STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–9.
Published online in International Academic Press (www.IAPress.org)

Linear Algebra-Based Solution of Trinomial Markov Chain-Random Walk Between an Absorbing and an Elastic Barrier

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Abstract Two trinomial Markov chain-random walk (MC-RW) problems involving nonnegative integers amidst an elastic and an absorbing barrier are considered. The first has an elastic barrier at the origin and an absorber barrier at the end-state N, while the second is the opposite. Employing an unconventional approach based on eigenvalues and eigenvectors, we derive explicit formulas for the probabilities of absorption, segregation, and annihilation at the barriers. We also extract simple closed-form expressions for specific scenarios, including the semi-infinite lattice segment case.

Keywords Markov chain-random walk, elastic barrier, eigenvalues and eigenvectors

AMS 2010 subject classifications 60G50, 60J10, 65F15

DOI: 10.19139/soic-2310-5070-2806

1. Introduction

The Markov chain-random walk (MC-RW) problem is one of the most well-known and historically significant issues in probability. It was researched and expanded upon by several probabilists in the early years, becoming a significant topic in the history of probability and introducing numerous new notions. Moreover, the scientific community finds it intriguing since it has a wide range of notable applications in many theoretical and applied areas. Besides, the MC-RW problems with barriers frequently act as simplified representations of considerably more intricate many-body phenomena. For a broad debate on MC-RW and their applications, we commend books, e.g., [4, 5, 8, 9] and papers, e.g., [1, 2, 7, 10, 13] and the papers therein. Despite the MC-RW's extensive literary history, fresh aspects continue emerging.

The present paper is concerned with finding explicit formulas for the absorption, segregation, and annihilation probabilities of the trinomial MC-RW problems between an absorbing and an elastic barrier based on linear algebra (eigenvalues and eigenvectors), using a methodology similar to that given by Orosi [11]. To specify precisely, we are investigating two trinomial MC-RW problems. The first has an elastic barrier at state 0 and an absorber barrier at state N, while the second is the opposite. The elastic barrier at state 0 (state N) implies the particle is either

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

annihilated, segregated, or reflects to state 1 (state $N-1$) with probabilities α, ρ_0, and ρ_1, respectively (β, ω_0, and ω_1, respectively). These two problems are generalizations of several MC-RW issues examined in previous works, e.g., [3, 4, 6, 11, 12, 14]. Besides, many explicit formulas for some intriguing special cases are derived from the results through certain choices of reflection and segregation probabilities, ρ_0 and ρ_1 (or ω_0 and ω_1). We also discuss the outcomes when the number of states, N, becomes large ($N \to \infty$). The primary motivation for employing the algebra-based method lies in its relative conceptual clarity, offering a straightforward solution framework for those proficient in linear algebra, even in the absence of familiarity with classical difference equation techniques. As far as we know, this work is not covered in existing literature.

Concerning this paper's structure, Section 2 presents the formulation of the two trinomial MC-RW problems between absorbing and elastic barriers. In Section 3, the closed-form expressions for the absorption, segregation, and annihilation probabilities of the two trinomial MC-RW problems are obtained based on linear algebra. Section 4 exhibits certain interesting particular cases and the case when $N \to \infty$. The final section concludes the paper.

2. Trinomial MC-RW problem with absorbing and elastic barriers

The trinomial MC-RW problem between an absorbing and an elastic barrier (see Figure 1) is essentially formulated as follows: Consider an MC-RW $\{X_n,\, n \geq 0\}$ on the state space $SS = \{0, 1, \ldots, N\}$, with one perfect absorber end-state and other elastic (i.e., partially reflector/ segregated/annihilator) end-state such that a particle when away from the end-states, moves one step to the right or left or stays in the same state with specified probabilities. Let us state the underlying presumptions:

(i) The particle starts at the state i, $1 \leq i \leq N-1$.

(ii) The one-step right-moving probability is $p_{ii+1} = p$, the one-step left-moving probability is $p_{ii-1} = q$, and thus, the staying probability in the same state is $p_{ii} = r = 1-(p+q)$, for $i \in SS \setminus \{0, N\}$, where $0 \leq r < 1$ and $0 < p, q < 1$.

