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The Epanechnikov-Rayleigh Distribution: Statistical Properties and Real-World Applications

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Abstract In this work, we present the Epanechnikov-Rayleigh Distribution (ERD), a new one-parameter lifespan distribution that is obtained by combining the standard Rayleigh distribution with the Epanechnikov kernel. The study investigates ERD's statistical characteristics, such as quantiles, moments, cumulative distribution function (CDF), and probability density function (PDF). Maximum likelihood estimation (MLE) is used for parameter estimation, and its consistency and effectiveness are confirmed by extensive simulation experiments. Lower Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values show that ERD has a better goodness-of-fit than conventional Rayleigh, Weibull, and Gamma distributions in real-world applications in reliability analysis and environmental modeling. The results demonstrate how ERD is a useful tool for real-world statistical analysis due to its adaptability and efficiency in simulating intricate data patterns.

Keywords Epanechnikov, Rayleigh distribution, Moments, Maximum likelihood.

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1. Introduction

In many real-world domains, such as science and engineering, statistical research relies heavily on the modeling and analysis of natural phenomena. A lot of work has been done in the last few decades to create statistical models that can faithfully capture the characteristics of natural phenomena [1]. The Rayleigh distribution is one of them that has been used extensively to model data in fields like environmental research, reliability analysis, and signal processing [2]. Despite its adaptability, the traditional Rayleigh distribution frequently fails to reflect intricate data patterns that are prevalent in real-world datasets, such as multi-modality, skewness, or heavy-tailed behavior.

Researchers have put forth a number of generalizations and extensions of the Rayleigh distribution in an effort to overcome these constraints. To improve the adaptability and flexibility of conventional models, for example, kernel-based techniques and compound distributions have been proposed [3]. Specifically, kernel density estimation has become a potent non-parametric method for modeling and smoothing intricate data structures [4]. The Epanechnikov kernel is well known among kernel functions for its effectiveness and ideal density estimation characteristics [5].

In this paper, we suggest a new distribution, the Epanechnikov-Rayleigh Distribution (ERD), which combines the Rayleigh distribution and the Epanechnikov kernel function. This new distribution improves the modeling of data with non-standard properties by combining the advantages of kernel-based techniques and the Rayleigh distribution. The suggested method is both economical and computationally efficient because it does not call for the insertion of extra parameters, in contrast to conventional techniques.

Many academics used the distribution theory to generalize certain distributions. [6] proposed the transmuted Janardan distribution using the quadratic transmutation map that [7] had produced. The same map is used

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to generalize the Ishita distribution [9] and the two-parameter Lindley distribution [8]. [10] generalized two distributions in her master's thesis: the transmuted Gamma-Gompertz Distributions and the transmuted Generalized Type-II Half-Logistics Distributions. [11] computed the transmuted reciprocal and the two-parameter weighted exponential distributions. On the other hand, [12] extended the Pareto distribution using the Epanechnikov kernel technique.

Applications where precise representation of data variability is essential, such as survival studies, environmental modeling, and dependability analysis, are especially well-suited for the Epanechnikov-Rayleigh Distribution. We derive the novel distribution's mathematical formulation, investigate its statistical characteristics, and use simulation studies and real-world data applications to show its usefulness. In terms of goodness-of-fit, as determined by metrics like the Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC), the results demonstrate that the Epanechnikov-Rayleigh Distribution performs better than the classical Rayleigh distribution and other rival models.

The probability density function of Rayleigh distribution is given by

$$f(x) = 2\theta x e^{-\theta x^2},\ x \geq 0$$

With cumulative distribution function given by

$$F(x) = 1 - e^{-\theta x^2},\ x \geq 0$$

2. Epanechnikov-Rayleigh Distribution

Kernel functions are employed in the theory of functions that solve particular differential equations in a specified domain. The Epanechnikov Kernel function (EKF) is one of these functions [1]. A common option for kernel density estimation, a non-parametric method of estimating the probability density function of a random variable, is the Epanechnikov kernel function. The definition of the Epanechnikov kernel is:

$$k(u) = \frac{3}{4}\left(1 - u^2\right), \qquad |u| \leq 1$$

(EKF) is a probability density function that is continuous and has the lowest mean square error (MSE).

