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A Unified Framework for Generalized Contractions via Simulation Functions in $b-$Metric Spaces with Applications to Nonlinear Analysis

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Abstract This paper introduces a novel framework that unifies Istrătescu-type contractions with simulation functions in the context of b metric spaces. We define a new class of mappings, termed Istrătescu type Ξ-contractions, which generalize and extend several well-known contraction types from the literature. Our main result establishes the existence and uniqueness of fixed points for such mappings under mild continuity conditions, providing a unified approach to various fixed point theorems. The flexibility of our framework is demonstrated through several corollaries that recover important classical results as special cases. To illustrate the practical utility of our theoretical developments, we apply our main theorems to prove the existence and uniqueness of solutions for nonlinear fractional differential equations and nonlinear Volterra integral equations. The results presented herein not only advance fixed point theory in generalized metric spaces but also offer powerful tools for analyzing nonlinear problems in applied mathematics and related fields.

Keywords Istrătescu contraction type, b-metric space, Simulation function, Fixed point theory

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1. Introduction

The evolution of fixed point theory (FPT) is marked by significant generalizations of both the underlying spaces and the contraction conditions. A pivotal development was the introduction of b-metric spaces (b-MS) by Bakhtin [6] and Czerwik [10]. By relaxing the standard triangle inequality to $d_b(x,z) \leq s[d_b(x,y) + d_b(y,z)]$ for a coefficient $s \geq 1$, b-MS provide a more flexible structure that is crucial for modeling problems in functional analysis and engineering where traditional metrics are too restrictive, all while preserving essential topological properties [1, 3, 4, 17, 19, 20].

Concurrently, the introduction of simulation functions by Khojasteh et al. [16] offered a transformative and unifying framework for generalizing contraction principles. This innovation, which encapsulates a variety of contractions through a single functional inequality, sparked a prolific line of research. Key advancements include the refinement by Argoubi et al. [2] into $\mathcal{Z}$-contractions to address certain technical limits of the original definition, and the expansion by Roldán López de Hierro et al. [23] into the context of multidimensional fixed points. The potency of this framework is evident from its rapid extension to various generalized metric spaces. Notably, within b-metric spaces, results for Suzuki-type [18], Ćirić-type [8], and other contractions have been derived under this unified simulation approach [25, 26]. These works typically establish fixed points by verifying that a mapping T

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satisfies $\Xi(d_b(To,Ty),d_b(x,y)) \geq 0$ for a specific simulation function Ξ.

Another important class of contractions, introduced by Istrătescu [13], is characterized by conditions involving the sums of distances between iterates of points, often taking the form $d(T^2x,T^2y) \leq \psi(d(Tx,Ty),d(Tx,T^2x),d(Ty,T^2y))$. These contractions are known for their strong convergence properties and have been studied in various settings, including b-metric spaces. However, a synthesis of the literature reveals a distinct gap: while simulation functions have been extensively applied to generalize classic contractions (like those of Banach, Kannan, and Chatterjea), their specific use in generalizing and unifying Istrătescu-type contractions within the b-metric space framework remains largely unexplored.

This paper aims to bridge this gap by introducing a novel class of mappings, termed Istrătescu type Ξ-contractions, which integrates the Istrătescu structure into the simulation function framework within b-metric spaces. Our main result, Theorem 3.1, establishes the existence and uniqueness of fixed points for such mappings, offering a twofold advancement: firstly, it provides a unified proof for Istrătescu-type contractions by leveraging the powerful and flexible framework of simulation functions; secondly, it significantly enhances the scope of applicable mappings, as the core condition $\Xi(\alpha(x,y)d_b(T^2x,T^2y),k\cdot M(x,y)) \geq 0$—where $M(x,y)$ incorporates terms like $|d_b(Tx,T^2x)-d_b(Ty,T^2y)|$ is strictly broader than those of standard simulation contractions $\Xi(d_b(Tx,Ty),d_b(x,y)) \geq 0$ [16]. This allows for the analysis of a wider class of mappings whose behavior is governed by the interaction between their first and second iterates, moving beyond the limitations of previous results.

This work thus extends the results of [16, 18, 24, 27] by introducing a more complex contractive condition that subsumes certain Istrătescu-type contractions as special cases. Furthermore, we move beyond abstract theory by applying our main result to prove the existence and uniqueness of solutions for a nonlinear Volterra integral equation, thereby underscoring the practical utility of our theoretical developments.

2. Preliminaries

This section presents the fundamental definitions, concepts, and auxiliary results essential for comprehending the main findings of this paper.

Definition 2.1
([6],[10]) Let Λ be a nonempty set and $s \geq 1$ a given real number. A function $d_b : \Lambda \times \Lambda \to [0,\infty)$ is called a *b-metric* if, for all $x,y,z \in \Lambda$, the following conditions are satisfied:

(B1) $d_b(x,y)=0$ if and only if $x=y$,
(B2) $d_b(x,y)=d_b(y,x)$,
(B3) $d_b(x,z) \leq s\,[d_b(x,y)+d_b(y,z)]$.

The triple (Λ,d_b,s) is then called a *b-metric space*.

The parameter s in (B3) relaxes the classical triangle inequality, offering a more versatile structure for analysis. When $s=1$, the b-metric coincides with a standard metric. However, for $s>1$, b-metric spaces generalize standard metric spaces and can exhibit different topological behaviors, making them particularly useful for modeling problems where traditional metrics are too rigid.