(iii) There are two barriers, one of which is perfectly absorbing at the end-state N (end-state 0). The other is elastic (i.e., partially reflecting/segregating/annihilating), at the end-state 0 (end-state N), that is, when the particle reaches the end-state 0 (end-state N), it is segregated with probability ρ_0 (probability ω_0), is reflected to the state 1 (state $N-1$) with probability ρ_1 (probability ω_1), or it is annihilated with probability $\alpha = 1-\rho_0-\rho_1$, for $0 \leq \alpha, \rho_0, \rho_1 \leq 1$ (probability $\beta = 1-\omega_0-\omega_1$, for $0 \leq \beta, \omega_0, \omega_1 \leq 1$)

Consider the first situation depicted in Figure 1(a). Let $U_0(k;\rho_0,\rho_1)$ (and $U_N(k;\rho_0,\rho_1)$) be the probability of the number of steps necessary for the particle to reach and segregate at 0 before reaching N (or reaches N before reaching and segregating at 0), given that the particle started in state k. In other words, $U_0(k;\rho_0,\rho_1)$ and $U_N(k;\rho_0,\rho_1)$ are the probabilities of partially segregated at state 0 and ultimate absorption at state N, respectively, given that the initial state is k. Then, by conditioning on the first move, it can be readily demonstrated that the probabilities $U_0(k;\rho_0,\rho_1)$ and $U_N(k;\rho_0,\rho_1)$ satisfy the following recurrence relations, respectively:

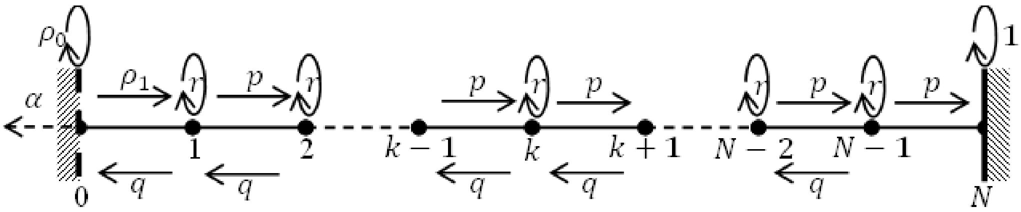
$$U_0(k;\rho_0,\rho_1) = p\,U_0(k+1;\rho_0,\rho_1) + r\,U_0(k;\rho_0,\rho_1) + q\,U_0(k-1;\rho_0,\rho_1), \tag{1}$$

for $k = 1, 2, \ldots, N-1$, with the boundary conditions

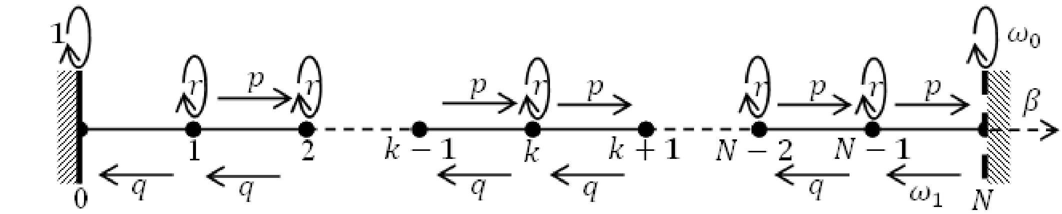
$$\begin{cases} U_0(0;\rho_0,\rho_1) = \rho_1\,U_0(1;\rho_0,\rho_1) + \rho_0, \\ U_0(N;\rho_0,\rho_1) = 0, \end{cases} \tag{2}$$

and

$$U_N(k;\rho_0,\rho_1) = p\,U_N(k+1;\rho_0,\rho_1) + r\,U_N(k;\rho_0,\rho_1) + q\,U_N(k-1;\rho_0,\rho_1), \tag{3}$$



(a) Elastic barrier at 0 and perfectly absorbing barrier at N



(b) Perfectly absorbing barrier at 0 and elastic barrier at N

Figure 1. Trinomial MC-RW between an absorbing and an elastic barrier

for $k = 1, 2, \ldots, N-1$, with the boundary conditions

$$\begin{cases} U_N(0;\rho_0,\rho_1) = \rho_1\, U_N(1;\rho_0,\rho_1), \\ U_N(N;\rho_0,\rho_1) = 1. \end{cases} \tag{4}$$