Definition 2.1. The probability density function of the Epanechnikov Rayleigh distribution (ERD) is given by the following theorem:

Theorem 2.2
A random variable X is said to have an ERD if its CDF and PDF are respectively given by

$$G(x) = \frac{3}{2}\left[\frac{2}{3} - e^{-2\theta x^2} + \frac{1}{3}e^{-3\theta x^2}\right]$$
$$g(x) = 3\theta(2xe^{-2\theta x^2} - xe^{-3\theta x^2})$$

Proof
The CDF $(G(x))$ is constructed by applying the Epanechnikov kernel smoothing technique to the CDF of the Rayleigh distribution:

$$F(x) = 1 - e^{-\theta x^2}$$

The Epanechnikov kernel, defined as

$$K(u) = \frac{3}{4}\left(1 - u^2\right), \quad |u| \leq 1$$

is integrated over the Rayleigh CDF:

$$G(x) = 2\int_0^{F(x)} k(u)du$$

$$G(x) = \frac{3}{2}\int_0^{F(x)} \left(1-u^2\right) du$$

$$G(x) = \frac{3}{2}\int_0^{1-e^{-\theta x^2}} \left(1-u^2\right) du$$

$$= \frac{3}{2}\left[1-e^{-\theta x^2} - \frac{1}{3}\left(1-e^{-\theta x^2}\right)^3\right]$$

By simplification we get

$$G(x) = \frac{3}{2}\left[\frac{2}{3} - e^{-2\theta x^2} + \frac{1}{3}e^{-3\theta x^2}\right]$$

Differentiating (2.1) with respect to X we get

$$g(x) = 3\theta(2xe^{-2\theta x^2} - xe^{-3\theta x^2})$$

□

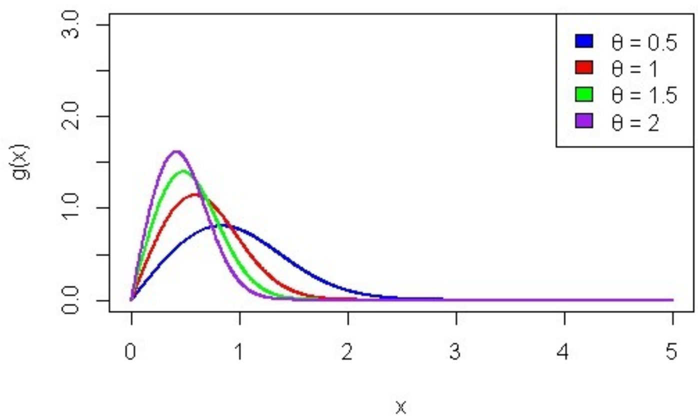

Figure 1. Plot of $g(x)$ for different θ values

3. Moment and Moment Generating Function

3.1. The Moment of the Distribution

Theorem 3.1
Let X be a random variable then the r^{th} moment is given by

$$E\left(x^r\right) = 3\theta\left[\frac{1}{(2\theta)^{\frac{r+2}{2}}}\Gamma\left(\frac{r}{2}+1\right) - \frac{1}{2(3\theta)^{\frac{r+1}{2}}}\Gamma\left(\frac{r}{2}+1\right)\right]$$

