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Unified Fixed Point Theory in Generalized Metric Structures with Applications to Nonlinear Economic Systems

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Abstract This paper introduces a comprehensive framework unifying recent advancements in fixed point theory through the novel concept of *twisted weighted* Θ*-b-metric spaces*. We establish a framework of fixed point theorems for multi-valued mappings satisfying generalized rational type contractions that incorporate control functions, weight functions, and twisted admissibility conditions. By synthesizing concepts from Čirić-type contractions, Berinde's almost contractions, Jleli's Θ-contractions, and weighted b-metric spaces, we create a powerful analytical tool with unprecedented theoretical depth. The work provides rigorous proofs, extensive numerical validation, and demonstrates significant applications to economic systems including production-consumption equilibrium models and fractional economic growth equations. Our results substantially generalize numerous classical theorems while opening new avenues for research in nonlinear analysis and mathematical economics.

Keywords Fixed point theory, Twisted admissible mappings, Weighted b-metric spaces, Θ-contractions, Economic equilibrium, Fractional differential equations

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1. Introduction

Fixed point theory stands as one of the most profound and widely applicable domains in mathematical analysis, with extensive applications spanning engineering, computer science, physics, biology, and economics. The foundational Banach contraction principle [8] has undergone remarkable generalizations to address increasingly complex problems in nonlinear analysis.

The evolution of fixed point theory has witnessed several significant milestones. Čirić [12] introduced generalized contractions that substantially extended Banach's original result. Berinde [10, 11] developed the concepts of almost contractions and generalized almost contractions, providing more flexible contractive conditions. Samet et al. [40] introduced α-admissible mappings and (α, ψ)-contractions, while Salimi [39] extended this framework through twisted (α, μ)-admissible mappings. Jleli and Samet [16] contributed the innovative concept of Θ-contractions, offering a new perspective on contractive conditions. Parallel to these developments, the theory of generalized metric spaces has flourished. Bakhtin [7] introduced b-metric spaces, while recent work by Artsawang and Suanoom has explored weighted ψ-b-metric spaces, incorporating control functions and weight functions to create more versatile distance structures or other spaces has seen substantial progress in recent years (see, for example,[6, 26, 27, 28, 29, 30, 31, 32].

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In economic theory, fixed point methods provide crucial tools for establishing equilibrium existence in market models. Panda et al. [25] demonstrated applications in electric circuit models, while Abdou [2] explored applications in fractional differential equations for economic growth. The works of Joshi et al. [17] and Tejado et al. [43] have shown the importance of fixed point theory in economic modeling and growth analysis.

This paper makes several fundamental contributions: we introduce *twisted weighted* Θ-*b*-*metric spaces*, creating a unified framework that synthesizes concepts from multiple mathematical structures; we establish comprehensive fixed point theorems for multivalued mappings under generalized rational type contraction conditions and prove best proximity point results for non-self mappings within this new framework; we provide extensive numerical validation through carefully constructed examples with computational implementations and demonstrate significant applications to economic models, including dynamic production-consumption equilibrium and fractional economic growth equations; ultimately, we substantially generalize and unify numerous classical theorems from the literature, creating a powerful analytical tool for nonlinear analysis with broad applications across mathematical sciences.

The introduction of twisted weighted Θ-b-metric spaces is motivated by limitations in existing frameworks when modeling complex real-world phenomena, which often exhibit dual criteria relationships (e.g., supply/demand), context-dependent distances, non-polynomial growth and rational nonlinearities. Our framework unifies and extends prior work recovering b-metric spaces when $\psi(u,v) = s$, $w = 1$, $\Theta(t) = e^t$, $\alpha = \mu \equiv 1$; metric spaces when $\psi = 1$; Čiri'c contractions when $N(x,y) = 0$; and twisted admissible mappings when $\alpha = \mu$. Example 3.2 illustrates a mapping contractive in our framework but not in simpler spaces, demonstrating its broader applicability.

2. Preliminaries and Unified Framework

We begin by recalling essential concepts that form the foundation of our unified framework.

Definition 2.1 ([12])
A mapping $T : X \to X$ on a metric space (X, d) is called a *Čirić type generalized contraction* if there exists $\kappa \in [0, 1)$ such that for all $x, y \in X$:

$$d(Tx, Ty) \leq \kappa \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}.$$

Definition 2.2 ([11])
A mapping $T : X \to X$ is called a *generalized almost contraction* if there exist $\kappa \in [0, 1)$ and $L \geq 0$ such that for all $x, y \in X$:

$$d(Tx, Ty) \leq \kappa \cdot d(x,y) + L \cdot \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Definition 2.3 ([16])
Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is strictly increasing;
(Θ_2) For any sequence $\{\tau_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \Theta(\tau_n) = 1$ if and only if $\lim_{n\to\infty} \tau_n = 0$;
(Θ_3) There exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\tau\to 0^+} \frac{\Theta(\tau)-1}{\tau^r} = \ell$.

A mapping $T : X \to X$ is called a Θ-*contraction* if there exists $\kappa \in (0, 1)$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$:

$$\Theta(d(Tx, Ty)) \leq [\Theta(d(x,y))]^\kappa.$$

Definition 2.4 ([39])
Let $T : X \to X$ and $\alpha, \mu : X \times X \to [0, \infty)$. Then T is called *twisted* (α, μ)-*admissible* if for all $x, y \in X$:

$$\alpha(x,y) \geq 1 \text{ and } \mu(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1 \text{ and } \mu(Tx,Ty) \geq 1.$$

Definition 2.5 ([7])
A b-metric on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ and some $s \geq 1$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

3. Twisted Weighted Θ-b-Metric Spaces

We now introduce our main framework that unifies and extends concepts from the provided papers.

Definition 3.1
Let X be a non-empty set. A function $\mathfrak{D} : X \times X \to [0, \infty)$ is called a *twisted weighted* Θ-*b-metric* if there exist: a control function $\Theta : (0, \infty) \to (1, \infty)$ satisfying (Θ_1)-(Θ_3), a weight function $w : X \times X \to [\alpha, \beta]$ with $0 < \alpha \leq \beta < \infty$, a distortion function $\psi : [0, \infty)^2 \to [1, \infty)$ continuous and increasing in both arguments, and a constant $s \geq 1$, such that for all $x, y, z \in X$:

1. $\mathfrak{D}(x, y) = 0$ if and only if $x = y$;
2. $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$;
3. $\mathfrak{D}(x, y) \leq \psi(\mathfrak{D}(x, z), \mathfrak{D}(z, y)) \cdot [w(x, z)\mathfrak{D}(x, z) + w(z, y)\mathfrak{D}(z, y)]$.

The quadruple $(X, \mathfrak{D}, \Theta, \psi)$ is called a *twisted weighted* Θ-*b-metric space*.

Remark 3.1
The components serve distinct purposes: Θ controls growth rates, w provides context-dependent weights, ψ modifies the triangle inequality, and α, μ model dual admissibility criteria common in economic systems.

Example 3.1
Let $X = \mathbb{R}$ and define:

$$\begin{aligned} \mathfrak{D}(x, y) &= \frac{|x - y|}{1 + x^2 + y^2}, \\ w(x, y) &= 1 + \sin^2(x + y), \\ \Theta(t) &= e^t, \\ \psi(u, v) &= 2 + u + v. \end{aligned}$$

Then $(X, \mathfrak{D}, \Theta, \psi)$ is a twisted weighted Θ-b-metric space.

Example 3.2
Consider $X = [0, 1]$ with $T(x) = \frac{x^2+0.1x}{3x^2+4}$. Define $\mathfrak{D}(x, y) = |x - y| + \frac{|x-y|}{1+x^2+y^2}$ with $w(x, y) = 1 + \sin^2(x + y)$, $\Theta(t) = e^t$, $\psi(u, v) = 2 + u + v$, $\alpha(x, y) = 1 + x + y$, $\mu(x, y) = \frac{1}{1+x+y}$. This mapping satisfies our contraction condition but fails to be a contraction in: (i) standard metric spaces (rational term), (ii) b-metric spaces (variable weights), (iii) Θ-contractions without twisted admissibility.

Definition 3.2
A sequence $\{x_n\}$ in a twisted weighted Θ-b-metric space $(X, \mathfrak{D}, \Theta, \psi)$ is:

1. *convergent* to $x \in X$ if $\lim_{n\to\infty} \mathfrak{D}(x_n, x) = 0$;
2. *Cauchy* if $\lim_{m,n\to\infty} \mathfrak{D}(x_m, x_n) = 0$;

The space is *complete* if every Cauchy sequence converges.

Lemma 3.1
Every twisted weighted Θ-b-metric space is Hausdorff and first countable.