Besides, the probability of partially annihilated at state 0, denoted by $U_0^h(k;\rho_0,\rho_1)$, fulfills

$$U_0^h(k;\rho_0,\rho_1) = 1 - \left[U_0(k;\rho_0,\rho_1) + U_N(k;\rho_0,\rho_1)\right]. \tag{5}$$

Similarly, the following recurrence relations can be deduced for the situation where there is a perfectly absorbing barrier at the origin and an elastic barrier at the end-state N, shown in Figure 1(b):

$$U_0(k;\omega_0,\omega_1) = p\, U_0(k+1;\omega_0,\omega_1) + r\, U_0(k;\omega_0,\omega_1) + q\, U_0(k-1;\omega_0,\omega_1), \tag{6}$$

for $k = 1, 2, \ldots, N-1$, with the boundary conditions

$$\begin{cases} U_0(0;\omega_0,\omega_1) = 1, \\ U_0(N;\omega_0,\omega_1) = \omega_1\, U_0(N-1;\omega_0,\omega_1), \end{cases} \tag{7}$$

and

$$U_N(k;\omega_0,\omega_1) = p\, U_N(k+1;\omega_0,\omega_1) + r\, U_N(k;\omega_0,\omega_1) + q\, U_N(k-1;\omega_0,\omega_1), \tag{8}$$

for $k = 1, 2, \ldots, N-1$, with the boundary conditions

$$\begin{cases} U_N(0;\omega_0,\omega_1) = 0, \\ U_N(N;\omega_0,\omega_1) = \omega_1\, U_0(N-1;\omega_0,\omega_1) + \omega_0. \end{cases} \tag{9}$$

where $U_0(k;\omega_0,\omega_1)$ (and $U_N(k;\omega_0,\omega_1)$) be the probability of the number of steps necessary for the particle to reach 0 before reaching and segregating at N (or reach and segregate at N before reaching 0), given that the particle started in state k. In other terms, $U_0(k;\omega_0,\omega_1)$ and $U_N(k;\omega_0,\omega_1)$ are the probabilities of ultimate absorption at state

0 and partially segregated at state N, respectively, given that the initial state is k, in situation (b) in Figure 1. While the probability of partially annihilated at state N, represented by $U_N^h(k;\omega_0,\omega_1)$, can be obtained by

$$U_N^h(k;\omega_0,\omega_1) = 1 - \left[U_0(k;\omega_0,\omega_1) + U_N(k;\omega_0,\omega_1)\right]. \tag{10}$$

Notice that the two cases (a) and (b) in Figure 1 can be easily derived from each other by reversing p and q, as well as substituting $N-k$, ρ_0, and ρ_1 with k, ω_0, and ω_1, respectively. Therefore, in the next section, we address the first of them in some detail and then briefly provide the results related to the second case.

3. Closed-form solution derivation using linear algebra

The main concern behind this section is to find explicit expressions for $U_0(k;\rho_0,\rho_1)$ and $U_N(k;\rho_0,\rho_1)$ by solving the difference equation (1) with (2) (or (3) with (4)) algebraically, using the known method of eigenvalues, eigenvectors, and matrix diagonalization. The pivotal idea is to employ a matrix representation for expressing $U_0(k+1;\rho_0,\rho_1)$ (or $U_N(k+1;\rho_0,\rho_1)$) in terms of $U_0(k;\rho_0,\rho_1)$ and $U_0(k-1;\rho_0,\rho_1)$ (or $U_N(k;\rho_0,\rho_1)$ and $U_N(k-1;\rho_0,\rho_1)$). Firstly, rearranging formula (1) as

$$U_0(k+1;\rho_0,\rho_1) = \frac{1-r}{p}\,U_0(k;\rho_0,\rho_1) - \frac{q}{p}\,U_0(k-1;\rho_0,\rho_1), \quad \text{for } k = 1,2,\ldots,N-1. \tag{11}$$

