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Locally D- and A-Optimal Design Framework for Poisson Regression with Square Root Link

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Abstract This study develops a locally D- and A-optimal design framework for Poisson regression models employing a square-root link function. Unlike the canonical log-link, the square-root link provides variance stabilization and is particularly useful for experimental situations involving low-count or zero-inflated data. Using the General Equivalence Theorem, we derive locally optimal designs and verify their optimality through analytical and numerical methods. Both D- and A-optimal design criteria are explored for two- and three-point designs across multiple design regions. The proposed framework extends classical local design theory to a non-canonical link, thereby broadening its practical applicability. Comprehensive numerical results illustrate how optimal support points and weights vary with model parameters and design spaces. Sensitivity and robustness analyses confirm that the locally optimal designs maintain high efficiency even under moderate parameter misspecification. The results provide practitioners with a structured approach to constructing efficient experimental designs under Poisson regression models with variance-stabilizing link functions.

Keywords D- & A-optimal design, Information matrix, Link function, Poisson regression model, Equivalence theorem

AMS 2010 subject classifications 62K05

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1. Introduction

Optimal experimental design plays a crucial role in maximizing the statistical efficiency of parameter estimation while minimizing resource use. In generalized linear models (GLMs), the link function connecting the mean response and the linear predictor is central to model behavior and inference quality. For Poisson regression, the canonical link is the logarithmic function, which ensures interpretability and simplicity. However, the square-root link has emerged as a viable alternative due to its desirable variance-stabilizing properties and numerical robustness when dealing with small or zero counts.

The square-root link is particularly attractive when the response variance is approximately proportional to its mean, as in many ecological, biomedical, and environmental studies. Under such conditions, the transformation $\eta = \sqrt{\mu}$ approximately stabilizes the variance and avoids undefined expressions for zero-valued responses, a common drawback of the canonical log-link. Moreover, designs based on the square-root link can provide higher efficiency and numerical stability in parameter estimation, especially in low-count regimes.

Despite its potential, the square-root link has received limited attention in the optimal design literature compared with canonical links. Earlier works, including those of McGree and Eccleston (2012) and Russell (2009), have

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primarily focused on log-linked Poisson regression models. The present study fills this gap by constructing and analyzing locally D- and A-optimal designs for Poisson regression under the square-root link. The local optimality criterion assumes known parameter values, which serve as prior estimates or pilot information. Though this approach can be sensitive to misspecification, it remains a foundational tool for design derivation and comparison.

This paper contributes to the literature in several important ways. First, it establishes a unified computational framework for deriving D- and A-optimal designs using the square-root link. Second, it provides analytical insights into the structure and limiting behavior of optimal designs through derivative-based conditions derived from the Fisher information matrix. Third, it performs an extensive numerical investigation across varying parameter values and design regions, producing a comprehensive reference catalog for practitioners. Finally, it introduces a sensitivity and robustness analysis to quantify the efficiency loss under parameter deviations, thereby addressing one of the main limitations of local optimality.

The remainder of this paper is organized as follows. Section 2 presents the model specification and the structure of the Fisher information matrix. Section 3 derives the locally D- and A-optimality criteria, along with analytical insights into their behavior. Section 4 presents the results, including sensitivity analyses and efficiency comparisons. Section 5 discusses the findings, and Section 6 concludes the paper with remarks on applicability and future research directions.

2. Model and Information Structure

Let the response variable Y follow a Poisson distribution with mean parameter μ, i.e.,

$$Y_j \sim \text{Poisson}(\mu_j), \quad j = 1, \dots, m. \tag{1}$$

The relationship between the mean response and the linear predictor is specified through a square-root link function,

$$\sqrt{\mu_j} = \eta_j = \boldsymbol{g}\left(\boldsymbol{z}_j\right)' \boldsymbol{\beta}, \tag{2}$$

where $\boldsymbol{z}_{\mathrm{j}} = (z_1, z_2, \dots, z_{\mathrm{p}})$, $\boldsymbol{g}\left(\boldsymbol{z}_{\mathrm{j}}\right) = \left(1, z_{\mathrm{j}}'\right)'$, $\boldsymbol{\beta} = \beta_1, \beta_2, \dots, \beta_p$ are p-dimensional vectors of the unknown parameters. The inverse link gives

$$\mu_j = (\boldsymbol{g}\left(\boldsymbol{z}_j\right)' \boldsymbol{\beta})^2, \quad \boldsymbol{g}\left(\boldsymbol{z}_j\right)' \boldsymbol{\beta} > 0. \tag{3}$$

For the model Equation (2) , the Fisher information matrix is $p \times p$ dimension at $\boldsymbol{z}$ and $\boldsymbol{\beta}$ can be defined as

$$\boldsymbol{M}(\boldsymbol{z}, \boldsymbol{\beta}) = \eta^2 \boldsymbol{g}(\boldsymbol{z}) \boldsymbol{g}'(\boldsymbol{z}) \tag{4}$$

where $\eta = \sqrt{\mu_j}$ is the Square root link or intensity function. To achieve the locally D - & A-optimal design for the model equation (2), consider the continuous design $\xi \in \Xi$ (Ξ the set of all continuous designs) of the form

$$\xi = \left\{ \begin{array}{cccc} \boldsymbol{z}_1 & \boldsymbol{z}_2 & \dots & \boldsymbol{z}_p \\ \delta_1 & \delta_2 & \dots & \delta_p \end{array} \right\}, \delta_j (> 0) \text{ and } \sum_{j=1}^{p} \delta_j = 1 \tag{5}$$

where $\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_{\mathrm{p}} \in \Omega \, (\Omega \subset R^p)$ are the 'p' distinct points and δ_j is the weight associated with the point $\boldsymbol{z}_j$ for $j = 1, 2, \dots, p$. For the model Equation (2), the Information matrix of a design ξ at parameter $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{M}(\xi, \boldsymbol{\beta}) = \sum_{j=1}^{p} \delta_{\mathrm{i}} \boldsymbol{M}\left(\boldsymbol{z}_j, \boldsymbol{\beta}\right) \tag{6}$$

For more information, refer to Russell (2018, p. 152).

A locally optimum design aims to minimize a convex criterion function of the information matrix at a certain parameter vector β. The determinant and trace of a matrix are denoted as "*det*" and "*tr*", respectively. In this article, the widely used D- and A-criterion are used.

D-optimal Design: A design $\xi^* \in \Xi$ with non-singular information matrix $\boldsymbol{M}(\xi^*)$ for model (2) is called D-optimal design if it minimizes $\det\left(\boldsymbol{M}^{-1}(\xi^*)\right)$ over Ω.

Equivalence Theorem 1: The design $\xi^* \in \Xi$ is locally D-optimal for model (2) if and only if

$$\Phi(\boldsymbol{z}, \xi^*) \leq p \quad \text{for all } \boldsymbol{z} \in \Omega \tag{7}$$

where $\Phi(\boldsymbol{z}, \xi^*) = \eta^2 \boldsymbol{g}'(\boldsymbol{z}) \boldsymbol{M}^{-1}(\xi^*) \boldsymbol{g}(\boldsymbol{z})$ and 'p' is the number of unknown parameters. Moreover, supremum exists at the support point of ξ^*.

A-optimal Design: A design $\xi^* \in \Xi$ with an information matrix $\boldsymbol{M}(\xi^*)$ for model (2) is called A-optimal design if it minimizes $\operatorname{tr}\left(\boldsymbol{M}^{-1}(\xi^*)\right)$ over Ω.

Fedorov's (1971) equivalence theorem defines the necessary and sufficient criteria for determining A-optimal design across the simplex region Ω.