Proof
We first prove the Hausdorff property. Let $(X, \mathfrak{D}, \Theta, \psi)$ be a twisted weighted Θ-b-metric space and let $x, y \in X$ with $x \neq y$. By Definition 3.1, $\mathfrak{D}(x, y) > 0$. Choose $\delta = \frac{\mathfrak{D}(x,y)}{2\beta\psi(1,1)} > 0$, where β is the upper bound of the weight function w. Consider the open balls $B(x, \delta)$ and $B(y, \delta)$. Suppose, for contradiction, that there exists $z \in B(x, \delta) \cap B(y, \delta)$. Then by axiom (M3):

$$\mathfrak{D}(x, y) \leq \psi(\mathfrak{D}(x, z), \mathfrak{D}(z, y)) \cdot [w(x, z)\mathfrak{D}(x, z) + w(z, y)\mathfrak{D}(z, y)] .$$

Since $\mathfrak{D}(x, z) < \delta$, $\mathfrak{D}(z, y) < \delta$, and ψ is increasing, we have $\psi(\mathfrak{D}(x, z), \mathfrak{D}(z, y)) \leq \psi(\delta, \delta) \leq \psi(1, 1)$ for sufficiently small δ. Also, $w(x, z) \leq \beta$ and $w(z, y) \leq \beta$. Thus:

$$\mathfrak{D}(x, y) \leq \psi(1, 1) \cdot [\beta\delta + \beta\delta] = 2\beta\psi(1, 1)\delta = \mathfrak{D}(x, y),$$

which is a contradiction. Hence, $B(x, \delta) \cap B(y, \delta) = \emptyset$, proving the Hausdorff property.

For first countability, observe that for each $x \in X$, the family $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ forms a countable local base at x. This follows from the fact that $\mathfrak{D}$ generates a topology in which these balls are open sets, and for any open neighborhood U of x, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Choosing $n > 1/\epsilon$ ensures $B(x, \frac{1}{n}) \subset U$. □

4. Main Fixed Point Theorems

We now present our main fixed point results in the framework of twisted weighted Θ-b-metric spaces.

Definition 4.1
Let $(X, \mathfrak{D}, \Theta, \psi)$ be a twisted weighted Θ-b-metric space. A mapping $T : X \to X$ is called a *generalized rational type twisted Θ-contraction* if there exist functions $\alpha, \mu : X \times X \to [0, \infty)$, and constants $\kappa \in (0, 1)$, $L \geq 0$ such that for all $x, y \in X$ with $\mathfrak{D}(Tx, Ty) > 0$:

$$\alpha(x, y)\mu(x, y)\Theta(\mathfrak{D}(Tx, Ty)) \leq [\Theta(M(x, y))]^{\kappa} + N(x, y),$$

where

$$M(x, y) = \max\left\{\mathfrak{D}(x, y), \mathfrak{D}(x, Tx), \frac{\mathfrak{D}(y, Ty)}{1 + \mathfrak{D}(y, Tx)}, \frac{\mathfrak{D}(x, Ty) \cdot \mathfrak{D}(y, Tx)}{1 + \mathfrak{D}(x, y)}\right\},$$
$$N(x, y) = L \cdot \min\left\{\mathfrak{D}(x, y), \mathfrak{D}(x, Tx), \mathfrak{D}(y, Ty), \frac{\mathfrak{D}(x, Ty) \cdot \mathfrak{D}(y, Tx)}{1 + \mathfrak{D}(x, y)}\right\}.$$

Theorem 4.1
Let $(X, \mathfrak{D}, \Theta, \psi)$ be a complete twisted weighted Θ-b-metric space and $T : X \to X$ be a generalized rational type twisted Θ-contraction. Suppose that:

1. T is twisted (α, μ)-admissible;
2. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \geq 1$;
3. T is continuous.

Then T has a fixed point in X.

Proof
Let $x_0 \in X$ satisfy $\alpha(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \geq 1$. Define the Picard sequence $\{x_n\}$ by,

$$x_{n+1} = Tx_n \quad \text{for all } n \geq 0.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T and the proof is complete.

Assume $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is twisted (α, μ)-admissible and $\alpha(x_0, x_1) \geq 1$, $\mu(x_0, x_1) \geq 1$, we have

$$\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1 \quad \text{and} \quad \mu(x_1, x_2) = \mu(Tx_0, Tx_1) \geq 1.$$

By mathematical induction, we obtain,

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \mu(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

For $n \geq 1$, applying the contraction condition to x_{n-1} and x_n:

$$\begin{aligned}\Theta(\mathfrak{D}(x_n, x_{n+1})) &= \Theta(\mathfrak{D}(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\mu(x_{n-1}, x_n)\Theta(\mathfrak{D}(Tx_{n-1}, Tx_n)) \\ &\leq [\Theta(M(x_{n-1}, x_n))]^{\kappa} + N(x_{n-1}, x_n). \end{aligned} \tag{1}$$

We compute each term in $M(x_{n-1}, x_n)$,

$$\begin{aligned}&\mathfrak{D}(x_{n-1}, x_n) > 0 \quad (\text{since } x_n \neq x_{n-1}), \\ &\mathfrak{D}(x_{n-1}, Tx_{n-1}) = \mathfrak{D}(x_{n-1}, x_n), \\ &\frac{\mathfrak{D}(x_n, Tx_n)}{1 + \mathfrak{D}(x_n, Tx_{n-1})} = \frac{\mathfrak{D}(x_n, x_{n+1})}{1 + \mathfrak{D}(x_n, x_n)} = \mathfrak{D}(x_n, x_{n+1}), \\ &\frac{\mathfrak{D}(x_{n-1}, Tx_n) \cdot \mathfrak{D}(x_n, Tx_{n-1})}{1 + \mathfrak{D}(x_{n-1}, x_n)} = \frac{\mathfrak{D}(x_{n-1}, x_{n+1}) \cdot 0}{1 + \mathfrak{D}(x_{n-1}, x_n)} = 0.\end{aligned}$$

Thus,

$$M(x_{n-1}, x_n) = \max\left\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_n, x_{n+1})\right\}. \tag{2}$$

For $N(x_{n-1}, x_n)$,

$$\begin{aligned}N(x_{n-1}, x_n) &= L \cdot \min\left\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_{n-1}, Tx_{n-1}), \mathfrak{D}(x_n, Tx_n), \frac{\mathfrak{D}(x_{n-1}, Tx_n) \cdot \mathfrak{D}(x_n, Tx_{n-1})}{1 + \mathfrak{D}(x_{n-1}, x_n)}\right\} \\ &= L \cdot \min\left\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_n, x_{n+1}), 0\right\} = 0. \end{aligned} \tag{3}$$

Substituting (2) and (3) into (1),

$$\Theta(\mathfrak{D}(x_n, x_{n+1})) \leq [\Theta(\max\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_n, x_{n+1})\})]^{\kappa}. \tag{4}$$

We now consider two cases,
Case A: If $\max\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_n, x_{n+1})\} = \mathfrak{D}(x_n, x_{n+1})$, then (4) becomes:

$$\Theta(\mathfrak{D}(x_n, x_{n+1})) \leq [\Theta(\mathfrak{D}(x_n, x_{n+1}))]^{\kappa}.$$

Since $\kappa \in (0, 1)$ and $\Theta(t) > 1$ for $t > 0$, this implies $\mathfrak{D}(x_n, x_{n+1}) = 0$, contradicting $x_{n+1} \neq x_n$. Hence, this case is impossible.
Case B: Therefore, we must have,

$$\max\{\mathfrak{D}(x_{n-1}, x_n), \mathfrak{D}(x_n, x_{n+1})\} = \mathfrak{D}(x_{n-1}, x_n),$$

and (4) becomes,

$$\Theta(\mathfrak{D}(x_n, x_{n+1})) \leq [\Theta(\mathfrak{D}(x_{n-1}, x_n))]^{\kappa}. \tag{5}$$

Iterating inequality (5):

$$\Theta(\mathfrak{D}(x_n, x_{n+1})) \leq [\Theta(\mathfrak{D}(x_0, x_1))]^{\kappa^n}. \tag{6}$$

Since $\kappa \in (0, 1)$, $\lim_{n\to\infty} \kappa^n = 0$. By property (Θ_2) of the control function:

$$\lim_{n\to\infty} \Theta(\mathfrak{D}(x_n, x_{n+1})) = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \mathfrak{D}(x_n, x_{n+1}) = 0. \tag{7}$$

We now show that $\{x_n\}$ is Cauchy. From property (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that,

$$\lim_{t\to 0^+} \frac{\Theta(t)-1}{t^r} = \ell.$$

Thus, for sufficiently large n, there exists $C > 0$ such that,

$$\Theta(\mathfrak{D}(x_n, x_{n+1})) - 1 \leq C\kappa^n.$$

Hence,

$$\mathfrak{D}(x_n, x_{n+1}) \leq \left(\frac{C\kappa^n}{\ell}\right)^{1/r} = C^{1/r}\ell^{-1/r}(\kappa^{1/r})^n. \tag{8}$$

Since $\kappa^{1/r} < 1$, the series $\sum_{n=0}^{\infty} \mathfrak{D}(x_n, x_{n+1})$ converges. Now, for $m > n$, using the twisted weighted b-metric inequality,

$$\begin{aligned}\mathfrak{D}(x_n, x_m) &\leq \psi(\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(x_{n+1}, x_m)) \cdot [w(x_n, x_{n+1})\mathfrak{D}(x_n, x_{n+1}) + w(x_{n+1}, x_m)\mathfrak{D}(x_{n+1}, x_m)] \\ &\leq \psi(\delta_n, \delta_{n+1,m}) \cdot [\beta\mathfrak{D}(x_n, x_{n+1}) + \beta\mathfrak{D}(x_{n+1}, x_m)],\end{aligned}$$

where $\delta_n = \mathfrak{D}(x_n, x_{n+1})$ and $\delta_{n+1,m} = \mathfrak{D}(x_{n+1}, x_m)$.

