STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 1–13.
Published online in International Academic Press (www.IAPress.org)

Some results of a random (b, θ)-enriched contraction with application on non-linear stochastic integral equations

Krissada Yajai [1], Orapan Janngam [2], Wiroj Mongkolthep [1], Phachara Saipara [1,*]

[1] *Division of Mathematics, Department of Science, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Nan, Thailand*
[2] *Division of Mathematics, Department of Science, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Lampang, Thailand*

Abstract In this paper, we propose a random (b, θ)-enriched contraction operator and prove an existence theorem of random fixed points for this operator. Moreover, we establish an existence result for a solution to a nonlinear stochastic integral equation of Hammerstein type.

Keywords a random (b, θ)-enriched contraction, stochastic fixed point, non-linear stochastic integral equation

AMS 2010 subject classifications 47H09, 47H10

DOI: 10.19139/soic-2310-5070-2714

1. Introduction

In 1922, Banach's contraction principle [1] developed as a major result for non-linear analysis. It has formed the basis of metric fixed point theory, and its importance lies in its various applicability over several mathematical fields. Banach's contraction principle is given by the following theorem.

Theorem 1
(Banach's contraction principle) If (X, d) be a complete metric space and $T : X \to X$ be a self-mapping such that,

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{1}$$

for each $x, y \in X$ for some $\alpha \in [0, 1)$, then T has a unique fixed point.

Note that while the mappings T satisfying the Banach contraction is continuous, but the following mappings T satisfying the contraction need not be continuous such as Kannan's contraction, Reich's contraction, Ćirić's contraction, Chatterjea's contraction, Zamfirescu contractive conditions, Hardy and Rogers's contraction and Ćirić (see, [2, 3]).

Fixed point theory has gained importance in modern analysis, optimization, differential equations and applied mathematics. Its foundation is the well-known Banach's contraction principle, one of mathematics greatest findings. This principle illustrates that a fixed point's existence and uniqueness in a complete metric space can be ensured by a straightforward contraction condition. For this reason, mathematicians have spent decades

*Correspondence to: Phachara Saipara (Email: splernn@gmail.com). Division of Mathematics, Department of Science, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Nan, Thailand (55000).

ISSN 2310-5070 (online) ISSN 2311-004X (print)

generalizing and extending the concept to encompass a broader class of mappings that exhibit fixed-point behavior without necessarily satisfying the strict contraction condition. The notion of enriched contractions has become a significant generalization among these extensions becuase enriched contraction is more general, flexible, and applicable but technically richer. Through a number of significant contributions, the idea of enriched contractions has been methodically developed. First proposed by Berinde et al.[4] in 2020, Banach contractions are a specific instance of the classical contractive mappings generalized by enriched contractions in Banach spaces, as follows

Definition 1
Let $(X, \|\cdot\|)$ be a linear normed space. The mappings $T : X \to X$ is said to be a (b, θ)-Enriched contraction if there exist $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that

$$\|b(x_1 - x_2) + Tx_1 - Tx_2\| \leq \theta \|x_1 - x_2\|, \forall x_1, x_2 \in X. \tag{2}$$

Furthermore, enriched contractions have been extensively investigated in various studies (see, for instance, [5, 6, 7, 8, 9]).

On the other hand, Spacek et al. originally proved random fixed point theorems for random contraction mappings on separable complete metric spaces (see, [10, 11, 12]). A stochastic extension of a classical fixed point referred to as a random fixed point. Additionally, the concept of random fixed point have been the subject of various studies (see, for example, [13, 14, 15, 16]).

In the study of random fixed points in a separable Banach spaces, the following approach is taken. Saha et al. [17, 18] proved some random fixed point theorems over a separable Banach space and a separable Hilbert space with a probability measure. On the other hand, Padgett [19] studied the existence and uniqueness of a random solution of a non-linear stochastic integral equation of the Hammerstein type. In 2012, Saha et al.[20] proved random fixed point theorems for $(\theta, L)-$weak contractions in a separable Banach space. Saha et al.[21] proved a random fixed point theorem in a separable Banach space equipped with a complete probability measure for a certain class of contractive mappings.

Recently, the notion of random fixed points, Plubtieng et al.[22] defined a random $\mathcal{P}$-contraction in a separable Banach spaces as follows.

Definition 2
Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T is said to be a random $\mathcal{P}$-contraction if we have

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \|x_1 - x_2\| - \varrho(\omega, x_1, x_2) \tag{3}$$

for all $x_1, x_2 \in X$ and $\varrho(\omega, \cdot, \cdot) : X \times X \to \mathbb{R}$ satisfies the condition of $\mathcal{P}$-contraction.

Moreover, they prove the existence of a random fixed point of T in X as follows.

Theorem 2
Let X be a separable partially ordered Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T satisfies a random $\mathcal{P}$-contraction in Definition 2. Then there exists a random fixed point of T.

Several works have also explored random fixed point contractions (see, [23, 24, 25]).

The purpose of this paper is to prove a random fixed point theorem for a random (b, θ)-Enriched contraction operator in a separable Banach spaces. The paper is organized as follows. Sections 1 and 2 contains Introduction and Preliminaries, respectively. The main results are presented in section 3. The last section contains some application to a non-linear stochastic integral equations.

2. Preliminaries

Let (X, β_X) be a separable Banach space, where β_X is a σ-algebra of Borel subsets of X, (Ω, β, μ) be a complete probability measure space. More details we refer to the paper of Joshi et.al. [26].