The difference equation (11) can be reformulated in the equivalent matrix representation as

$$\begin{pmatrix} U_0(k+1;\rho_0,\rho_1) \\ U_0(k;\rho_0,\rho_1) \end{pmatrix} = \mathbf{M}\begin{pmatrix} U_0(k;\rho_0,\rho_1) \\ U_0(k-1;\rho_0,\rho_1) \end{pmatrix}, \quad \text{for } k = 1,2,\ldots,N-1, \tag{12}$$

where $\mathbf{M} = \begin{pmatrix} \frac{1-r}{p} & -\frac{q}{p} \\ 1 & 0 \end{pmatrix}$, $0 \le r < 1$ and $r = 1-(p+q)$. Furthermore,

$$\begin{pmatrix} U_0(k+1;\rho_0,\rho_1) \\ U_0(k;\rho_0,\rho_1) \end{pmatrix} = \mathbf{M}^k \begin{pmatrix} U_0(1;\rho_0,\rho_1) \\ U_0(0;\rho_0,\rho_1) \end{pmatrix}, \tag{13}$$

Hence, by diagonalizing the matrix $\mathbf{M}$, a simple form for $\mathbf{M}^k$ can be obtained. Let λ_i be the i^{th}-eigenvalue of $\mathbf{M}$, and $\mathbf{u}_i$ is the corresponding eigenvector, for $i = 1,2$.

Case I: When $p \neq q$
The eigenvalues of $\mathbf{M}$ are the solutions of the following characteristic equation

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (\lambda - 1)\left(\lambda - \frac{1-r-p}{p}\right) = 0. \tag{14}$$

Thus, there are two distinct eigenvalues of $\mathbf{M}$: $\lambda_1 = \frac{1-r-p}{p}$ and $\lambda_2 = 1$. Subsequently, we have

$$\mathbf{M} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & -\frac{1-r-p}{p} \\ 1 & -\frac{1-r-p}{p} \end{pmatrix} \text{ and } \mathbf{M} - \lambda_2\mathbf{I} = \begin{pmatrix} \frac{1-r-p}{p} & -\frac{1-r-p}{p} \\ 1 & -1 \end{pmatrix}$$

that can be transformed into the reduced Echelon forms $\begin{pmatrix} 1 & -\frac{1-r-p}{p} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, respectively. Therefore, the eigenvectors corresponding to $\lambda_1 = \frac{1-r-p}{p}$ and $\lambda_2 = 1$ are given by $\mathbf{u}_1 = \begin{pmatrix} \frac{1-r-p}{p} \\ 1 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. Consequently, the diagonal decomposition of $\mathbf{M}$ is given by

$$\mathbf{M} = \mathbf{PDP}^{-1} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{q}{p} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{q}{p} & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{q-p}\begin{pmatrix} p & -p \\ -p & q \end{pmatrix}, \tag{15}$$

where $q = 1 - r - p$, and after some matrix computations, a simple form for $\mathbf{M}^k$ can be derived

$$\mathbf{M}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} = \frac{p}{q-p}\begin{pmatrix} \left(\frac{q}{p}\right)^{k+1} - 1 & -\left(\frac{q}{p}\right)^{k+1} + \frac{q}{p} \\ \left(\frac{q}{p}\right)^{k} - 1 & -\left(\frac{q}{p}\right)^{k} + \frac{q}{p} \end{pmatrix}. \tag{16}$$

Substituting from (16) in formula (13), we get

$$\begin{pmatrix} U_0(k+1;\rho_0,\rho_1) \\ U_0(k;\rho_0,\rho_1) \end{pmatrix} = \frac{1}{q-p}\begin{pmatrix} p\left[\left(\frac{q}{p}\right)^{k+1} - 1\right] & -p\left(\frac{q}{p}\right)^{k+1} + q \\ p\left[\left(\frac{q}{p}\right)^{k} - 1\right] & -p\left(\frac{q}{p}\right)^{k} + q \end{pmatrix}\begin{pmatrix} U_0(1;\rho_0,\rho_1) \\ U_0(0;\rho_0,\rho_1) \end{pmatrix}. \tag{17}$$

Thus

$$U_0(k;\rho_0,\rho_1) = \frac{1}{q-p}\left\{p\left[\left(\frac{q}{p}\right)^{k} - 1\right]U_0(1;\rho_0,\rho_1) - \left[p\left(\frac{q}{p}\right)^{k} - q\right]U_0(0;\rho_0,\rho_1)\right\}. \tag{18}$$