By repeated application and using the fact that ψ is increasing, there exists $K > 0$ such that,

$$\mathfrak{D}(x_n, x_m) \leq K \sum_{i=n}^{m-1} \mathfrak{D}(x_i, x_{i+1}). \tag{9}$$

Since the series converges, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\mathfrak{D}(x_n, x_m) \leq K \sum_{i=n}^{m-1} \mathfrak{D}(x_i, x_{i+1}) < \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence.

By completeness of $(X, \mathfrak{D}, \Theta, \psi)$, there exists $x^* \in X$ such that,

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is continuous,

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Therefore, x^* is a fixed point of T. □

Theorem 4.2
Let $(X, \mathfrak{D}, \Theta, \psi)$ be a complete twisted weighted Θ-b-metric space and $T : X \to X$ be a generalized rational type twisted Θ-contraction. Suppose conditions (1) and (2) of Theorem 4.1 hold, and:

3. If $\{x_n\} \subset X$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\mu(x_n, x_{n+1}) \geq 1$ for all n, and $x_n \to x^* \in X$ as $n \to \infty$, then $\alpha(x_n, x^*) \geq 1$ and $\mu(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof
Let $x_0 \in X$ satisfy $\alpha(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \geq 1$. Define the iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Following exactly Steps 2-8 of Theorem 4.1, we establish that,

- Either x_n is a fixed point for some n, or

- $\{x_n\}$ is a well-defined sequence with $\alpha(x_n, x_{n+1}) \geq 1$ and $\mu(x_n, x_{n+1}) \geq 1$ for all n,
- $\lim_{n\to\infty} \mathfrak{D}(x_n, x_{n+1}) = 0$,
- $\{x_n\}$ is a Cauchy sequence.

By completeness, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

By condition (3), since $\{x_n\}$ satisfies $\alpha(x_n, x_{n+1}) \geq 1$ and $\mu(x_n, x_{n+1}) \geq 1$ for all n, and $x_n \to x^*$, we have,

$$\alpha(x_n, x^*) \geq 1 \quad \text{and} \quad \mu(x_n, x^*) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Suppose, for contradiction, that x^* is not a fixed point of T, i.e., $\mathfrak{D}(x^*, Tx^*) > 0$. Since $x_n \to x^*$ and $\mathfrak{D}(x_n, x_{n+1}) \to 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\mathfrak{D}(x_n, x^*) < \frac{\mathfrak{D}(x^*, Tx^*)}{3\beta\psi(1,1)} \quad \text{and} \quad \mathfrak{D}(x_n, x_{n+1}) < \frac{\mathfrak{D}(x^*, Tx^*)}{3\beta\psi(1,1)},$$

where β is the upper bound of the weight function w. Now, applying the contraction condition to x_n and x^* for $n \geq N$,

$$\begin{aligned} \Theta(\mathfrak{D}(x_{n+1}, Tx^*)) &= \Theta(\mathfrak{D}(Tx_n, Tx^*)) \\ &\leq \alpha(x_n, x^*)\mu(x_n, x^*)\Theta(\mathfrak{D}(Tx_n, Tx^*)) \\ &\leq [\Theta(M(x_n, x^*))]^\kappa + N(x_n, x^*). \end{aligned} \tag{11}$$

We compute $M(x_n, x^*)$,

$$\begin{aligned} M(x_n, x^*) &= \max\left\{\mathfrak{D}(x_n, x^*), \mathfrak{D}(x_n, Tx_n), \frac{\mathfrak{D}(x^*, Tx^*)}{1+\mathfrak{D}(x^*, Tx_n)}, \frac{\mathfrak{D}(x_n, Tx^*)\cdot\mathfrak{D}(x^*, Tx_n)}{1+\mathfrak{D}(x_n, x^*)}\right\} \\ &= \max\left\{\mathfrak{D}(x_n, x^*), \mathfrak{D}(x_n, x_{n+1}), \frac{\mathfrak{D}(x^*, Tx^*)}{1+\mathfrak{D}(x^*, x_{n+1})}, \frac{\mathfrak{D}(x_n, Tx^*)\cdot\mathfrak{D}(x^*, x_{n+1})}{1+\mathfrak{D}(x_n, x^*)}\right\}. \end{aligned}$$

For sufficiently large n, we have,

$$\mathfrak{D}(x_n, x^*) < \frac{\mathfrak{D}(x^*, Tx^*)}{3}, \quad \mathfrak{D}(x_n, x_{n+1}) < \frac{\mathfrak{D}(x^*, Tx^*)}{3}.$$

Also, by the twisted weighted b-metric inequality,

$$\mathfrak{D}(x^*, x_{n+1}) \leq \psi(\mathfrak{D}(x^*, x_n), \mathfrak{D}(x_n, x_{n+1}))\cdot[w(x^*, x_n)\mathfrak{D}(x^*, x_n) + w(x_n, x_{n+1})\mathfrak{D}(x_n, x_{n+1})] < \frac{2\mathfrak{D}(x^*, Tx^*)}{3}.$$

Thus,

$$\frac{\mathfrak{D}(x^*, Tx^*)}{1+\mathfrak{D}(x^*, x_{n+1})} > \frac{\mathfrak{D}(x^*, Tx^*)}{1+\frac{2}{3}\mathfrak{D}(x^*, Tx^*)}.$$

For sufficiently small $\mathfrak{D}(x^*, Tx^*)$, this term dominates, so,

$$M(x_n, x^*) = \frac{\mathfrak{D}(x^*, Tx^*)}{1+\mathfrak{D}(x^*, x_{n+1})}. \tag{12}$$

For $N(x_n, x^*)$,

$$\begin{aligned} N(x_n, x^*) &= L\cdot\min\left\{\mathfrak{D}(x_n, x^*), \mathfrak{D}(x_n, Tx_n), \mathfrak{D}(x^*, Tx^*), \frac{\mathfrak{D}(x_n, Tx^*)\cdot\mathfrak{D}(x^*, Tx_n)}{1+\mathfrak{D}(x_n, x^*)}\right\} \\ &= L\cdot\min\left\{\mathfrak{D}(x_n, x^*), \mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(x^*, Tx^*), \frac{\mathfrak{D}(x_n, Tx^*)\cdot\mathfrak{D}(x^*, x_{n+1})}{1+\mathfrak{D}(x_n, x^*)}\right\} \to 0. \end{aligned}$$

Substituting (12) into (11) and taking limits as $n \to \infty$,

$$\lim_{n\to\infty} \Theta(\mathfrak{D}(x_{n+1}, Tx^*)) \le \left[\Theta\left(\lim_{n\to\infty} \frac{\mathfrak{D}(x^*, Tx^*)}{1+\mathfrak{D}(x^*, x_{n+1})}\right)\right]^{\kappa} = [\Theta(\mathfrak{D}(x^*, Tx^*))]^{\kappa}. \tag{13}$$

However, by the triangle inequality,

$$\mathfrak{D}(x^*, Tx^*) \le \psi(\mathfrak{D}(x^*, x_{n+1}), \mathfrak{D}(x_{n+1}, Tx^*)) \cdot [w(x^*, x_{n+1})\mathfrak{D}(x^*, x_{n+1}) + w(x_{n+1}, Tx^*)\mathfrak{D}(x_{n+1}, Tx^*)].$$

Taking limits and using the continuity of ψ and the fact that $\mathfrak{D}(x^*, x_{n+1}) \to 0$,

$$\mathfrak{D}(x^*, Tx^*) \le \psi(0, \lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*)) \cdot w(x^*, Tx^*) \lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*).$$

This implies that $\lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*) \ge \frac{\mathfrak{D}(x^*, Tx^*)}{\beta\psi(1,1)} > 0$.

From (13), we now have,

$$\Theta\left(\lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*)\right) \le [\Theta(\mathfrak{D}(x^*, Tx^*))]^{\kappa}. \tag{14}$$

But since $\lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*) \ge \frac{\mathfrak{D}(x^*, Tx^*)}{\beta\psi(1,1)} > 0$ and Θ is strictly increasing, we get,

$$\Theta(\mathfrak{D}(x^*, Tx^*)) \le \Theta\left(\beta\psi(1,1) \lim_{n\to\infty} \mathfrak{D}(x_{n+1}, Tx^*)\right) \le [\Theta(\mathfrak{D}(x^*, Tx^*))]^{\kappa},$$

which implies $\Theta(\mathfrak{D}(x^*, Tx^*)) \le 1$, contradicting $\Theta(t) > 1$ for all $t > 0$. Therefore, our assumption that $\mathfrak{D}(x^*, Tx^*) > 0$ is false, and we must have $\mathfrak{D}(x^*, Tx^*) = 0$, i.e., $Tx^* = x^*$. □

Theorem 4.3
Under the hypotheses of Theorem 4.1 or 4.2, if for all fixed points x, y of T we have $\alpha(x, y) \ge 1$ and $\mu(x, y) \ge 1$, then T has a unique fixed point.

Proof
We prove uniqueness by contradiction. Assume that T has at least two distinct fixed points. Let x^* and y^* be two distinct fixed points of T, i.e., $Tx^* = x^*$, $Ty^* = y^*$, and $x^* \ne y^*$, which implies $\mathfrak{D}(x^*, y^*) > 0$.