Definition 3
A mapping $x : \Omega \to X$ is called

1. An *X-valued random variable* if $x^{-1}(B) \in \beta$ for any $B \in \beta_X$.
2. A *finitely valued random variable* if it is constant on any finite number of disjoint sets $A_i \in \beta$ and is equal to 0 over $\Omega \setminus (\bigcup_{i=1}^{n} A_i)$. The mapping x is said to be a *simple random variable* if it's finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.
3. A *strong random variable* if there is a sequence of simple random variables $\{x_n(\omega)\}$ converges to $x(\omega)$ almost surely, that is, there is a set $A_0 \in \beta$ with $\mu(A_0) = 0$ so that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$ for any $\omega \in \Omega \setminus A_0$.
4. A *weak random variable* if the function $x^*(x(\cdot))$ is a real valued random variable for any $x^* \in X^*$, where X^* denotes the first normed dual space of X.

The concepts of strong and weak random variables coincide in a separable Banach space X (see, [26]).

Theorem 3
([26]) Let $x, y : \Omega \to X$ be strong random variables and α, β be constants. Then the following statements hold:
(1) $\alpha x(\omega) + \beta y(\omega)$ is a strong random variable.
(2) If $f(\omega)$ is a real-valued random variable and $x(\omega)$ is a strong random variable, then $f(\omega)x(\omega)$ is a strong random variable.
(3) If $x_n(\omega)$ is a sequence of strong random variables converging strongly to $x(\omega)$ almost surely, that is, if there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that

$$\lim_{n\to\infty} \|x_n(\omega) - x(\omega)\| = 0$$

for any $\omega \notin A_0$, then $x(\omega)$ is a strong random variable.

Observes that all strong and weak random variables are measurable in the context of Definition 3 if X is a separable Banach space. Let Y be another Banach space. Also, we need to provide the following definitions (see, [26]).

Definition 4
A mapping $F : \Omega \times X \to Y$ is called

1. A *random mapping* if $F(\cdot, x)$ is a Y-valued random variable $\forall x \in X$.
2. A *continuous random mapping* if $\mu(\{\omega \in \Omega : F(\omega, x)$ is a continuous function of $x\}) = 1$.
3. A *demicontinuous* at $x \in X$ if $\|x_n - x\| \to 0$ implies $F(\cdot, x_n) \rightharpoonup F(\cdot, x)$ almost surely.

Theorem 4
([26]) Let $F : \Omega \times X \to Y$ be a demicontinuous random mapping where Y is a separable Banach space. Then, for any X-valued random variable x, the function $F(\cdot, x(\cdot))$ is a Y-valued random variable.

Remarks that Theorem 3 is also true for a continuous random mapping as it is a demicontinuous random mapping(see, [26]).

We discuss some important definitions and findings in accordance with Joshi et al. [26].

Definition 5
An equation $F(\omega, x(\omega)) = x(\omega)$ is said to be a *random fixed point equation*, where F is a random mapping.

Definition 6
For each $x : \Omega \to X$ which satisfies the random fixed point equation almost surely is called a *wide sense solution* of the fixed point equation.

Definition 7
For each X-valued random variable x which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$ is called a random fixed point of $F : \Omega \to X$.

Observes that a random solution is a fixed point equation solution in the widest sense. However, this isn't always the situation. This is evident from an example, under some Remarks, in the work of Joshi et al.(see, [26]).

3. The main results

In this section, we provide the following definition of a random (b, θ)-enriched contraction, which is motivated and influenced by Definitions 1 and 2.

Definition 8
Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T is said to be a random (b, θ)-Enriched contraction if

$$\|b(\omega)(x_1 - x_2) + T(\omega, x_1) - T(\omega, x_2)\| \le \theta(\omega)\|x_1 - x_2\| \tag{4}$$

for all $x_1, x_2 \in X$, $b : \Omega \to [0, \infty)$ and $\theta : \Omega \to [0, b(\omega) + 1)$ are random variables, meaning their values depend on ω.

The next example demonstrates a valid case of Definition 8.

Example 1
Let $X = \mathbb{R}$, $\Omega = [0, 1]$, $b(\omega) = \omega$, $\theta(\omega) = \frac{\omega}{2}$ and $T(\omega, x) = -\frac{\omega}{2}x$. For all $x_1, x_2 \in X = \mathbb{R}$, we get

$$\begin{aligned}
&\|b(\omega)(x_1 - x_2) + T(\omega, x_1) - T(\omega, x_2)\| \\
&= \|b(\omega)(x_1 - x_2) + T(\omega, x_1) - T(\omega, x_2)\| \\
&= \|\omega(x_1 - x_2) + (-\frac{\omega}{2}x_1) - T(-\frac{\omega}{2}x_2)\| \\
&= |\omega - \frac{\omega}{2}|\|x_1 - x_2\| \\
&= \frac{\omega}{2}\|x_1 - x_2\| \\
&\le \theta(\omega)\|x_1 - x_2\|,
\end{aligned}$$

which $\theta(\omega) = \frac{\omega}{2} < \omega + 1 = b(\omega) + 1$. This confirms that T is a random $(b, \theta)-$ enriched contraction as defined in Definition 8.

Next, we prove the existence of a random fixed point for a random (b, θ)-enriched contraction in a separable Banach space.

Theorem 5
Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T satisfies a random enriched contraction in Definition 8. Then there exists a random fixed point of T.