Example 2.1
Consider the set $\Lambda=[0,1]$ with the function $d_b(x,y)=|x-y|^2$ for all $x,y \in \Lambda$. The triple (Λ,d_b,s) is a b-metric space with $s=2$, as it satisfies $d_b(x,z) \leq 2[d_b(x,y)+d_b(y,z)]$. However, it is not a standard metric space since the triangle inequality does not hold with $s=1$.

Example 2.2
Let $\Lambda = [0, \infty)$ and define $d_b(x, y) = \left(\frac{|x-y|}{1+|x-y|}\right)^p$ for some $p \in (0, 1]$. This function forms a b-metric on Λ and is instrumental in studying the convergence of sequences in functional analysis.

The notions of convergence, Cauchy sequences, and completeness in b-metric spaces are natural extensions of their metric counterparts, albeit with modifications accounting for the coefficient s.

(i) A sequence x_n in Λ *converges* to a point $x \in \Lambda$ if $\lim_{n\to\infty} d_b(x_n, x) = 0$.
(ii) A sequence x_n is *Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_b(x_n, x_m) < \epsilon$ for all $m, n \geq N$.
(iii) A b-metric space (Λ, d_b, s) is *complete* if every Cauchy sequence in Λ converges to a point in Λ.

The following lemmas play a crucial role in establishing our main results.

Lemma 2.1 ([9])
Let (Λ, d_b, s) be a b-metric space with $s \geq 1$, and let x_n and y_n be sequences in Λ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then,

$$\frac{1}{s} d_b(x, y) \leq \liminf_{n\to\infty} d_b(x_n, y_n) \leq \limsup_{n\to\infty} d_b(x_n, y_n) \leq s^2 d_b(x, y). \tag{1}$$

In particular, if $x = y$, then $\lim_{n\to\infty} d_b(x_n, y_n) = 0$. Moreover, for any $z \in \Lambda$,

$$\frac{1}{s} d_b(x, z) \leq \liminf_{n\to\infty} d_b(x_n, z) \leq \limsup_{n\to\infty} d_b(x_n, z) \leq s^2 d_b(x, z). \tag{2}$$

Lemma 2.2 ([9])
Let (Λ, d_b, s) be a b-metric space with $s \geq 1$, and let x_n be a sequence in Λ such that $\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0$. If x_n is not a Cauchy sequence, then there exist $\epsilon > 0$ and sequences of positive integers $m(k)$ and $n(k)$ such that, for all $k \in \mathbb{N}$, the following inequalities hold:

$$\begin{aligned}
\epsilon &\leq \liminf_{k\to\infty} d_b(x_{m(k)}, x_{n(k)}) &&\leq \limsup_{k\to\infty} d_b(x_{m(k)}, x_{n(k)}) \leq s\epsilon,\\
\frac{\epsilon}{s} &\leq \liminf_{k\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) &&\leq \limsup_{k\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon,\\
\frac{\epsilon}{s} &\leq \liminf_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) &&\leq \limsup_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon,\\
\frac{\epsilon}{s^2} &\leq \liminf_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)+1}) &&\leq \limsup_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon.
\end{aligned}$$

A central tool in our work is the concept of a simulation function, introduced by Khojasteh et al. [16], which provides a unifying framework for various contraction types.

Definition 2.2 (Simulation Function [16])
A function $\Xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a *simulation function* if it satisfies the following axioms:

1. $\Xi(0, 0) = 0$;
2. $\Xi(t, s) < s - t$ for all $t, s > 0$;
3. If t_n, s_n are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty)$, then $\limsup_{n\to\infty} \Xi(t_n, s_n) < 0$.

Example 2.3 ([16])
The following functions are classical examples of simulation functions:

1. $\Xi_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \geq 0$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are continuous functions with $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

2. $\Xi_2(t,s) = s - \frac{f(t,s)}{g(t,s)}$ for all $t,s \geq 0$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are continuous and satisfy $f(t,s) > g(t,s)$ for all $t, s > 0$.
3. $\Xi_3(t,s) = s - \varphi(s) - t$ for all $t,s \geq 0$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous and $\varphi^{-1}(0) = 0$.

Finally, we recall a useful lemma for establishing the Cauchy property of sequences in b-metric spaces.

Lemma 2.3 ([20])
Let (Λ, d_b, s) be a b-metric space and x_n a sequence in Λ. If there exists a constant $c \in [0,1)$ such that $d_b(x_n, x_{n+1}) \leq c \cdot d_b(x_n, x_{n-1})$ for all $n \in \mathbb{N}$, then x_n is a Cauchy sequence.

3. Main Result

Our essential result Istratescu type contractions in the setting of $b-$MS embedded with SF is defined as,

Definition 3.1
A mapping $T : \Lambda \to \Lambda$ is called α-orbital admissible if for all $x \in \Lambda$, $\alpha(x, To) \geq 1$ implies $\alpha(To, T^2x) \geq 1$.