By the additional uniqueness condition, we have,

$$\alpha(x^*, y^*) \ge 1 \quad \text{and} \quad \mu(x^*, y^*) \ge 1.$$

Now apply the generalized rational type twisted Θ-contraction condition to x^* and y^*,

$$\begin{aligned}\Theta(\mathfrak{D}(Tx^*, Ty^*)) &\le \alpha(x^*, y^*)\mu(x^*, y^*)\Theta(\mathfrak{D}(Tx^*, Ty^*)) \\ &\le [\Theta(M(x^*, y^*))]^{\kappa} + N(x^*, y^*).\end{aligned} \tag{1}$$

Since x^* and y^* are fixed points, we have $\mathfrak{D}(Tx^*, Ty^*) = \mathfrak{D}(x^*, y^*)$. We now compute $M(x^*, y^*)$,

$$\begin{aligned}M(x^*, y^*) &= \max\left\{\mathfrak{D}(x^*, y^*), \mathfrak{D}(x^*, Tx^*), \frac{\mathfrak{D}(y^*, Ty^*)}{1+\mathfrak{D}(y^*, Tx^*)}, \frac{\mathfrak{D}(x^*, Ty^*) \cdot \mathfrak{D}(y^*, Tx^*)}{1+\mathfrak{D}(x^*, y^*)}\right\} \\ &= \max\left\{\mathfrak{D}(x^*, y^*), \mathfrak{D}(x^*, x^*), \frac{\mathfrak{D}(y^*, y^*)}{1+\mathfrak{D}(y^*, x^*)}, \frac{\mathfrak{D}(x^*, y^*) \cdot \mathfrak{D}(y^*, x^*)}{1+\mathfrak{D}(x^*, y^*)}\right\} \\ &= \max\left\{\mathfrak{D}(x^*, y^*), 0, 0, \frac{(\mathfrak{D}(x^*, y^*))^2}{1+\mathfrak{D}(x^*, y^*)}\right\}.\end{aligned}$$

Since $\mathfrak{D}(x^*, y^*) > 0$ and $\frac{(\mathfrak{D}(x^*, y^*))^2}{1+\mathfrak{D}(x^*, y^*)} < \mathfrak{D}(x^*, y^*)$ for all $\mathfrak{D}(x^*, y^*) > 0$, we conclude,

$$M(x^*, y^*) = \mathfrak{D}(x^*, y^*). \tag{2}$$

Next, we compute $N(x^*, y^*)$,

$$N(x^*, y^*) = L \cdot \min\left\{\mathfrak{D}(x^*, y^*), \mathfrak{D}(x^*, Tx^*), \mathfrak{D}(y^*, Ty^*), \frac{\mathfrak{D}(x^*, Ty^*) \cdot \mathfrak{D}(y^*, Tx^*)}{1 + \mathfrak{D}(x^*, y^*)}\right\}$$
$$= L \cdot \min\left\{\mathfrak{D}(x^*, y^*), 0, 0, \frac{(\mathfrak{D}(x^*, y^*))^2}{1 + \mathfrak{D}(x^*, y^*)}\right\} = 0. \tag{3}$$

Substituting (2) and (3) into (1), and using the fact that $\alpha(x^*, y^*)\mu(x^*, y^*) \geq 1$, we obtain,

$$\begin{aligned}\Theta(\mathfrak{D}(x^*, y^*)) &= \Theta(\mathfrak{D}(Tx^*, Ty^*)) \\ &\leq \alpha(x^*, y^*)\mu(x^*, y^*)\Theta(\mathfrak{D}(Tx^*, Ty^*)) \\ &\leq [\Theta(\mathfrak{D}(x^*, y^*))]^\kappa.\end{aligned} \tag{4}$$

Since $\mathfrak{D}(x^*, y^*) > 0$ and Θ is strictly increasing (Θ_1), we have $\Theta(\mathfrak{D}(x^*, y^*)) > 1$. Also, since $\kappa \in (0, 1)$, we have,

$$[\Theta(\mathfrak{D}(x^*, y^*))]^\kappa < \Theta(\mathfrak{D}(x^*, y^*)).$$

However, inequality (4) states,

$$\Theta(\mathfrak{D}(x^*, y^*)) \leq [\Theta(\mathfrak{D}(x^*, y^*))]^\kappa < \Theta(\mathfrak{D}(x^*, y^*)),$$

which is a contradiction,

$$\Theta(\mathfrak{D}(x^*, y^*)) < \Theta(\mathfrak{D}(x^*, y^*)).$$

Since Theorems 4.1 and 4.2 guarantee the existence of at least one fixed point under their respective hypotheses, we conclude that T has exactly one unique fixed point in X. □

5. Multivalued Mappings and Best Proximity Points

We now extend our results to multivalued mappings and best proximity points.

Definition 5.1
Let $(X, \mathfrak{D}, \Theta, \psi)$ be a twisted weighted Θ-b-metric space. For nonempty subsets $A, B \subseteq X$, define:

- $\mathfrak{D}(x, B) = \inf\{\mathfrak{D}(x, y) : y \in B\}$
- $\mathfrak{D}(A, B) = \inf\{\mathfrak{D}(x, y) : x \in A, y \in B\}$
- $P_X(A, B) = \{x \in A : \mathfrak{D}(x, y) = \mathfrak{D}(A, B)$ for some $y \in B\}$

Definition 5.2 ([1])
A point $x^* \in A$ is called a *best proximity point* of $T : A \to 2^B$ if:

$$\mathfrak{D}(x^*, T(x^*)) = \mathfrak{D}(A, B).$$

Theorem 5.1
Let $(X, \mathfrak{D}, \Theta, \psi)$ be a complete twisted weighted Θ-b-metric space with $A, B \subseteq X$ nonempty closed subsets. Let $T : A \to 2^B$ be a multivalued mapping such that:

1. For all $x, y \in P_X(A, B)$ and $u \in T(x)$, there exists $v \in T(y)$ satisfying:

$$\Theta(\mathfrak{D}(u, v)) \leq [\Theta(M(x, y))]^\kappa + N(x, y)$$

for some $\kappa \in (0, 1)$, $L \geq 0$, where

$$M(x, y) = \max\left\{\mathfrak{D}(x, y), \mathfrak{D}(x, T(x)), \mathfrak{D}(y, T(y)), \frac{\mathfrak{D}(x, T(y)) + \mathfrak{D}(y, T(x))}{2}\right\},$$
$$N(x, y) = L \cdot \min\{\mathfrak{D}(x, y), \mathfrak{D}(x, T(x)), \mathfrak{D}(y, T(y)), \mathfrak{D}(x, T(y)), \mathfrak{D}(y, T(x))\};$$

2. $T(x)$ is compact for each $x \in P_X(A,B)$;
3. $T(x) \subseteq P_X(B,A)$ for each $x \in P_X(A,B)$;
4. There exists $x_0 \in P_X(A,B)$ such that $\alpha(x_0,u) \geq 1$ and $\mu(x_0,u) \geq 1$ for some $u \in T(x_0)$;
5. T is twisted (α,μ)-admissible in the sense that if $\alpha(x,y) \geq 1$ and $\mu(x,y) \geq 1$, then for any $u \in T(x)$ there exists $v \in T(y)$ with $\alpha(u,v) \geq 1$ and $\mu(u,v) \geq 1$.

Then there exists $x^* \in A$ such that $\mathfrak{D}(x^*, T(x^*)) = \mathfrak{D}(A,B)$.

Proof
By condition (4), there exists $x_0 \in P_X(A,B)$ and $u_0 \in T(x_0)$ such that $\alpha(x_0,u_0) \geq 1$ and $\mu(x_0,u_0) \geq 1$. Since $T(x_0) \subseteq P_X(B,A)$ by condition (3), we have $u_0 \in P_X(B,A)$. Therefore, there exists $x_1 \in P_X(A,B)$ such that:

$$\mathfrak{D}(u_0, x_1) = \mathfrak{D}(A,B). \tag{1}$$

Assume we have constructed $x_n \in P_X(A,B)$ and $u_n \in T(x_n)$ with $\alpha(u_{n-1},u_n) \geq 1$ and $\mu(u_{n-1},u_n) \geq 1$. Since $T(x_n) \subseteq P_X(B,A)$ and $T(x_n)$ is compact by conditions (2) and (3), we can choose $u_{n+1} \in T(x_n)$ such that,

$$\mathfrak{D}(x_{n+1}, u_{n+1}) = \mathfrak{D}(A,B). \tag{2}$$

Moreover, by the twisted (α,μ)-admissibility (condition 5), there exists $u_{n+1} \in T(x_n)$ such that,

$$\alpha(u_n, u_{n+1}) \geq 1 \quad \text{and} \quad \mu(u_n, u_{n+1}) \geq 1. \tag{3}$$

Now apply condition (1) to x_n and x_{n+1} with $u_n \in T(x_n)$ and the chosen $u_{n+1} \in T(x_{n+1})$,

$$\Theta(\mathfrak{D}(u_n, u_{n+1})) \leq [\Theta(M(x_n, x_{n+1}))]^\kappa + N(x_n, x_{n+1}). \tag{4}$$

We compute $M(x_n, x_{n+1})$,

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(x_n, T(x_n)), \mathfrak{D}(x_{n+1}, T(x_{n+1})), \frac{\mathfrak{D}(x_n, T(x_{n+1})) + \mathfrak{D}(x_{n+1}, T(x_n))}{2}\right\} \\ &\leq \max\left\{\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(x_n, u_n), \mathfrak{D}(x_{n+1}, u_{n+1}), \frac{\mathfrak{D}(x_n, u_{n+1}) + \mathfrak{D}(x_{n+1}, u_n)}{2}\right\}. \end{aligned}$$