Equivalence Theorem 2: A design $\xi^* \in \Xi$ is locally A-optimal for model (2) if and only if

$$\underset{\boldsymbol{z}\in\Omega}{\operatorname{Max}} \Psi(\boldsymbol{z}, \xi^*) \leq \operatorname{tr}\left(\boldsymbol{M}^{-1}(\xi^*)\right) \tag{8}$$

where $\Psi(\boldsymbol{z}, \xi^*) = \eta^2 \boldsymbol{g}'(\boldsymbol{z}) \boldsymbol{M}^{-2}(\xi^*) \boldsymbol{g}(\boldsymbol{z})$, with equality attained at the support points ξ^*.

3. D-optimal designs for two parameters

We are going to examine the simple Poisson regression model in two parameters, random intercept and random slope, respectively: $\mathbf{g}'(\boldsymbol{z})\boldsymbol{\beta} = \beta_0 + \beta_1 z > 0$, for all $z \in \mathrm{R}$. Note that, we limit our study with two-, and three-support points design by considering discrete values of β_0, β_1 in the randomly chosen intervals $[1, 5]$ with region space $[0, 1], [0, 5] \& [0, 10]$.

3.1. D-optimal design

Let us consider a 2 -point design ξ of the structure

$$\xi = \begin{Bmatrix} u & v \\ 1/2 & 1/2 \end{Bmatrix}. \tag{9}$$

Lemma 3.1.1. The design ξ^* that allocates equal weights to the support points u^* and v^* in Ω is D-optimal design where u^* and v^* are given in Table 1, Table 2 & Table 3 (*see* Appendix).

Proof

The Fisher information matrix for the model Equation (9) at the two-point design ξ described in Equation (6) is given by

$$\boldsymbol{M}(\xi) = \frac{1}{2}\begin{bmatrix} \left((\beta_0 + u\beta_1)^2 + (\beta_0 + v\beta_1)^2\right) & \left(u(\beta_0 + u\beta_1)^2 + v(\beta_0 + v\beta_1)^2\right) \\ \left(u(\beta_0 + u\beta_1)^2 + v(\beta_0 + v\beta_1)^2\right) & \left(u^2(\beta_0 + u\beta_1)^2 + v^2(\beta_0 + v\beta_1)^2\right) \end{bmatrix}. \tag{10}$$

Using equation (10), we get the inverse of the information matrix, which is as follows

$$\boldsymbol{M}^{-1}(\xi) = \frac{2}{(u-v)^2}\begin{bmatrix} \left(\frac{v^2}{(\beta_0+u\beta_1)^2} + \frac{u^2}{(\beta_0+v\beta_1)^2}\right) & \left(\frac{-v}{(\beta_0+u\beta_1)^2} - \frac{u}{(\beta_0+v\beta_1)^2}\right) \\ \left(\frac{-v}{(\beta_0+u\beta_1)^2} - \frac{u}{(\beta_0+v\beta_1)^2}\right) & \left(\frac{1}{(\beta_0+u\beta_1)^2} + \frac{1}{(\beta_0+v\beta_1)^2}\right) \end{bmatrix}. \tag{11}$$

From Equation (11), we obtain determinant the function

$$\det \boldsymbol{M}^{-1}(\xi) = \frac{4}{(u-v)^2 (\beta_0 + u\beta_1)^2 (\beta_0 + v\beta_1)^2}. \tag{12}$$

Now, the problem is to minimize the function $\det \boldsymbol{M}^{-1}(\xi)$ with respect to u^* and v^* for given values of β_0 and β_1. This is prepared by using the "*fminsearch*" function of Matlab software and getting the optimum values u^* and v^*. The numerical values of u^* and v^* are provided in Table 1, Table 2 &Table 3 (See Appendix). Next, by using Equation (11), the quadratic form as specified in Equation (7) which is as follows:

$$\Phi(z,\xi^*) = \frac{\begin{array}{c} 2\left(\beta_0+z\beta_1\right)^2\left(\left(u^2+v^2-2(u+v)z+2z^2\right)\beta_0^2+2\left(u^3+v^3-2\left(u^2+v^2\right)z\right.\right. \\ \left.\left.+(u+v)z^2\right)\beta_0\beta_1+\left(u^4+v^4-2\left(u^3+v^3\right)z+\left(u^2+v^2\right)z^2\right)\beta_1^2\right) \end{array}}{(u-v)^2\left(\beta_0+u\beta_1\right)^2\left(\beta_0+v\beta_1\right)^2}. \tag{13}$$

Replacing the numerical values of u^* and v^* in Equation (13) using the "*fminsearch*" function of Matlab software, we get $\underset{z\in\Omega}{\operatorname{Sup}}\Phi(z,\xi^*) = 2$. Hence, the necessary and sufficient condition of the equivalence theorem is established. □

Remarks: In 3-point design settings having equal weights, we could not obtain the optimal values which satisfy the D-optimality Criterion.

3.2. A-optimal design

Let ξ is any design with two points in the experimental setup, i.e.

$$\xi = \left\{\begin{array}{cc} u & v \\ \delta & 1-\delta \end{array}\right\} \text{ where } 0<\delta<1. \tag{14}$$

Lemma 3.2.1 The design ξ^* that allocates a weight δ^* to the point u^* and $1-\delta^*$ to the point v^* in Ω is an A-optimal design where u^*, v^*, and δ^* are given in Table 4, Table 5, & Table 6 (See Appendix).

Proof

The Fisher information matrix for the model Equation (14) at the two-point design ξ defined in Equation (6) is given by

$$\boldsymbol{M}(\xi) = \left[\begin{array}{cc} \delta\left(\beta_0+u\beta_1\right)^2+(1-\delta)\left(\beta_0+v\beta_1\right)^2 & \delta\left(\beta_0+u\beta_1\right)^2u+v(1-\delta)\left(\beta_0+v\beta_1\right)^2 \\ \delta\left(\beta_0+u\beta_1\right)^2u+v(1-\delta)\left(\beta_0+v\beta_1\right)^2 & u^2\delta\left(\beta_0+u\beta_1\right)^2+v^2(1-\delta)\left(\beta_0+v\beta_1\right)^2 \end{array}\right]. \tag{15}$$

The inverse of the above Fisher information matrix is given by

$$\boldsymbol{M}^{-1}(\xi) = \frac{1}{(u-v)^2}\left[\begin{array}{cc} \frac{v^2}{\delta(\beta_0+u\beta_1)^2}-\frac{u^2}{(-1+\delta)(\beta_0+u\beta_1)^2} & \frac{-v}{\delta(\beta_0+u\beta_1)^2}+\frac{u}{(-1+\delta)(\beta_0+v\beta_1)^2} \\ \frac{-v}{\delta(\beta_0+u\beta_1)^2}+\frac{u}{(-1+\delta)(\beta_0+v\beta_1)^2} & \frac{1}{\delta(\beta_0+u\beta_1)^2}-\frac{1}{(-1+\delta)(\beta_0+v\beta_1)^2} \end{array}\right]. \tag{16}$$

Using the above Equation (16), we obtain the trace function

$$\operatorname{tr}\boldsymbol{M}^{-1}(\xi) = \frac{1}{(u-v)^2}\left(\frac{1+v^2}{\delta\left(\beta_0+u\beta_1\right)^2}-\frac{1+u^2}{(-1+\delta)\left(\beta_0+v\beta_1\right)^2}\right). \tag{17}$$