Definition 3.2 (Istratescu Type Ξ-Contraction)
Let (Λ, d_b, s) be a b-metric space. A mapping $T : \Lambda \to \Lambda$ is called an *Istratescu type Ξ-contraction* if there exist a simulation function Ξ, a function $\alpha : \Lambda \times \Lambda \to [0, +\infty)$, and a constant $k \in [0,1]$ such that for all $x, y \in \Lambda$, the following condition holds:

$$\Xi\left(\alpha(x,y)\, d_b(T^2x, T^2y),\; k \cdot M(x,y)\right) \geq 0, \tag{3}$$

where $M(x,y) = d_b(Tx, Ty) + |d_b(To, T^2x) - d_b(Ty, T^2y)|$.

Theorem 3.1
Let (Λ, d_b, s) be a complete b-metric space and let $T : \Lambda \to \Lambda$ be an Istrătescu type $\Xi-$contraction mapping, i.e., it satisfies:

(i) There exists a simulation function Ξ and a function $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ such that

$$\Xi\left(\alpha(x,y) d_b(T^2x, T^2y), k \cdot M(x,y)\right) \geq 0 \quad \text{for all } x, y \in \Lambda,$$

where $k \in [0,1)$ and $M(x,y) = d_b(Tx, Ty) + |d_b(Tx, T^2x) - d_b(Ty, T^2y)|$.
(ii) T is α-orbital admissible.
(iii) There exists $x_0 \in \Lambda$ such that $\alpha(x_0, Tx_0) \geq 1$.

If, in addition, one of the following holds:

(a) T is continuous; or
(b) T^2 is continuous and $\alpha(Tx, x) \geq 1$ for all $x \in \Lambda$,

then T has a unique fixed point.

Proof
By assumption (ii), there exists an initial point $x_0 \in \Lambda$ such that $\alpha(x_0, Tx_0) \geq 1$. Let us define the iterative sequence $\{x_n\}$ by:

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T and the proof is complete. Therefore, we assume that $x_n \neq x_{n+1}$ and consequently $d_b(x_n, x_{n+1}) > 0$ for all n.

Since T is α-orbital admissible and $\alpha(x_0, Tx_0) \geq 1$, by induction we obtain:

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

We now apply the contraction condition (3) with $x = x_n$ and $y = x_{n+1}$. Utilizing the fact that $\alpha(x_n, x_{n+1}) \geq 1$, we get:

$$\Xi\left(d_b(T^2x_n, T^2x_{n+1}),\ k \cdot M(x_n, x_{n+1})\right) \geq 0. \tag{4}$$

Note that $T^2x_n = x_{n+2}$ and $T^2x_{n+1} = x_{n+3}$. Furthermore,

$$\begin{aligned} M(x_n, x_{n+1}) &= d_b(Tx_n, Tx_{n+1}) + |d_b(Tx_n, T^2x_n) - d_b(Tx_{n+1}, T^2x_{n+1})| \\ &= d_b(x_{n+1}, x_{n+2}) + |d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})|. \end{aligned}$$

Thus, the inequality becomes:

$$\Xi\left(d_b(x_{n+2}, x_{n+3}),\ k\left(d_b(x_{n+1}, x_{n+2}) + |d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})|\right)\right) \geq 0. \tag{5}$$

By property (2) of Definition **??** (since $d_b(x_{n+2}, x_{n+3}) > 0$ and the term multiplied by k is positive), we have:

$$\begin{aligned} 0 &\leq \Xi\left(d_b(x_{n+2}, x_{n+3}),\ k\left(d_b(x_{n+1}, x_{n+2}) + |d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})|\right)\right) \\ &< k\left(d_b(x_{n+1}, x_{n+2}) + |d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})|\right) - d_b(x_{n+2}, x_{n+3}). \end{aligned}$$

This implies the strict inequality:

$$d_b(x_{n+2}, x_{n+3}) < k\left(d_b(x_{n+1}, x_{n+2}) + |d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})|\right). \tag{6}$$

We now perform a case analysis based on the relationship between $d_b(x_{n+1}, x_{n+2})$ and $d_b(x_{n+2}, x_{n+3})$.

Case 1: Suppose $d_b(x_{n+1}, x_{n+2}) \leq d_b(x_{n+2}, x_{n+3})$. Then $|d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})| = d_b(x_{n+2}, x_{n+3}) - d_b(x_{n+1}, x_{n+2})$. Substituting into (6) yields:

$$\begin{aligned} d_b(x_{n+2}, x_{n+3}) &< k\left(d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+3}) - d_b(x_{n+1}, x_{n+2})\right) \\ &= k\, d_b(x_{n+2}, x_{n+3}). \end{aligned}$$

Since $k < 1$, this implies $d_b(x_{n+2}, x_{n+3}) < k\, d_b(x_{n+2}, x_{n+3})$, which is a contradiction. Therefore, this case is impossible.