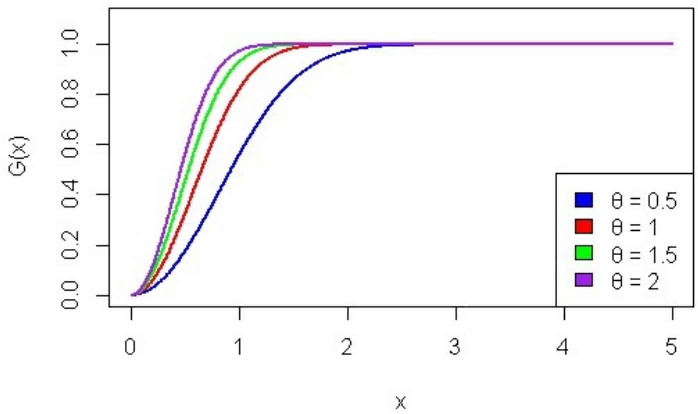

Figure 2. Plot of $G(x)$ for different θ values

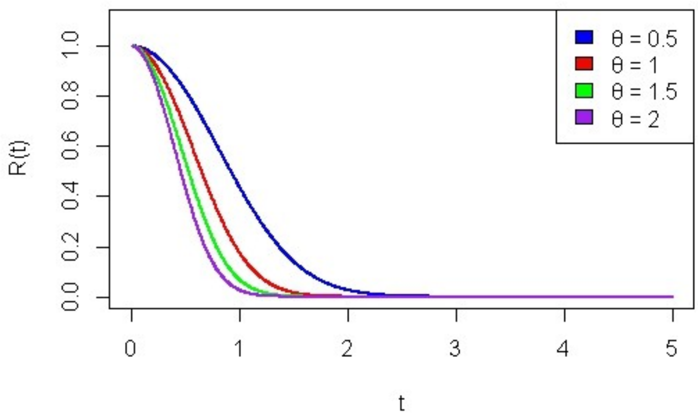

Figure 3. Plot of $R(t)$ for different θ values

Proof
We have

$$E\left(x^{r}\right) = \int_{-\infty}^{\infty} x^{r} g(x) dx$$

$$= 3\theta \int_{0}^{\infty} x^{r}(2xe^{-2\theta x^{2}} - xe^{-3\theta x^{2}})dx$$

$$= 3\theta \left[\int 2x^{r+1}e^{-2\theta x^{2}}dx - \int x^{r+1}e^{-3\theta x^{2}}dx\right]$$

Let $u_1 = 2\theta x^2 \Longrightarrow x = \frac{u_1^{\frac{1}{2}}}{\sqrt{2}\theta^{\frac{1}{2}}} \Longrightarrow dx = \frac{u_1^{\frac{-1}{2}}}{2\sqrt{2}\theta^{\frac{1}{2}}} du_1$

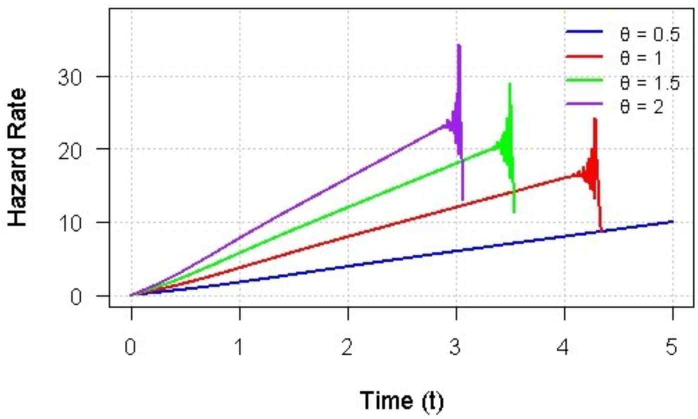

Figure 4. Plot of $H(t)$ for different θ values

And $u_2 = 3\theta x^2 \Longrightarrow x = \frac{u_2^{\frac{1}{2}}}{\sqrt{3}\theta^{\frac{1}{2}}} \Longrightarrow dx = \frac{u_2^{\frac{-1}{2}}}{2\sqrt{3}\theta^{\frac{1}{2}}} du_2$
By substituting the value of u_1 and u_2 in (3.1) we get

$$E\left(x^r\right) = 3\theta\left[\frac{1}{(2\theta)^{\frac{r+2}{2}}}\Gamma\left(\frac{r}{2}+1\right) - \frac{1}{2(3\theta)^{\frac{r+2}{2}}}\Gamma\left(\frac{r}{2}+1\right)\right]$$

For $r = 1$:

$$E(x) = 3\theta\left[\frac{\sqrt{\pi}}{2(2\theta)^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{2(3\theta)^{\frac{3}{2}}}\right]$$

For $r = 2$:

$$E\left(x^2\right) = 3\theta\left[\frac{1}{(2\theta)^2} - \frac{1}{2(3\theta)^2}\right] = \frac{7}{12\theta}$$

Then the variance of x is

$$v(x) = E\left(x^2\right) - (E(x))^2$$

$$= \frac{7}{12\theta} - \left(3\theta\left[\frac{3}{(2\theta)^{\frac{3}{2}}}\Gamma\left(\frac{1}{2}+1\right) - \frac{1}{2(3\theta)^{\frac{r+2}{2}}}\Gamma\left(\frac{1}{2}+1\right)\right]\right)^2$$

$$v(x) = \frac{7}{12\theta} - \frac{\sqrt{\pi}}{2}\left(\frac{3^{\frac{3}{2}} - 2^{\frac{3}{2}}}{2^{\frac{3}{2}}\sqrt{3\theta}}\right)$$