Proof
Let A, B and C be three sets defined by

$$A = \{\omega \in \Omega : T(\omega, x) \quad \text{is a continuous function of} \quad x\},$$

$$B = \{\omega \in \Omega : b(\omega) \in [0, \infty)\} \cap \{\omega \in \Omega : \theta(\omega) \in [0, b(\omega) + 1)\}$$

and for $x_1, x_2 \in X$

$$C_{x_1,x_2} = \{\omega \in \Omega : \|b(\omega)(x_1 - x_2) + T(\omega, x_1) - T(\omega, x_2)\| \leq \theta(\omega)\|x_1 - x_2\|\}.$$

Let S be a countable dense subset of X. Since the use of the countable dense subset S guarantees measurability, while the continuity of T allows the contraction condition verified on S to extend to the entirety of X. Now, we prove that

$$\bigcap_{x_1,x_2 \in X}(C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in S}(C_{s_1,s_2} \cap A \cap A \cap B).$$

Now, for all $s_1, s_2 \in S$, we have

$$\begin{aligned}&\|b(\omega)(s_1 - s_2) + T(\omega, s_1) - T(\omega, s_2)\| \\ &\leq b(\omega)\|s_1 - s_2\| + \|T(\omega, s_1) - T(\omega, s_2)\|,\end{aligned}$$

thus,

$$\begin{aligned}&b(\omega)\|s_1 - s_2\| + \|T(\omega, s_1) - T(\omega, s_2)\| \\ &\leq \theta(\omega)\|s_1 - s_2\| - b(\omega)\|s_1 - s_2\| \\ &\leq (\theta(\omega) - b(\omega))\|s_1 - s_2\|.\end{aligned} \tag{5}$$

Since S is dense subset of X, for any $\delta_i(x_i) > 0$, there exist $s_1, s_2 \in S$ such that $\|x_i - s_i\| < \delta_i(x_i)$ for each $i = 1, 2$. Note that, for any $x_1, x_2 \in X$, we get

$$\|s_1 - s_2\| \leq \|s_1 - x_1\| + \|x_1 - x_2\| + \|s_2 - s_2\|. \tag{6}$$

Suppose that

$$\|T(\omega, s_1) - T(\omega, s_2)\|\|\| \leq (\theta(\omega) - b(\omega))\|s_1 - s_2\|.$$

Since

$$\begin{aligned}&\|T(\omega, x_1) - T(\omega, x_2)\| \\ &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_1) - T(\omega, s_2)\| \\ &+\|T(\omega, s_2) - T(\omega, x_2)\|,\end{aligned} \tag{7}$$

substituting (6) in (7), we get

$$\begin{aligned}&\|T(\omega, x_1) - T(\omega, x_2)\| \\ &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\ &+(\theta(\omega) - b(\omega))(\|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\| \\ &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\ &+(\theta(\omega) - b(\omega))\|s_1 - x_1\| + (\theta(\omega) - b(\omega))\|x_1 - x_2\| \\ &+(\theta(\omega) - b(\omega))\|x_2 - s_2\|.\end{aligned}$$

From (5), (6) and (7), it follows that

$$\begin{aligned}&\|T(\omega, x_1) - T(\omega, x_2)\| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + (\theta(\omega) - b(\omega))\|s_1 - x_1\| + (\theta(\omega) - b(\omega))\|x_1 - x_2\| \\ &+(\theta(\omega) - b(\omega))\|x_2 - s_2\|.\end{aligned} \tag{8}$$

For any $\omega \in \Omega$, since $T(\omega, x)$ is a continuous function of $x(\omega)$, for any $\varepsilon > 0$, there exists $\delta_i(x_i) > 0$, for $i = 1, 2$, such that

$$\|T(\omega, x_1) - T(\omega, s_1)\| < \frac{\varepsilon}{4} \tag{9}$$

whenever $\|x_1 - s_1\| < \delta_1(x_1)$ and

$$\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\varepsilon}{4} \tag{10}$$

whenever $\|x_2 - s_2\| < \delta_2(x_2)$. Now, we choosing

$$\delta_1 = \min\{\delta_1(x_1), \frac{\varepsilon}{4}\} \tag{11}$$

and

$$\delta_2 = \min\{\delta_2(x_2), \frac{\varepsilon}{4}\}. \tag{12}$$

By (8), we get

$$\begin{aligned} &\|T(\omega, x_1) - T(\omega, x_2)\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}(\theta(\omega) - b(\omega)) + (\theta(\omega) - b(\omega))\|x_1 - x_2\| \\ &\leq \frac{\varepsilon}{2}(1 + \theta(\omega) - b(\omega)) + (\theta(\omega) - b(\omega))\|x_1 - x_2\|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq (\theta(\omega) - b(\omega))\|x_1 - x_2\|,$$

so,

$$\|b(\omega)(x_1 - x_2) + T(\omega, x_1) - T(\omega, x_2)\| \leq \theta(\omega)\|x_1 - x_2\|.$$

Thus, we have $\omega \in \bigcap_{x_1,x_2 \in X}(C_{x_1,x_2} \cap A \cap B)$, which implies that

$$\bigcap_{s_1,s_2 \in S}(C_{s_1,s_2} \cap A \cap A \cap B) \subset \bigcap_{x_1,x_2 \in X}(C_{x_1,x_2} \cap A \cap B).$$

Also, we have

$$\bigcap_{x_1,x_2 \in X}(C_{x_1,x_2} \cap A \cap B) \subset \bigcap_{s_1,s_2 \in S}(C_{s_1,s_2} \cap A \cap A \cap B).$$

Therefore, we get

$$\bigcap_{x_1,x_2 \in X}(C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in S}(C_{s_1,s_2} \cap A \cap A \cap B).$$

Let $N' = \bigcap_{s_1,s_2 \in S}(B_{s_1,s_2} \cap A)$. Then $\mu(N') = 1$. Since N' is a measurable full-measure set ensuring that the desired properties of $T(\omega, x)$ hold almost surely, extending $x(\omega)$ outside N' guarantees it is a well-defined random variable on all of Ω. Next, we prove that $\forall \omega \in N'$, $T(\omega, x)$ is a deterministic continuous operators satisfying the mapping referred in [4].

