-1}(h(Q_1(u_2)) - Q_2^{-1}(h(Q_1(u_1))}{Q_1^{-1}(h(Q_1(u_2)) - Q_1^{-1}(h(Q_1(u_1))}\right),$$

where $h(.) = \phi^{-1}(.)$.

Proof
Let $f_{\phi_1}(Q_1(u)), F_{\phi_1}(Q_1(u))$ and $g_{\phi_2}(Q_2(u)), G_{\phi_2}(Q_2(u))$ denote the probability distribution function and cumulative distribution function of random variable respectively. By (32),

$$KL_{\phi_1(X),\phi_2(Y)}(Q;u_1,u_2) = \int_{u_1}^{u_2}\frac{f(h(Q_1(u)))h'(Q_1(u))}{F(h(Q_1(u_2))) - F(h(Q_1(u_1)))}\log\left(\frac{\frac{f(h(Q_1(u))h'(Q_1(u))}{F(h(Q_1(u_2)))-F(h(Q_1(u_1)))}}{\frac{g(h(Q_1(u)))h'(Q_1(u))}{G(h(Q_1(u_2)))-G(h(Q_1(u_1)))}}\right)d(Q_1(u)). \tag{40}$$

By quantile function (5) we have $F^{-1}(u) = Q_1(u)$ and therefore

$$G(h(Q_1(u))) = G(h(F^{-1}(u))) = Q_2^{-1}(h(Q_1(u))). \tag{41}$$

Similarly we have

$$F(h(Q_1(u))) = F(h(F^{-1}(u))) = Q_1^{-1}(h(Q_1(u))). \tag{42}$$

On differentiating (41) and (42) with respect to u, we get

$$g(h(Q_1(u)))h'(Q_1(u))q_1(u) = \frac{d}{du}(Q_2^{-1}(h(Q_1(u)))), \tag{43}$$

and

$$f(h(Q_1(u)))h'(Q_1(u))q_1(u) = \frac{d}{du}(Q_1^{-1}(h(Q_1(u)))). \tag{44}$$

Substituting (43) and (44) in (40) we get desire result. □

Example 3.1
If X follows exponential distribution with *QF* $Q(u) = -\frac{1}{\lambda}\log(1-u)$, then $X^{\frac{1}{\alpha}}$ follows Weibull distribution with $Q(u) = (-\frac{1}{\lambda}\log(1-u))^{\frac{1}{\alpha}}$. Let X and Y be two exponential distribution with *QF* respectively by $Q_1(u) = -\frac{1}{\lambda_1}\log(1-u)$ and $Q_2(u) = -\frac{1}{\lambda_2}\log(1-u)$. Then the quantile- based Kullback Leibler divergence for two Weibull distribution can be obtain from propositon (3.2) by taking $\phi_1(X) = X^{\frac{1}{\alpha}}$ and $\phi_2(Y) = Y^{\frac{1}{\alpha}}$. From these transformation we have $\phi^{-1}(Q_1(u)) = h(Q_1(u) = (Q_1(u))^{\alpha}$ which implies that $h'(Q_1(u))q_1(u) = \alpha((Q_1(u))^{\alpha-1}q_1(u)$. Consequently, $Q_2^{-1}(h(Q_1(u))) = 1 - e^{-\lambda_2(Q_1(u))^{\alpha}}$ and $Q_1^{-1}(h(Q_1(u))) = 1 - e^{-\lambda_1(Q_1(u))^{\alpha}}$ and correspondingly

$$\frac{d}{du}Q_2^{-1}(h(Q_1(u))) = \lambda_2\alpha(Q_1(u))^{\alpha-1}e^{-\lambda_2(Q_1(u))^{\alpha}}q_1(u),$$

and

$$\frac{d}{du}Q_1^{-1}(h(Q_1(u))) = \lambda_1\alpha(Q_1(u))^{\alpha-1}e^{-\lambda_1(Q_1(u))^{\alpha}}q_1(u).$$

