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The Log-Exponentiated Polynomial G Family: Properties, Characterizations and Risk Analysis under Different Estimation Methods

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Abstract This work presents a new class of probability distributions called the Log-Exponentiated Polynomial (LEP) G family. We explore its fundamental properties and provide characterizations. The paper focuses on risk analysis using different estimation methods. The LEP G family offers flexibility for modeling various data types. We derive useful expansions of the new family, these expansions facilitate the calculation of moments and other statistical measures. The model's parameters are estimated using several methods, including Maximum Likelihood Estimation (MLE). We also employ Cramer-von Mises (CVM), Anderson-Darling (ADE), Right Tail Anderson-Darling (RTADE), and Length Bias Extended (LEADE) estimation techniques. A simulation study illustrates the performance of these estimation methods. Bias, Root Mean Square Error (RMSE), and Anderson-Darling distance metrics are assessed. The LEP Weibull model is applied to insurance claims data for risk measurement. Key Risk Indicators (KRIs) like Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean Variance (TMV) and Expected Loss (EL) are calculated. We also analyze artificial data to demonstrate the model's behavior under controlled conditions.

Keywords Value-at-Risk; Weibull model; Claims Data; Risk Analysis; Characterizations

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

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1. Introduction

In recent years, significant advancements have been made in the development of generalized statistical distributions to better capture the complexities of real data across various domains such as finance, insurance, medicine, and engineering (Abiad et al., 2025; Afify et al., 2018). These efforts have focused on enhancing classical models by introducing additional shape parameters or combining existing distribution families to improve flexibility, accuracy, and applicability (Alizadeh et al., 2018; Abouelmagd et al., 2019). Notable among these developments are the Odd Log-Logistic Topp–Leone G family, which provides greater modeling capabilities for skewed and bimodal datasets, and the Zero Truncated Poisson Burr X family, designed to simultaneously model count and continuous data (Alizadeh et al., 2018; Abouelmagd et al., 2019). Other notable contributions include the Transmuted Weibull-G family, Exponential Lindley Odd Log-Logistic-G family, and the Odd Log-Logistic Weibull-G family, which extend the utility of classical distributions in survival and reliability analysis (Korkmaz et al., 2018; Rasekhi et al., 2022). Additionally, Alizadeh et al. (2023) explored copula-based extensions of the XGamma distribution, while

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Mansour et al. (2020f) introduced copulas into the modeling of acute bone cancer data. Ibrahim et al. (2025a, 2025b) further expanded this field by applying Clayton copulas to validate flexible Weibull models. The current paper builds upon these developments by proposing a novel G-family that incorporates closed-form expressions for moments, quantile functions, and entropy measures, allowing for a broader range of shapes and tail behaviors.This new framework enables more accurate modeling of complex data structures and improves interpretability and predictive performance.

In recent years, the development of flexible and general families of probability distributions has gained significant attention in statistical research. These models aim to enhance the modeling capabilities of classical distributions by introducing additional parameters that control shape, skewness, tail behavior, and other distributional characteristics. One effective approach involves transforming a baseline cumulative distribution function (CDF), $G(x)$, using carefully designed generator functions. In this context, we introduce a novel class of continuous distributions called the LEP family and characterized by a unique combination of logarithmic and exponential transformations. The proposed model is defined by a CDF

$$F(x;\alpha,\beta) = C\log\left[1+G(x)^{\alpha}\right]e^{\beta G(x)^{\alpha}}, \quad x\in\mathbb{R}, \tag{1}$$

with the corresponding probability density function (PDF):

$$f(x;\alpha,\beta) = \alpha C g(x)\,G(x)^{\alpha-1}\,e^{\beta G(x)^{\alpha}}\tau(x), \qquad x\in\mathbb{R}, \tag{2}$$

where $\alpha>0,\beta>0$ are parameters, $C=\frac{1}{e^{\beta}\log(2)}$ is a normalization constant and

$$\tau(x) = \frac{1}{1+G(x)^{\alpha}} + \beta\log\left[1+G(x)^{\alpha}\right]$$

is a shaping function that controls the behavior of the distribution and $G(x)^{\alpha}$ is the experimented baseline CDF with the corresponding PDF $\alpha g(x)\,G(x)^{\alpha-1}$. A key feature of this family lies in the structure of the PDF, where the exponential term $e^{\beta G(x)}$ introduces a form of exponential weighting dependent on the baseline CDF. Crucially, this exponential component is further modulated by the function $\tau(x)$, which incorporates a logarithmic adjustment of $G(x)$. The LEP acts as a flexible weight function that influences the tail behavior, skewness, and kurtosis of the resulting distribution. By emphasizing the interaction between the exponential generator and the logarithmic polynomial weight $\tau(x)$, this family offers enhanced flexibility over traditional models. It allows for a wide range of hazard rate shapes, including increasing, decreasing, bathtub, and upside-down bathtub forms, making it suitable for applications in reliability analysis, survival modeling, actuarial science, and other fields requiring nuanced data fitting. As $x\to-\infty$, $G(x)\to 0$, $\log\left[1+G(x)^{\alpha}\right]=G(x)^{\alpha}$, $e^{\beta G(x)^{\alpha}}=1+\beta G(x)^{\alpha}$. Then,

$$F(x;\alpha,\beta)\approx CG(x)^{\alpha}\left[1+\beta G(x)^{\alpha}\right]\to 0 \text{ as } x\to-\infty. \tag{3}$$

As $x\to+\infty$, $G(x)\to 1$, $\log\left[1+G(x)^{\alpha}\right]=\log(2)$, $e^{\beta G(x)^{\alpha}}=e^{\beta}$. Then,

$$F(x;\alpha,\beta)\approx C\log(2)\,e^{\beta}=1 \text{ as } x\to+\infty. \tag{4}$$

As $x\to-\infty$, $f(x;\alpha,\beta)\to 0$.As $x\to+\infty$, $f(x;\alpha,\beta)\to 0$. The tail index of the proposed family is the same as that of the baseline distribution, and this can be easily proven mathematically.