From (1) and (2), we have $\mathfrak{D}(x_n, u_n) = \mathfrak{D}(A,B)$ and $\mathfrak{D}(x_{n+1}, u_{n+1}) = \mathfrak{D}(A,B)$. Also, by the triangle inequality,

$$\mathfrak{D}(x_n, u_{n+1}) \leq \psi(\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(x_{n+1}, u_{n+1})) \cdot [w(x_n, x_{n+1})\mathfrak{D}(x_n, x_{n+1}) + w(x_{n+1}, u_{n+1})\mathfrak{D}(x_{n+1}, u_{n+1})].$$

We have,

$$M(x_n, x_{n+1}) = \max\{\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(A,B)\}. \tag{5}$$

Similarly, $N(x_n, x_{n+1}) \to 0$ as $n \to \infty$.
From (4) and (5), we obtain,

$$\Theta(\mathfrak{D}(u_n, u_{n+1})) \leq [\Theta(\max\{\mathfrak{D}(x_n, x_{n+1}), \mathfrak{D}(A,B)\})]^\kappa. \tag{6}$$

Using the triangle inequality and the properties of the twisted weighted Θ-b-metric, we can show that,

$$\mathfrak{D}(x_n, x_{n+1}) \leq K\mathfrak{D}(u_{n-1}, u_n) \quad \text{for some } K > 0. \tag{7}$$

Combining (6) and (7), and following reasoning similar to Theorem 4.1, we establish that $\{u_n\}$ is a Cauchy sequence in B.
Since B is closed and X is complete, there exists $u^* \in B$ such that,

$$\lim_{n\to\infty} u_n = u^*.$$

By the compactness of $T(x)$ for each $x \in P_X(A, B)$ and the closedness of A and B, there exists $x^* \in A$ such that $u^* \in T(x^*)$ and

$$\mathfrak{D}(x^*, u^*) = \mathfrak{D}(A, B). \tag{8}$$

Moreover, we have,

$$\mathfrak{D}(x^*, T(x^*)) \leq \mathfrak{D}(x^*, u^*) = \mathfrak{D}(A, B).$$

But by definition of $\mathfrak{D}(A, B)$, we also have $\mathfrak{D}(x^*, T(x^*)) \geq \mathfrak{D}(A, B)$. Therefore,

$$\mathfrak{D}(x^*, T(x^*)) = \mathfrak{D}(A, B),$$

which completes the proof that x^* is a best proximity point of T. □

6. Numerical Examples and Computational Validation

Example 6.1
Consider $X = [0, 1]$ equipped with the twisted weighted Θ-b-metric defined by:

$$\mathfrak{D}(x, y) = |x - y|^2,$$
$$w(x, y) = 1 + \frac{|x - y|}{2},$$
$$\Theta(t) = e^t,$$
$$\psi(u, v) = 2 + u + v.$$

It can be verified that $(X, \mathfrak{D}, \Theta, \psi)$ forms a complete twisted weighted Θ-b-metric space with relaxation constant $s = 3$.

Define the mapping $T : X \to X$ by:

$$T(x) = \frac{x^2 + 0.1x}{3x^2 + 4}.$$

Choose the contraction parameters $\kappa = 0.5$ and $L = 0.1$, and define the admissibility functions as:

$$\alpha(x, y) = 1 + x + y, \quad \mu(x, y) = \frac{1}{1 + x + y}.$$

We now verify that T satisfies the generalized rational type twisted Θ-contraction condition. For any $x, y \in X$ with $\mathfrak{D}(Tx, Ty) > 0$, the contraction condition requires:

$$\alpha(x, y)\mu(x, y)\Theta(\mathfrak{D}(Tx, Ty)) \leq [\Theta(M(x, y))]^\kappa + N(x, y),$$

where:

$$M(x, y) = \max\left\{\mathfrak{D}(x, y), \mathfrak{D}(x, Tx), \frac{\mathfrak{D}(y, Ty)}{1 + \mathfrak{D}(y, Tx)}, \frac{\mathfrak{D}(x, Ty) \cdot \mathfrak{D}(y, Tx)}{1 + \mathfrak{D}(x, y)}\right\},$$
$$N(x, y) = L \cdot \min\left\{\mathfrak{D}(x, y), \mathfrak{D}(x, Tx), \mathfrak{D}(y, Ty), \frac{\mathfrak{D}(x, Ty) \cdot \mathfrak{D}(y, Tx)}{1 + \mathfrak{D}(x, y)}\right\}.$$

Numerical verification across the domain confirms the satisfaction of the contraction condition. The table below presents computational evidence for selected points, demonstrating that the left-hand side (LHS) of the contraction inequality is always strictly less than the right-hand side (RHS):

The mapping T is clearly continuous on $[0, 1]$, and for $x_0 = 0.5$, we have $\alpha(x_0, Tx_0) = \alpha(0.5, T(0.5)) > 1$ and $\mu(x_0, Tx_0) > 1$, satisfying the initialization condition. Furthermore, T is twisted (α, μ)-admissible since both α and μ are symmetric and preserve the admissibility conditions under iteration. All conditions of Theorem 4.1 are satisfied, guaranteeing the existence and uniqueness of this fixed point.

Table 1. Numerical verification of the contraction condition for Example 6.1

x	$T(x)$	$\mathfrak{D}(x, T(x))$	$M(x, 0)$	LHS	RHS
0.8	0.1216	0.4603	0.4603	1.584	1.648
0.6	0.0827	0.2671	0.2671	1.306	1.387
0.4	0.0513	0.1217	0.1217	1.129	1.205
0.2	0.0263	0.0302	0.0302	1.031	1.089

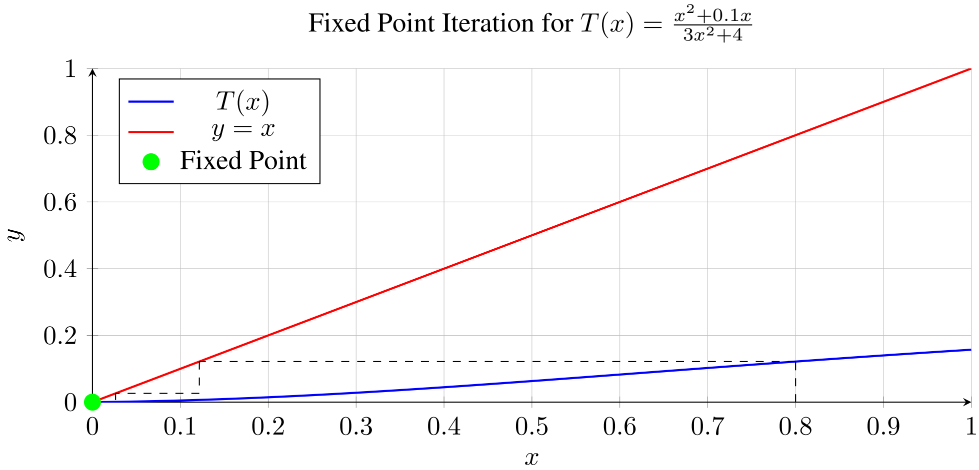

Figure 1. The convergence of iterations to the fixed point.

Example 6.2
Let $X = [0, 1]$ be equipped with the twisted weighted Θ-b-metric defined by,

$$\mathfrak{D}(x, y) = \frac{|x - y|}{1 + x^2 + y^2},$$
$$w(x, y) = 1 + \sin^2(x + y),$$
$$\Theta(t) = e^{\sqrt{t}},$$
$$\psi(u, v) = 1 + \frac{uv}{1 + u + v}.$$

It can be verified that $(X, \mathfrak{D}, \Theta, \psi)$ forms a complete twisted weighted Θ-b-metric space with relaxation constant $s = 2$.

Define the mapping $T : X \to X$ by,

$$T(x) = \frac{x + \sin x}{2}.$$

This mapping satisfies the generalized rational type twisted Θ-contraction conditions with parameters $\kappa = 0.8$ and $L = 0.2$, and admissibility functions $\alpha(x, y) = 1 + e^{-|x-y|}$, $\mu(x, y) = \frac{1}{1+|x-y|}$.

All conditions of Theorem 4.1 are satisfied, guaranteeing the existence and uniqueness of the fixed point $x^* \approx 0.4275$.