Now using in (40) becomes

$$KL_{\phi_1(X),\phi_2(Y)}(Q;u_1,u_2) = \log\left(\frac{e^{-\lambda_2(Q_1(u_1))^{\alpha}} - e^{-\lambda_2(Q_1(u_1))^{\alpha}}}{e^{-\lambda_1(Q_1(u_1))^{\alpha}} - e^{-\lambda_1(Q_1(u_2))^{\alpha}}}\right) - \frac{1}{e^{-\lambda_1(Q_1(u_1))^{\alpha}} - e^{-\lambda_1(Q_1(u_2))^{\alpha}}}$$
$$\int_{u_1}^{u_2}\log\frac{e^{-\lambda_1(Q_1(u_1))^{\alpha}}}{e^{-\lambda_2(Q_1(u_1))^{\alpha}}}(1 - e^{-\lambda_1(Q_1(u))^{\alpha}})du.$$

Example 3.2
For two independent exponential distribution with quantile function $Q_1(u) = \frac{-\lambda}{\log u}$ and $Q_2(u) = \frac{-\lambda_2}{\log u}$ where $\lambda_1, \lambda_2 > 0$ and $Q_3(u) = e^{\frac{-\lambda_2}{\lambda_1}} \log u$, $q_3(u) = \frac{\lambda_2}{\lambda_1} u^{\frac{\lambda_2}{\lambda_1}-1}$, we get our result from

$$KL_{X,Y}(Q; u_1, u_2) = \log \frac{Q_3(u_2)) - Q_3(u_1)}{u_2 - u_1} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log(q_3(u))du.$$

$$KL_{X,Y}(Q; u_1, u_2) = -log(\frac{\lambda_2}{\lambda_1}) - (\frac{1}{u_2 - u_1})(\frac{\lambda_2}{\lambda_1} - 1)[u_2 \log u_2 - u_2 - u_1 \log u_1 + u_1] + \log[u_2^{\frac{\lambda_2}{\lambda_1}} - u_1^{\frac{\lambda_2}{\lambda_1}}] - \log(u_2 - u_1).$$

and

$$KL_{Y,X}(Q; u_2, u_1) = -log(\frac{\lambda_1}{\lambda_2}) - (\frac{1}{u_2 - u_1})(\frac{\lambda_1}{\lambda_2} - 1)[u_2 \log u_2 - u_2 - u_1 \log u_1 + u_1] + \log[u_2^{\frac{\lambda_1}{\lambda_2}} - u_1^{\frac{\lambda_1}{\lambda_2}}] - \log(u_2 - u_1).$$

This explicitly shows that the quantile-based Kullback-Leibler discrimination measure is not symmetric, that is $KL_{X,Y}(Q; u_1, u_2) \neq KL_{Y,X}(Q; u_2, u_1)$.
In figure 4, assuming $u_2 = 0.8$, and $\lambda_1 = 1$, $\lambda_2 = 2$ the quantile-based interval Kullback-Leibler divergence measures obtain in example (3.1) are plotted with respect to u_1. Both dark(thin) line represent the graph of $KL_{X,Y}(Q; u_1, u_2), KL_{Y,X}(Q; u_2, u_1)$. When $u_1 = u_2 < 0.6$ then $KL_{X,Y}(Q; u_1, u_2) = KL_{Y,X}(Q; u_2, u_1)$.