Using the boundary condition at 0 in (2) and by some simple calculations, (18) becomes

$$U_0(k;\rho_0,\rho_1) = \frac{1}{q-p}\left\{\left[p\left(\frac{q}{p}\right)^{k}(1-\rho_1) - p + q\rho_1\right]U_0(1;\rho_0,\rho_1) - \rho_0\left[p\left(\frac{q}{p}\right)^{k} - q\right]\right\}. \tag{19}$$

Applying the second condition in (2), one can conclude that

$$U_0(1;\rho_0,\rho_1) = \frac{\rho_0\left(\frac{q}{p}\right)\left[1 - \left(\frac{q}{p}\right)^{N-1}\right]}{1 - \rho_1\left(\frac{q}{p}\right) - (1-\rho_1)\left(\frac{q}{p}\right)^{N}}. \tag{20}$$

Finally, substituting from (20) into (19), we obtain a formal solution to the difference equation (1) that meets the boundary conditions (2). It is a closed-form formula for the probability $U_0(k;\rho_0,\rho_1)$ of the trinomial MC-RW problem between an elastic barrier at 0 and an absorbing barrier at N, in the case $p \neq q$, where $q = 1 - r - p$:

$$U_0(k;\rho_0,\rho_1) = \frac{\rho_0\left[\left(\frac{q}{p}\right)^{k} - \left(\frac{q}{p}\right)^{N}\right]}{1 - \rho_1\left(\frac{q}{p}\right) - (1-\rho_1)\left(\frac{q}{p}\right)^{N}}. \tag{21}$$

Case II: When $p = q$

The matrix $\mathbf{M}$ becomes $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. It has one repeated eigenvalue, $\eta_1 = 1 = \eta_2$, which are the roots of the following characteristic equation

$$\det(\mathbf{M} - \eta\mathbf{I}) = (\eta - 1)^2 = 0. \tag{22}$$

Consequently, there exists a single eigenvector that corresponds to the duplicated eigenvalue, $\eta_1 = 1$, given by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To continue diagonalizing the matrix $\mathbf{M}$, a generalized eigenvector must be acquired by solving $(\mathbf{M} - \eta_2\mathbf{I})\,\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$. Straightforward calculations led to $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, the Jordan decomposition of $\mathbf{M}$ is given by

$$\mathbf{M} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}\begin{pmatrix} \eta_1 & 1 \\ 0 & \eta_2 \end{pmatrix}\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \tag{23}$$

After some matrix algebra, a simple form for $\mathbf{M}^k$ can be obtained

$$\mathbf{M}^k = \mathbf{P}\mathbf{J}^k\mathbf{P}^{-1} = \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix}. \tag{24}$$

Substituting from (24) in formula (13), we have

$$\begin{pmatrix} U_0(k+1;\rho_0,\rho_1) \\ U_0(k;\rho_0,\rho_1) \end{pmatrix} = \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix} \begin{pmatrix} U_0(1;\rho_0,\rho_1) \\ U_0(0;\rho_0,\rho_1) \end{pmatrix}. \tag{25}$$

Hence,

$$U_0(k;\rho_0,\rho_1) = k\, U_0(1;\rho_0,\rho_1) + (1-k)\, U_0(0;\rho_0,\rho_1). \tag{26}$$

The first boundary condition in (2) implies that (26) becomes

$$U_0(k;\rho_0,\rho_1) = [(1-\rho_1)k + \rho_1]\, U_0(1;\rho_0,\rho_1) - \rho_0(k-1). \tag{27}$$

Employing the second condition in (2), one can deduce that

$$U_0(1;\rho_0,\rho_1) = \frac{\rho_0(N-1)}{(1-\rho_1)N + \rho_1}. \tag{28}$$

Substituting from (28) into (27), we finally obtain the following closed-form formula for the partially segregated probability at state 0 of the considered MC-RW problem in the case $p = q$, for $0 \le r < 1$:

$$U_0(k;\rho_0,\rho_1) = \frac{\rho_0(N-k)}{(1-\rho_1)N + \rho_1}. \tag{29}$$

Notice that the same form of (29) (in the case $p = q$) can be obtained from (21) (in the case $p \neq q$) by using L'Hospital's rule with limit as $a = \frac{1-r-p}{p} \to 1$.