Now, the problem is to minimize the function $\operatorname{tr}\boldsymbol{M}^{-1}(\xi)$ with respect to u^*, v^*, and δ^* for given values of β_0 and β_1. This is done using the "*fminsearch*" function of Matlab software and getting the optimal values u^*, v^*, and δ^*. The numerical values of u^*, v^*, and δ^* are given in Table 4, Table 5, Table 6 (See Appendix). Next, by using Equation (16), the quadratic form as specified in Equation (8) which is as follows:

$$\Psi(z,\xi^*) = \frac{\left(\beta_0+z\beta_1\right)^2}{(u-v)^4}\left(\frac{k_1+k_2-k_3}{k_4}\right) \tag{18}$$

where $$k_1 = \left(\frac{v^4-2v^3z+z^2+v^2\left(1+z^2\right)}{\delta^2(\beta_0+u\beta_1)^4}\right),$$

$$k_2 = \left(\frac{\left(1+u^2\right)(u-z)^2}{(-1+\delta)^2\left(\beta_0+v\beta_1\right)^4}\right),$$

$$k_3 = \left(\begin{array}{c} 2(uv(1+uv)\delta-(v+u(1+v(u+v))\delta))+(1+uv)\delta v^2\big)\beta_0^2+ \\ 4\left(u^3v\delta(v-z)+v^2(-1+\delta)z+u\delta(-v+z)z-u^2\delta(v-z)(-1+vz)\right)\beta_0\beta_1+ \\ 2\left(u^4v\delta(v-z)+v^3(-1+\delta)z+u^2\delta z(-v+z)-u^3\delta(v-z)(-1+vz)\right)\beta_1^2 \end{array}\right),$$

$$k_4 = \left((-1+\delta)\delta^2\left(\beta_0+u\beta_1\right)^4\left(\beta_0+v\beta_1\right)^2\right).$$

Now, replacing the numerical values of u^*, v^*, and δ^* in Equation (18) using the "*fminsearch*" function of Matlab software, we find $\underset{z\in\Omega}{\operatorname{Max}}\Psi\left(z,\xi^*\right) \leq \operatorname{tr}\left(\boldsymbol{M}^{-1}\left(\xi^*\right)\right)$. Thus, the necessary and sufficient condition of the general equivalence theorem is established. □

Designs derived from three support points

Let us consider a 3-point design ξ of the form

$$\xi = \left\{\begin{array}{ccc} u & v & w \\ \delta/2 & 1-\delta & \delta/2 \end{array}\right\} \text{ where } 0<\delta<1. \tag{19}$$

Lemma 3.2.2 The design ξ^* that allocates a weight $\delta^*/2$ to the point u^*, $(1-\delta^*)$ to the point v^*, and $\delta^*/2$ to the point w^* in Ω is an A-optimal design are provided in Table 7, Table 8, & Table 9 (See Appendix).

Proof
Using Equation (6), the Fisher information matrix for the model Equation (19) at the three-point design ξ will be

$$\boldsymbol{M}(\xi) = \left[\begin{array}{cc} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{array}\right] \tag{20}$$

with

$$a_{11}^* = \frac{1}{2}\delta\left(\beta_0+u\beta_1\right)^2+(1-\delta)\left(\beta_0+v\beta_1\right)^2+\frac{1}{2}\delta\left(\beta_0+w\beta_1\right)^2,$$
$$a_{12}^* = a_{21}^* = \frac{1}{2}\delta u\left(\beta_0+u\beta_1\right)^2+v(1-\delta)\left(\beta_0+v\beta_1\right)^2+\frac{1}{2}\delta w\left(\beta_0+w\beta_1\right)^2,$$
$$a_{22}^* = \frac{1}{2}\delta u^2\left(\beta_0+u\beta_1\right)^2+v^2(1-\delta)\left(\beta_0+v\beta_1\right)^2+\frac{1}{2}\delta w^2\left(\beta_0+w\beta_1\right)^2.$$

The inverse of the above Fisher information matrix is given by

$$\boldsymbol{M}^{-1}(\xi) = \left[\begin{array}{cc} c_{11}^+ & c_{12}^+ \\ c_{21}^+ & c_{22}^+ \end{array}\right] \tag{21}$$

with

$$c_{11}^{+} = \frac{1}{b}\left(2\left(u^2\delta\left(\beta_0+u\beta_1\right)^2 - 2v^2(-1+\delta)\left(\beta_0+v\beta_1\right)^2 + w^2\delta\left(\beta_0+w\beta_1\right)^2\right)\right),$$

$$b = \left(\begin{array}{l} -\left(\delta u\left(\beta_0+u\beta_1\right)^2 - 2v(-1+\delta)\left(\beta_0+v\beta_1\right)^2 + \delta w\left(\beta_0+w\beta_1\right)^2\right. \\ +\left(\delta\left(\beta_0+u\beta_1\right)^2 - 2(-1+\delta)\left(\beta_0+v\beta_1\right)^2 + \delta\left(\beta_0+w\beta_1\right)^2\right) \\ \left(u^2\delta\left(\beta_0+u\beta_1\right)^2 - 2v^2(-1+\delta)\left(\beta_0+v\beta_1\right)^2 + w^2\delta\left(\beta_0+w\beta_1\right)^2\right) \end{array}\right),$$

$$c_{12}^{+} = c_{21}^{+} = \frac{1}{b}\left(4\left(-(1/2)\delta u\left(\beta_0+u\beta_1\right)^2 + v(-1+\delta)\left(\beta_0+v\beta_1\right)^2 - (1/2)\delta w\left(\beta_0+w\beta_1\right)^2\right)\right),$$

$$c_{22}^{+} = \frac{1}{b}\left(2\left(\delta\left(\beta_0+u\beta_1\right)^2 - 2(-1+\delta)\left(\beta_0+v\beta_1\right)^2 + \delta\left(\beta_0+w\beta_1\right)^2\right)\right).$$

Using Equation (21), we obtain the trace function

$$\begin{aligned} \operatorname{tr}\boldsymbol{M}^{-1}(\xi) = \frac{1}{b^2}\Big[&2\big(\delta(\beta_0+u\beta_1)^2 + u^2\delta(\beta_0+u\beta_1)^2 - 2(1-\delta)(\beta_0+v\beta_1)^2 \\ &- 2v^2(1-\delta)(\beta_0+v\beta_1)^2 + \delta(\beta_0+w\beta_1)^2\big) + w^2\delta(\beta_0+w\beta_1)^2\Big]. \end{aligned} \tag{22}$$

Now, the problem is need to minimize the function $\operatorname{tr}\boldsymbol{M}^{-1}(\xi)$ with respect to u^*, v^*, w^* and δ^* for given values of β_0 and β_1. This is achieved by using the "*fminsearch*" function of Matlab software and getting the optimal values u^*, v^*, w^* and δ^*. The numerical values of u^*, v^*, w^* and δ^* are given in Table 7, Table 8, Table 9 (See Appendix). Next, by using Equation (22), the quadratic form as specified in Equation (8) which is as follows:

$$\Psi\left(z,\xi^*\right) = \left(\beta_0+z\beta_1\right)^2\left\{\begin{array}{l} {c_{12}^{+}}^2 + {c_{11}^{+}}^2 + z\left(\left(c_{12}^{+}\times c_{22}^{+}\right) + \left(c_{12}^{+}+c_{11}^{+}\right)\right) + \\ z\left(\left(c_{22}^{+}\times c_{12}^{+}\right) + \left(c_{12}^{+}\times c_{11}^{+}\right)\right) + z\left(\left({c_{22}^{+}}^2\times {c_{12}^{+}}^2\right)\right. \end{array}\right\} \tag{23}$$

Now, replacing the numerical values of u^*, v^*, w^* and δ^* in Equation (3.15) using the "*fminsearch*" function of Matlab software, we get $\eta^2\boldsymbol{g}'(z)\boldsymbol{M}^{-2}\left(\xi^*\right)\boldsymbol{g}(z) \leq \operatorname{tr}\left(\boldsymbol{M}^{-1}\left(\xi^*\right)\right)$. Thus, the inequality condition of the general equivalence theorem is verified. □

4. Numerical Results and Sensitivity Analysis

This section presents the numerical findings for the locally D- and A-optimal designs under various parameter configurations and design spaces. The results are obtained using the optimization framework with $\beta_0, \beta_1 \in [1,5]$ and design regions $\Omega_1 = [0,1]$, $\Omega_2 = [0,5]$, and $\Omega_3 = [0,10]$. For each configuration, the optimal support points and their associated weights are reported. The accuracy of all numerical results was verified using the equivalence theorem conditions for the corresponding criterion.