2. Properties

In this section, we investigate some mathematical properties of the LEP family. We derive useful expansions for the CDF and PDF, which facilitate the calculation of moments and other statistical measures. The tail behavior of the distribution is examined, showing that the tail index matches that of the baseline distribution. We also explore the quantile function, which is expressed using the Lambert W function, enabling random number generation and calculation of risk measures like VaR and TVaR. Additionally, we derive expressions for incomplete moments and the moment generating function, providing a comprehensive set of tools for statistical analysis based on this family.

2.1. Useful expansions

By expanding $e^{\beta G(x)}$, the new CDF can be expressed as

$$F(x;\alpha,\beta) = C\log\left[1+G(x)^{\alpha}\right]\sum_{k=0}^{+\infty}\frac{\beta^{k}}{k!}G(x)^{\alpha k}, \qquad x\in\mathbb{R}. \tag{5}$$

Then, by expanding $\log\left[1+G(x)^{\alpha}\right]$, we have

$$\log\left[1+G(x)^{\alpha}\right] = \sum_{j=1}^{+\infty}\frac{(-1)^{1+j}}{j!}G(x)^{\alpha j} \tag{6}$$

Inserting (6) into (5), the new CDF can be simplified as

$$F(x;\alpha,\beta) = \sum_{k=0}^{+\infty}\sum_{j=1}^{+\infty} d_{k,j}W_{k,j}(x), \qquad x\in\mathbb{R}, \tag{7}$$

where

$$d_{k,j} = C\frac{1}{k!j!}(-1)^{1+k}\beta^{k},$$

and $W_{k,j}(x) = [G(x)]^{(k+j)\alpha}$ refers to the CDF of the exponentiated G family.By differentiating (7), we have

$$f(x;\alpha,\beta) = \sum_{k=0}^{+\infty}\sum_{j=1}^{+\infty} d_{k,j}w_{k,j}(x), \qquad x\in\mathbb{R}, \tag{8}$$

where

$$w_{k,j}(x) = dW_{k,j}(x)/dx = \alpha(k+j)\,g(x)\,[G(x)]^{(k+j)\alpha-1},$$

which refers to the PDF of the exponentiated G family. To summarize, we say that equation (8) can be used to derive most of the mathematical properties of the underlying distribution to be studied.

2.2. Quantile function

The quantile function (QF) of X can be determined by inverting $F(x) = u$ in (1), where

$$ue^{\beta}\log(2) = \log\left[1+G(x)^{\alpha}\right]e^{\beta G(x)^{\alpha}},$$

let $G(x)^{\alpha} = y$, then

$$ue^{\beta}\log(2) = \log(1+y)\,e^{\beta y},$$

For small values of y y, we can approximate $\log(1+y)\approx y$, so

$$u\beta e^{\beta}\log(2) = \beta y e^{\beta y},$$

Let $z=\beta y$, then using Lambert $W[.]$

$$\begin{aligned}
u\beta e^{\beta}\log(2) &= ze^{z}\\
&\Rightarrow\ z = W\left(u\beta e^{\beta}\log(2)\right)\\
&\Rightarrow\ \beta y = W\left[u\beta e^{\beta}\log(2)\right]\\
&\Rightarrow\ y = \frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right]\\
&\Rightarrow\ G(x)^{\alpha} = \frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right]\\
&\Rightarrow\ G(x) = \left\{\frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right]\right\}^{\frac{1}{\alpha}}.
\end{aligned}$$

Finally,

$$x_u = G^{-1}\left\{\frac{1}{\beta}W\left[u\beta e^{\beta}\log(2)\right]\right\}^{\frac{1}{\alpha}}. \tag{9}$$

2.3. Moments

Let $Y_{k,j}$ be a rv having density $w_{k,j}(x)$. The r^{th} ordinary moment of X, say μ'_r, follows from (8) as

$$\mu'_r = E\left(X^r\right) = \sum_{k=0}^{+\infty}\sum_{j=1}^{+\infty}\left[d_{k,j}\; E\left(Y^r_{(k+j)\alpha}\right)\right], \tag{10}$$

where

$$E(Y^r_{(k+j)\alpha}) = (k+j)\,\alpha\int_{-\infty}^{\infty} x^r\; g(x)\,G(x)^{(k+j)\alpha-1}\,dx$$

can be evaluated numerically in terms of the baseline qf

$$Q_G(u) = G^{-1}(u)$$

as

$$E(Y^n_{(k+j)\alpha}) = (k+j)\,\alpha\int_0^1 Q_G(u)^n\; u^{(k+j)\alpha-1}du.$$

Setting $r=1$ in (10) gives the mean of X.

2.4. Incomplete moments

The r^{th} incomplete moment of X is given by

$$m_r(y) = \int_{-\infty}^{y} x^r\, f\left(x;\alpha,\beta\right)dx.$$

Using (8), the r^{th} incomplete moment of LEP family is

$$m_r(y) = \sum_{k=0}^{+\infty}\sum_{j=1}^{+\infty}\left[d_{k,j}\; m_{r,(k+j)\alpha}(y)\right], \tag{11}$$

where

$$m_{r,(k+j)\alpha}(y) = \int_0^{G(y)} Q^r_G(u)\,u^{(k+j)\alpha-1}\,du.$$

The $m_{r,(k+j)\alpha}(y)$ can be calculated numerically by using the software such as **Matlab**, **R**, **Mathematica** etc.

2.5. Moment generating function

The moment generating function (MGF) of X, say $M(t)=E(e^{t\,X})$, is obtained from (8) as

$$M(t) = \sum_{k=0}^{+\infty}\sum_{j=1}^{+\infty}\left[d_{k,j}\; M_{(k+j)\alpha}\left(t\right)\right],$$

where $M_{(k+j)\alpha}(t)$ is the generating function of $Y_{(k+j)\alpha}$ given by

$$\begin{aligned} M_{(k+j)\alpha}(t) &= (k+j)\,\alpha\int_{-\infty}^{\infty} e^{t\,x}g(x)G(x)^{(k+j)\alpha-1}\,dx \\ &= (k+j)\,\alpha\int_0^1 \exp[t\,Q_G(u;(k+j)\,\alpha\,)]\,u^{(k+j)\alpha-1}du. \end{aligned}$$

The last two integrals can be computed numerically for most parent distributions.