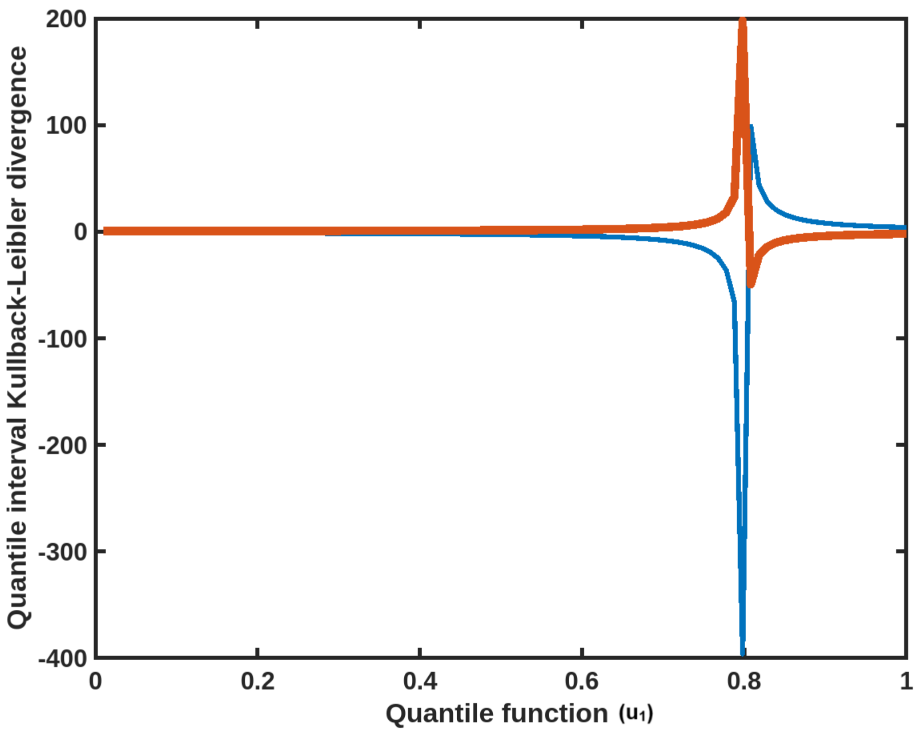

Figure 4. Plot a Quantile-based Interval Kullback-Leibler Divergence vs Quantile function

Example 3.3
Let us consider two non-negative exponential distribution with quantile functions where $Q_1(u) = -\frac{1}{\lambda_1}\log(1-u)$; $\lambda_1 > 0$ and $Q_2(u) = -\frac{1}{\lambda_2}\log(1-u)$; $\lambda_2 > 0$, Then $Q_3(u) = Q_2^{-1}(Q_1(u)) = 1-(1-u)^{\frac{\lambda_2}{\lambda_1}}$ and $q_3(u) = \frac{\lambda_2}{\lambda_1}(1-u)^{(\frac{\lambda_2}{\lambda_1})} - 1$. Thus from (32), we get

$$KL_{X,Y}(Q; u_1, u_2) = -\frac{1}{u_2-u_1}(\frac{\lambda_2}{\lambda_1} - 1)(u_2\log(1-u_2) - u_2 + \log(1-u_2) - u_1\log(1-u_1) + u_1 - \log(1-u_1)) + \log((1-u_1)^{\frac{\lambda_2}{\lambda_1}} - (1-u_2)^{\frac{\lambda_2}{\lambda_1}}) - \log(u_2-u_1) - \log(\frac{\lambda_2}{\lambda_1}).$$

Example 3.4
Assume that there are two distribution function with quantile function $Q_1(u) = c_1u^{\lambda_1}(1-u)^{-\lambda_2}$; $c_1, \lambda_2 > 0$ and $Q_2(u) = c_2u^{\frac{1}{\lambda_3}}$; $c_2, \lambda_3 > 0$. Here $Q_1(u)$ and $Q_2(u)$ are the quantile function of Davis and Power distribution respectively. $Q_3(u) = (\frac{c_1}{c_2})^{\lambda_3}u^{\lambda_1\lambda_3}(1-u)^{-\lambda_2\lambda_3}$ Then from (32) we get

$$KL_{X,Y}(Q; u_1, u_2) = -\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\lambda_3 du - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log(\frac{c_1}{c_2})^{\lambda_3}u^{\lambda_3}(1-u)^{-\lambda_3} \left[\frac{1}{u} + \frac{1}{1-u}\right] du + \log\left[\frac{(\frac{c_1}{c_2})^{\lambda_3}u_2^{\lambda_3}(1-u_2)^{-\lambda_3}}{u_2-u_1}\right].$$

Example 3.5
Consider the QF of Govindarajulu and Inverted reciprocal exponential distribution as $Q_1(u) = 2u - u^2$ and $Q_2(u) = \frac{-\lambda}{\log u}, ; \lambda > 0$. Then $Q_3(u)$ and $q_3(u)$ can be obtained as $Q_3(u) = \exp(\frac{-\lambda}{2u-u^2})$, $q_3(u) = (\frac{2\lambda(1-u)}{(2u-u^2)^2})(\exp-(\frac{\lambda}{2u-u^2}))$.Then from (32) we obtain

$$KL_{X,Y}(Q; u_1, u_2) = -\log 2\lambda - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\left(\frac{(1-u)}{(2u-u^2)^2}\exp(\frac{-\lambda}{2u-u^2})\right)du + \log\left[\exp\frac{-\lambda}{2u_2-{u_2}^2} - \exp\frac{-\lambda}{2u_1-{u_1}^2}\right] - \log(u_2-u_1).$$