4.1. D-Optimal Designs

The D-optimal designs aim to maximize the determinant of the Fisher information matrix $\det[\boldsymbol{M}(\xi;\beta)]$. Table 10 summarizes the optimal two-point designs for selected parameter values and design regions. The table presents the optimal support points (z_1, z_2) and corresponding weights (δ_1, δ_2), along with the determinant values for comparison.

As expected from the theoretical discussion in Section 3.1, the support points in all cases lie at or near the boundaries of the design region. The relative weights are approximately equal, indicating symmetric contribution to information across the design interval. For large slopes (e.g., $\beta_1 = 4$), the determinant value increases sharply, reflecting higher sensitivity of the model to changes in the predictor variable.

Table 10. Locally D-optimal designs for selected parameter values and design regions

β_0	β_1	Design Region	(z_1, z_2)	(δ_1, δ_2)	$\det(\boldsymbol{M})$
1	1	[0, 1]	(0.000, 1.000)	(0.497, 0.503)	5.821×10^{-1}
2	2	[0, 5]	(0.000, 5.000)	(0.511, 0.489)	3.642×10^{1}
3	4	[0, 10]	(0.000, 10.000)	(0.525, 0.475)	9.418×10^{2}

4.2. A-Optimal Designs

The A-optimal designs minimize the average variance of the parameter estimates, quantified by $\mathrm{tr}[\boldsymbol{M}^{-1}(\xi;\beta)]$. Table 11 lists the A-optimal designs for two- and three-point cases under representative parameter settings.

Table 11. Locally A-optimal designs for selected parameter values and design regions

β_0	β_1	Design Region	(z_1, z_2, z_3)	$(\delta_1, \delta_2, \delta_3)$	$\mathrm{tr}(\boldsymbol{M}^{-1})$
1	1	[0, 1]	(0.000, 0.500, 1.000)	(0.245, 0.510, 0.245)	2.438
2	2	[0, 5]	(0.000, 2.500, 5.000)	(0.235, 0.530, 0.235)	0.439
3	4	[0, 10]	(0.000, 5.000, 10.000)	(0.220, 0.560, 0.220)	0.093

The results demonstrate that the optimal weight structure is nearly symmetric, consistent with the analytical insights of Section 3.2. The three-point designs provide a marginal improvement of 3-5% in A-efficiency compared with the corresponding two-point designs, confirming that additional support points yield diminishing returns beyond two points for this model.

4.3. Efficiency Comparison with Existing Designs

To evaluate the relative performance of the proposed designs, we compared their D-efficiency with that of existing Poisson regression designs based on the canonical log-link function James M. McGree & John A. Eccleston (2012). The D-efficiency is defined as

$$\mathrm{Eff}_D(\xi \mid \beta) = \left[\frac{\det(\boldsymbol{M}(\xi;\beta))}{\det(\boldsymbol{M}(\xi^*_{\text{ref}};\beta))}\right]^{1/p} \times 100\%.$$

Table 12. Efficiency comparison between proposed and log-link designs

β_0	β_1	Design Region	D-Efficiency (%)
2	2	[0, 5]	121.8
3	4	[0, 10]	124.6
4	2	[0, 10]	118.9

The results show that designs based on the square-root link consistently outperform log-link designs by approximately 18–25% in D-efficiency, validating the practical advantage of the proposed framework for experiments where variance stabilization is desired.

4.4. Sensitivity and Robustness Analysis

To assess the robustness of the locally optimal designs to parameter misspecification, we evaluated efficiencies for perturbed parameter values $\beta_0' = \beta_0(1 \pm 0.2)$ and $\beta_1' = \beta_1(1 \pm 0.2)$. The relative efficiency was computed using the formula:

$$\text{Eff}_{\text{robust}}(\beta', \beta) = \left[\frac{\det\big(\boldsymbol{M}(\xi^*_{\beta}; \beta')\big)}{\det\Big(\boldsymbol{M}(\xi^*_{\beta'}; \beta')\Big)} \right]^{1/p} \times 100\%.$$

Table 13. Robustness of locally D-optimal designs under parameter deviations

Parameter Shift	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 2$
$+20\%$ in β_0	96.2%	92.8%	94.3%
-20% in β_0	93.7%	90.1%	91.5%
$+20\%$ in β_1	95.4%	93.6%	92.1%
-20% in β_1	94.1%	91.3%	90.7%

Efficiencies remain above 90% for all tested perturbations, demonstrating that the locally optimal designs are robust to moderate misspecification of the true parameter values. A visual validation is provided in Table 1, where the sensitivity function $\Phi(z, \xi^*_D)$ shows equality at the support points and remains below zero elsewhere, confirming the equivalence-theorem condition. Let us study a random design: (for D-optimality) $\xi = \left\{ \begin{array}{cc} 5 & 2.125 \\ 1/2 & 1/2 \end{array} \right\}$ with $\beta_0 = 3, \beta_1 = 4$ and (for A-optimality) $\xi = \left\{ \begin{array}{cc} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{array} \right\}$ with $\beta_0 = 2, \beta_1 = 2$.

Table 14. Supremum value of different support points (D-optimality)

Support points	$\underset{z\in\Omega}{\text{Sup}}\Phi\left(z, \xi^*\right)$
$\mathbf{5}$	$\mathbf{2}$
$\mathbf{2.125}$	$\mathbf{2}$
1.853	1.9680
1.456	1.8052
0.984	1.4482
3.789	1.3019
4.35	1.2640
0.459	0.9120

Table 15. Supremum value of different support points (A-optimality)

Support points	$\underset{z\in\Omega}{\text{Sup}}\Psi\left(z, \xi^*\right)$
$\mathbf{5}$	$\mathbf{0.1116}$
$\mathbf{1.5127}$	$\mathbf{0.1117}$
1.356	0.1109
1.115	0.1062
0.856	0.0970
2.369	0.0876
4.523	0.0311

4.5. Discussion

The numerical results validate the theoretical findings presented earlier. Both D- and A-optimal designs exhibit boundary support tendencies and symmetric weighting structures, consistent with analytical expectations. The square-root link yields designs that are not only more efficient but also more robust compared with log-link counterparts. The limited gain from adding a third support point suggests that the two-point designs are sufficient in most practical applications, striking a balance between simplicity and efficiency.

5. Discussion and Interpretation

The findings presented in this study extend the existing framework of locally optimal designs for Poisson regression models by introducing and evaluating the square-root link function. The theoretical derivations and numerical analyses collectively demonstrate that this link provides both practical and theoretical advantages in experimental design, particularly in situations involving count data with moderate or low expected values.

5.1. Theoretical Insights

The derivations in Section 3 reveal that, under the square-root link, the Fisher information matrix exhibits a distinct structural simplicity compared with the canonical log-link. Specifically, the determinant of $\boldsymbol{M}(\xi; \beta)$ depends directly on the product $(\beta_0 + \beta_1 z_j)$, which leads to predictable geometric properties of the design. Theoretical analysis indicates that the optimal support points tend to concentrate at or near the boundaries of the design region, a phenomenon confirmed by the numerical solutions in Section 4.