Similarly, by considering the difference equation (3) with (4), we can derive a closed-form formula for $U_N(k;\rho_0,\rho_1)$ (the probability of ultimate absorption at N given that the initial state is k for $0 \le r < 1$). Then, by substituting for $U_0(k;\rho_0,\rho_1)$ and $U_N(k;\rho_0,\rho_1)$ into (5), we can obtain a closed-form formula for $U_0^h(k;\rho_0,\rho_1)$ (the probability of partially annihilated at state 0 given that the initial state is k). Theorem 1 presents the findings.

Theorem 1. *In the trinomial MC-RW between an elastic barrier at state* 0 *and an absorbing barrier at state* N*, the probability of*

(i) *segregating partially at* 0*, given that* k *was its initial position, is given by*

$$U_0(k;\rho_0,\rho_1) = \begin{cases} \rho_0 \left[\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N \right] \mathcal{K}^{-1}, & \text{for } p \neq q \\ \rho_0(N-k)\Lambda^{-1}, & \text{for } p = q \end{cases} \tag{30}$$

(ii) *annihilating partially at* 0*, given that* k *was its initial position, is given by*

$$U_0^h(k;\rho_0,\rho_1) = \begin{cases} (1-\rho_0-\rho_1) \left[\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N \right] \mathcal{K}^{-1}, & \text{for } p \neq q \\ (1-\rho_0-\rho_1)(N-k)\Lambda^{-1}, & \text{for } p = q \end{cases} \tag{31}$$

(iii) *ultimate absorption at N, given that k was its initial position, is given by*

$$U_N(k;\rho_0,\rho_1) = \begin{cases} \left[1-\rho_1\left(\dfrac{q}{p}\right)-(1-\rho_1)\left(\dfrac{q}{p}\right)^k\right]\mathcal{K}^{-1}, & \text{for } p \neq q \\ \left[(1-\rho_1)k+\rho_1\right]\Lambda^{-1}, & \text{for } p = q \end{cases} \tag{32}$$

where $\mathcal{K} = 1-\rho_1\left(\frac{q}{p}\right)-(1-\rho_1)\left(\frac{q}{p}\right)^N$ *and* $\Lambda = (1-\rho_1)N+\rho_1$*, for* $q = 1-r-p$ *and* $0 \leq r < 1$.

For the trinomial MC-RW problem between an absorbing barrier at state 0 and an elastic barrier at state N, depicted in Figure 1(b), one can follow the same algebraic approach but employ formulas (6)-(10) to derive closed-form formulas for the probabilities $U_0(k;\omega_0,\omega_1)$, $U_N(k;\omega_0,\omega_1)$, and $U_0^h(k;\omega_0,\omega_1)$, defined before. However, as mentioned before, these probabilities can be directly obtained from the probabilities $U_0(k;\rho_0,\rho_1)$, $U_N(k;\rho_0,\rho_1)$, and $U_0^h(k;\rho_0,\rho_1)$, by replacing $N-k$, ρ_0, and ρ_1 with k, q, ω_0, and ω_1, respectively. The results are provided by Theorem 2.

Theorem 2. *In the trinomial MC-RW between an absorbing barrier at state* 0 *and an elastic barrier at state* N*, the probability of*

(i) *ultimate absorption at 0, given that k was its initial position, is given by*

$$U_0(k;\omega_0,\omega_1) = \begin{cases} \left[(1-\omega_1)\left(\dfrac{q}{p}\right)^k-\left(\dfrac{q}{p}-\omega_1\right)\left(\dfrac{q}{p}\right)^{N-1}\right]\Gamma^{-1}, & \text{for } p \neq q \\ \left[(1-\omega_1)(N-k)+\omega_1\right]\Delta^{-1}, & \text{for } p = q \end{cases} \tag{33}$$