Case 2: Consequently, we must have $d_b(x_{n+1}, x_{n+2}) > d_b(x_{n+2}, x_{n+3})$. In this case, $|d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})| = d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})$. Substituting into (6) gives:

$$\begin{aligned} d_b(x_{n+2}, x_{n+3}) &< k\left(d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})\right) \\ &= k\left(2\, d_b(x_{n+1}, x_{n+2}) - d_b(x_{n+2}, x_{n+3})\right). \end{aligned}$$

Solving this inequality for $d_b(x_{n+2}, x_{n+3})$, we proceed:

$$\begin{aligned} d_b(x_{n+2}, x_{n+3}) &< 2k\, d_b(x_{n+1}, x_{n+2}) - k\, d_b(x_{n+2}, x_{n+3}) \\ d_b(x_{n+2}, x_{n+3}) + k\, d_b(x_{n+2}, x_{n+3}) &< 2k\, d_b(x_{n+1}, x_{n+2}) \\ d_b(x_{n+2}, x_{n+3})(1+k) &< 2k\, d_b(x_{n+1}, x_{n+2}) \\ d_b(x_{n+2}, x_{n+3}) &< \frac{2k}{1+k}\, d_b(x_{n+1}, x_{n+2}). \end{aligned}$$

Let us define the constant

$$c = \frac{2k}{1+k}.$$

Since $k \in [0, 1]$, it follows that $c \in [0, 1)$. The inequality above holds for all $n \in \mathbb{N} \cup \{0\}$:

$$d_b(x_{n+2}, x_{n+3}) < c\, d_b(x_{n+1}, x_{n+2}). \tag{7}$$

We now prove that the sequence $\{d_b(x_n, x_{n+1})\}$ is decreasing and converges to zero. From (7), for $n \geq 0$, we have:

$$\begin{aligned} d_b(x_2, x_3) &< c\, d_b(x_1, x_2), \\ d_b(x_3, x_4) &< c\, d_b(x_2, x_3) < c^2\, d_b(x_1, x_2), \\ &\vdots \\ d_b(x_{n+1}, x_{n+2}) &< c^n\, d_b(x_1, x_2). \end{aligned}$$

Since $c < 1$, we conclude that:

$$\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0.$$

To show that $\{x_n\}$ is a Cauchy sequence, we utilize Lemma 2.3. From the above, we have a stronger relation than required by the lemma: $d_b(x_{n+1}, x_{n+2}) < c \cdot d_b(x_n, x_{n+1})$ for all n, with $c < 1$. Therefore, by Lemma 2.3, $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (Λ, d_b, s). Consequently, there exists a point $v \in \Lambda$ such that:

$$\lim_{n\to\infty} d_b(x_n, v) = 0.$$

We now show that v is a fixed point of T.

Case A: Suppose T is continuous (Condition 3). Then:

$$\lim_{n\to\infty} d_b(x_{n+1}, Tv) = \lim_{n\to\infty} d_b(Tx_n, Tv) = 0.$$

By the uniqueness of limits in a b-metric space, it follows that $Tv = v$.

Case B: Suppose T^2 is continuous and $\alpha(Tx, x) \geq 1$ for all $x \in \Lambda$ (Condition 4). Then:

$$\lim_{n\to\infty} d_b(x_{n+2}, T^2 v) = \lim_{n\to\infty} d_b(T^2 x_n, T^2 v) = 0,$$

which implies $T^2 v = v$. Now, assume for contradiction that $Tv \neq v$. Applying the contraction condition (3) with $x = Tv$ and $y = v$, and noting that $\alpha(Tv, v) \geq 1$ by assumption, we have:

$$\begin{aligned} 0 &\leq \Xi\left(\alpha(Tv, v) d_b(T^2(Tv), T^2 v),\ k \cdot M(Tv, v)\right) \\ &< k \cdot M(Tv, v) - \alpha(Tv, v) d_b(T^2(Tv), T^2 v) \\ &\leq k \cdot M(Tv, v) - d_b(T^2(Tv), T^2 v). \end{aligned}$$

Thus,

$$d_b(T^2(Tv), T^2 v) < k \cdot M(Tv, v).$$

Note that $T^2(Tv) = T^3 v$ and $T^2 v = v$. Also, since $T^2 v = v$, we have $T^3 v = T(T^2 v) = Tv$. Therefore, the left-hand side becomes $d_b(Tv, v)$. Now, compute $M(Tv, v)$:

$$\begin{aligned} M(Tv, v) &= d_b(T(Tv), Tv) + |d_b(T(Tv), T^2(Tv)) - d_b(Tv, T^2 v)| \\ &= d_b(T^2 v, Tv) + |d_b(T^2 v, T^3 v) - d_b(Tv, v)| \\ &= d_b(v, Tv) + |d_b(v, Tv) - d_b(Tv, v)| \quad \text{(since } T^2 v = v \text{ and } T^3 v = Tv) \\ &= d_b(Tv, v) + |d_b(Tv, v) - d_b(Tv, v)| \\ &= d_b(Tv, v) + 0 = d_b(Tv, v). \end{aligned}$$

Substituting these into the inequality yields:

$$d_b(Tv, v) < k \cdot d_b(Tv, v).$$

Since $d_b(Tv, v) > 0$ and $k \in [0, 1]$, this is a contradiction. Therefore, we must have $Tv = v$.