□

3.2. Moment Generating Function

Theorem 3.2
Let X be a random variable belonging to ERD; then the moment generating function of the random variable is given by

$$m_x(t) = 3\theta\sum_{n=0}^{\infty}\left[\frac{t^n}{n!}\left(\frac{1}{(2\theta)^{\frac{n+2}{2}}}\Gamma\left(\frac{n}{2}+1\right) - \frac{1}{(3\theta)^{\frac{n+2}{2}}}\Gamma\left(\frac{n}{2}+1\right)\right)\right]$$

Proof

$$m_x(t) = \int_0^\infty e^{tx} g(x)dx$$

$$= 3\theta \left(\int_0^\infty e^{tx} 2xe^{-2\theta x^2} dx - \int_0^\infty e^{tx} xe^{-3\theta x^2} dx \right)$$

$$= 3\theta \left(\int_0^\infty \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} 2xe^{-2\theta x^2} dx - \int_0^\infty \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} xe^{-3\theta x^2} dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(3\theta \int 2x^{n+1} e^{-2\theta x^2} dx - \int x^{n+1} e^{-3\theta x^2} dx \right)$$

$$= 3\theta \sum_{n=0}^{\infty} \left[\frac{t^n}{n!} \left(\frac{1}{(2\theta)^{\frac{n+2}{2}}} \Gamma\left(\frac{n}{2}+1\right) - \frac{1}{2(3\theta)^{\frac{n+2}{2}}} \Gamma\left(\frac{n}{2}+1\right) \right) \right]$$

□

4. Maximum Likelihood Estimator

One popular technique for calculating the distribution parameters is the maximum likelihood approach of estimate [12].

Let $x_1, x_2, \ldots, x_n$ be a random sample from ERD that is independent and identically distributed. Where

$$g(x_i) = 3\theta(2x_i e^{-2\theta x_i^2} - x_i e^{-3\theta x_i^2})$$

Then, the joint pdf of $x_1, x_2, \ldots,$ and x_n is given by

$$g(x_1, x_2, \ldots, x_n \setminus \theta) = \prod_{i=1}^{n} 3\theta(2x_i e^{-2\theta x_i^2} - x_i e^{-3\theta x_i^2})$$

The likelihood function is

$$l(\theta) = (3\theta)^n \prod_{i=1}^{n} (2x_i e^{-2\theta x_i^2} - x_i e^{-3\theta x_i^2})$$

The log-likelihood function is

$$\log(l(\theta)) = n\log(3) + n\log(\theta) + \sum_{i=1}^{n} \log(2x_i e^{-2\theta x_i^2} - x_i e^{-3\theta x_i^2})$$

Now, deriving the log-likelihood function with respect to the parameter θ, we get

$$\frac{d\log(l(\theta))}{d\theta} = \frac{n}{\theta} + \frac{\frac{1}{x}\, e^{-2\theta x_i^2} - \frac{1}{3x}\, e^{-3\theta x_i^2}}{2x_i e^{-2\theta x_i^2} - x_i e^{-3\theta x_i^2}}$$

$$\frac{n}{\theta} = \frac{1 - \frac{1}{3}e^{-\theta x^2}}{x^2\left(2 - e^{-\theta x^2}\right)}$$

The solution of (4.1) is the maximum likelihood estimator for ERD parameters. Since the Newton-Raphson approach is effective in approximating the roots of nonlinear equations, we will utilize numerical methods to solve (4.1) as there is no exact solution.

5. Reliability Analysis

Definition 5.1. The reliability function is defined to be the probability that an item's life lasts longer than t units. Therefore, mathematically, it is defined as

$$R(t) = p(T \geq t) = 1 - G(t)$$

$$= 1 - \frac{3}{2}\left[\frac{2}{3} - e^{-2\theta t^2} + \frac{1}{3}e^{-3\theta t^2}\right]$$

$$R(t) = \frac{3}{2}e^{-2\theta t^2} - \frac{1}{2}e^{-3\theta t^2}$$

Definition 5.2. The hazard rate is the probability that the life of the item ends in the next moment if it remains alive till time t. It is a practical approach to explaining the distribution.

$$h(t) = \frac{g(t)}{R(t)}$$

$$= \frac{3\theta(2te^{-2\theta t^2} - te^{-3\theta t^2})}{\frac{3}{2}e^{-2\theta t^2} - \frac{1}{2}e^{-3\theta t^2}}$$

$$h(t) = \frac{2\theta t(2 - e^{-\theta t^2})}{1 - \frac{1}{3}e^{-\theta t^2}}$$

6. Ordered Statistics

In real-world life, order statistics are essential, especially in reliability engineering, quality control, and extreme-value analysis. For example, failure thresholds in materials or devices may be represented by the maximum value, whereas the weakest component in a system may be indicated by the minimum value among a sample of lifetimes. Time to first failure or time to system breakdown in systems with several identical components can be modeled using order statistics in the framework of the ERD.