(ii) *segregating partially at N, given that k was its initial position is given by*

$$U_N(k;\omega_0,\omega_1) = \begin{cases} \omega_0\left[1-\left(\dfrac{q}{p}\right)^k\right]\Gamma^{-1}, & \text{for } p \neq q \\ \omega_0 k\Delta^{-1}, & \text{for } p = q \end{cases} \tag{34}$$

(iii) *annihilating partially at N, given that k was its initial position, is given by*

$$U_N^h(k;\omega_0,\omega_1) = \begin{cases} (1-\omega_0-\omega_1)\left[1-\left(\dfrac{q}{p}\right)^k\right]\Gamma^{-1}, & \text{for } p \neq q \\ (1-\omega_0-\omega_1)\,k\Delta^{-1}, & \text{for } p = q \end{cases} \tag{35}$$

where $\Gamma = 1-\omega_1-\left(\frac{q}{p}-\omega_1\right)\left(\frac{q}{p}\right)^{N-1}$ *and* $\Delta = (1-\omega_1)N+\omega_1$*, for* $q = 1-r-p$ *and* $0 \leq r < 1$.

4. Some particular cases

The trinomial MC-RW problem between an absorbing and an elastic barrier being investigated is beneficial as it seems to be a generalization to several MC-RW issues that can occur in different applied science domains. Thus, many interesting explicit formulas for absorption, segregation, or annihilating probabilities can be derived from Theorems 1 and 2 as particular cases through an appropriate choice of the reflecting and segregating probabilities, ρ_0 and ρ_1 (or ω_0 and ω_1). Table 1 presents some formulas for the probabilities $U_0(k;\rho_0,\rho_1)$ extracted from Theorem 1 when $p+q \leq 1$ (or $r \neq 0$).

Table 1. The probabilities $U_0(k;\rho_0,\rho_1)$ for particular cases from Theorem 1, when $p+q\leq 1$ (or $r\neq 0$)

Particular Case	**If**	**The probability $U_0(k;\rho_0,\rho_1)$ given by (30) becomes**
Case I	$\rho_0+\rho_1=1$	$U_0(k;1-\rho_1,\rho_1)=\begin{cases}\dfrac{(1-\rho_1)\left[(\frac{q}{p})^k-(\frac{q}{p})^N\right]}{1-\rho_1(\frac{q}{p})-(1-\rho_1)(\frac{q}{p})^N}, & \text{for } p\neq q\\ \dfrac{(1-\rho_1)(N-k)}{(1-\rho_1)N+\rho_1}, & \text{for } p=q\end{cases}$
Case II	$\rho_0+\rho_1=1$ and $\rho_1=0$	$U_0(k;1,0)=\begin{cases}\dfrac{(\frac{q}{p})^k-(\frac{q}{p})^N}{1-(\frac{q}{p})^N}, & \text{for } p\neq q\\ \dfrac{N-k}{N}, & \text{for } p=q\end{cases}$
Case III	$\rho_0+\rho_1=1$ and $\rho_1=1$	$U_0(k;0,1)=0$
Case IV	$\rho_0+\rho_1=1$ and $\rho_1=p$	$U_0(k;1-p,p)=\begin{cases}\dfrac{(1-p)\left[(\frac{q}{p})^k-(\frac{q}{p})^N\right]}{1-q-(1-p)(\frac{q}{p})^N}, & \text{for } p\neq q\\ \dfrac{(1-p)(N-k)}{(1-p)N+p}, & \text{for } p=q\end{cases}$

Note. $q=1-r-p$

Strikingly, for all particular cases, the probabilities $U_N(k;\rho_0,\rho_1)=1-U_0(k;\rho_0,\rho_1)$ when $\rho_0+\rho_1=1$. Besides, the probabilities $U_N(k;\rho_0,\rho_1)$ (when $\rho_0+\rho_1\leq 1$) are identical to the corresponding probabilities $U_N(k;\rho_0,\rho_1)$ (when $\rho_0+\rho_1=1$). But the probabilities $U_0(k;\rho_0,\rho_1)$ (when $\rho_0+\rho_1\leq 1$) are $\frac{\rho_0}{1-\rho_1}$ times the corresponding probabilities $U_0(k;\rho_0,\rho_1)$ (when $\rho_0+\rho_1=1$). When $p+q=1$, we obtain the same forms for all cases but with $q=1-p$. Furthermore, the formulas of the corresponding particular cases from Theorem 2 can be reached by interchanging $N-k$, p, ρ_0, and ρ_1 with k, q, ω_0, and ω_1, respectively.