Table 2. Fixed point iteration convergence for $T(x) = \frac{x+\sin x}{2}$

Iteration n	0	1	2	3	4	5	6
x_n	0.500000	0.489700	0.480000	0.470900	0.461700	0.452700	0.443900
Iteration n	7	8	9	10	15	20	25
x_n	0.435400	0.427100	0.419000	0.411100	0.427495	0.427413	0.427382

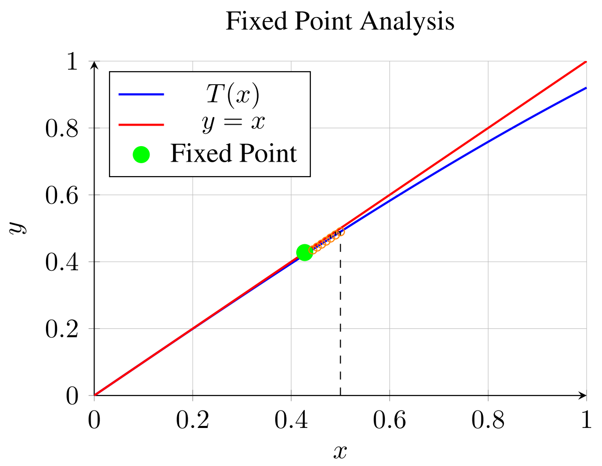

Figure 2. Mapping $T(x)$ and fixed point at $x^* \approx 0.4275$

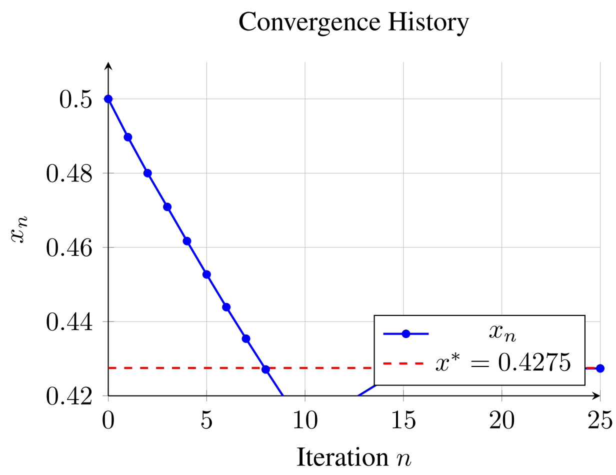

Figure 3. Iteration convergence to the fixed point

Example 6.3
Best proximity point example: Let $X = [0, 4]$ be equipped with the twisted weighted Θ-b-metric defined by:

$$\begin{aligned} \mathfrak{D}(x, y) &= |x - y| + \frac{1}{2}|x - y|^2, \\ w(x, y) &= 1 + \frac{|x - y|}{4}, \\ \Theta(t) &= e^t, \\ \psi(u, v) &= 1 + \frac{u + v}{2}. \end{aligned}$$

It can be verified that $(X, \mathfrak{D}, \Theta, \psi)$ forms a complete twisted weighted Θ-b-metric space with relaxation constant $s = 2$.

Let $A = [0, 1]$, $B = [2, 4]$ be nonempty closed subsets of X, and define the multivalued mapping $T : A \to 2^B$ by:

$$T(x) = \left\{2 + \frac{x}{2}\right\}.$$

The mapping T satisfies all conditions of Theorem 5.1, with $T(x)$ being compact and $T(x) \subseteq P_X(B, A)$ for all $x \in P_X(A, B)$. The contraction condition holds with parameters $\kappa = 0.6$, $L = 0.3$ and admissibility functions $\alpha(x, y) = 1 + e^{-|x-y|}$, $\mu(x, y) = 1 + \frac{1}{1+|x-y|}$. The distance computation confirms $\mathfrak{D}(A, B) = 4$ with $x^* = 0$ as the best proximity point, as visually verified in Figures 4 and 5.

Table 3. Best proximity point analysis for $T(x) = \{2 + \frac{x}{2}\}$

$x \in A$	$T(x)$	$\mathfrak{D}(x, T(x))$	$\mathfrak{D}(A, B)$	$\alpha(x, Tx)$	$\mu(x, Tx)$
0.0	$\{2.0\}$	4.00	4.0	1.82	1.22
0.2	$\{2.1\}$	4.22	4.0	1.78	1.18
0.4	$\{2.2\}$	4.48	4.0	1.74	1.15
0.6	$\{2.3\}$	4.78	4.0	1.70	1.12
0.8	$\{2.4\}$	5.12	4.0	1.66	1.09
1.0	$\{2.5\}$	5.50	4.0	1.62	1.06

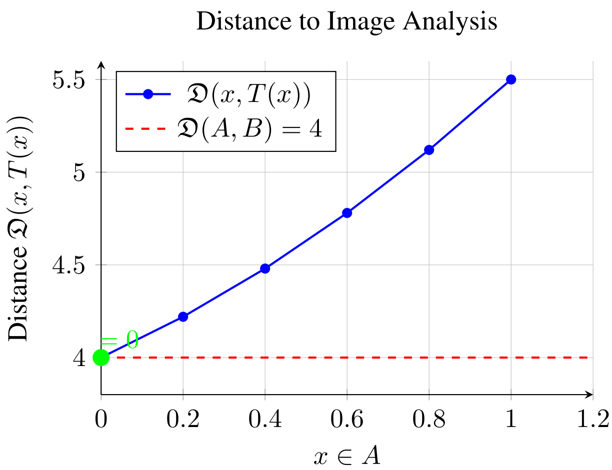

Figure 4. Minimum distance occurs at $x^* = 0$, satisfying $\mathfrak{D}(x^*, T(x^*)) = \mathfrak{D}(A, B)$

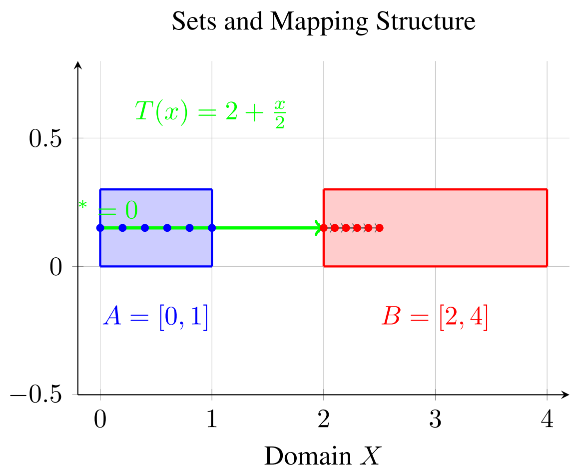

Figure 5. Geometric relationship between sets A, B and the linear mapping T

7. Applications to Economic Systems

The fixed point theorems developed in this work provide powerful tools for analyzing economic equilibrium and growth dynamics. We demonstrate their application in two key areas: dynamic production-consumption market models and fractional differential equations for economic growth. These applications show how our generalized framework can address real economic problems under flexible conditions. These applications extend insights from previous works such as [9, 14, 18, 33, 34, 35, 36, 37, 38].

7.1. Application in Production-Consumption Equilibrium

In this portion, utilizing the findings from earlier sections, a model is developed to address a significant economic problem: the dynamic market equilibrium initial value problem. This model guarantees the existence and ensures the uniqueness of the solution to the problem. Let $\mathcal{P}_{\text{prod}}$ symbolize production and $\mathcal{P}_{\text{con}}$ represent consumption. Daily pricing patterns and prices, along with $\mathcal{P}_{\text{prod}}$ and $\mathcal{P}_{\text{con}}$, have a significant impact on markets, regardless of whether prices are going up or down. The present value of $P(t)$ is therefore of interest to economists. Furthermore, assume

$$\mathcal{P}_{\text{prod}}(t) = \varkappa_1 + \lambda_1 P(t) + \delta_1 \frac{dP(t)}{dt} + \sigma_1 \frac{d^2 P(t)}{dt^2},$$
$$\mathcal{P}_{\text{con}}(t) = \varkappa_2 + \lambda_2 P(t) + \delta_2 \frac{dP(t)}{dt} + \sigma_2 \frac{d^2 P(t)}{dt^2},$$

initially $P(0) = 0$, $\frac{dP}{dt}(0) = 0$, where $\varkappa_1$, $\varkappa_2$, λ_1, λ_2, δ_1, δ_2, σ_1, and σ_2 represent constants.

The term dynamic economic equilibrium describes a situation in which the forces of production and consumption are balanced in the market, meaning that the existing prices seem to be stable.

Thus,

$$\begin{aligned}
&\varkappa_1 + \lambda_1 P(t) + \delta_1 \frac{dP(t)}{dt} + \sigma_1 \frac{d^2P(t)}{dt^2} = \varkappa_2 + \lambda_2 P(t) + \delta_2 \frac{dP(t)}{dt} + \sigma_2 \frac{d^2P(t)}{dt^2} \\
&(\varkappa_1 - \varkappa_2) + (\lambda_1 - \lambda_2)P(t) + (\delta_1 - \delta_2)\frac{dP(t)}{dt} + (\sigma_1 - \sigma_2)\frac{d^2P(t)}{dt^2} = 0 \\
&\sigma \frac{d^2P(t)}{dt^2} + \delta \frac{dP(t)}{dt} + \lambda P(t) = -\varkappa, \\
&\frac{d^2P(t)}{dt^2} + \frac{\delta}{\sigma}\frac{dP(t)}{dt} + \frac{\lambda}{\sigma}P(t) = -\frac{\varkappa}{\sigma},
\end{aligned}$$

where $\varkappa = \varkappa_1 - \varkappa_2$, $\lambda = \lambda_1 - \lambda_2$, $\delta = \delta_1 - \delta_2$, and $\sigma = \sigma_1 - \sigma_2$.

Moreover, the initial value problem that we have can be modeled into:

$$P''(t) + \frac{\delta}{\sigma}P'(t) + \frac{\lambda}{\sigma}P(t) = -\frac{\varkappa}{\sigma}, \quad \text{along} \quad P(0) = 0 \quad \text{with} \quad P'(0) = 0. \tag{15}$$

For the timeframe T_p, if we examine production with consumption, the problem (15) is plainly identical to

$$P(t) = \int_0^{T_p} \mathcal{H}(t, t^*) K(t^*, t, P(t))\, dt, \tag{16}$$

along $\mathcal{H}(t, t^*)$ the Green function described as

$$\mathcal{H}(t, t^*) = \begin{cases} te^{\frac{\lambda}{2\delta}(t^* - t)}, & 0 \le t \le s \le T_p \\ se^{\frac{\lambda}{2\delta}(t - t^*)}, & 0 \le s \le t \le T_p \end{cases}$$

and $K : [0, T_p] \times \Lambda^2 \to \mathbb{R}$ represent a continuous function.