Example 3.6
Suppose X and Y be follow Pareto II distribution with quantile function respectively, $Q_1(u) = b((1-u)^{\frac{-1}{p_1}} - 1); p_1 > 0\,, b > 0$ and $Q_2(u) = b((1-u)^{\frac{-1}{p_1}} - 1); p_1 > 0\,, b > 0$. Then $Q_3(u) = 1-(1-u)^{\frac{p_2}{p_1}}$ and $q_3(u) = \frac{p_2}{p_1}(1-u)^{\frac{p_2}{p_1}-1}$. Then from (32) we get

$$KL_{X,Y}(Q; u_1, u_2) = -\frac{1}{u_2-u_1}\left(\frac{p_2}{p_1} - 1\right)(u_2\log(1-u_2) - u_2 + \log(1-u_2) - u_1\log(1-u_1) + u_1 - \log(1-u_1)) + \log((1-u_1)^{\frac{p_2}{p_1}} - (1-u_2)^{\frac{p_2}{p_1}}) - \log(u_2-u_1) - \log\left(\frac{p_2}{p_1}\right).$$

Example 3.7
Consider Van- Staden loots and uniform distribution with quantile function where $Q_1(u) = \lambda_1 + \lambda_2((\frac{1-\lambda_3}{\lambda_4})(u^{\lambda_4} - 1) - (\frac{\lambda_3}{\lambda_4})((1-u)^{\lambda_4} - 1))$ and $Q_2(u) = u$, respectively where $\lambda_i > 0$ for $i = 1, 2, 3, 4$. Here $Q_3(u) = \lambda_1 + \lambda_2((\frac{1-\lambda_3}{\lambda_4})(u^{\lambda_4} - 1) - (\frac{\lambda_3}{\lambda_4})((1-u)^{\lambda_4} - 1))$ and $q_3(u) = \lambda_2((1-\lambda_3)u^{\lambda_4-1} + \lambda_3(1-u)^{\lambda_4-1})$. Then from (32) we

get

$$KL_{X,Y}(Q;u_1,u_2) = -\log\lambda_2 - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log((1-\lambda_3)u^{\lambda_4-1}+\lambda_3(1-u)^{\lambda_4-1})du - \log(u_2-u_1)$$
$$+\log[(\lambda_1+\lambda_2((\frac{1-\lambda_3}{\lambda_4})({u_2}^{\lambda_4}-1)-(\frac{\lambda_3}{\lambda_4})((1-u_2)^{\lambda_4}-1)))-(\lambda_1+\lambda_2)((\frac{1-\lambda_3}{\lambda_4})({u_1}^{\lambda_4}-1)$$
$$-(\frac{\lambda_3}{\lambda_4})((1-u_1)^{\lambda_4}-1))].$$

Example 3.8

Let X and Y where X follow generalized lambda distribution and Y follow the uniform distribution with their quantile functions $Q_1(u) = \lambda_1 + \frac{1}{\lambda_2}(u^{\lambda_3} - (1-u)^{\lambda_4})$ and $Q_2(u) = \frac{2u}{\lambda_2} + (\lambda_1 - \frac{1}{\lambda_2})$, where $\lambda_i > 0;\ i = 1,2,3,4$, respectively with support $(\lambda_1 + \frac{1}{\lambda_2}, \lambda_1 - \frac{1}{\lambda_2})$. Simple calculation lead to $Q_3 = \frac{1}{2}(u^{\lambda_3} - (1-u)^{\lambda_4} + 1)$ and $q_3(u) = \frac{1}{2}(\lambda_3 u^{\lambda_3-1} + \lambda_4(1-u)^{\lambda_4-1})$. From (32) we get

$$KL_{X,Y}(Q;u_1,u_2) = -\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\frac{1}{2}[(\lambda_3 u^{\lambda_3-1}+\lambda_4(1-u)^{\lambda_4-1})]du$$
$$+\log\left[\frac{\frac{1}{2}\left(u_2^{\lambda_3}-(1-u_2)^{\lambda_4}+1\right)-\frac{1}{2}\left(u_1^{\lambda_3}-(1-u_1)^{\lambda_4}+1\right)}{u_2-u_1}\right].$$