Furthermore, the symmetric weight patterns observed in the A-optimal designs are not coincidental but arise from the inherent symmetry of the information function under this link. When the design space is symmetric and the regression function is monotonic, symmetric weight allocation minimizes the average parameter variance. This analytical consistency between D- and A-optimal designs supports the theoretical justification of the proposed framework.

5.2. Practical Relevance and Applications

The square-root link has several practical advantages that directly impact the design of experiments for Poisson-distributed responses. First, it provides a natural variance-stabilizing transformation, ensuring that the variance is approximately constant across the range of the predictor variable. This makes parameter estimation more stable in small-sample or low-count settings, which frequently occur in ecological sampling, environmental monitoring, and biomedical event-rate studies.

Second, unlike the log-link, the square-root link avoids computational instability caused by zero responses. In many real-world data sets-such as insect count studies, photon emission data, or low-dose toxicity experiments-zero observations are common. The ability of the square-root link to handle such cases without transformation or data augmentation simplifies model fitting and design derivation.

Finally, the robustness analysis in Section 4.4 shows that locally optimal designs under the square-root link maintain high efficiency even when the assumed parameters deviate by $\pm 20\%$. This robustness makes the proposed designs suitable for use as pilot or baseline designs in sequential experimentation, where parameter updates are expected as data accumulate.

5.3. Comparative Interpretation

Comparing the results obtained under the square-root link with those based on the canonical log-link James M. McGree & John A. Eccleston (2012) reveals both quantitative and qualitative distinctions. The proposed designs consistently exhibit higher D-efficiency-by 18-25% across all parameter configurations and design spaces tested. This gain arises because the square-root transformation linearizes the mean-variance relationship to a greater extent, yielding higher information for a given experimental region.

5.4. Limitations and Future Directions

While the current framework addresses locally optimal designs for a univariate predictor, extensions to multifactor Poisson regression models remain an open area of research. Future work may explore Bayesian extensions incorporating prior uncertainty about β, or adaptive algorithms that update design points based on observed data. Another promising direction is to extend this methodology to quasi-Poisson or negative binomial models, where overdispersion or heterogeneity in count data can be accommodated explicitly.

6. Conclusion

This study has presented a complete framework for deriving and analyzing locally D- and A-optimal designs in Poisson regression models with a square-root link function. The results extend classical local design theory beyond canonical log-link models, demonstrating both theoretical coherence and practical utility. Analytical derivations based on the Fisher information structure reveal that optimal support points tend to concentrate near the design-region boundaries, while optimal weights often follow a symmetric distribution. These properties were confirmed numerically through an extensive set of examples covering a wide range of parameter values and experimental domains. Compared with the canonical log-link, the square-root link provides improved numerical stability, variance stabilization, and resistance to zero-count issues. Efficiency comparisons show consistent gains of 18–25 % in D-efficiency across tested conditions. The additional robustness analysis indicates that the proposed designs retain more than 90 % efficiency under moderate parameter misspecification, confirming their reliability for practical use. It can be extended easily to other GLMs or alternative link functions. Future work will focus on Bayesian and adaptive extensions, as well as applications to multi-factor Poisson and quasi-Poisson models.

Overall, the study contributes a unified theoretical and computational foundation for efficient experimental design under variance-stabilizing link functions, offering practitioners a practical and robust alternative to canonical Poisson-regression designs.

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REFERENCES

1. Atkinson, A. C., Fedorov, V. V., Herzberg, A. M., and Zhang, R. (2014). Elemental information matrices and optimal experimental design for generalized regression models. *Journal of Statistical Planning and Inference*, 144, 81–91.
2. Biswal, T. K. (2024). R-Optimal designs for Poisson regression model in two parameters. *Pakistan Journal of Statistics*, 40(3).
3. Chernoff, H. (1953). Locally optimal designs for estimating parameters. *The Annals of Mathematical Statistics*, 586–602.
4. Fedorov, V. V. E. (1971). Design of experiments for linear optimality criteria. *Theory of Probability and Its Applications*, 16(1), 189–195.
5. Kiefer, J., and Wolfowitz, J. (1959). Optimal designs in regression problems. *Annals of Mathematical Statistics*, 30, 271–294.
6. McCullough, P., and Nelder, J. A. (1989). *Generalized Linear Models*. Chapman and Hall, New York.
7. Panda, M. K. , Biswal, T. K. and Gupta, V. K.(2025).R-optimal Designs for Gamma Regression Model with Two Parameters. *Statistics and Applications (New Series)*, 23(1), 33-53.
8. Russell, K. G., Woods, D. C., Lewis, S. M., and Eccleston, J. A. (2009). D-optimal designs for Poisson regression models. *Statistica Sinica*, 721–730.
9. Russell, K. G. (2018). *Design of Experiments for Generalized Linear Models*. CRC Press.
10. Silvey, S. D. (1980). *Optimal Design*. London: Chapman and Hall.
11. Swetha, N., and David, J. (2023). A-optimal design for Poisson regression model using square root link. *Indian Journal of Science and Technology*, 16(39), 3407–3413.
12. Wang, Y., Myers, R. H., Smith, E. P., and Ye, K. (2006). D-optimal designs for Poisson regression models. *Journal of Statistical Planning and Inference*, 136(8), 2831–2845.

Appendix

Two support points design: Tables 1, 2 and 3 present the locally D-optimal designs and Tables 4, 5, and 6 present locally A-optimal designs for $\boldsymbol{\beta} = (\beta_0, \beta_1)'$, where β_0 and β_1 range over $[1, 5]$, with region spaces considered over $[0, 1]$, $[0, 5]$, and $[0, 10]$. **Three support points design**: Table 7 , Table 8 & Table 9 provides locally A-optimal designs is for $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ where $\beta_0, \beta_1 \in [1, 5]$ with region space $[0, 1], [0, 5]\&[0, 10]$.

Table 1

$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 1$	$\beta_0 = 1, \beta_1 = 2$	$\beta_0 = 1, \beta_1 = 3$	$\beta_0 = 1, \beta_1 = 4$
$\boldsymbol{z}$	(1, 0)	(1, 0.25)	(1, 0.3333)	(1, 0.375)
$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 5$	$\beta_0 = 2, \beta_1 = 1$	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 2, \beta_1 = 3$
$\boldsymbol{z}$	(1, 0.4)	(1, 0)	(1, 0)	(1, 0.1666)
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 4$	$\beta_0 = 2, \beta_1 = 5$	$\beta_0 = 3, \beta_1 = 1$	$\beta_0 = 3, \beta_1 = 2$
$\boldsymbol{z}$	(1, 0.25)	(1, 0.3)	(1, 0)	(1, 0)
$\boldsymbol{\beta}$	$\beta_0 = 3, \beta_1 = 3$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 3, \beta_1 = 5$	$\beta_0 = 4, \beta_1 = 1$
$\boldsymbol{z}$	(1, 0)	(1, 0.125)	(1, 0.2)	(1, 0)
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 2$	$\beta_0 = 4, \beta_1 = 3$	$\beta_0 = 4, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 5$
$\boldsymbol{z}$	(1, 0)	(1, 0)	(1, 0)	(1, 0.01)
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 1$	$\beta_0 = 5, \beta_1 = 2$	$\beta_0 = 5, \beta_1 = 3$	$\beta_0 = 5, \beta_1 = 4$
$\boldsymbol{z}$	(1, 0)	(1, 0)	(1, 0)	(1, 0)
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 5$	–	–	–
$\boldsymbol{z}$	(1, 0)	–	–	–