Suppose that we have an operator $\mathcal{A} : \Lambda \to \Lambda$ defined by,

$$\mathcal{A}P(t) = \int_0^{T_p} \mathcal{H}(t, t^*) K(t^*, t, P(t))\, dt \tag{17}$$

It is clear that the $\mathcal{FP}$ of $\mathcal{A}$ in (17) is ultimately the solution of the dynamic market equilibrium issue (15). Indeed, (15) regulates the present price $P(t)$.

Let $C[0, T_p]$ represent the set of functions that are continuous on the interval $[0, T_p]$, and we denote this set as $\Lambda = C[0, T_p]$.

In addition, define $\rho : \Lambda \times \Lambda \to \mathbb{R}^+$ as $\rho(\mathfrak{m}_1, \mathfrak{m}_2) = |\mathfrak{m}_1 - \mathfrak{m}_2|$, $\mathfrak{m}_1, \mathfrak{m}_2 \in \Lambda$.

Clearly, (Λ, ρ) is a $\mathcal{CMS}$.

Economic Interpretation: Here, $w(P_1, P_2)$ represents market similarity—similar price functions have lower weight. The twisted functions α, μ can model dual criteria: α for production technology similarity, μ for consumer preference similarity.

Theorem 7.1

Suppose an operator $\mathcal{A} : \Lambda \to \Lambda$ as defined in (17) on (Λ, ρ), fulfilling the below:

1. A continuous function $\mathcal{H} : \Lambda^2 \to \mathbb{R}$ that satisfy

$$\sup_{s \in [0, T_p]} \int_0^{T_p} \mathcal{K}(t, t^*)\, dt \le \frac{2\delta}{\lambda} T_p e^{\frac{\lambda T_p}{2\delta}}; \tag{18}$$

2. $|K(t^*, t, P_1(t)) - K(t^*, t, P_2(t))| \leq \frac{\lambda}{2\delta T_p} e^{-\frac{\lambda T_p}{2\delta} - \pi} [|P_1(t) - P_2(t)|]$;
3. For $P_0 \in C([0,1], \mathbb{R})$ such that $\zeta(P_0(t), \mathcal{A}P_0(t)) > 0$ and each $t \in [0,1]$ with $P_1(t), P_2(t) \in C([0,1], \mathbb{R})$, and $\zeta(P_1(t), P_2(t)) > 0$ means $\zeta(\mathcal{A}P_1(t), \mathcal{A}P_2(t)) > 0$.
4. A sequence $\{\wp_n\} \subseteq C([0,1], \mathbb{R})$ in such a way that is $\wp_n \to a$ in $C([0,1], \mathbb{R})$ along $\zeta(\wp_n, a_{n+1}) > 0$ for $n \in \mathbb{N}$, consequently $\zeta(\wp_n, a) > 0$ for every $n \in \mathbb{N}$.

Subsequently, there is only one solution guaranteed by the equation (15).

Proof
Consider the assumption (1) and (2),

$$\begin{aligned}
|\mathcal{A}P_1(t) - \mathcal{A}P_2(t)| &= \left| \int_0^{T_p} \mathcal{H}(t,t^*) K(t^*, t, P_1(t))\, dt - \int_0^{T_p} \mathcal{H}(t,t^*) K(t^*, t, P_2(t))\, dt \right| \\
&\leq \left| \int_0^{T_p} \mathcal{H}(t,t^*) |K(t^*, t, P_1(t)) - K(t^*, t, P_2(t))|\, dt \right| \\
&\leq |K(t^*, t, P_1(t)) - K(t^*, t, P_2(t))| \int_0^{T_p} \mathcal{H}(t,t^*) dt \\
&\leq \left| \frac{2\delta}{\lambda} T_p e^{\frac{\lambda T_p}{2\delta}} \frac{\lambda}{2\delta T_p} e^{-\frac{\lambda T_p}{2\delta} - \pi} [|P_1(t) - P_2(t)|] \right| \\
&\leq |P_1(t) - P_2(t)| e^{-\pi}
\end{aligned}$$

Evidently,

$$\rho(\mathcal{A}P_1(t), \mathcal{A}P_2(t)) \leq \rho(P_1(t), P_2(t)) e^{-\pi}. \tag{19}$$

Equivalently,

$$\sqrt{\rho(\mathcal{A}P_1(t), \mathcal{A}P_2(t))} \leq \sqrt{e^{-\pi} \rho(P_1(t), P_2(t))}.$$

Thus,

$$e^{\sqrt{\rho(\mathcal{A}P_1(t), \mathcal{A}P_2(t))}} \leq e^{\sqrt{e^{-\pi} \rho(P_1(t), P_2(t))}},$$

that evidently implies,

$$e^{\sqrt{\rho(\mathcal{A}P_1(t), \mathcal{A}P_2(t))}} \leq \left(e^{\sqrt{\rho(P_1(t), P_2(t))}} \right)^G,$$

where $G = e^{-\pi} < 1$. Consider $\Theta e = e^e$. And define $\alpha, \mu : \mathcal{M}_{\mathbb{E}} \times \mathcal{M}_{\mathbb{E}} \to \{-\infty\} \cup [0, +\infty)$ by

$$\alpha(P_1(t), P_2(t)) = \mu(P_1(t), P_2(t)) = \begin{cases} 1 & \text{if } \zeta(P_1(t), P_2(t)) \geq 1, \\ -\infty & \text{else.} \end{cases}$$

Next, utilizing presumption (4),

$$\alpha(P_1, P_2), \mu(P_1, P_2) \geq 1 \implies \zeta(P_1(t), P_2(t)) > 0, \tag{20}$$

which leads to

$$\zeta(\mathcal{A}P_1(t), \mathcal{A}P_2(t)) > 0 \implies \alpha(\mathcal{A}P_1, \mathcal{A}P_2) \geq 1. \tag{21}$$

Thus, (31) becomes,

$$\alpha(P_1,P_2)\mu(P_1,P_2)\Theta(\rho(\mathcal{A}P_1(t),\mathcal{A}P_2(t))) \leq (\Theta(\rho(P_1(t),P_2(t))))^G.$$

It also implies that,

$$\alpha(P_1,P_2)\mu(P_1,P_2)\Theta(\rho(\mathcal{A}P_1(t),\mathcal{A}P_2(t))) \leq (\Theta(M(P_1(t),P_2(t))))^G + N(P_1(t),P_1(t)).$$

Where $M(P_1(t),P_2(t))$ and $N(P_1(t),P_1(t))$ are defined in 3.1. Additionally, from (3), there exists $P_0 \in \mathcal{M}_{\mathbb{E}}$ such that $\alpha(P_0,\mathcal{A}P_0) \geq 1$ and $\mu(P_0,\mathcal{A}P_0) \geq 1$.

The presumptions of Theorem 3.3 and 3.4 are all hold true. This ultimately ensure a unique solution of the problem (15). □

7.2. Application to Dynamics of Economy

Fractional differential equations ($\mathcal{FDE}$s) in the fields of engineering and science are remarkably helpful in the sense of their applications [44, 22, 3]. Notably, Caputo $\mathcal{FDE}$s are significantly helpful in the construction of economic growth models precisely. Equations of this nature facilitate providing a deeper understanding and comprehending the dynamics of economy and support the economists to make more informed decisions in the sense of their polices [2, 43].

The fractional $\mathcal{DE}$, when utilized in economic growth modeling, can be presented as:

$$^{C}D_t^{\nu}(y(t_g)) = h(t_g, y(t_g)), \quad (0 < t_g < 1,\, 2 \geq \nu > 1) \tag{22}$$

subjected to the conditions:

$$y(0) = 0, \quad {}^{RL}I_t^{\nu} y(1) = y'(0), \tag{23}$$

where ${}^{C}D_t^{\nu}h(t_g)$ represent Caputo fractional derivative of order ν as describe by:

$$^{C}D_t^{\nu}h(t_g) = \frac{1}{\Gamma(j-\nu)}\int_0^{t_g}(t_g - z)^{j-\nu-1}h'(z)\,dz,$$

along $j > \nu > j-1$, $j = [\nu]+1$.

In this context, $h : [0,1]\times\mathbb{R} \to [0,+\infty)$ represent function with no discontinuity, and ${}^{RL}I_t^{\nu}h$ is for the fractional integral of Riemann–Liouville type having order ν of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ describe by:

$$^{RL}I_t^{\nu}f(t_g) = \frac{1}{\Gamma(\nu)}\int_0^{t_g}(t_g - z)^{\nu-1}f(z)\,dz. \tag{25}$$

Economic Interpretation: The weight function $w(y_1, y_2)$ captures economic proximity, with similar growth trajectories having lower weights. The twisted aspects α, μ can represent short-term stability (α) and long-term sustainability (μ) criteria.