3. Characterizations

This section deals with various characterizations of the proposed distribution. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) reverse hazard function and (iii) conditional expectation of certain function of the random variable. It should be mentioned that for the characterization (i) the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

3.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the LEP family, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem 1 below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right] \eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right) q_1\left(u\right) - q_2\left(u\right)}\right| \exp\left(-s\left(u\right)\right)\, du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_1}{\eta\, q_1\, -\, q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark 3.1.1. The goal is to have $\eta\left(x\right)$ as simple as possible.

Proposition 3.1.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1\left(x\right) = \left[\tau\left(x\right)\right]^{-1}$ and $q_2\left(x\right) = q_1\left(x\right) e^{\beta G(x)^{\alpha}}$ for $x \in \mathbb{R}$. The random variable X has pdf (2) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right) = \frac{1}{2}\left\{e^{\beta} + e^{\beta G(x)^{\alpha}}\right\}, \quad x \in \mathbb{R}.$$

Proof. Let X be a random variable with pdf (2), then

$$\begin{aligned}\left(1 - F\left(x\right)\right) E\left[q_1\left(X\right) \mid X \geq x\right] &= \int_x^{\infty} C\alpha g\left(u\right) G\left(u\right)^{\alpha-1} e^{\beta G(x)^{\alpha}} du \\ &= \frac{C}{\beta}\left\{e^{\beta} - e^{\beta G(x)^{\alpha}}\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}\left(1 - F\left(x\right)\right) E\left[q_2\left(X\right) \mid X \geq x\right] &= \int_x^{\infty} C\alpha g\left(u\right) G\left(u\right)^{\alpha-1} e^{2\beta G(x)^{\alpha}} du \\ &= \frac{C}{2\beta}\left\{e^{2\beta} - e^{2\beta G(x)^{\alpha}}\right\}, \qquad x \in \mathbb{R},\end{aligned}$$

and finally

$$\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)=\frac{q_{1}\left(x\right)}{2}\left\{e^{\beta}-e^{\beta G(x)^{\alpha}}\right\}>0\quad for\ \ x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'\left(x\right)=\frac{\eta'\left(x\right)q_{1}\left(x\right)}{\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)}=\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}},\quad x\in\mathbb{R},$$

and hence

$$s\left(x\right)=-\log\left\{e^{\beta}-e^{\beta G(x)^{\alpha}}\right\},\qquad x\in\mathbb{R}.$$

Now, in view of Theorem 1, X has density (2).

Corollary 3.1.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1\left(x\right)$ be as in Proposition 3.1.1. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'\left(x\right)q_{1}\left(x\right)}{\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)}=\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}},\quad x\in\mathbb{R}.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 3.1.1 is

$$\eta\left(x\right)=\left\{e^{\beta}-e^{\beta G(x)^{\alpha}}\right\}^{-1}\left[-\int\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right)dx+D\right],$$

where D is a constant.

Proof. If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'\left(x\right)=\left(\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}}\right)\eta\left(x\right)-\left(\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}}\right)\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right),$$

or

$$\eta'\left(x\right)-\left(\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}}\right)\eta\left(x\right)$$
$$=-\left(\frac{\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}}{e^{\beta}-e^{\beta G(x)^{\alpha}}}\right)\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right),$$

or

$$\frac{d}{dx}\left\{\left(e^{\beta}-e^{\beta G(x)^{\alpha}}\right)\eta\left(x\right)\right\}=-\left(\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}\right)\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right),$$

from which we arrive at

$$\eta\left(x\right)=\left\{e^{\beta}-e^{\beta G(x)^{\alpha}}\right\}^{-1}\left[-\int\alpha\beta g\left(x\right)G\left(x\right)^{\alpha-1}e^{\beta G(x)^{\alpha}}\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right)dx+D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 3.1.1, is given in Proposition 3.1.1 with $D=\frac{e^{2\beta}}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

3.2. Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \;\; x \in support\ of\ F.$$

In this subsection we present characterizations of five distributions in terms of the reverse hazard function.

Proposition 3.2.1. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x) - (\alpha - 1)\frac{g(x)}{G(x)} r_F(x) = \alpha G(x)^{\alpha-1} \frac{d}{dx}\left\{\frac{g(x)\, e^{\beta G(x)^\alpha} \tau(x)}{\log\left[1 + G(x)^\alpha\right]}\right\}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = \frac{\alpha \lim_{x\to\infty} g(x) e^{\beta}(1+2\beta\log(2))}{2\log(2)}$.

Proof. Multiplying both sides of the above equation by $G(x)^{-(\alpha-1)}$, we have

$$\frac{d}{dx}\left\{G(x)^{-(\alpha-1)} r_F(x)\right\} = \alpha \frac{d}{dx}\left\{\frac{g(x)\, e^{\beta G(x)^\alpha} \tau(x)}{\log\left[1 + G(x)^\alpha\right]}\right\},$$

or

$$r_F(x) = \alpha G(x)^{\alpha-1} \left\{\frac{g(x)\, e^{\beta G(x)^\alpha} \tau(x)}{\log\left[1 + G(x)^\alpha\right]}\right\},$$

which is the reverse hazard function corresponding to the pdf (2).

3.3. Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize the LEP distribution.

Proposition 3.2.1. Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F . Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x\to f^-} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E\left[\psi(X) \mid X \le x\right] = \delta\psi(x), \qquad x \in (e, f)$$

implies

$$\psi(x) = (F(x))^{\frac{1}{\delta}-1}. \qquad x \in (e, f)$$

Remarks 3.2.1. For the LEP we have $(e, f) = \mathbb{R}$, $\psi(x) = C^{1/\beta}\left(\log\left[1 + G(x)^\alpha\right]\right)^{1/\beta} e^{G(x)^\alpha}$ and $\delta = \frac{\beta}{\beta+1}$.