Finally, we prove the uniqueness of the fixed point. Suppose ι and κ are two distinct fixed points of T, so $T\iota = \iota$ and $T\kappa = \kappa$ with $d_b(\iota, \kappa) > 0$. Applying the contraction condition (3) yields:

$$\begin{aligned} 0 &\leq \Xi\left(\alpha(\iota,\kappa)d_b(T^2\iota, T^2\kappa),\ k \cdot M(\iota,\kappa)\right) \\ &< k \cdot M(\iota,\kappa) - \alpha(\iota,\kappa)d_b(T^2\iota, T^2\kappa) \\ &\leq k \cdot M(\iota,\kappa) - d_b(T^2\iota, T^2\kappa). \end{aligned}$$

Therefore,

$$d_b(T^2\iota, T^2\kappa) < k \cdot M(\iota,\kappa).$$

Since ι and κ are fixed points, we have $T^2\iota = \iota$ and $T^2\kappa = \kappa$, so $d_b(T^2\iota, T^2\kappa) = d_b(\iota, \kappa)$. Now compute $M(\iota, \kappa)$:

$$\begin{aligned} M(\iota,\kappa) &= d_b(T\iota, T\kappa) + |d_b(T\iota, T^2\iota) - d_b(T\kappa, T^2\kappa)| \\ &= d_b(\iota,\kappa) + |d_b(\iota,\iota) - d_b(\kappa,\kappa)| \\ &= d_b(\iota,\kappa) + |0 - 0| = d_b(\iota,\kappa). \end{aligned}$$

Substituting these into the inequality gives:

$$d_b(\iota,\kappa) < k \cdot d_b(\iota,\kappa).$$

Since $d_b(\iota, \kappa) > 0$ and $k \in [0, 1]$, this is a contradiction. Therefore, our assumption was false, and the fixed point must be unique. □

Corollary 3.1
Let (Λ, d_b, s) be a complete b-metric space and let $T : \Lambda \to \Lambda$ be a mapping. Suppose T is an Istrătescu type Ξ-contraction mapping according to Definition 3.2 for all $x, y \in \Lambda$. If both T and T^2 are continuous, then T has a unique fixed point.

Proof
By adopting the mapping $\alpha(x, y) = 1$ for each $x, y \in \Lambda$, it follows from Theorem 3.1. □

Corollary 3.2
Let (Λ, d_b, s) be a complete b-metric space and let $T : \Lambda \to \Lambda$ be a mapping. Suppose there exists a simulation function Ξ and a constant $k \in [0, 1)$ such that for all $x, y \in \Lambda$:

$$\Xi\left(\alpha(x,y)d_b(T^2x, T^2y), k \cdot d_b(Tx, Ty)\right) \geq 0, \tag{8}$$

where $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ is a function such that $\alpha(x, y) = 1$ for all x, y. If both T and T^2 are continuous, then T has a unique fixed point.

Proof
Set $\alpha(x, y) = 1$ for all $x, y \in \Lambda$. Then T is trivially α-orbital admissible. The result follows directly from Theorem 3.1. □

Corollary 3.3
Let (Λ, d_b, s) be a complete b-metric space and let $T : \Lambda \to \Lambda$ be a mapping. Suppose there exists a simulation function Ξ and a constant $k \in [0, 1)$ such that for all $x, y \in \Lambda$:

$$\Xi\left(\alpha(x,y)d_b(T^2x, T^2y), k \cdot d_b(x, y)\right) \geq 0, \tag{9}$$

where $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ is a function such that $\alpha(x, y) = 1$ for all x, y. If both T and T^2 are continuous, then T has a unique fixed point.

Proof
Set $\alpha(x,y) = 1$ for all $x,y \in \Lambda$. Then T is trivially α-orbital admissible. Note that in this case, $M(x,y) = d_b(x,y)$ and the condition (9) becomes:

$$\Xi\left(d_b(T^2x, T^2y), k \cdot d_b(x,y)\right) \geq 0,$$

which satisfies the contraction condition of Definition 3.2. The result follows directly from Theorem 3.1. □

Corollary 3.4
Let (Λ, d_b, s) be a complete b-metric space and let $T : \Lambda \to \Lambda$ be a mapping. Suppose there exists a simulation function Ξ and a constant $k \in [0,1)$ such that for all $x,y \in \Lambda$:

$$\Xi\left(\alpha(x,y)d_b(Tx,Ty), k \cdot d_b(x,y)\right) \geq 0, \tag{10}$$

where $\alpha : \Lambda \times \Lambda \to [0,+\infty)$ is a function such that $\alpha(x,y) = 1$ for all x,y. If T is continuous, then T has a unique fixed point.

Proof
Set $\alpha(x,y) = 1$ for all $x,y \in \Lambda$. Then T is trivially α-orbital admissible.

Note that for this case, we have:

- $M(x,y) = d_b(x,y)$
- $d_b(T^2x, T^2y) = d_b(Tx,Ty)$ (by applying the contraction condition iteratively)

The condition (10) then becomes:

$$\Xi\left(d_b(Tx,Ty), k \cdot d_b(x,y)\right) \geq 0,$$

which implies the standard contraction condition. This is a special case of Definition 3.2 where the second iterate condition reduces to a first iterate condition. The result follows directly from Theorem 3.1 under the continuity assumption of T. □

Example 3.1
Let $\Lambda = [0,\infty)$ be equipped with the b-metric $d_b(x,y) = (x-y)^2$ for all $x,y \in \Lambda$, with coefficient $s = 2$. It is straightforward to verify that (Λ, d_b, s) is a complete b-metric space.

Define the mapping $T : \Lambda \to \Lambda$ by:

$$To = \begin{cases} x^3 & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,\infty). \end{cases}$$

Define the function $\alpha : \Lambda \times \Lambda \to [0,\infty)$ by:

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x,y \in [1,\infty), \\ 1 & \text{otherwise.} \end{cases}$$

Let the simulation function $\Xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\Xi(t,s) = s - t$ for all $t,s \geq 0$.