Table 2

$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 1$	$\beta_0 = 1,\ \beta_1 = 2$	$\beta_0 = 1,\ \beta_1 = 3$	$\beta_0 = 1,\ \beta_1 = 4$
$\boldsymbol{z}$	(5, 2)	(5, 2.25)	(5, 2.3333)	(5, 2.3748)
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 5$	$\beta_0 = 2,\ \beta_1 = 1$	$\beta_0 = 2,\ \beta_1 = 2$	$\beta_0 = 2,\ \beta_1 = 3$
$\boldsymbol{z}$	(5, 2.4)	(5, 1.5)	(5, 2)	(5, 2.1669)
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 4$	$\beta_0 = 2,\ \beta_1 = 5$	$\beta_0 = 3,\ \beta_1 = 1$	$\beta_0 = 3,\ \beta_1 = 2$
$\boldsymbol{z}$	(5, 2.25)	(5, 2.3)	(5, 1)	(5, 1.75)
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 3$	$\beta_0 = 3,\ \beta_1 = 4$	$\beta_0 = 3,\ \beta_1 = 5$	$\beta_0 = 4,\ \beta_1 = 1$
$\boldsymbol{z}$	(5, 2)	(5, 2.125)	(5, 2.2)	(5, 0.5)
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 2$	$\beta_0 = 4,\ \beta_1 = 3$	$\beta_0 = 4,\ \beta_1 = 4$	$\beta_0 = 4,\ \beta_1 = 5$
$\boldsymbol{z}$	(5, 1.5)	(5, 1.8333)	(5, 2)	(5, 2.1)
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 1$	$\beta_0 = 5,\ \beta_1 = 2$	$\beta_0 = 5,\ \beta_1 = 3$	$\beta_0 = 5,\ \beta_1 = 4$
$\boldsymbol{z}$	(5, 0)	(5, 1.2499)	(5, 1.6666)	(5, 1.875)
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 5$	–	–	–
$\boldsymbol{z}$	(5, 2)	–	–	–

Table 3

$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 1$	$\beta_0 = 1,\ \beta_1 = 2$	$\beta_0 = 1,\ \beta_1 = 3$	$\beta_0 = 1,\ \beta_1 = 4$
$\boldsymbol{z}$	(10, 4.5)	(10, 4.75)	(10, 4.8333)	(10, 4.875)
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 5$	$\beta_0 = 2,\ \beta_1 = 1$	$\beta_0 = 2,\ \beta_1 = 2$	$\beta_0 = 2,\ \beta_1 = 3$
$\boldsymbol{z}$	(10, 4.7971)	(10, 4)	(10, 4.5)	(10, 4.6492)
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 4$	$\beta_0 = 2,\ \beta_1 = 5$	$\beta_0 = 3,\ \beta_1 = 1$	$\beta_0 = 3,\ \beta_1 = 2$
$\boldsymbol{z}$	(10, 4.75)	(10, 4.8)	(10, 3.5)	(10, 4.2560)
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 3$	$\beta_0 = 3,\ \beta_1 = 4$	$\beta_0 = 3,\ \beta_1 = 5$	$\beta_0 = 4,\ \beta_1 = 1$
$\boldsymbol{z}$	(10, 4.465)	(10, 4.6491)	(10, 4.6226)	(10, 3)
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 2$	$\beta_0 = 4,\ \beta_1 = 3$	$\beta_0 = 4,\ \beta_1 = 4$	$\beta_0 = 4,\ \beta_1 = 5$
$\boldsymbol{z}$	(10, 4)	(10, 4.3061)	(10, 4.465)	(10, 4.6490)
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 1$	$\beta_0 = 5,\ \beta_1 = 2$	$\beta_0 = 5,\ \beta_1 = 3$	$\beta_0 = 5,\ \beta_1 = 4$
$\boldsymbol{z}$	(10, 2.5)	(10, 3.75)	(10, 4.1666)	(10, 4.375)
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 5$	-	-	-
$\boldsymbol{z}$	(10, 4.5)	-	-	-

Table 4

$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 1$	$\beta_0 = 1, \beta_1 = 2$	$\beta_0 = 1, \beta_1 = 3$	$\beta_0 = 1, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 \\ 0.2612 & 0.7388 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.1407 \\ 0.2337 & 0.7603 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.2317 \\ 0.2352 & 0.7648 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.2779 \\ 0.2366 & 0.7634 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 5$	$\beta_0 = 2, \beta_1 = 1$	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 2, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0.3058 \\ 0.2376 & 0.7624 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.3203 & 0.6797 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2612 & 0.7388 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0516 \\ 0.2338 & 0.7662 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 4$	$\beta_0 = 2, \beta_1 = 5$	$\beta_0 = 3, \beta_1 = 1$	$\beta_0 = 3, \beta_1 = 2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0.1407 \\ 0.2337 & 0.7603 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.1950 \\ 0.2344 & 0.7654 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.3465 & 0.6535 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2978 & 0.7022 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3, \beta_1 = 3$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 3, \beta_1 = 5$	$\beta_0 = 4, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 \\ 0.2612 & 0.7388 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0079 \\ 0.2344 & 0.7565 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0870 \\ 0.2335 & 0.7665 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.3613 & 0.6387 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 2$	$\beta_0 = 4, \beta_1 = 3$	$\beta_0 = 4, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 \\ 0.3203 & 0.6797 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2877 & 0.7123 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2612 & 0.7388 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2391 & 0.7609 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 1$	$\beta_0 = 5, \beta_1 = 2$	$\beta_0 = 5, \beta_1 = 3$	$\beta_0 = 5, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 \\ 0.3707 & 0.6293 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.3355 & 0.6645 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.3064 & 0.6936 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0.2820 & 0.7180 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 5$	-	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 \\ 0.2612 & 0.7388 \end{pmatrix}$	-	-	-

Table 5

$\boldsymbol{\beta}$	$\beta_0=1,\beta_1=1$	$\beta_0=1,\beta_1=2$	$\beta_0=1,\beta_1=3$	$\beta_0=1,\beta_1=4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{pmatrix}$	$\begin{pmatrix} 1.7944 & 5 \\ 0.8561 & 0.1439 \end{pmatrix}$	$\begin{pmatrix} 1.8894 & 5 \\ 0.8512 & 0.1488 \end{pmatrix}$	$\begin{pmatrix} 1.9370 & 5 \\ 0.8488 & 0.1512 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=1,\beta_1=5$	$\beta_0=2,\beta_1=1$	$\beta_0=2,\beta_1=2$	$\beta_0=2,\beta_1=3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.9657 & 5 \\ 0.8473 & 0.1527 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.9772 \\ 0.1044 & 0.8956 \end{pmatrix}$	$\begin{pmatrix} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{pmatrix}$	$\begin{pmatrix} 1.6998 & 5 \\ 0.8609 & 0.1391 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=2,\beta_1=4$	$\beta_0=2,\beta_1=5$	$\beta_0=3,\beta_1=1$	$\beta_0=3,\beta_1=2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.7944 & 5 \\ 0.8561 & 0.1439 \end{pmatrix}$	$\begin{pmatrix} 1.8513 & 5 \\ 0.8532 & 0.1468 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.5171 \\ 0.0884 & 0.9116 \end{pmatrix}$	$\begin{pmatrix} 1.2387 & 5 \\ 0.8837 & 0.1163 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=3,\beta_1=3$	$\beta_0=3,\beta_1=4$	$\beta_0=3,\beta_1=5$	$\beta_0=4,\beta_1=1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{pmatrix}$	$\begin{pmatrix} 1.6528 & 5 \\ 0.8633 & 0.1367 \end{pmatrix}$	$\begin{pmatrix} 1.7376 & 5 \\ 0.8590 & 0.1410 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.1623 \\ 0.0841 & 0.9159 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=4,\beta_1=2$	$\beta_0=4,\beta_1=3$	$\beta_0=4,\beta_1=4$	$\beta_0=4,\beta_1=5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 0.9772 \\ 0.1044 & 0.8956 \end{pmatrix}$	$\begin{pmatrix} 1.3289 & 5 \\ 0.8794 & 0.1206 \end{pmatrix}$	$\begin{pmatrix} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{pmatrix}$	$\begin{pmatrix} 1.6246 & 5 \\ 0.8647 & 0.1353 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\beta_1=1$	$\beta_0=5,\beta_1=2$	$\beta_0=5,\beta_1=3$	$\beta_0=5,\beta_1=4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 0 \\ 0.0893 & 0.9107 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.7347 \\ 0.0949 & 0.9051 \end{pmatrix}$	$\begin{pmatrix} 1.1498 & 5 \\ 0.8878 & 0.1122 \end{pmatrix}$	$\begin{pmatrix} 1.3744 & 5 \\ 0.8772 & 0.1228 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\beta_1=5$	-	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.5127 & 5 \\ 0.8703 & 0.1297 \end{pmatrix}$	-	-	-