Example 3.9

Let X and Y two distribution function where X follow Power-Pareto distribution with quantile function $Q_1(u) = \frac{cu^\lambda}{1-u},\ \ c,\lambda > 0$ and Y have quantile function $Q_2(u) = \alpha u^\beta,\ \ \alpha,\beta > 0$ where $Q_3(u) = \frac{c^{\frac{1}{\beta}}u^{\frac{\lambda}{\beta}}}{(1-u)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}$ and $q_3(u) = \frac{c^{\frac{1}{\beta}}}{\alpha^{\frac{1}{\beta}}}[\frac{\lambda}{\beta}(u)^{\frac{\lambda}{\beta}-1}(1-u)^{\frac{-1}{\beta}} - \frac{1}{\beta}(u)^{\frac{\lambda}{\beta}}(1-u)^{\frac{1}{\beta}-1}]$. Then from (32) we get

$$KL_{X,Y}(Q;u_1,u_2) = -\log(\frac{c^{\frac{1}{\beta}}}{\alpha^{\frac{1}{\beta}}}) - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\frac{\lambda}{\beta}(u_2)^{\frac{\lambda}{\beta}-1}(1-u_2)^{\frac{-1}{\beta}} - \frac{1}{\beta}(u_1)^{\frac{\lambda}{\beta}}(1-u_1)^{\frac{1}{\beta}-1}du$$
$$+\log\left(\frac{c^{\frac{1}{\beta}}u_2^{\frac{\lambda}{\beta}}}{(1-u_2)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}} - \frac{c^{\frac{1}{\beta}}u_1^{\frac{\lambda}{\beta}}}{(1-u_1)^{\frac{1}{\beta}}\alpha^{\frac{1}{\beta}}}\right) - \log(u_2-u_1).$$

Example 3.10

Let X follows a generalized lambda distribution and Y follows power distribution where $Q_1(u) = \frac{1}{\theta_1} + (\frac{1}{\theta_2})(u^{\theta_3} - (1-u)^{\theta_4})$ and $Q_2(u) = u^\alpha,\ \ \theta_1,\theta_2 > 0\ Q_3(u) = (\frac{1}{\theta_1} + (\frac{1}{\theta_2})(u^{\theta_3} - (1-u)^{\theta_4}))^{\frac{1}{\alpha}}$ where $q_3(u) = \frac{1}{\theta_2}[\theta_3(u)^{\theta_3-1} + \theta_4(1-u)^{\theta_4-1}]$. Then from (32) we get

$$KL_{X_2,X_1}(Q;u_1,u_2) = -\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log\left(\frac{1}{\theta_2}\left(\theta_3(u)^{\theta_3-1}+\theta_4(1-u)^{\theta_4-1}\right)\right)du$$
$$+\log((\frac{1}{\theta_1}+(\frac{1}{\theta_2})(u_2^{\theta_3}-(1-u_2)^{\theta_4}))^{\frac{1}{\alpha}} - (\frac{1}{\theta_1}+(\frac{1}{\theta_2})(u_1^{\theta_3}-(1-u_1)^{\theta_4}))^{\frac{1}{\alpha}}) - \log(u_2-u_1).$$

In figure 5, assuming the different value of u_1 and $\theta_1 = 2,\ \theta_2 = 1,\ \theta_3 = 2.8,\ \theta_4 = 0.4,\ \ \alpha = 0.3$ in $KL_{X_1,X_2}(Q;u_1,u_2)$ as obtain in example (3.10).

In figure 6, assuming the different value of u_2 and $\theta_1 = 2,\ \theta_2 = 1,\ \theta_3 = 2.8,\ \theta_4 = 0.4,\ \ \alpha = 0.3$ in $KL_{X_2,X_1}(Q;u_1,u_2)$ as obtain in example (3.10).