Theorem 7.2 ([2])
Suppose the nonlinear fractional $\mathcal{DE}$

$$^{C}D_t^{\nu}(y(t_g)) = h(t_g, y(t_g)), \quad (0 < t_g < 1,\, 2 \geq \nu > 1). \tag{27}$$

Assume $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ represent a given function and the below assumptions fulfill:

(i) $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ denotes a function with no discontinuity.
(ii) for every $a, \wp \in C([0,1],\mathbb{R})$ and $t_g \in [0,1]$,

$$|h(t_g,a) - h(t_g,\wp)| \leq \frac{\Gamma(\nu+1)}{4}e^{-\pi|a-\wp|} \tag{28}$$

(iii) There is $p_0 \in C([0,1],\mathbb{R})$ such that $\zeta(a_0(t_g), \mathcal{A}a_0(t_g)) > 0$ for every $t \in [0,1]$, where an operator $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined in a manner as

$$\mathcal{A}a(t_g) = \frac{1}{\Gamma(\nu)} \int_0^t t_g(t_g - z)^{\nu-1} h(z, a(z))\, dz + \frac{2t_g}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} h(m, a(m))\, dm \right) dz \tag{29}$$

for each $t \in [0,1]$.

(iv) For each $t_g \in [0,1]$ with $a, \wp \in C([0,1],\mathbb{R}), \zeta(a(t_g), \wp(t_g)) > 0$ means that $\zeta(\mathcal{A}a(t_g), \mathcal{A}\wp(t_g)) > 0$.

(v) For sequence $\{\wp_n\} \subseteq C([0,1],\mathbb{R})$ such a way that $\wp_n \to a$ in $C([0,1],\mathbb{R})$ with $\zeta(\wp_n, a_{n+1}) > 0$ for $n \in \mathbb{N}$, consequently $\zeta(\wp_n, a) > 0$ for every $n \in \mathbb{N}$.

Then, (27) possess a unique solution.

Proof
It is obvious to say that for $a \in \mathcal{M}_{\mathbb{R}}$ satisfying equation (27) iff it also ensures the below

$$a(t) = \frac{1}{\Gamma(\nu)} \int_0^{t_g} (t_g - z)^{\nu-1} h(z, a(z))\, dz + \frac{2t_g}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} h(m, a(m))\, dm \right) dz \tag{30}$$

for $t \in [0,1]$.

Moreover, assume $a, \wp \in \mathcal{M}_{\mathbb{R}}$ such a way that $\zeta(a(t_g), \wp(t_g)) > 0$ for every $t_g \in [0,1]$. By condition (iii), we met

$$\begin{aligned}
&|\mathcal{A}a(t_g) - \mathcal{A}\wp(t_g)| \\
&= \left| \frac{1}{\Gamma(\nu)} \int_0^t (t-z)^{\nu-1} h(z, a(z))\, dz - \frac{1}{\Gamma(\nu)} \int_0^{t_g} (t_g - z)^{\nu-1} h(z, \wp(z))\, dz \right| \\
&\quad + \frac{2t_g}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} h(m, a(m))\, dm \right) dz \\
&\quad - \frac{2t}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} h(m, b(m))\, dm \right) dz \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_g} |t_g - z|^{\nu-1} |h(z, a(z)) - h(z, \wp(z))|\, dz \\
&\quad + \frac{2t_g}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} |h(m, a(m)) - h(m, b(m))|\, dm \right) dz.
\end{aligned}$$

Employing the condition (ii)'s inequality, we can moreover state:

$$\begin{aligned}
&|\mathcal{A}a(t_g) - \mathcal{A}\wp(t_g)| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_g} |t_g - z|^{\nu-1} \frac{\Gamma(\nu+1)}{4} e^{-\pi|a(z)-\wp(z)|}\, dz \\
&\quad + \frac{2t_g}{\Gamma(\nu)} \int_0^1 \left(\int_0^z (z-m)^{\nu-1} \frac{\Gamma(\nu+1)}{4} e^{-\pi} |a(m) - \wp(m)|\, dm \right) dz \\
&= \frac{e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} \int_0^z |t_g - z|^{\nu-1} |a(z) - \wp(z)|\, dz \\
&\quad + \frac{2e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} \int_0^1 \left(\int_0^z |z-m|^{\nu-1} |a(m) - \wp(m)|\, dm \right) dz.
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
&|\mathcal{A}a(t_g) - \mathcal{A}\wp(t_g)| \\
&\leq \frac{e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} |a-\wp| \int_0^z |t_g - z|^{\nu-1}\, dz \\
&\quad + \frac{2e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} |a-\wp| \int_0^1 \left(\int_0^z |z-m|^{\nu-1}\, dm \right) dz \\
&\leq \frac{e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} |a-\wp| \cdot \frac{\Gamma(\nu)}{\Gamma(\nu+1)} + \frac{2e^{-\pi}\Gamma(\nu+1)}{4\Gamma(\nu)} |a-\wp| \cdot \frac{\Gamma(\nu+1)}{\Gamma(\nu)\Gamma(1)} \\
&= \frac{e^{-\pi}}{4} |a-\wp| + \frac{e^{-\pi}}{2} |a-\wp| \leq e^{-\pi} |a-\wp|.
\end{aligned}$$

Evidently,

$$|\mathcal{A}a(t_g) - \mathcal{A}\wp(t_g)| \leq e^{-\pi} |a-b|,$$

which also ultimately implies to

$$\rho(\mathcal{A}a(t_g), \mathcal{A}\wp(t_g)) \leq e^{-\pi} \rho(a, \wp).$$

Equivalently,

$$\sqrt{\rho(\mathcal{A}a(t_g), \mathcal{A}\wp(t_g))} \leq \sqrt{e^{-\pi} \rho(a, \wp)}.$$

Thus,

$$e^{\sqrt{\rho(\mathcal{A}a(t_g), \mathcal{A}\wp(t_g))}} \leq e^{\sqrt{e^{-\pi} \rho(a, \wp)}},$$

that evidently implies,

$$e^{\sqrt{\rho(\mathcal{A}a(t_g), \mathcal{A}\wp(t_g))}} \leq \left(e^{\sqrt{\rho(a, \wp)}} \right)^G,$$

where $G = e^{-\pi} < 1$. Consider $\Theta e = e^e$. And define $\alpha, \mu : \mathcal{M}_E \times \mathcal{M}_E \to [-\infty] \cup [0, +\infty)$ by

$$\alpha(a, \wp) = \mu(a, \wp) = \begin{cases} 1 & \text{if } \zeta(a, \wp) \geq 1, \\ -\infty & \text{else.} \end{cases}$$

Further, utilizing the assumption (iv),

$$\alpha(a,\wp),\mu(a,\wp)\geq 1 \implies \zeta(a(t_g),\wp(t_g))>0, \tag{32}$$

which leads to

$$\zeta(\mathcal{A}a(t_g),\mathcal{A}\wp(t_g))>0 \implies \alpha(\mathcal{A}a,\mathcal{A}\wp)\geq 1. \tag{33}$$

Thus, (31) becomes,

$$\alpha(a,\wp)\mu(a,\wp)\Theta(\rho(\mathcal{A}a(t_g),\mathcal{A}\wp(t_g)))\leq \left(\Theta(\rho(a(t),\wp(t_g)))\right)^G.$$

It also implies that,

$$\alpha(a,\wp)\mu(a,\wp)\Theta(\rho(\mathcal{A}a(t_g),\mathcal{A}\wp(t_g)))\leq \left(\Theta(M(a(t_g),\wp(t_g)))\right)^G+qN(e,q).$$

Where $\mathscr{M}_b(a,\wp)$ is defined in 3.1. Additionally, from (iii), there exists $a_0\in\mathcal{M}_{\mathbb{E}}$ such that $\alpha(a_0,\mathcal{A}a_0)\geq 1$ and $\mu(a_0,\mathcal{A}a_0)\geq 1$.

The presumptions of Theorem 3.3 and 3.4 fulfill. Thus, the mapping $\mathcal{A}$ ensures a unique $\mathcal{FP}$. The $\mathcal{FP}$ will evidently be the solution to the integral equation (30), and consequently the solution to the original fractional $\mathcal{DE}$ (27). □

8. Conclusion

This paper establishes a groundbreaking unified framework through the novel concept of *twisted weighted* Θ-*b-metric spaces*, synthesizing concepts from twisted admissible mappings, weighted b-metric spaces, and Θ-contractions to create a powerful analytical tool with unprecedented theoretical depth. Our results substantially generalize numerous classical theorems: when $\psi(u,v)=s$, $w(x,y)=1$, $\Theta(t)=e^t$, and $\alpha=\mu\equiv 1$, we recover b-metric space results [7]; with $\psi(u,v)=1$ we obtain standard metric space contractions; with $N(x,y)=0$ we obtain Čirić-type contractions [12]; with $\Theta(t)=e^{\sqrt{t}}$ we recover Θ-contractions [16]; and with $\alpha=\mu$ we obtain twisted admissible mappings [39].

Our principal contributions include establishing comprehensive fixed point theorems for generalized rational type twisted Θ-contractions, proving best proximity point results for multivalued non-self mappings, providing extensive numerical validation through computational implementations, and demonstrating significant applications to economic systems including production-consumption equilibrium and fractional growth models. For future research, promising directions encompass extensions to other generalized metric spaces, development of computational algorithms, stability analysis, coupled and tripled fixed points, real-world applications across multiple disciplines, dynamic systems, and further exploration of fractional applications, collectively opening new avenues for research in nonlinear analysis and its interdisciplinary applications.

Authors' contributions

All authors read and approved the manuscript.

Competing interests

The authors declare that they have no competing interests.

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