Let $x : \Omega \to X$ be a random variable defined for some $x^* \in X$ by

$$x(\omega) = \begin{cases} x_\omega, & \omega \in N' \\ x^*, & \omega \notin N'. \end{cases}$$

Next, we show that $x(\omega)$ is the random variable. The following is how we construct a sequence of random variables $x_n(\omega)$. Let $x_0(\omega)$ be an arbitrary random variable and $x_1(\omega) = T(\omega, x_0(\omega))$. Thus $x_1(\omega)$ is a random variable. Next, we get $x_{n+1}(\omega) = T(\omega, x_n(\omega))$, by repeated generating, it gives that $\{x_n(\omega)\}_{n=1,2,...}$ is a random variables sequence converge to $x(\omega)$. Therefore, $x(\omega)$ is a random variable.

Lastly, we prove the uniqueness of $x(\omega)$. Let $y : \Omega \to X$ be another random fixed point. We want to prove that $x(\omega) = y(\omega)$ almost surely. Define

$$M = \{\omega \in N' : x(\omega) = y(\omega)\}.$$

It is sufficient to demonstrate that $\mu(M) = 0$ in order to prove uniqueness. Suppose, for proof by contradiction, that $\mu(M) > 0$. Then there exists $\omega \in M$ such that $x(\omega) \neq y(\omega)$. But $x(\omega)$ and $y(\omega)$ are fixed point of $T(\omega, \cdot) : X \to X$. Since $T(\omega, \cdot)$ admits a unique fixed point for every $\omega \in N'$, it follows that $x(\omega) = y(\omega)$, contradicting the assumption that $\omega \in M$. Hence, the set $M \neq \emptyset$, i.e., $\mu(M) = 0$ which is contradiction. Thus, $x(\omega) = y(\omega)$ almost surely, which shows that $x(\omega)$ is a unique. Therefore, $x(\omega)$ is a unique random fixed point of T. This completes the proof. □

The next example demonstrates a valid case of Theorem 5.

Example 2
Let $X = \mathbb{R}$ be the Banach space, $\Omega = [0, 1]$ be the probability space with Lebesgue measure, $b(\omega) = \omega \in [0, 1]$, $\theta(\omega) = \frac{\omega+\omega^2}{2}$ and $T(\omega, x) = \frac{\theta(\omega)-\omega}{b(\omega)+1}x = \frac{\omega^2-\omega}{2(\omega+1)}x$ be the random operator satisfying Definition 8 and $\xi(\omega)$ be an explicit random fixed point. For all $x, y \in X = \mathbb{R}$, we get

$$
\begin{aligned}
&\|b(\omega)(x-y) + T(\omega, x) - T(\omega, y)\| \\
&= \|b(\omega)(x-y) + T(\omega, x) - T(\omega, y)\| \\
&= \|(\omega + \frac{\omega^2-\omega}{2(\omega+1)})(x-y)\| \\
&= |\omega + \frac{\omega^2-\omega}{2(\omega+1)}||x-y| \\
&= |\frac{3\omega^2+\omega}{2(\omega+1)}||x-y| \\
&\leq \theta(\omega)\|x-y\|.
\end{aligned}
$$

This inequality holds since $3\omega + 1 \leq (\omega + 1)^2$ for $\omega \in [0, 1]$. Next,we find the fixed point of $T(\omega, \cdot)$. From $T(\omega, \xi(\omega)) = \xi(\omega)$, we get $T(\omega, x) = \frac{\omega^2-\omega}{2(\omega+1)}\xi(\omega)$. This has the solution $\xi(\omega) = 0$ for all $\omega \in [0, 1] = \Omega$. The zero function is trivially measurable. The operator $T(\omega, x)$ and its fixed point $\xi(\omega) = 0$ are shown in Figure 1. as explained Example 2 for $\omega \in (0, 1)$ and $x \in [-1, 1]$.

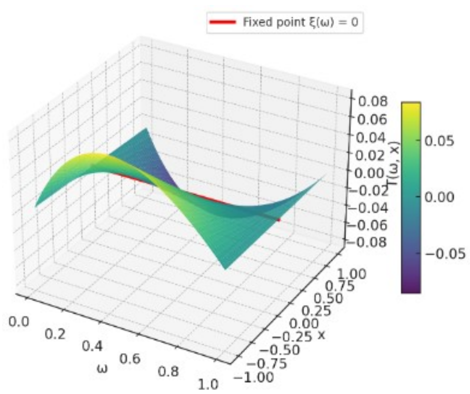

Figure 1. The operator $T(\omega, x)$ and its fixed point $\xi(\omega) = 0$

The behavior of the operator $T(\omega, x)$ as a function of the variable x and the stochastic parameter ω is shown in Figure 1. The surface plot shows how $T(\omega, x)$ varies across the domain, with the color gradient indicating its magnitude according to the scale on the right. The fixed point $\xi(\omega) = 0$, where the operator value and the input x coincide, is highlighted by the red line. This visualization provides an intuitive understanding of the operator's structure and the location of its fixed point.