On the other side, it is of interest to see what happens when the number of states, N, becomes large ($N\to\infty$). Theorems 1 and 2 can be easily modified to the semi-infinite lattice segment case by taking the limit as ($N\to\infty$). The following corollaries formulate our results.

Corollary 1. *In the semi-infinite lattice segment case of the trinomial MC-RW between an elastic barrier at state* 0 *and an absorbing barrier at state* N*, the probability of*

(i) *segregating partially at the origin, starting from state* k*, is given by*

$$\lim_{N\to\infty}U_0(k;\rho_0,\rho_1)=\begin{cases}\rho_0\left(\dfrac{q}{p}\right)^k\left[1-\rho_1\dfrac{q}{p}\right]^{-1}, & \text{for } p\neq q\\ \rho_0(1-\rho_1)^{-1}, & \text{for } p=q\end{cases}\tag{36}$$

(ii) *annihilating partially at the origin, starting from state* k*, is given by*

$$\lim_{N\to\infty} U_0^h(k;\rho_0,\rho_1) = \begin{cases} (1-\rho_0-\rho_1)\left(\frac{q}{p}\right)^k\left[1-\rho_1\frac{q}{p}\right]^{-1}, & \text{for } p\neq q \\ (1-\rho_0-\rho_1)(1-\rho_1)^{-1}, & \text{for } p=q \end{cases} \tag{37}$$

(iii) *escapes to infinity, starting from state k, is given by*

$$\begin{aligned} U_\infty(k;\rho_0,\rho_1) &= 1-\left[\lim_{N\to\infty} U_0(k;\rho_0,\rho_1)+\lim_{N\to\infty} U_0^h(k;\rho_0,\rho_1)\right] \\ &= \begin{cases} \left[1-\rho_1\left(\frac{q}{p}\right)-(1-\rho_1)\left(\frac{q}{p}\right)^k\right]\left[1-\rho_1\left(\frac{q}{p}\right)\right]^{-1}, & \text{for } p\neq q \\ 0, & \text{for } p=q \end{cases} \end{aligned} \tag{38}$$

where $\frac{q}{p}<1$, $q=1-r-p$, *and* $0\leq r<1$.

Corollary 2. *In the semi-infinite lattice segment case of the trinomial MC-RW between an absorbing barrier at state* 0 *and an elastic barrier at state* N, *the probability of*

(i) *ultimate absorption at the origin, starting from state k, is given by*

$$\lim_{N\to\infty} U_0(k;\omega_0,\omega_1) = \begin{cases} \left(\frac{q}{p}\right)^k, & \text{for } p\neq q \\ 1, & \text{for } p=q \end{cases} \tag{39}$$

(ii) *escapes to infinity, starting from state k, is given by*

$$U_\infty(k;\omega_0,\omega_1) = \lim_{N\to\infty} U_N(k;\omega_0,\omega_1)+\lim_{N\to\infty} U_N^h(k;\omega_0,\omega_1) = \begin{cases} 1-\left(\frac{q}{p}\right)^k, & \text{for } p\neq q \\ 0, & \text{for } p=q \end{cases} \tag{40}$$

where $\frac{q}{p}<1$, $q=1-r-p$, *and* $0\leq r<1$.

5. Conclusions

This study underscores the potential viability of solving the difference equations that govern the motion of the trinomial MC-RW problem with absorbing and elastic barriers using principles of linear algebra. Closed-form expressions for the absorption, segregation, and annihilation probabilities of two trinomial MC-RW problems between an absorbing and an elastic barrier have been derived utilizing eigenvalues and eigenvectors. Moreover, some precise and clear explicit formulas have been obtained for some particular cases in addition to the semi-infinite lattice segment case. While some of the findings were established in existing literature using traditional methods, others seem "new" and have not been discussed. Notwithstanding that the derivation process based on linear algebra may be more intricate than traditional methods, this research effectively illustrates how straightforward the problem can be solved. Further research could explore applying the method to other MC-RW problems, including those in higher dimensions.

Funding

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2504)

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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