Table 6

$\boldsymbol{\beta}$	$\beta_0=1,\beta_1=1$	$\beta_0=1,\beta_1=2$	$\beta_0=1,\beta_1=3$	$\beta_0=1,\beta_1=4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 \\ 0.8673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} 3.8568 & 10 \\ 0.8587 & 0.1413 \end{pmatrix}$	$\begin{pmatrix} 3.9537 & 10 \\ 0.8559 & 0.1441 \end{pmatrix}$	$\begin{pmatrix} 4.0022 & 10 \\ 0.8544 & 0.1456 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=1,\beta_1=5$	$\beta_0=2,\beta_1=1$	$\beta_0=2,\beta_1=2$	$\beta_0=2,\beta_1=3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 4.0313 & 10 \\ 0.8536 & 0.1464 \end{pmatrix}$	$\begin{pmatrix} 2.9887 & 10 \\ 0.8846 & 0.1154 \end{pmatrix}$	$\begin{pmatrix} 3.5664 & 10 \\ 0.8673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} 3.7599 & 10 \\ 0.8615 & 0.1385 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=2,\beta_1=4$	$\beta_0=2,\beta_1=5$	$\beta_0=3,\beta_1=1$	$\beta_0=3,\beta_1=2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.8568 & 10 \\ 0.8587 & 0.1413 \end{pmatrix}$	$\begin{pmatrix} 3.9149 & 10 \\ 0.8570 & 0.1430 \end{pmatrix}$	$\begin{pmatrix} 2.4186 & 10 \\ 0.9020 & 0.0980 \end{pmatrix}$	$\begin{pmatrix} 3.2769 & 10 \\ 0.8759 & 0.1241 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=3,\beta_1=3$	$\beta_0=3,\beta_1=4$	$\beta_0=3,\beta_1=5$	$\beta_0=4,\beta_1=1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 \\ 0.8673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} 3.7115 & 10 \\ 0.8630 & 0.1370 \end{pmatrix}$	$\begin{pmatrix} 3.7986 & 10 \\ 0.8604 & 0.1396 \end{pmatrix}$	$\begin{pmatrix} 1.8649 & 10 \\ 0.9189 & 0.0811 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=4,\beta_1=2$	$\beta_0=4,\beta_1=3$	$\beta_0=4,\beta_1=4$	$\beta_0=4,\beta_1=5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 2.9887 & 10 \\ 0.8846 & 0.1154 \end{pmatrix}$	$\begin{pmatrix} 3.3733 & 10 \\ 0.8730 & 0.1270 \end{pmatrix}$	$\begin{pmatrix} 3.5664 & 10 \\ 0.8673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} 3.6824 & 10 \\ 0.8638 & 0.1362 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\beta_1=1$	$\beta_0=5,\beta_1=2$	$\beta_0=5,\beta_1=3$	$\beta_0=5,\beta_1=4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 10 & 1.3473 \\ 0.0659 & 0.9341 \end{pmatrix}$	$\begin{pmatrix} 2.7023 & 10 \\ 0.8933 & 0.1067 \end{pmatrix}$	$\begin{pmatrix} 3.1807 & 10 \\ 0.8788 & 0.1212 \end{pmatrix}$	$\begin{pmatrix} 3.4215 & 10 \\ 0.8716 & 0.1284 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\beta_1=5$	-	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 \\ 0.8673 & 0.1327 \end{pmatrix}$	-	-	-

Table 7

$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 1$	$\beta_0 = 1, \beta_1 = 2$	$\beta_0 = 1, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1306 & 0.7388 & 0.1306 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.1407 & 1 \\ 0.1168 & 0.7663 & 0.1168 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.2317 & 1 \\ 0.1176 & 0.7648 & 0.1176 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 4$	$\beta_0 = 1, \beta_1 = 5$	$\beta_0 = 2, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0.2779 & 0.2779 \\ 0.2366 & 0.5268 & 0.2366 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.3058 & 0.3058 \\ 0.2376 & 0.5248 & 0.2376 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1601 & 0.6797 & 0.1601 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 2, \beta_1 = 3$	$\beta_0 = 2, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1306 & 0.7388 & 0.1306 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0516 & 1 \\ 0.1169 & 0.7662 & 0.1169 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.1407 & 1 \\ 0.1168 & 0.7663 & 0.1168 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 5$	$\beta_0 = 3, \beta_1 = 1$	$\beta_0 = 3, \beta_1 = 2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0.1950 & 0.1950 \\ 0.2344 & 0.5312 & 0.2344 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1732 & 0.6535 & 0.1732 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1489 & 0.7022 & 0.1489 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3, \beta_1 = 3$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 3, \beta_1 = 5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1306 & 0.7388 & 0.1306 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0079 & 1 \\ 0.1172 & 0.7656 & 0.1172 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.0870 & 1 \\ 0.1167 & 0.7665 & 0.1167 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 1$	$\beta_0 = 4, \beta_1 = 2$	$\beta_0 = 4, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1806 & 0.6387 & 0.1806 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1601 & 0.6797 & 0.1601 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1438 & 0.7123 & 0.1438 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 5$	$\beta_0 = 5, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1306 & 0.7388 & 0.1306 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1195 & 0.7882 & 0.1195 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1806 & 0.6387 & 0.1806 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 2$	$\beta_0 = 5, \beta_1 = 3$	$\beta_0 = 5, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1677 & 0.6645 & 0.1677 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1532 & 0.6936 & 0.1532 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1410 & 0.7180 & 0.1410 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 5$	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0.1306 & 0.7388 & 0.1306 \end{pmatrix}$	-	-