From Theorem 5, if $b(\omega) = 0$ and $\theta(\omega) = 1$, we obtain the following corollary.

Corollary 1
Assume that (Ω, β, μ) be a complete probability measure space and T be a operator satisfying

$$\|T(\omega, x_1) - T(\omega, x_2)\| < \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$, where X be a separable Banach space. Then a random fixed point of T exists in X.

Proof
Suppose A and B be two sets defined by

$$A = \{\omega \in \Omega : T(\omega, x) \quad \text{is a continuous of} \quad x\}$$

and

$$B_{x_1,x_2} \quad = \quad \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| < \|x_1 - x_2\|\}.$$

Suppose S be a set of countable dense, $S \subset X$. Now, we prove that

$$\textstyle\bigcap_{x_1,x_2 \in X} (B_{x_1,x_2} \cap A) = \bigcap_{s_1,s_2 \in S} (B_{s_1,s_2} \cap A).$$

Then for all $s_1, s_2 \in S$, we get

$$\|T(\omega, s_1) - T(\omega, s_2)\| < \|s_1 - s_2\|. \tag{13}$$

We then arrive at the following result after proving Theorem 5. □

4. Application to a non-linear stochastic integral equation

We now demonstrate that a solution to a non-linear stochastic integral equation exists in a Banach space via Theorem 5. Let (Ω, β, μ) be the probability measure space, β being σ-algebra, and μ the probability measure, and let S be a locally compact metric space. This Hammerstein-type equation (see, [19]) can be represented as follows:

$$x(t_1; \omega) = h(t_1; \omega) + \int_S k(t_1; t_2; \omega) f(t_2; x(t_2; \omega)) d\mu(t_2), \tag{14}$$

where

1. d is a metric imposed on product cartesian of S;
2. μ_0 is a complete σ-finite measure imposed on the collection of Borel subsets of S;
3. $\omega \in \Omega$ where Ω is the supporting set of (Ω, β, μ);
4. $x(t_1; \omega)$ is the unknown vector valued random variable for any $t_1 \in S$;
5. $h(t_1; \omega)$ is the stochastic free term imposed for $t_1 \in S$;
6. $k(t_1, t_2; \omega)$ is the stochastic kernel imposed for $t_1, t_2 \in S$;
7. $f(t_1, x)$ is a vector valued function for $t_1 \in S$ and x.

Note that (14) is called the Bochner integral (see, [27]).

Next, we assume that $C_{n+1} \subset C_n$ is the union of a countable family $\{C_n\}$ of compact sets, such that for every other compact set in S, there exists C_i that contains it (see, [28]).

By using $C = C(S, L_2(\Omega, \beta, \mu))$ and the topology of uniform convergence on compact sets of S, we impose a space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$. This implies that for each fixed $t_1 \in S$, $x(t_1; \omega)$ is a vector valued random variable

$$\|x(t_1;\omega)\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega |x(t_1;\omega)|^2 d\mu(\omega) < \infty.$$

Notice that $C(S, L_2(\Omega, \beta, \mu))$ is a locally convex space and that its topology is provided by

$$\|x(t_1;\omega)\|_n = \sup_{t_1 \in C_n} \|x(t_1;\omega)\|_{L_2(\Omega,\beta,\mu)} \tag{15}$$

which, for each $n \geq 1$, is the countable family of semi-norms. Furthermore, $C(S, L_2(\Omega, \beta, \mu))$ is complete in relation to (15) when $L_2(\Omega, \beta, \mu)$ is complete.

Later, we use $BC = BC(S, L_2(\Omega, \beta, \mu)$ to impose a Banach space containing all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ by the norm

$$\|x(t_1;\omega)\|_{BC} = \sup_{t_1 \in S} \|x(t_1;\omega)\|_{L_2(\Omega,\beta,\mu)}.$$

$BC \subset C$ is a space of all second order vector valued stochastic processes imposed on S which are bounded and continuous in mean square.

Now, we consider the functions $h(t_1;\omega)$ and $f(t_1, x(t_1;\omega))$ to belong to space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We also suppose that, for every pair (t_1, t_2), $k(t_1, t_2; \omega) \in L_\infty(\Omega, \beta, \mu)$ with the norm denoted by

$$|||k(t_1, t_2;\omega)||| = \|k(t_1, t_2;\omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - ess\sup_{\omega\in\Omega} |k(t_1, t_2;\omega)|.$$

Also, $k(t_1, t_2; \omega) \in L_\infty(\Omega, \beta, \mu)$ is assumed to be such that

$$|||k(t_1, t_2;\omega)||| = \|x(t_2;\omega)\|_{L_2(\Omega,\beta,\mu)}$$

is μ-integrable by respect to t_2 for any $t_1 \in S$ and $x(t_2;\omega) \in C(S, L_2(\Omega, \beta, \mu))$ and there is a real valued function G μ-a.e. on S so that $G(S)\|x(t_2;\omega)\|_{L_2(\Omega,\beta,\mu))}$ is μ-integrable and, for any (t_1, t_2) in $S \times S$,

$$|||k(t_1, u;\omega) - k(t_2, u;\omega)||| \cdot \|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)} \leq G(u)\|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)} \quad \mu - a.e..$$

Assume later that $k(t_1, t_2; \omega)$ is continuous in t_1 from S into $L_\infty(\Omega, \beta, \mu)$ for almost everywhere $t_2 \in S$.