4. The LEP Weibull case

The LEP Weibull distribution is derived by substituting the baseline cumulative distribution function (CDF) of the Weibull distribution into the general LEP-G framework. The Weibull distribution, known for its versatility in modeling time-to-event data, is defined by the CDF $G(x) = 1 - e^{-x^\lambda}$. Then, the CDF OF THE LEP Weibull can be expressed as

$$F(x; \alpha, \beta) = C \log\left\{1 + \left[1 - e^{-x^\lambda}\right]^\alpha\right\} e^{\beta\left[1-e^{-x^\lambda}\right]^\alpha}, \qquad x > 0. \tag{12}$$

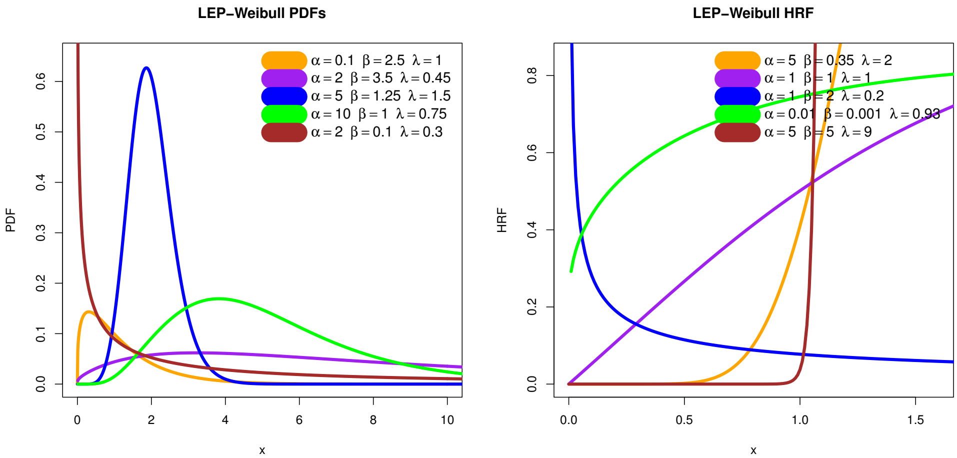

Figure 1. Plots of the new LEP Weibull PDF (right) and HRF (left) for selected values of the parameter.

The new PDF can be easily derivedby differentiating (12). The LEP Weibull model can accommodate various hazard rate behaviors, making it suitable for modeling complex real phenomena such as equipment failure rates, insurance claim arrivals, and survival times in medical studies. The model excels in capturing heavy-tailed and asymmetric data, which are common in insurance claims and financial risk modeling. The additional parameters α and β allow for greater control over tail thickness and skewness.The LEP Weibull distribution provides closed-form expressions for moments, quantile functions, and entropy measures. This is particularly advantageous for analytical work and computational efficiency.The quantile function is derived using the Lambert W function, enabling the generation of random samples and the calculation of risk measures such as VaR and TVaR. Moments are expressed in terms of infinite series expansions, facilitating the computation of expected values and higher-order statistics.

Figure 1 presents plots of the LEP Weibull PDF and hazard rate function (HRF) for selected parameter values. These plots demonstrate the model's ability to adapt to different shapes, from highly skewed to symmetric distributions, and from monotonic to bathtub-shaped hazard rates.

Figure 1 presents some plots of the new LEP Weibull PDF (right) and HRF (left) for selected values of the parameter. Based on Figure 1 (right) , which displays the HRF of the LEP Weibull distribution for selected parameter values, we observe that the model exhibits a wide range of hazard rate shapes, including increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped patterns. This flexibility highlights the LEP Weibull distribution's ability to adapt to diverse real scenarios, particularly in reliability engineering and survival analysis, where failure rates may not follow a strictly monotonic trend. The variation in the HRF with different combinations of the parameters α, β, and λ demonstrates the model's sensitivity and versatility in capturing complex failure behaviors, making it a valuable tool for modeling lifetime data and extreme events. Based on Figure 1 (left) , which displays the probability density function (PDF) of the LEP Weibull distribution for selected parameter values, we observe that the model exhibits a wide variety of density shapes, including symmetric, right-skewed, left-skewed, and heavy-tailed forms. This flexibility is influenced by the parameters α , β , and λ , where α controls the overall shape, β affects tail weight and skewness, and λ determines the spread. The LEP Weibull distribution is particularly effective in modeling asymmetric and heavy-tailed data , making it suitable for applications in insurance, finance, and reliability analysis . The ability to capture complex distributional shapes enhances its utility in fitting real data with non-standard behavior. This visual representation confirms the model's adaptability and reinforces its value in risk modeling, survival analysis, and extreme event prediction. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)).

5. Simulations for assessing estimation methods under the LEP Weibull case

Section 5 delves into the practical implications of our proposed family by focusing on risk analysis. Understanding the behavior of statistical models under different estimation paradigms is crucial for real applications, particularly in fields like finance and insurance where accurate risk assessment is paramount. Here, we shift our focus from theoretical properties to evaluating how well our model performs in estimating key risk metrics. We utilize the LEP Weibull distribution as a specific example within our family to demonstrate this analysis. This section aims to illustrate the impact of choosing different estimation methods on the resulting risk evaluations. We employ several estimation techniques. To achieve this, we analyze real insurance claims data, a domain where precise risk measurement directly influences financial decisions. The performance of the estimation methods is assessed using key risk indicators such as VaR and TVaR. These metrics are fundamental tools for quantifying potential losses and understanding the tail behavior of loss distributions. Furthermore, we examine the performance of these methods using simulated, or artificial, data. This dual approach allows us to observe the methods' behavior under controlled conditions, where the true parameters are known, as well as in a realistic scenario with actual data complexities. By comparing the estimated risk measures derived from different estimation techniques, we gain insights into their relative strengths and weaknesses in practical settings. This analysis helps identify which methods might be more suitable for specific risk management tasks, considering factors like bias and sensitivity to the distribution's tail. The results presented in this section are intended to guide practitioners in selecting appropriate estimation strategies for risk modeling. Ultimately, this section underscores the importance of methodological choices in statistical modeling and their direct consequences on risk assessment, providing a valuable perspective for applied statisticians and risk analysts working with complex datasets. The findings contribute to a deeper understanding of the LEP family's utility in risk analysis. We begin by detailing the analysis using the real insurance data. Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study. Table 1, Table 3 and Table 3 present a comprehensive simulation study assessed the performance of these estimators, evaluating bias, Root Mean Square Error (RMSE), and Anderson-Darling distance metrics (Dabs, Dmax) across different sample sizes and parameter settings.