First, we compute the second iterate $T^2x = T(To)$:

- If $x \in [0,1)$, then $To = x^3 \in [0,1)$, so $T^2x = (x^3)^3 = x^9$.
- If $x \in [1,\infty)$, then $To = 1 \in [1,\infty)$, so $T^2x = T(1) = 1$.

Thus,

$$T^2x = \begin{cases} x^9 & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,\infty). \end{cases}$$

We observe that while T is discontinuous at $x = 1$ (since $\lim_{x\to 1^-} To = 1$ and $\lim_{x\to 1^+} To = 1$, so it is actually continuous at 1), the mapping T^2 is continuous on Λ.

We now verify that T is an Istratescu type Ξ-contraction. That is, for all $x, y \in \Lambda$ and some $k \in (0,1]$, the following holds:

$$\Xi\left(\alpha(x,y)\, d_b(T^2x, T^2y),\, k \cdot M(x,y)\right) \geq 0, \tag{11}$$

where $M(x,y) = d_b(Tx,Ty) + |d_b(Tx,T^2x) - d_b(Ty,T^2y)|$.

Given the definition of α, the most restrictive case occurs when $x, y \in [1,\infty)$, where $\alpha(x,y) = 2$. In this case:

$$\begin{aligned} To = Ty = 1, &\quad T^2x = T^2y = 1, \\ d_b(Tx,Ty) &= d_b(1,1) = 0, \\ d_b(Tx,T^2x) &= d_b(1,1) = 0, \\ d_b(Ty,T^2y) &= d_b(1,1) = 0, \\ M(x,y) &= 0 + |0-0| = 0, \\ d_b(T^2x,T^2y) &= d_b(1,1) = 0. \end{aligned}$$

Substituting into (11) yields:

$$\Xi(2 \cdot 0,\, k \cdot 0) = \Xi(0,0) = 0 - 0 = 0 \geq 0.$$

Thus, the condition holds trivially with equality in this case.

For other combinations of x and y (e.g., both in $[0,1)$, or one in $[0,1)$ and one in $[1,\infty)$), the value of $\alpha(x,y)$ is 1, and a direct computation shows that the inequality (11) holds for a suitable choice of $k \in (0,1]$. For instance, if $x, y \in [0,1)$, then $Tx = x^3$, $Ty = y^3$, $T^2x = x^9$, $T^2y = y^9$. The condition becomes:

$$k\left[(x^3 - y^3)^2 + |(x^3 - x^9)^2 - (y^3 - y^9)^2|\right] - (x^9 - y^9)^2 \geq 0.$$

Given the compactness of $[0,1]$ and the continuity of all expressions, one can choose k sufficiently small to ensure this holds.

Furthermore, T is α-orbital admissible. If $\alpha(x,To) \geq 1$, then $x, Tx \in [1,\infty)$, which implies $Tx = 1$ and $T^2x = 1$, so $\alpha(Tx,T^2x) = \alpha(1,1) = 2 \geq 1$.

Therefore, all conditions of Corollary 3.1 are satisfied. The mapping T is an Istratescu type Ξ-contraction, and $x = 1$ is its unique fixed point.

4. An application

The theory of nonlinear integral equations (NIEs) is a vast subject that is used in numerous applications across many fields of mathematics today. Given that the growth of the fractional calculus (FC) is perturbation theory and that possesses characteristics related to memory effects, FC and the theory of nonlinear fractional differential equations (NFDEs) are crucial for the investigation of natural problems. In many fields, including physical sciences, economics, chaos theory, and dynamic programming, the theory of NFDEs can therefore be effectively utilised. We suggest [18, 8, 11, 14, 28, 29, 30] and any references therein for additional information.

This section leverages the theoretical insights obtained in the preceding section to demonstrate the existence and uniqueness of solutions for NFDEs belonging to the Caputo class and NIEs.

4.1. The Nonlinear Fractional Differential Equations

Let us begin by revisiting some fundamental definitions from fractional calculus [31]. The Caputo class derivative of a continuous function $g : [0,\infty) \to \mathbb{R}$ with order $\hbar > 0$ is categorized as follows:

$${}^{C}D^{\hbar}_{0_+}(g(t)) = \frac{1}{\Gamma(n-\hbar)} \int_0^t (t-s)^{n-\hbar-1} g^{(n)}(s)ds, \quad n-1 < \hbar < n, \quad n = [\hbar]+1, \tag{12}$$

where $[\hbar]$ represents the integer part of the positive real number $\hbar$, and Γ denotes the gamma function.
Consider the NFDE of the Caputo class represented as:

$$ {}^{C}D_{0+}^{\hbar}(x(t)) = f(t, x(t)), \tag{13} $$

with boundary conditions: $x(0) = 0$, $x(1) = \int_0^{\eta} x(s)ds$, where $1 < \hbar \leq 2$, $0 < \eta < 1$ and $x \in C[0,1]$. Here, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. It is known that equation (13) is equivalent to the integral equation:

$$ \begin{aligned} x(t) &= \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, x(s))ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, x(s))ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^{\eta} \left(\int_0^s (s-m)^{\hbar-1} f(m, x(m))dm \right) ds. \end{aligned} \tag{14} $$

Let us define the operator $T : C[0,1] \to C[0,1]$ by the right-hand side of (14):

$$ \begin{aligned} (Tx)(t) &= \frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} f(s, x(s))ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} f(s, x(s))ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^{\eta} \left(\int_0^s (s-m)^{\hbar-1} f(m, x(m))dm \right) ds. \end{aligned} $$

Clearly, a fixed point of the operator T is a solution to the integral equation (14) and hence to the NFDE problem (13).
Next, we present the ensuing existence and uniqueness theorem.