Now, we defined the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t_1;\omega) = \int_S k(t_1, t_2;\omega)x(t_2;\omega)d\mu(t_2), \tag{16}$$

this is referred to as a Bochner integral. By the assumptions on $k(t_1, t_2; \omega)$, it follows that, for each $t_1 \in S$, $(Tx)(t_1;\omega) \in L_2(\Omega, \beta, \mu)$ and $(Tx)(t_1;\omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem, that is, $(Tx)(t_1;\omega) \in C(S, L_2(\Omega, \beta, \mu))$.

Lemma 1
([19]) The linear operator T defined by (16) is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.

Definition 9
([29], [30]) Let B and D be Banach spaces. The pair (B, D) is called *admissible* by respect to a linear operator T if $T(B) \subset D$.

Lemma 2
([19]) If T is a continuous linear operator from $C(S, L_2(\Omega,\beta,\mu))$ into itself and $B, D \subset C(S, L_2(\Omega,\beta,\mu))$ are Banach spaces stronger than $C(S, L_2(\Omega,\beta,\mu))$ so that (B, D) is admissible by respect to T, then T is continuous from B into D.

By a *random solution* of (14), we mean a function

$$x(t_1;\omega) \in C(S, L_2(\Omega,\beta,\mu))$$

which satisfies (14) $\mu - a.e.$.

The following is the state proof of the theorem using Theorem 5.

Theorem 6
If (14) is subject to the assumptions as follows:
(1) B and D are Banach spaces stronger than $C(S, L_2(\Omega,\beta,\mu))$ so that (B, D) is admissible by respect to the integral operator imposed by (16);
(2) $x(t_1;\omega) \mapsto f(t_1, x(t_1;\omega))$ is an operator from $Q(\rho) = \{x(t_1;\omega) : x(t_1;\omega) \in D, \|x(t_1;\omega)\|_D \leq \rho\}$ into B satisfying

$$\begin{aligned} & \|f(t_1, x_1(t_1,\omega)) - f(t_1, x_2(t_1,\omega))\|_B \\ \leq \; & (\theta(\omega) - b(\omega))\|x_1(t_1,\omega) - x_2(t_1,\omega)\|_D \end{aligned} \tag{17}$$

for any $x_1(t_1,\omega), x_2(t_1,\omega) \in Q(\rho)$;
(3) $h(t_1;\omega) \in D$,

then a unique stochastic solution of (14) exist in $Q(\rho)$ provided $\frac{l(\omega)}{1-\theta(\omega)+b(\omega)}) < 1$ and

$$\|h(t_1,\omega)\|_D + l(\omega)\|f(t_1,0)\|_B[\frac{1}{1-\theta(\omega)+b(\omega)}] \leq \rho(1 - \frac{l(\omega)}{1-\theta(\omega)+b(\omega)}),$$

where the norm of $T(\omega)$ is denoted by $l(\omega)$.

Proof
Let $\mathcal{U}(\omega) : Q(\rho) \to D$ be a mapping defined by

$$(\mathcal{U}x)(t_1,\omega) = h(t_1,\omega) + \int_S k(t_1,t_2,\omega) f(s, x(t_2,\omega)) d_{\mu_0}(s).$$

Then we get

$$\begin{aligned} \|(\mathcal{U}x)(t_1,\omega)\|_D &\leq \|h(t_1,\omega)\|_D + l(\omega)\|f(t_1, x(t_1,\omega))\|_B \\ &= \|h(t_1,\omega)\|_D + l(\omega)\|f(t_1,0) + f(t_1,x(t_1,\omega)) - f(t_1,0)\|_B \\ &\leq \|h(t_1,\omega)\|_D + l(\omega)\|f(t_1,0)\|_B + l(\omega)\|f(t_1,x(t_1,\omega)) - f(t_1,0)\|_B. \end{aligned}$$

Thus, it follows by (17) that

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < (\theta(\omega) - b(\omega))\|x(t_1,\omega)\|_D$$

which implies that

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < \|x(t_1,\omega)\|_D.$$

Therefore, we obtained

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < \rho. \tag{18}$$

Thus, by (18), we have

$$
\begin{aligned}
& \|(\mathcal{U}x)(t_1,\omega)\|_D \\
\leq\ & \|h(t_1,\omega)\|_D + (\frac{l(\omega)}{1-\theta(\omega)+b(\omega)})\|f(t_1,0)\|_B \\
& +(\frac{l(\omega)}{1-\theta(\omega)+b(\omega)})\|f(t_1,x(t_1,\omega))-f(t_1,0)\|_B \\
=\ & \|h(t_1,\omega)\|_D + l(\omega)\|f(t_1,0)\|_B[\frac{1}{1-\theta(\omega)+b(\omega)}] + (\frac{\rho l(\omega)}{1-\theta(\omega)+b(\omega)}) \\
<\ & \rho(1-\frac{l(\omega)}{1-\theta(\omega)+b(\omega)}) + (\frac{\rho l(\omega)}{1-\theta(\omega)+b(\omega)}) \\
=\ & \rho \qquad (19)
\end{aligned}
$$

and so, by (19), $(\mathcal{U}x)(t_1,\omega)\in Q(\rho)$. Thus, for any $x_1(t_1,\omega)$, $x_2(t_1,\omega)\in Q(\rho)$ and, by condition (2), we get

$$
\begin{aligned}
& \|(\mathcal{U}x_1)(t_1,\omega)-(\mathcal{U}x_2)(t_1,\omega)\|_D \\
=\ & \Big\|\int_S k(t_1,t_2,\omega)[f(t_2,x_1(t_2,\omega))-f(t_2,x_2(t_2,\omega))]d\mu_0(s)\Big\|_D \\
\leq\ & l(\omega)\|f(t_2,x_1(t_2,\omega))-f(t_2,x_2(t_2,\omega))\|_B \\
\leq\ & (\theta(\omega)-b(\omega))\|x_1(t_1,\omega)-x_2(t_1,\omega)\|_D.
\end{aligned}
$$