Consider the random variables $X_1, X_2, \cdots, X_n$. They almost assume to be independent and identically distributed. Arrange them in an ascending order, so we get, where

$$X_{(1)} = \min(X_1, X_2, \cdots, X_n)$$

$X_{(i)}$ is the i^{th} order statistic $(i = 1, \ldots, n)$, and

$$X_{(n)} = \max(X_1, X_2, \cdots, X_n)$$

assuming the ERD is being followed by these random variables. The characteristics of these random variables and their uses are covered in the topic of order statistics [10]. As a result, the general case $x_{(i)}, 1 < i < n$, and the pdf of $x_{(1)}, x_{(n)}$ are defined by

$$g_1(x) = ng(x)\left[1 - G(x)\right]^{n-1}$$

$$= 3n\theta[2te^{-2\theta t^2} - te^{-3\theta t^2}]\left[1 - \frac{3}{2}\left[\frac{2}{3} - e^{-2\theta t^2} + \frac{1}{3}e^{-3\theta t^2}\right]\right]^{n-1}$$

$$g_1(x) = 3n\theta[2te^{-2\theta t^2} - te^{-3\theta t^2}]\left[\frac{3}{2}e^{-2\theta t^2} - \frac{1}{2}e^{-3\theta t^2}\right]^{n-1}$$

$$g_n(x) = 3n\theta[2te^{-2\theta t^2} - te^{-3\theta t^2}]\left[\frac{3}{2}\left[\frac{2}{3} - e^{-2\theta t^2} + \frac{1}{3}e^{-3\theta t^2}\right]\right]^{n-1}$$

$$g_n(x) = 3n\theta[2te^{-2\theta t^2} - te^{-3\theta t^2}]\left[1 - \frac{3}{2}e^{-2\theta t^2} + \frac{1}{2}e^{-3\theta t^2}\right]^{n-1}$$

We use R to simulate $X_{(1)}$ for $n = 10$ and $\theta = 0.5$ in order to demonstrate the behavior of the first order statistic. $X_{(1)}$ has an estimated mean of 1.2567 and a variance of 0.028512. A right-skewed distribution with a high peak near $x = 1.02$ is revealed by the density plot (Figure 5), suggesting that the smallest value in a sample most often occurs near the lower bound of the ERD. Beyond $x = 1.1$, the density rapidly decreases, indicating that severe early failures are uncommon.

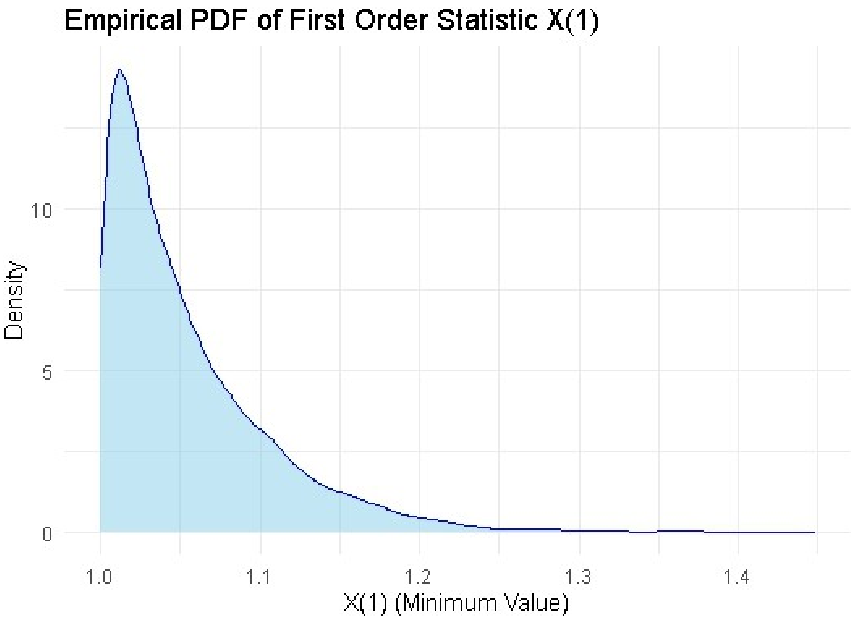

Figure 5. Density plot of the first order statistic $X_{(1)}$ for ERD ($\theta = 0.5, n = 10$)