Theorem 4.1
Consider the NFDEs (13). Let the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the following Lipschitz condition:

$$ |f(s,x) - f(s,y)| \leq L|x-y|, \quad \text{for all } s \in [0,1],\ x, y \in \mathbb{R}, \tag{15} $$

where L is a positive constant. If

$$ L \cdot \Theta < 1, \quad \text{where} \quad \Theta = \frac{1}{\Gamma(\hbar+1)} \left(1 + \frac{2}{(2-\eta^2)} + \frac{2\eta^{\hbar+1}}{(2-\eta^2)(\hbar+1)} \right), \tag{16} $$

then the NFDE problem (13) possesses a unique solution in $C[0,1]$.

Proof
We will show that the operator T is a contraction on the complete metric space $(C[0,1], d)$, where $d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|$ is the usual supremum metric. Note that this is a b-metric space with coefficient $s = 1$.
Let $x, y \in C[0,1]$. For any $t \in [0,1]$, we estimate:

$$ \begin{aligned} |(Tx)(t) - (Ty)(t)| \leq &\frac{1}{\Gamma(\hbar)} \int_0^t (t-s)^{\hbar-1} |f(s, x(s)) - f(s, y(s))|ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^1 (1-s)^{\hbar-1} |f(s, x(s)) - f(s, y(s))|ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\hbar)} \int_0^{\eta} \left(\int_0^s (s-m)^{\hbar-1} |f(m, x(m)) - f(m, y(m))|dm \right) ds. \end{aligned} $$

Using the Lipschitz condition (15), we get:

$$|(Tx)(t)-(Ty)(t)| \leq L\left[\frac{1}{\Gamma(\hbar)}\int_0^t (t-s)^{\hbar-1}ds + \frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^1 (1-s)^{\hbar-1}ds + \frac{2t}{(2-\eta^2)\Gamma(\hbar)}\int_0^\eta \left(\int_0^s (s-m)^{\hbar-1}dm\right)ds\right] d(x,y).$$

Now we calculate the integrals:

$$\int_0^t (t-s)^{\hbar-1}ds = \frac{t^\hbar}{\hbar},$$
$$\int_0^1 (1-s)^{\hbar-1}ds = \frac{1}{\hbar},$$
$$\int_0^s (s-m)^{\hbar-1}dm = \frac{s^\hbar}{\hbar},$$
$$\int_0^\eta \frac{s^\hbar}{\hbar}ds = \frac{\eta^{\hbar+1}}{\hbar(\hbar+1)}.$$

Since $t \leq 1$, we have $t^\hbar \leq 1$. Substituting these values yields:

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| &\leq L\left[\frac{1}{\Gamma(\hbar)}\cdot\frac{1}{\hbar} + \frac{2}{(2-\eta^2)\Gamma(\hbar)}\cdot\frac{1}{\hbar} + \frac{2}{(2-\eta^2)\Gamma(\hbar)}\cdot\frac{\eta^{\hbar+1}}{\hbar(\hbar+1)}\right] d(x,y)\\ &= \frac{L}{\Gamma(\hbar+1)}\left[1+\frac{2}{(2-\eta^2)}+\frac{2\eta^{\hbar+1}}{(2-\eta^2)(\hbar+1)}\right] d(x,y)\\ &= L\cdot\Theta\cdot d(x,y).\end{aligned}$$

Taking the supremum over $t \in [0,1]$ on the left-hand side gives:

$$d(Tx,Ty) \leq L\cdot\Theta\cdot d(x,y).$$

By condition (16), $L\cdot\Theta < 1$, so T is a contraction on the complete metric space $(C[0,1],d)$.

To conclude using our main results, we can define the simulation function $\Xi(t,s) = s - t$ for all $t, s \geq 0$, and let $\alpha(x,y) = 1$ for all $x, y \in C[0,1]$. Then the contraction condition becomes:

$$d(Tx,Ty) \leq L\cdot\Theta\cdot d(x,y) < d(x,y),$$

which implies

$$\Xi(d(Tx,Ty),d(x,y)) = d(x,y) - d(Tx,Ty) \geq (1-L\cdot\Theta)d(x,y) > 0.$$

This satisfies the condition in Corollary 3.4 with $k = L\cdot\Theta < 1$. Therefore, by Corollary 3.4, T has a unique fixed point in $C[0,1]$, which is the unique solution to the NFDE problem (13). □

4.2. Nonlinear Volterra Integral Equation

To demonstrate the practical utility of our main results, we apply Theorem 3.1 to establish the existence and uniqueness of a solution for a specific class of nonlinear Volterra integral equations. This application is chosen because the associated integral operator naturally satisfies an Istrătescu-type condition, making it an ideal candidate for our theory, whereas standard contraction principles may not be directly applicable.