Since $\frac{l(\omega)}{1-\theta(\omega)+b(\omega)}) < 1$. Consequently, $\mathcal{U}(\omega)$ is a random contraction mapping over $Q(\rho)$. Therefore, by Theorem 5, there is a unique $x^*(t_1,\omega)\in Q(\rho)$, which is a random fixed point of $\mathcal{U}$, i.e., x^* is a stochastic solution of equation (14). This completes the proof. □

Example 3
Consider the non-linear stochastic integral equation as follows:

$$x(t_1;\omega) = sin(t_1) + \omega^2 + \int_0^1 \frac{e^{-|t_1-t_2|}}{(1+|x(t_2;\omega)|)}dt_2. \qquad (20)$$

Next, we compare between equations (14) and (20), we get that $h(t_1;\omega) = sin(t_1)+\omega^2$, the kernel $k(t_1;t_2;\omega) = e^{-|t_1-t_2|}$ is deterministic and integrable over $s\in[0,1]$ and the non-linearity is $f(t_2;x(t_2;\omega)) = \frac{1}{1+|x(t_2;\omega)|}$. Then, the equation (17) is hold.

Also, comparing with integral equation (16), we get that $l(\omega)=\frac{\omega}{2}$ which $l(\omega)$ is the norm of $T(\omega)$. Thus, all assumption of Theorem 6 are satisfied and therefore, random operator T has a random fixed point. The numerical solution of the non-linear stochastic integral equation provided in Example 3 is displayed in Table 1 and Figure 2.

Table 1. The calculation of the solution for the non-linear stochastic integral equation for $x(t_1;\omega)$ of Example 3

Table 1 and Figure 2 present the calculation of the solution for the non-linear stochastic integral equation for$x(t_1;\omega)$ in Example 3. Table 1 presents the calculated values of $x(t_1;\omega)$ for different values of t_1 ranging from 0 to 2, and for several values of the stochastic parameter ω between 0 and 1. The results show that $x(t_1;\omega)$ generally increases with both t_1 and ω reaching a peak around $t_1 = 1.5$ to 1.75 depending on ω, before slightly decreasing. Figure 2 visually illustrates these trends, showing that for each fixed ω, the solution $x(t_1;\omega)$ follows a smooth curve that rises to a maximum and then gently declines, with higher values of ω producing correspondingly higher solution curves. Taken together, the table and figure clearly illustrate how the solution varies with the stochastic parameter ω and the variable t_1.

t_1	Approximate $x(t_1, \omega)$					
	$\omega = 0.00$	$\omega = 0.25$	$\omega = 0.50$	$\omega = 0.75$	$\omega = 0.875$	$\omega = 1.00$
0.00	0.183	0.278	0.550	0.945	1.183	1.278
0.25	0.415	0.498	0.747	1.128	1.351	1.576
0.50	0.632	0.704	0.929	1.297	1.504	1.793
0.75	0.820	0.881	1.082	1.438	1.629	1.964
1.00	0.953	0.999	1.191	1.536	1.722	2.080
1.25	1.027	1.074	1.259	1.597	1.787	2.130
1.50	1.046	1.096	1.277	1.611	1.804	2.134
1.75	1.011	1.066	1.254	1.574	1.771	2.087
2.00	0.923	0.981	1.169	1.486	1.686	2.000

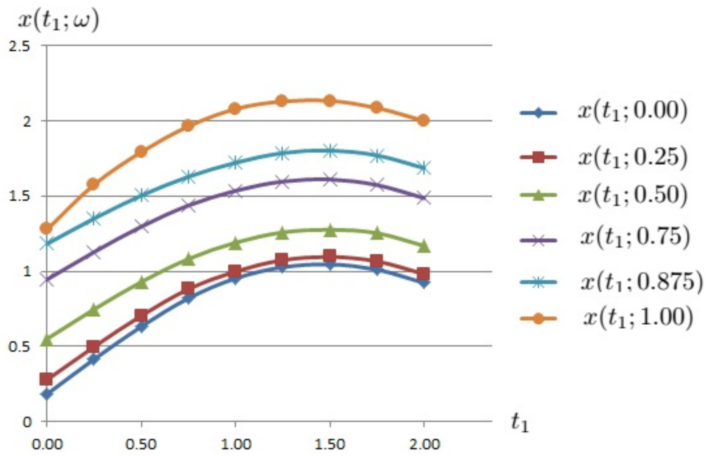

Figure 2. The calculation of the solution for the non-linear stochastic integral equation for $x(t_1; \omega)$

5. Discussion

The results of this work extend and complement those of [22], [23], [24] and [25] by introducing the random (b, θ)-enriched contraction operator, which provides a more general and flexible framework for random fixed point theory. Random fixed point theorems for Hardy–Rogers self-random operators were established by [23], while random $\mathcal{P}$-contractions were examined as a stochastic counterpart of the Banach Contraction Principle by [22], random Hardy–Rogers almost contractions were addressed by [25] and random Z-contractions were introduced by [24]. Compared to these previous contraction types, the enriched contraction in this work allows for wider applicability. Furthermore, this work focuses on non-linear stochastic integral equations of Hammerstein type, expanding the range of possible applications in stochastic analysis.