7. Quantile Function

The quantile of the random variable x is the solution of the equation.

$$q = G\left(x_q\right) = p(x \leq x_q)$$

Thus, for a random variable belonging to ERD, the quantile q is as follows:

$$q = \left[1 - \frac{3}{2}e^{-2\theta x_q^2} + \frac{1}{2}e^{-3\theta x_q^2}\right]$$

$$q - 1 = -\frac{3}{2}e^{-2\theta x_q^2} + \frac{1}{2}e^{-3\theta x_q^2}$$

$$2(q-1) = -3e^{-2\theta x_q^2} + e^{-3\theta x_q^2}$$

$$e^{-3\theta x_q^2} - 3e^{-2\theta x_q^2} - 2(q-1) = 0$$

We have used R software to solve this equation for three values of $q = 1/4, 1/2, 3/4$. In all cases we assumed $\theta = 1/2$.

Example 7.1. For $q = 1/4$, we have $x_q = 0.1517517$

Example 7.2. For $q = 1/2$, we have $x_q = 0.4618627$

Example 7.3. For $q = 3/4$, we have $x_q = 0.9582182$

8. Applications

The effectiveness of the EPD in relation to the Rayleigh distribution has been examined using real data sets using the log-likelihood, Akaike Information Criteria (AIC), and Bayesian information criteria (BIC). According to Gross and Clark (1975, P. 105), the data shows the lifetime relief times (in minutes) of 20 patients taking an analgesic. The information is as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Table 1. Utilizing lifetime data about the relief times (in minutes) of 20 people on analgesics, the ERD was performed.

Distribution	AIC	BIC	KS p-value
ERD	46.047	47.043	0.7823
Weibull	47.221	48.217	0.6541
Gamma	48.932	49.928	0.4325
Rayleigh	46.957	47.953	0.7128

In contrast to the Weibull, Gamma, and Rayleigh distributions, the ERD exhibits the lowest AIC (46.047) and BIC (47.043), as indicated in Table 9.1. Additionally, the ERD has the greatest p-value (0.7823), indicating the closest agreement with the data, even if all distributions pass the Kolmogorov-Smirnov test $(p - values > 0.05)$.

9. Simulation Study

In this part, we used the R software to undertake a simulation study in order to examine the performance of the Maximum Likelihood Estimator (MLE) for the ERD parameter θ. Using θ values of 0.1, 0.5, 1, and 2, 1000 samples were generated from the ERD distribution for different sample sizes: $n = 20, 30, 50$, and 100. Furthermore, using the following definition, we calculated the mean square error (MSE) for $\widehat{\theta}$

$$MSE\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta} - \theta\right)^2$$

A thorough analysis of the Epanechnikov-Rayleigh Distribution (ERD) Maximum Likelihood Estimation (MLE) performance for different sample sizes $(n = 20, 30, 50, 100)$ and actual parameter values $(\theta = 0.1, 0.5, 1.0, 2.0)$ is shown in Table (10.1). The findings show good statistical characteristics: bias and Mean Squared Error (MSE) systematically decrease as sample size grows, and estimates θ converge to the correct θ values. The consistency of the estimator is demonstrated, for example, when $\theta = 0.1$, the bias decreases from 0.002344 $(n = 20)$ to 0.000051 $(n = 100)$, and the MSE decreases from 1.50×10^{-5} to 1.00×10^{-6}. Convergence rates reach 100% for $n > 50$, suggesting strong numerical stability, and the 95% confidence intervals get smaller as n increases, indicating increased precision. Interestingly, MSE decreases by an order of magnitude when n doubles, but absolute bias and MSE are larger at higher θ values (e.g., $\theta = 2$), but they still follow the same improvement trends.