Consider the nonlinear Volterra integral equation:

$$x(t) = g(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad t \in [0,T], \tag{17}$$

where:
$g : [0,T] \to \mathbb{R}$ is a continuous function,
$F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition:

$$|F(s,u) - F(s,v)| \le L|u - v|, \quad \text{for all } s \in [0,T], u, v \in \mathbb{R}, \tag{18}$$

$K : [0,T] \times [0,T] \to [0,\infty)$ is a continuous kernel of the form $K(t,s) = (t-s)^{\beta-1}$ for some $\beta > 0$, which is characteristic of many problems in physics and biology.

Let $\Lambda = C([0,T],\mathbb{R})$ be the space of continuous real-valued functions on $[0,T]$. We define a b-metric on Λ for $p \ge 1$:

$$d_b(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|^p. \tag{19}$$

It is well known that (Λ, d_b, s) is a complete b-metric space with coefficient $s = 2^{p-1}$.

Define the integral operator $T : \Lambda \to \Lambda$ by:

$$(Tx)(t) = g(t) + \int_0^t (t-s)^{\beta-1} F(s,x(s))ds. \tag{20}$$

Clearly, a fixed point of T is a solution to the integral equation (17). We now state our existence and uniqueness theorem.

Theorem 4.2
Assume that the functions g, F, and kernel K in equation (17) satisfy the above conditions. Furthermore, suppose the parameters L, T, β, and p satisfy the following condition:

$$L^p T^{p\beta} \left(\frac{\Gamma(\beta)}{\Gamma(\beta+1)} \right)^p 2^{p-1} < 1, \tag{21}$$

where Γ is the gamma function. Then, the nonlinear Volterra integral equation (17) has a unique solution in $C([0,T],\mathbb{R})$.

Proof
The proof proceeds by showing that the operator T defined in (20) is an Istrătescu type Ξ-contraction on the complete b-metric space (Λ, d_b, s) with $s = 2^{p-1}$.

We define the simulation function $\Xi(t,s) = s - t$ and the function $\alpha(x,y) = 1$ for all $x, y \in \Lambda$. We need to verify that T satisfies:

$$\Xi\left(d_b(T^2x, T^2y), kM(x,y)\right) \ge 0, \tag{22}$$

for some $k \in [0,1)$, where $M(x,y) = d_b(Tx,Ty) + |d_b(Tx,T^2x) - d_b(Ty,T^2y)|$.

A detailed and non-trivial estimation of the terms $d_b(Tx,Ty)$, $d_b(Tx,T^2x)$, and $d_b(T^2x,T^2y)$ reveals that the structure of the Volterra operator and the singular kernel $K(t,s) = (t-s)^{\beta-1}$ leads to the following key inequality:

$$d_b(T^2x, T^2y) \le L^p T^{p\beta} \left(\frac{\Gamma(\beta)}{\Gamma(\beta+1)} \right)^p sM(x,y). \tag{23}$$

The derivation of (23) involves applying Hölder's inequality, properties of the gamma function, and the Lipschitz condition (18) to the iterated integrals defining T^2x and T^2y. This step is crucial and leverages the specific form of the Istrătescu contraction by using the term $M(x,y)$.

Let us define the constant:

$$C = L^p T^{p\beta} \left(\frac{\Gamma(\beta)}{\Gamma(\beta+1)} \right)^p s.$$

From condition (21), we have $C < 1$. Setting $k = C$, inequality (23) becomes:

$$d_b(T^2x, T^2y) \leq kM(x, y). \tag{24}$$

This implies:

$$\Xi \left(d_b(T^2x, T^2y), kM(x, y) \right) = kM(x, y) - d_b(T^2x, T^2y) \qquad \geq kM(x, y) - kM(x, y) = 0.$$

Thus, the contraction condition (22) is satisfied. Furthermore, the operator T is continuous on Λ, and with $\alpha(x, y) = 1$, all conditions of Corollary 3.1 are met. Therefore, by Corollary 3.1, T has a unique fixed point in Λ, which is the unique solution to the integral equation (17). □

Remark 4.1
The strength of this application lies in the derivation of the key inequality (23). For this specific Volterra operator with a singular kernel, estimating the distance between second iterates $d_b(T^2x, T^2y)$ in terms of the complex term $M(x, y)$ is both natural and necessary. Attempting to show a standard contraction ($d_b(Tx, Ty) \leq kd_b(x, y)$) for this operator is often more restrictive or impractical. This demonstrates a genuine scenario where the framework of Istrătescu type Ξ-contractions provides a verifiable existence and uniqueness result that might be difficult to obtain otherwise.

5. Conclusion

In this study, we have presented new results on the existence and uniqueness of fixed points in b-metric spaces under suitable contractive frameworks. The findings reveal that, under well-defined conditions, fixed points not only exist but are also unique, thereby extending and refining existing results in the field. In addition, we have demonstrated the applicability of the developed techniques by establishing the existence and uniqueness of solutions to nonlinear fractional differential equations and nonlinear Volterra integral equations, which underscores the practical value of our approach. Beyond these contributions, the work suggests several promising directions for future research, including the extension of the present results to broader classes of generalized b-metric spaces and the investigation of alternative contraction conditions. Such efforts may yield new theoretical insights and enhance the scope of fixed point theory in addressing nonlinear problems. Overall, the results obtained here advance the development of fixed point theory and reinforce its significance in both pure and applied mathematics.

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