Table 1 below presents the simulation results for the LEP Weibull model when the true parameters are α =2, β =0.5, and λ =1.5. The results are displayed for different sample sizes (n =20, 50, 100, 300) and various estimation methods (MLE, CVM, ADE, RTADE, LTADE). As expected, the performance of all methods improves with increasing sample size, evidenced by decreasing bias and RMSE values. For small samples (n =20), the bias and RMSE are relatively high for all parameters and methods, indicating the challenges of estimation with limited data. The MLE generally shows moderate performance across metrics for small samples. RTADE exhibits the highest bias and RMSE for α and β at n =20, suggesting less accuracy for these parameters with this method initially. LTADE shows the lowest bias for α at n =20 but has higher RMSE for λ. CVM and ADE display intermediate performance. As the sample size grows to 300, all methods show significantly reduced bias and RMSE, approaching zero. LTADE demonstrates very low bias for all parameters at n =300. The Dabs and Dmax distances also decrease with larger samples, indicating better overall distributional fit. RTADE shows the largest Dmax at n =20, while LTADE has the smallest Dmax at n =300.

However, Table 2 shows simulation results for a different set of true parameters: α =1.2, β =0.9, and λ =0.5. Similar to Table 1, the evaluation covers sample sizes from 20 to 300 and the same five estimation methods. The general trend of improving performance with larger sample sizes holds for this parameter set as well. However, the magnitude of bias and RMSE differs compared to Table 1, reflecting the influence of the true parameter values. For n =20, biases are generally larger than in Table 1, particularly for β and λ, suggesting these parameters might be harder to estimate accurately for this specific configuration. RTADE again shows relatively high bias for α and β at the smallest sample size. LTADE exhibits negative bias for λ across most sample sizes, which is distinct from Table 1. The RMSE patterns are consistent, decreasing with sample size for all methods. MLE performs reasonably well for λ in terms of RMSE even at n =20. CVM shows relatively stable performance for β across sample sizes. ADE tends to have lower bias for β compared to other methods at n =20. At n =300, all methods demonstrate low bias and RMSE, confirming consistency. The Dabs and Dmax distances are generally smaller than in Table 1 for the larger sample sizes, indicating potentially better fit for this parameter set. LTADE shows competitive performance for the distributional distance metrics at larger samples.

Finally, Table 3 presents simulation results for the LEP Weibull model with true parameters α =5, β =1.5, and λ =0.9, representing a third parameter configuration. The evaluation follows the same methodology as Tables 1 and 2, examining sample sizes from 20 to 300 across five estimation methods. The results continue to show the expected improvement in estimation accuracy as sample size increases, with bias and RMSE values decreasing toward zero. For small samples (n =20), the estimation challenges are again evident, with relatively high bias and RMSE values across all methods and parameters. RTADE consistently shows higher bias and RMSE for α and β at n =20 compared to other methods, reinforcing the pattern seen in previous tables. LTADE demonstrates mixed performance at n =20, showing low bias for α but higher RMSE for β and λ. MLE maintains moderate performance across all parameters even with small samples, providing reliable baseline estimates. The convergence behavior as sample size increases is clear, with all methods achieving negligible bias and RMSE at n =300. LTADE achieves particularly low bias for α and β at the largest sample size, demonstrating strong parameter recovery capabilities. ADE shows consistently good performance for λ estimation across all sample sizes, with relatively low bias and RMSE. CVM exhibits stable behavior for β estimation, maintaining consistent performance metrics as sample size increases. The distributional distance measures (Dabs and Dmax) decrease with larger samples, indicating improved overall fit. RTADE shows the largest Dmax at n =20, while LTADE achieves the smallest Dmax at n =300, consistent with findings from Tables 1 and 2. The results across all three tables demonstrate the robustness and consistency of the estimation methods for different parameter configurations.

Table 1: Simulation results for parameter $\alpha = 2, \beta = 0.5$ and $\lambda = 1.5$.

	n	BIAS α	BIAS β	BIAS λ	RMSE α	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.123721	0.060619	0.043527	0.29538	0.195935	0.058849	0.019550	0.033167
CVM		0.111547	0.06021	0.019834	0.388502	0.206093	0.13629	0.019953	0.03114
ADE		0.081809	0.056636	-0.044375	0.302726	0.190870	0.06322	0.021967	0.031712
RTADE		0.170766	0.099887	0.003148	0.55673	0.261661	0.087528	0.033664	0.049528
LTADE		0.056585	0.030683	-0.072305	0.276289	0.194571	0.092197	0.017734	0.029649
MLE	50	0.019503	0.021802	0.041722	0.079889	0.018948	0.100458	0.006451	0.011588
CVM		0.043414	0.021557	0.014215	0.140488	0.082565	0.043942	0.0072	0.01204
ADE		0.029164	0.018331	-0.011083	0.119277	0.079502	0.025109	0.007157	0.010149
RTADE		0.057531	0.03093	0.008927	0.17229	0.090552	0.03023	0.01060	0.016354
LTADE		0.020715	0.010043	-0.031122	0.108662	0.082232	0.035695	0.00669	0.011687
MLE	100	0.015178	0.014694	0.032206	0.037762	0.009216	0.048733	0.005076	0.008923
CVM		0.027278	0.014747	0.008794	0.06289	0.038041	0.020941	0.004749	0.007855
ADE		0.021731	0.014103	-0.002796	0.05574	0.037572	0.012522	0.004949	0.007057
RTADE		0.032174	0.017549	0.005524	0.079231	0.042867	0.015249	0.005983	0.009264
LTADE		0.019673	0.012771	-0.009878	0.050594	0.038243	0.015491	0.005109	0.007314
MLE	300	0.009061	0.004429	0.004044	0.015573	0.011926	0.003015	0.001465	0.00256
CVM		0.00454	0.001802	0.001771	0.019685	0.012187	0.006366	0.000687	0.001192
ADE		0.006237	0.003991	-0.001021	0.017429	0.011876	0.004169	0.001431	0.002036
RTADE		0.007774	0.004241	0.000143	0.023342	0.012983	0.004536	0.001553	0.002281
LTADE		0.001285	-0.000232	-0.005869	0.016409	0.01268	0.005206	0.000793	0.001608