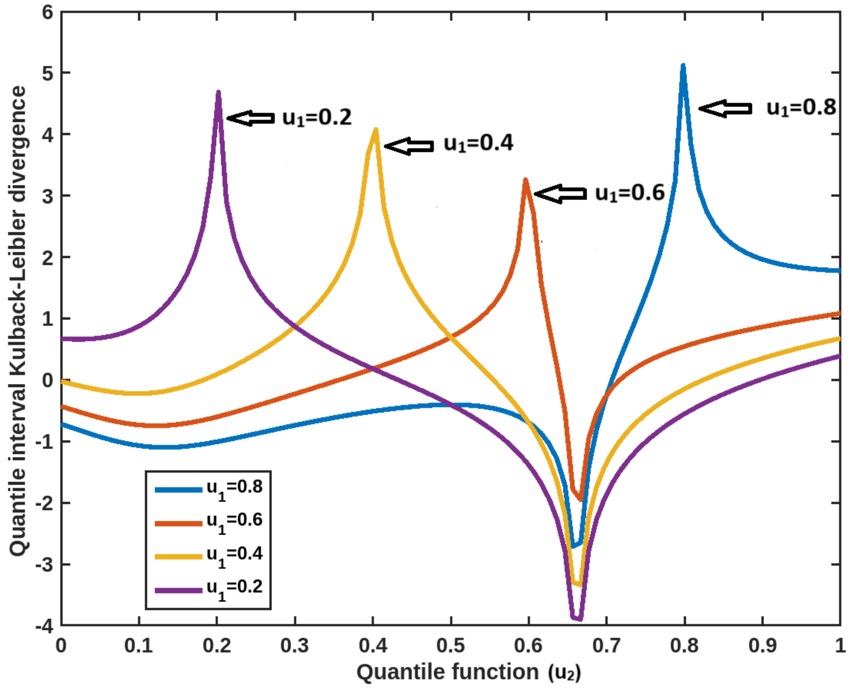

Figure 5. Quantile-based Interval Kullback-Leibler Divergence vs Quantile function

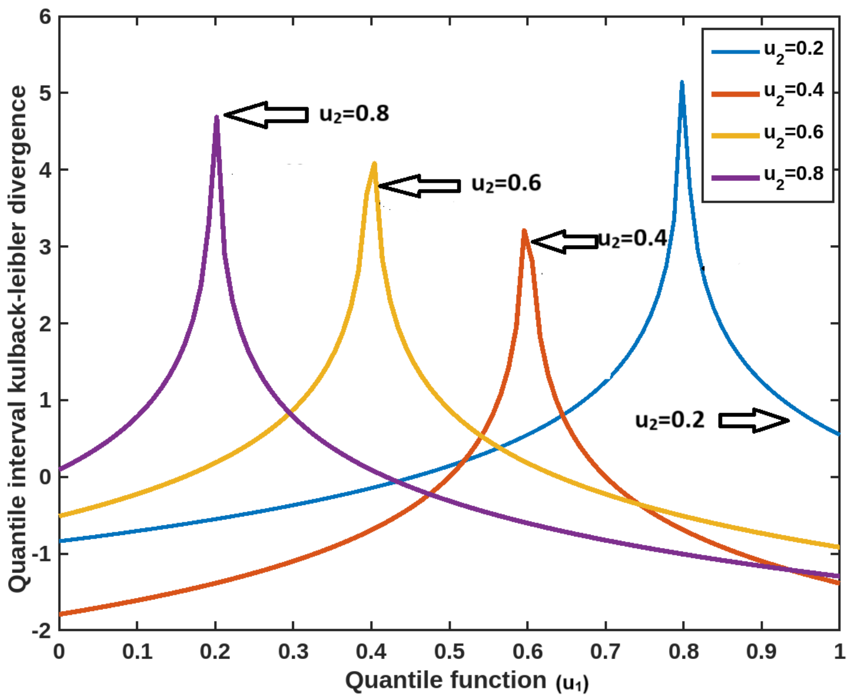

Figure 6. Quantile-based Interval Kullback-Leibler Divergence vs Quantile function

4. Conclusion

In the present paper we highlights the significance of using quantile-based methods in statistical analysis, particularly for doubly truncated random variables. Since many models have easily expressible quantile functions but lack explicit distribution forms, the proposed quantile-based inaccuracy measure and Kullback–Leibler divergence offer practical and flexible tools. These measures not only enhance our understanding of information loss and divergence in truncated settings but also open up new avenues for research and application where traditional methods fall short due to intractable distribution functions. The study provides a foundation for extending quantile-based techniques to more complex data structures and inference problems.

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