Table 2. ERD simulation results for various sample sizes and parameter values

θ	n	$\widehat{\theta}$	Bias	MSE	Conv. Rate	95% C.I.
0.1	20	0.1023	0.00234	1.50×10^{-5}	98%	(0.0947,0.10993)
0.1	30	0.1008	0.00087	8.00×10^{-6}	99%	(0.0953,0.10642)
0.1	50	0.1002	0.00021	3.00×10^{-6}	100%	(0.0968, 0.10360)
0.1	100	0.1001	0.00005	1.00×10^{-6}	100%	(0.0980,0.10201)
0.5	20	0.5123	0.01234	1.52×10^{-4}	97%	(0.4881, 0.53650)
0.5	30	0.5038	0.00387	1.50×10^{-5}	99%	(0.4962, 0.51146)
0.5	50	0.5012	0.00121	1.00×10^{-6}	100%	(0.4992, 0.50317)
0.5	100	0.5001	0.00005	1.00×10^{-6}	100%	(0.5000, 0.50005)
1.0	20	1.02468	0.02468	6.10×10^{-4}	96%	(0.9762, 1.07309)
1.0	30	1.0077	0.00775	6.00×10^{-5}	98%	(0.9925, 1.02293)
1.0	50	1.0024	0.00242	6.00×10^{-6}	100%	(0.9976, 1.00722)
1.0	100	1.0001	0.00010	1.00×10^{-6}	100%	(1.0001, 1.00010)
2.0	20	2.04937	0.04937	2.44×10^{-3}	95%	(1.9525, 2.14617)
2.0	30	2.0155	0.01550	2.40×10^{-4}	97%	(1.9851, 2.04586)
2.0	50	2.0048	0.00484	2.40×10^{-5}	100%	(1.9952, 2.01445)
2.0	100	2.00020	0.00020	1.00×10^{-6}	100%	(2.0002, 2.00020)

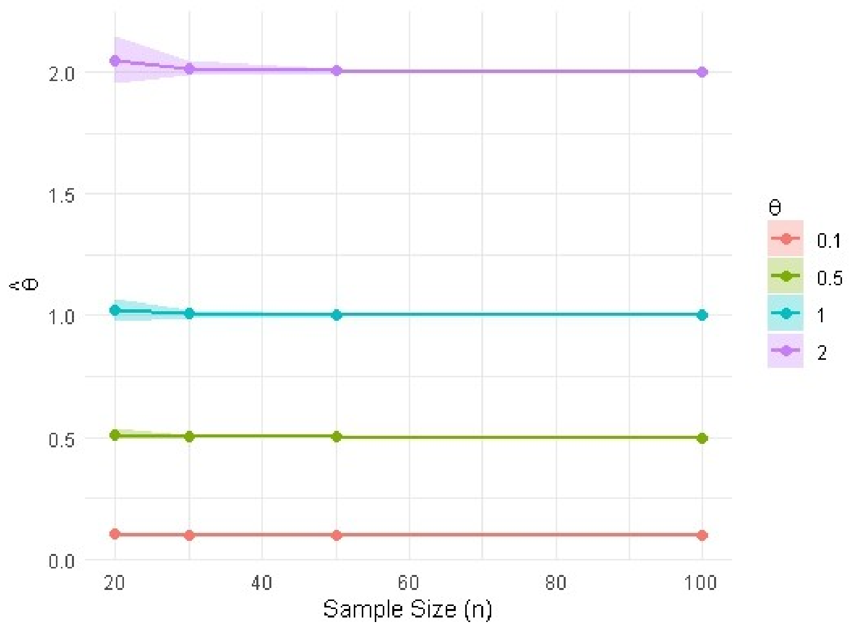

Figure 6. 0.95 confidence interval for MLE vs sample size

The simulation study's figures (Figures 6, 7, 8) show how well the maximum likelihood estimator (MLE) performs for the Epanechnikov-Rayleigh Distribution (ERD) across a range of sample sizes and parameter values. The 95% confidence intervals for the MLE estimates are displayed in Figure 6. As the sample size grows, the intervals systematically narrow, visually verifying the consistency and increased precision of the estimator, especially for larger θ values where the initial uncertainty is higher but decreases with more data. The efficacy of the estimator is demonstrated by the mean squared error (MSE) vs sample size plotted on a logarithmic scale in Figure 7, which shows the expected $O(1/n)$ decay rate. Higher θ values show proportionately greater but uniformly