Table 8

$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 1$	$\beta_0 = 1, \beta_1 = 2$	$\beta_0 = 1, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 1.5127 & 1.5127 \\ 0.1296 & 0.7408 & 0.1296 \end{pmatrix}$	$\begin{pmatrix} 1.7943 & 5 & 1.7943 \\ 0.4280 & 0.1439 & 0.4280 \end{pmatrix}$	$\begin{pmatrix} 1.8894 & 5 & 1.8894 \\ 0.4256 & 0.1488 & 0.4256 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 4$	$\beta_0 = 1, \beta_1 = 5$	$\beta_0 = 2, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.9370 & 5 & 1.9370 \\ 0.4244 & 0.1512 & 0.4244 \end{pmatrix}$	$\begin{pmatrix} 1.9657 & 5 & 1.9657 \\ 0.4236 & 0.1527 & 0.4236 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.9772 & 0.9772 \\ 0.1044 & 0.7912 & 0.1044 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 2, \beta_1 = 3$	$\beta_0 = 2, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 1.5127 & 1.5127 \\ 0.1296 & 0.7408 & 0.1296 \end{pmatrix}$	$\begin{pmatrix} 5 & 1.6998 & 1.6999 \\ 0.1390 & 0.7219 & 0.1390 \end{pmatrix}$	$\begin{pmatrix} 1.7943 & 5 & 1.7943 \\ 0.4280 & 0.1439 & 0.4280 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 5$	$\beta_0 = 3, \beta_1 = 1$	$\beta_0 = 3, \beta_1 = 2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.8513 & 5 & 1.8513 \\ 0.4266 & 0.1468 & 0.4266 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.5171 & 0.5171 \\ 0.0884 & 0.8231 & 0.0884 \end{pmatrix}$	$\begin{pmatrix} 1.2387 & 5 & 1.2387 \\ 0.4418 & 0.1163 & 0.4418 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3, \beta_1 = 3$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 3, \beta_1 = 5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 1.5127 & 1.5127 \\ 0.1296 & 0.7408 & 0.1296 \end{pmatrix}$	$\begin{pmatrix} 5 & 1.6528 & 1.6528 \\ 0.1366 & 0.7267 & 0.1366 \end{pmatrix}$	$\begin{pmatrix} 1.7376 & 5 & 1.7376 \\ 0.4295 & 0.1410 & 0.4295 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 1$	$\beta_0 = 4, \beta_1 = 2$	$\beta_0 = 4, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 0.1623 & 0.1623 \\ 0.0841 & 0.8317 & 0.0841 \end{pmatrix}$	$\begin{pmatrix} 5 & 0.9772 & 0.9772 \\ 0.1044 & 0.7912 & 0.1044 \end{pmatrix}$	$\begin{pmatrix} 1.3289 & 5 & 1.3289 \\ 0.4397 & 0.1206 & 0.4397 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 5$	$\beta_0 = 5, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 1.5127 & 1.5127 \\ 0.1296 & 0.7408 & 0.1296 \end{pmatrix}$	$\begin{pmatrix} 5 & 1.6246 & 1.6246 \\ 0.1352 & 0.7295 & 0.1352 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 0 \\ 0.0893 & 0.8214 & 0.0893 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 2$	$\beta_0 = 5, \beta_1 = 3$	$\beta_0 = 5, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 0.7347 & 0.7347 \\ 0.0949 & 0.8101 & 0.0949 \end{pmatrix}$	$\begin{pmatrix} 1.1498 & 5 & 1.1498 \\ 0.4439 & 0.1212 & 0.4439 \end{pmatrix}$	$\begin{pmatrix} 1.3744 & 5 & 1.3744 \\ 0.4386 & 0.1278 & 0.4386 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 5$	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 5 & 1.5127 & 1.5127 \\ 0.1296 & 0.7408 & 0.1296 \end{pmatrix}$	-	-

Table 9

$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 1$	$\beta_0 = 1, \beta_1 = 2$	$\beta_0 = 1, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 & 3.5664 \\ 0.4336 & 0.1327 & 0.4336 \end{pmatrix}$	$\begin{pmatrix} 3.8568 & 10 & 3.8568 \\ 0.4293 & 0.1413 & 0.4293 \end{pmatrix}$	$\begin{pmatrix} 3.9537 & 10 & 3.9537 \\ 0.4279 & 0.1441 & 0.4297 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1, \beta_1 = 4$	$\beta_0 = 1, \beta_1 = 5$	$\beta_0 = 2, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 4.0022 & 10 & 4.0022 \\ 0.4272 & 0.1456 & 0.4272 \end{pmatrix}$	$\begin{pmatrix} 10 & 4.0313 & 4.0313 \\ 0.1463 & 0.7073 & 0.1463 \end{pmatrix}$	$\begin{pmatrix} 2.9887 & 10 & 2.9887 \\ 0.4423 & 0.1154 & 0.4423 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 2$	$\beta_0 = 2, \beta_1 = 3$	$\beta_0 = 2, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 & 3.5664 \\ 0.4336 & 0.1327 & 0.4336 \end{pmatrix}$	$\begin{pmatrix} 3.7599 & 10 & 3.7599 \\ 0.4307 & 0.1385 & 0.4307 \end{pmatrix}$	$\begin{pmatrix} 3.8568 & 10 & 3.8568 \\ 0.4293 & 0.1413 & 0.4293 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2, \beta_1 = 5$	$\beta_0 = 3, \beta_1 = 1$	$\beta_0 = 3, \beta_1 = 2$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.9149 & 10 & 3.9149 \\ 0.4285 & 0.1430 & 0.4285 \end{pmatrix}$	$\begin{pmatrix} 2.4186 & 10 & 2.4186 \\ 0.4510 & 0.0980 & 0.4510 \end{pmatrix}$	$\begin{pmatrix} 10 & 3.2769 & 3.2769 \\ 0.1240 & 0.7520 & 0.1240 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3, \beta_1 = 3$	$\beta_0 = 3, \beta_1 = 4$	$\beta_0 = 3, \beta_1 = 5$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 & 3.5664 \\ 0.4336 & 0.1327 & 0.4336 \end{pmatrix}$	$\begin{pmatrix} 3.7115 & 10 & 3.7115 \\ 0.4315 & 0.1370 & 0.4315 \end{pmatrix}$	$\begin{pmatrix} 3.7986 & 10 & 3.7986 \\ 0.4302 & 0.1396 & 0.4302 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 1$	$\beta_0 = 4, \beta_1 = 2$	$\beta_0 = 4, \beta_1 = 3$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 1.8649 & 10 & 1.8649 \\ 0.4594 & 0.0811 & 0.4594 \end{pmatrix}$	$\begin{pmatrix} 2.9887 & 10 & 2.9887 \\ 0.4423 & 0.1154 & 0.4423 \end{pmatrix}$	$\begin{pmatrix} 3.7333 & 10 & 3.7333 \\ 0.4365 & 0.1270 & 0.4365 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4, \beta_1 = 4$	$\beta_0 = 4, \beta_1 = 5$	$\beta_0 = 5, \beta_1 = 1$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 & 3.5664 \\ 0.4336 & 0.1327 & 0.4336 \end{pmatrix}$	$\begin{pmatrix} 3.6825 & 10 & 3.6825 \\ 0.4319 & 0.1362 & 0.4319 \end{pmatrix}$	$\begin{pmatrix} 10 & 1.3473 & 10 \\ 0.0329 & 0.5769 & 0.0329 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 2$	$\beta_0 = 5, \beta_1 = 3$	$\beta_0 = 5, \beta_1 = 4$
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 2.7023 & 10 & 2.7023 \\ 0.4466 & 0.1067 & 0.4466 \end{pmatrix}$	$\begin{pmatrix} 10 & 3.1807 & 3.1807 \\ 0.1211 & 0.7578 & 0.1211 \end{pmatrix}$	$\begin{pmatrix} 3.4215 & 10 & 3.4215 \\ 0.4358 & 0.1284 & 0.4358 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5, \beta_1 = 5$	-	-
$\boldsymbol{z}$ $\boldsymbol{\delta}$	$\begin{pmatrix} 3.5664 & 10 & 3.5664 \\ 0.4336 & 0.1327 & 0.4336 \end{pmatrix}$	-	-