Table 2: Simulation results for parameter $\alpha = 1.2, \beta = 0.9$ and $\lambda = 0.5$.

	n	BIAS α	BIAS β	BIAS λ	RMSE α	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.065205	0.054032	0.019255	0.114043	0.275488	0.006681	0.017179	0.030292
CVM		0.056378	0.057218	-0.009154	0.132504	0.279289	0.012067	0.018797	0.026374
ADE		0.04213	0.050997	-0.022788	0.110838	0.25751	0.00747	0.017325	0.027677
RTADE		0.103132	0.121576	0.001581	0.258711	0.494132	0.011978	0.033274	0.049323
LTADE		0.031893	0.034091	-0.018195	0.114358	0.274196	0.013603	0.012723	0.020766
MLE	50	0.027386	0.022843	0.004188	0.041744	0.109194	0.00224	0.007471	0.012249
CVM		0.025032	0.025976	-0.000663	0.055078	0.11763	0.00454	0.00805	0.011657
ADE		0.018956	0.02197	-0.007797	0.049379	0.114236	0.002984	0.007343	0.010996
RTADE		0.035893	0.040805	0.003157	0.068527	0.138966	0.004485	0.011396	0.017612
LTADE		0.005495	0.000562	-0.008064	0.042246	0.106162	0.005341	0.002885	0.006011
MLE	100	0.01616	0.01240	0.004128	0.019741	0.051598	0.001144	0.004251	0.007429
CVM		0.017717	0.021741	-0.002039	0.02217	0.04860	0.00207	0.00633	0.008953
ADE		0.013643	0.019207	-0.004775	0.019394	0.046136	0.001418	0.005606	0.008103
RTADE		0.023857	0.030017	-0.001048	0.026697	0.055495	0.001798	0.008378	0.012103
LTADE		0.00761	0.008342	-0.003795	0.020422	0.052161	0.002607	0.002968	0.004655
MLE	300	0.010456	0.012071	0.001356	0.005942	0.016054	0.000374	0.003327	0.005287
CVM		0.007677	0.009452	0.000753	0.007512	0.016635	0.000689	0.002554	0.00396
ADE		0.006803	0.009059	-0.000159	0.006781	0.016218	0.000478	0.002454	0.003568
RTADE		0.008088	0.009774	0.000848	0.009181	0.019331	0.000652	0.002661	0.004145
LTADE		0.005832	0.008071	-0.000612	0.006172	0.015818	0.000835	0.002198	0.003124

Table 3: Simulation results for parameter $\alpha = 5, \beta = 1.5$ and $\lambda = 0.9$.

	n	BIAS α	BIAS β	BIAS λ	RMSE α	RMSE β	RMSE λ	Dabs	Dmax
MLE	20	0.333295	0.106541	0.012705	2.5858	0.498103	0.005623	0.015824	0.026893
CVM		0.261741	0.089148	0.012555	2.940441	0.537328	0.009491	0.011595	0.020535
ADE		0.220073	0.080447	0.00338	2.42529	0.460211	0.006715	0.015132	0.023003
RTADE		0.386337	0.143745	0.00277	3.91218	0.707178	0.006423	0.028525	0.042591
LTADE		0.195312	0.070001	0.008968	2.138973	0.412142	0.011979	0.009278	0.016201
MLE	50	0.136255	0.047038	0.006145	0.828712	0.163787	0.002175	0.006481	0.011276
CVM		0.171233	0.065722	0.000518	0.959027	0.180457	0.003157	0.013648	0.020205
ADE		0.087400	0.031377	0.002247	0.874328	0.170442	0.002691	0.00543	0.008555
RTADE		0.210519	0.082405	-0.001413	1.188876	0.221892	0.002401	0.018336	0.026747
LTADE		0.090277	0.033395	0.003781	0.849271	0.167027	0.004098	0.004711	0.008012
MLE	100	0.050596	0.014354	0.003284	0.359404	0.075108	0.001011	0.001728	0.003302
CVM		0.067272	0.024294	0.001802	0.499338	0.094947	0.001666	0.004144	0.006576
ADE		0.023763	0.006979	0.00156	0.381908	0.075082	0.001228	0.000828	0.001581
RTADE		0.082748	0.031193	0.000364	0.599148	0.112982	0.001227	0.006509	0.009671
LTADE		0.026816	0.008546	0.002904	0.364527	0.07296	0.001847	0.000947	0.001454
MLE	300	0.021069	0.007329	0.000661	0.119837	0.025199	0.000358	0.001211	0.001965
CVM		0.003798	0.000171	0.001441	0.141260	0.027242	0.000499	0.000819	0.001417
ADE		0.013442	0.00503	0.000071	0.131901	0.026135	0.000427	0.001051	0.001566
RTADE		0.006836	0.001546	0.000906	0.163156	0.03129	0.000366	0.000302	0.000572
LTADE		0.007754	0.002569	0.000609	0.124901	0.025195	0.000633	0.000268	0.000513