Acknowledgement

The first, third, and last authors received support from Rajamangala University of Technology Lanna, Nan (RMUTL Nan), while the second author was supported by Rajamangala University of Technology Lanna, Lampang (RMUTL Lampang).

REFERENCES

1. S.Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fundam. Math., vol.3, 133–181, 1922.
2. B.E.Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226, 257–290, 1997.
3. C.Lj.Ćirić, *On a generalization of a Gregus fixed point theorem*, Czechoslov. Math. J., 50, 449–458, 2000.
4. V.Berinde and M.Pacurar, *Approximating fixed points of enriched contractions in Banach spaces*, Fixed Point Theory and Appl. vol.22, no.2, 1–10, 2020.
5. V.Berinde, *Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces*, Carpathian J. Math. vol.35, no. 3, 293–304, 2019.
6. V.Berinde and M.Pacurar, *Existence and approximation of fixed points of enriched contractions and enriched $\varphi-$ contractions*, Symmetry, vol.13, no.3, 498, 2021.
7. P.Dechboon and K. Khammahawong, *Best proximity point for generalized cyclic enriched contractions*, Math. Meth. Appl. Sci., vol.47, 4573–4591, 2024.
8. M.Abbas, R.Anjum and S.Riasat, *Solution of integral equation involving interpolative enriched cyclic Kannan contraction mappings*, Bangmod JMCS, vol.9, 1–9, 2023.
9. S.Salisu, L.Hashim, A.U.Inuwa and A.U.Saje, *Implicit Midpoint Scheme for Enriched Nonexpansive Mappings*, Nonlinear Convex Analysis and Optimization: An International Journal on Numerical, Computation and Applications, 1(2), (2022), 211–225.
10. A.Spacek, *Zufallige Gleichungen*, Czechoslov. Math. J., vol.5, no.80, 462–466, 1955.
11. O.Hans, *Random operator equations*, In: Proceedings of 4th Berkeley Sympos. Math. Statist. Prob., vol. II, part I, pp. 185–202. University of California Press, Berkeley, 1961.
12. O.Hans, *Reduzierende zufallige transformationen*, Czechoslov. Math. J., vol.7, no.82, 154–158, 1957.
13. A.Mukherjee, *Transformation aleatoires separable theorem all point fixed aleatoire*, C. R. Acad. Sci. Paris, Ser. A-B 263, 393–395, 1966.
14. A.T.Bharucha-Reid, *Fixed point theorems in probabilistic analysis*, Bull. Am. Math. Soc. vol.82, no.5, 641–657, 1976.
15. S.Itoh, *Random fixed-point theorems with an application to random differential equations in Banach spaces*, J. Math.Anal. Appl., vol.67, no.2, 261–273, 1979.
16. V.M.Sehgal and C. Waters, *Some random fixed point theorems for condensing operators*, Proc. Am. Math. Soc., vol.90, no.1, 425–429, 1984.
17. M.Saha, *On some random fixed point of mappings over a Banach space with a probability measure*, Proc. Natl. Acad.Sci., India, Sect. A, 76, 219–224, 2006.
18. M.Saha and L.Debnath, *Random fixed point of mappings over a Hilbert space with a probability measure*, Adv. Stud.Contemp. Math., 1, 79–84, 2007.
19. W.J.Padgett, *On a nonlinear stochastic integral equation of the hammerstein type*, Proc. Am. Soc., 38(3), 625–631, 1973.
20. M.Saha and D.Dey, *Some random fixed point theorems for $(\theta, L)-$weak contractions*, Hacet. J. Math. Stat. (accepted and to appear).
21. M.Saha and A.Ganguly, *Random fixed point theorem on a Ciric-type contractive mapping and its consequence*, Fixed Point Theory Appl., Article ID 209, 2012.
22. S.Plubtieng, P.Kumam, S.Dhompongsa and P.Saipara, *On the Result of P-contraction Operators. Nonlinear Convex Analysis and Optimization*, Nonlinear Convex Analysis and Optimization, 1(2), 113–129, 2022.
23. P.Saipara, P.Kumam, P and Y.J.Cho, *Random fixed point theorems for Hardy-Rogers self-random operators with applications to random integral equations*, Stochastics, 90(2), 297-311, 2018.
24. P.Saipara, P.Kumam, A.Sombat, A.Padcharoen and W.Kumam, *Stochastic fixed point theorems for a random Z contraction in a complete probability measure space with application to nonlinear stochastic integral equations*, Mathematics in Natural Science, 1(01), 40–48, 2017.
25. P.Saipara, D.Gopal and W.Kumam, *Random Fixed Point of Random Hardy-Roger Almost Contraction for Solving Nonlinear Stochastic Integral Equations*, Thai J. Math., 379–395, 2018.
26. M.C.Joshi and R.K.Bose, *Some Topics in Nonlinear Functional Analysis*, Wiley, New York, 1984.
27. K.Yosida, *Functional analysis*, Die Grundlehren der math. Wissenschaften, Band 123, Academic Press, New York; Springer-Verlag, Berlin, 1965.
28. R.F.Arens, *A topology for spaces of transformations*, Ann. Math., (2)47, 480–495, 1946.
29. J.Achari, *On a pair of random generalized nonlinear contractions*, Internat. J. Math. Math. Sci., 6, 467–475, 1983.
30. A.C.H.Lee and W.J.Padgett, *On random nonlinear contraction*, Math. Syst. Theory, 11, 77–84, 1977.