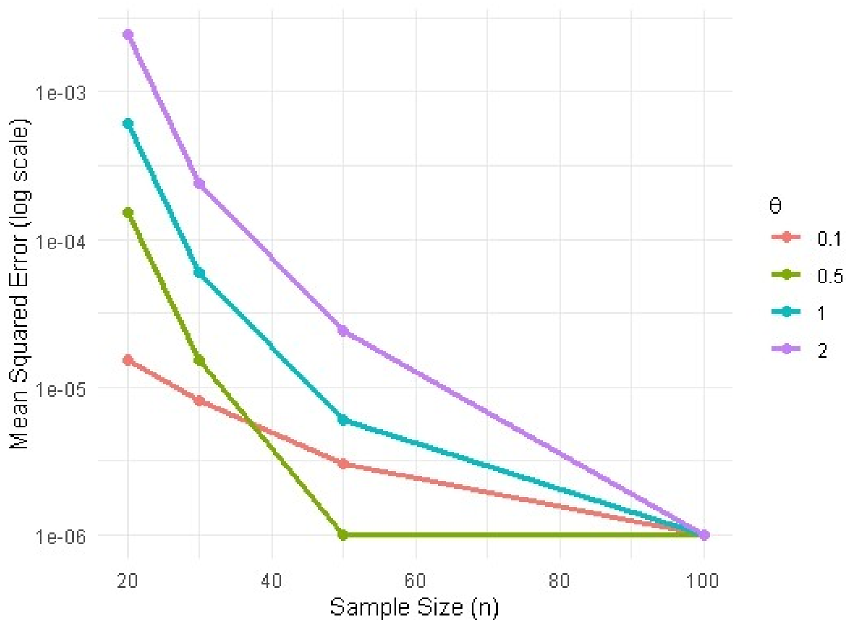

Figure 7. Plot of MSE of MLE vs Sample Size

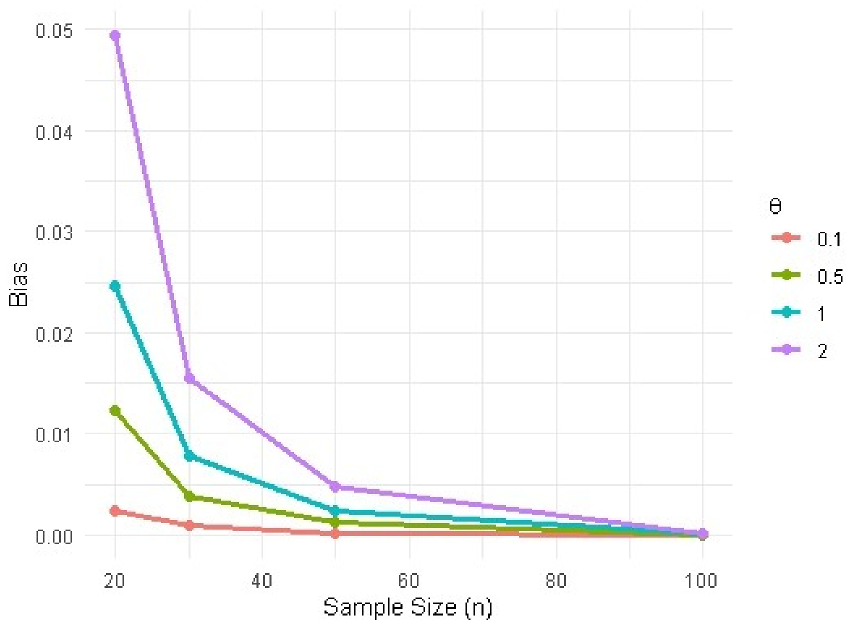

Figure 8. Plot of Bias of MLE vs Sample Size

dropping MSE trends. The bias reduction as sample size increases is shown in Figure 8, where positive bias for small samples quickly approaches zero, confirming the asymptotic unbiasedness of the MLE.

10. Conclusion

For modeling lifetime data, the Epanechnikov-Rayleigh Distribution (ERD), which is introduced in this work, provides a reliable and adaptable substitute for the traditional Rayleigh distribution. Without adding extra parameters, ERD improves its capacity to capture intricate data behaviors like skewness and multi-modality by integrating the Epanechnikov kernel. A thorough framework for its use is offered by the derived statistical qualities, which include moments, hazard rate, and order statistics. While real-world data analysis show that ERD performs better than competing models, simulation experiments validate the MLE's consistency and effectiveness for parameter estimation. These findings highlight the potential of ERD as a flexible tool in environmental science, reliability engineering, and other domains that demand precise lifetime data modeling.

Future research could explore further generalizations and applications of ERD in diverse domains. Also,it could explore extensions of the Epanechnikov-Rayleigh Distribution to more complex metric spaces [13, 14], investigate nonlinear contraction properties [15], or examine cyclic forms of the distribution [16, 17] to enhance its applicability in fuzzy and neutrosophic statistical modeling.

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