The simulation studies across Tables 1, 2, and 3 demonstrate that all five estimation methods for the LEP Weibull model are consistent, with performance improving significantly as sample size increases from 20 to 300. The choice of true parameters affects the magnitude of bias and RMSE, indicating that some parameter configurations are inherently more challenging to estimate accurately, especially with small samples. No single estimation method uniformly outperforms the others across all scenarios, metrics, and parameter sets, suggesting that method selection should consider specific application requirements. RTADE consistently shows higher bias and RMSE for certain parameters at small sample sizes, while LTADE often demonstrates competitive performance, particularly at larger sample sizes. MLE provides reliable and moderate performance across different parameter configurations, serving as a robust baseline method. The decreasing pattern of distributional distance measures (Dabs and Dmax) with increasing sample size confirms the methods' ability to recover the true distribution accurately. These comprehensive simulation results provide valuable guidance for practitioners in selecting appropriate estimation methods based on sample characteristics and desired accuracy levels. The findings underscore the importance of adequate sample sizes for reliable parameter estimation in the LEP Weibull model.

It is also noted that the MLE minimizes negative log-likelihood, prioritizing overall fit; CVM minimizes integrated squared distance, balancing global fit. ADE weights distribution tails more heavily than CVM, making it sensitive to extreme values; RTADE focuses only on the right tail, ideal for skewed loss data. LTADE incorporates length-biased sampling logic, amplifying sensitivity to large observations, perfect for conservative VaR/TVaR estimation. Simulations show MLE is most stable for small samples ($n < 50$), while LTADE and ADE excel in tail accuracy but need n¿100 for stability. RTADE often underperforms in tail risk despite its name, likely due to overfitting the extreme tail at the expense of body fit. For regulatory capital, use LTADE (if $n > 100$); for general modeling, use MLE or ADE. Avoid RTADE unless tail-only focus is explicitly required and sample size is very large. CVM offers a good middle ground for moderate samples. A practitioner's choice should align with their risk objective: overall fit (MLE), balanced tail sensitivity (ADE), or extreme conservatism (LTADE). Sample size alone is insufficient guidance, estimation purpose must drive selection. Table:$n < 50 \rightarrow$ MLE; $50 < n < 100$ $\rightarrow$ ADE/CVM; $n > 100$ $\rightarrow$LTADE (for tail risk), MLE (for stability). This transforms arbitrary comparison into actionable.

6. Risk analysis under artificial data and LEP Weibull case

Getting model parameters right is crucial when using statistical models in practice, especially in fields where risk assessment and prediction accuracy drive important decisions (see Mansour et al., 2020e; Ibrahim et al., 2020). Because different situations might call for different estimation methods, researchers have explored various techniques to see which ones work best under specific conditions (Hashem et al., 2024; Yousof et al., 2025a). For instance, Yousof et al. (2025a) looked at several methods within the generalized gamma distribution framework, figuring out how each method's performance changes based on the data's characteristics and whether some observations are censored. Similarly, Ibrahim et al. (2025a, 2025b) examined how to best estimate parameters for reciprocal Weibull models used in medical and reliability research. Risk analysis has gained significant importance, particularly in actuarial science and finance, where tools like VaR, TVaR, and KRIs are vital for gauging potential financial losses (Elbatal et al., 2024; Yousof et al., 2024). Researchers like Mohamed et al. (2024) and Ibrahim et al. (2025c) have applied sophisticated estimation methods to analyze risk in insurance data that shows negative skewness and over-dispersion. Elbatal et al. (2024) also introduced a new loss-revenue model using entropy-based analysis to refine VaR and mean-of-order-P assessments. Following these advancements, our current study broadens the use of estimation techniques, like the MLE, CVM, and Bayesian methods, to assess key risk indicators with actual insurance claims data. This deeper dive helps us understand how different estimation strategies influence risk measurement and interpretation, providing practical insights for building better risk models and making informed decisions in complex real settings.

Table 4 below presents the KRIs for the LEP Weibull model fitted to artificial data with a sample size of n =20. The estimated parameters (α, β, λ) show noticeable variation across the five estimation methods, reflecting the inherent uncertainty and sensitivity of parameter estimation with small samples. This parameter variability directly translates into differences in the calculated risk measures. For the 70% quantile, VaR ranges from 1.57 (MLE) to 1.64 (RTADE), indicating modest differences at lower risk levels. The variation becomes more pronounced at higher quantiles, with 90% VaR ranging from 2.14 (MLE, ADE) to 2.26 (LTADE), showing a wider spread in tail risk estimates. Similar patterns are observed for TVaR, TV, and TMV, with substantial differences across methods. RTADE consistently produces higher VaR estimates, while LTADE tends to yield the highest TVaR and TV values at the 90% level. The EL also varies across methods, though the differences are smaller compared to tail risk measures. These results highlight the significant impact of estimation method choice on risk assessment when working with limited data.

Table 5 displays KRIs for the LEP Weibull model with n =50, showing improved consistency compared to the n =20 results in Table 4. The variation in parameter estimates across estimation methods is visibly reduced, indicating better agreement between techniques as sample size increases. The differences in VaR estimates across methods are smaller than in Table 4, with 70% VaR ranging narrowly from 1.58 (RTADE) to 1.59 (LTADE). At the 90% quantile, VaR values range from 2.16 (RTADE) to 2.26 (LTADE), showing reduced spread compared to the smaller sample size results. TVaR and TV metrics also exhibit less variation across methods than observed with n =20. LTADE continues to produce relatively higher estimates for most risk measures, while RTADE provides intermediate values. The EL differences across methods are minimal at this sample size, indicating stable baseline risk assessment. The overall pattern demonstrates the beneficial effect of increased sample size on the stability and consistency of risk estimates across different estimation methods.

Table 6 shows KRIs for the LEP Weibull model with n =100, continuing the trend of improved consistency with larger sample sizes. Parameter estimates across methods show even less variation compared to n =50, indicating further convergence of the estimation techniques toward stable solutions. The range of VaR estimates at the 70% quantile is very narrow, from 1.58 (RTADE) to 1.59 (LTADE), demonstrating close agreement among methods for moderate risk levels. At the 90% quantile, VaR values range from 2.16 (RTADE) to 2.20 (LTADE), maintaining the pattern of reduced spread with increasing sample size. TVaR and TV metrics continue to show diminishing differences across estimation methods, suggesting convergence in tail risk assessment. The EL values remain nearly identical across all methods, indicating highly stable baseline risk measurement. LTADE still produces slightly higher estimates for tail risk measures, but the differences are becoming less pronounced. RTADE maintains its position with intermediate risk estimates, showing consistent behavior. The results reinforce the advantage of larger sample sizes for achieving reliable and consistent risk measurement.

Table 7 presents KRIs for the LEP Weibull model with n =300, representing the largest sample size examined in the artificial data analysis. The parameter estimates across methods show minimal variation, indicating strong convergence and stability of the estimation techniques at this sample size. VaR estimates at the 70% quantile are tightly clustered between 1.58 (RTADE) and 1.58 (LTADE), showing excellent agreement among all methods. At the 90% quantile, VaR values range from 2.16 (RTADE) to 2.17 (LTADE), demonstrating remarkable consistency across methods for high-risk quantiles. TVaR, TV, and TMV metrics show virtually identical values across all estimation techniques, indicating that the sample size is sufficiently large for stable tail risk assessment. The EL values are completely consistent across methods, reflecting the achievement of reliable baseline risk measurement. The differences between LTADE and RTADE estimates have diminished significantly compared to smaller sample sizes, with all methods producing nearly identical risk profiles. All methods converge to nearly identical parameter estimates, confirming the reliability of inference with adequate sample sizes. The results validate the asymptotic properties of the estimation methods and demonstrate their practical utility.

Table 4: KRIs under artificial data for $n = 20$.

Method	$\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}$	VaR(X)	TVaR(X)	TV(X)	TMV(X)	EL(X)
MLE	2.123721, 0.56062, 1.543527					
70%		1.59671	2.07436	0.17587	2.16230	0.47765
80%		1.81857	2.26029	0.15806	2.33933	0.44173
90%		2.14979	2.55152	0.13754	2.62029	0.40172
CVM	2.111547, 0.56021, 1.519834					
70%		1.60550	2.09606	0.18644	2.18928	0.49056
80%		1.83281	2.28717	0.16796	2.37115	0.45436
90%		2.17298	2.58702	0.14664	2.66035	0.41404
ADE	2.081809, 0.55664, 1.455625					
70%		1.63078	2.15959	0.21984	2.26951	0.52880
80%		1.87412	2.36605	0.19942	2.46577	0.49193
90%		2.24077	2.69169	0.17579	2.77958	0.45091
RTADE	2.170766, 0.59989, 1.503148					
70%		1.64140	2.13942	0.19282	2.23583	0.49802
80%		1.87187	2.33351	0.17402	2.42052	0.46165
90%		2.21708	2.63841	0.15238	2.71460	0.42133
LTADE	2.056585, 0.53068, 1.427695					
70%		1.62989	2.17757	0.23756	2.29635	0.54768
80%		1.88100	2.39166	0.21622	2.49977	0.51067
90%		2.26078	2.73020	0.19144	2.82592	0.46942

Table 5: KRIs under artificial data for $n = 50$.

Method	$\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}$	VaR(X)	TVaR(X)	TV(X)	TMV(X)	EL(X)
MLE	2.041722, 0.5195, 1.521802					
70%		1.57388	2.06548	0.18721	2.15908	0.49160
80%		1.80160	2.25701	0.16861	2.34132	0.45541
90%		2.14263	2.55754	0.14709	2.63108	0.41490
CVM	2.043414, 0.52156, 1.514215					
70%		1.57861	2.07429	0.19065	2.16961	0.49568
80%		1.80805	2.26746	0.17184	2.35338	0.45941
90%		2.1519	2.57073	0.15008	2.64577	0.41884
ADE	2.029164, 0.51833, 1.488917					
70%		1.58622	2.09656	0.20326	2.19819	0.51034
80%		1.82180	2.29562	0.18371	2.38748	0.47382
90%		2.17581	2.60878	0.16104	2.68930	0.43297
RTADE	2.057531, 0.53093, 1.508927					
70%		1.5879	2.08607	0.19278	2.18246	0.49817
80%		1.8184	2.28024	0.17386	2.36717	0.46184
90%		2.16394	2.58519	0.15198	2.66118	0.42125
LTADE	2.020715, 0.51004, 1.468878					
70%		1.59095	2.11347	0.21412	2.22053	0.52252
80%		1.83158	2.31745	0.19397	2.41443	0.48586
90%		2.19405	2.63888	0.17056	2.72416	0.44483

Table 6: KRIs under artificial data for $n = 100$.

Method	$\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}$	VaR(X)	TVaR(X)	TV(X)	TMV(X)	EL(X)
MLE	2.032206, 0.51518, 1.514694					
70%		1.57339	2.06914	0.19069	2.16449	0.49575
80%		1.80286	2.26235	0.17188	2.34828	0.45949
90%		2.14678	2.56567	0.15008	2.64071	0.41889
CVM	2.027278, 0.51475, 1.508794					
70%		1.57484	2.07396	0.19355	2.17074	0.49912
80%		1.80572	2.26852	0.17456	2.35580	0.46280
90%		2.15198	2.57411	0.15256	2.65039	0.42213
ADE	2.021731, 0.51410, 1.497204					
70%		1.57879	2.08458	0.19927	2.18421	0.50578
80%		1.81247	2.08458	0.17994	2.37178	0.46935
90%		2.16334	2.59189	0.15754	2.67066	0.42855
RTADE	2.032174, 0.51755, 1.505524					
70%		1.57861	2.07941	0.19500	2.17691	0.50081
80%		1.81020	2.27465	0.17593	2.36261	0.46445
90%		2.15761	2.58137	0.15383	2.65829	0.42376
LTADE	2.019673, 0.51277, 1.490122					
70%		1.58122	2.09114	0.20288	2.19258	0.50992
80%		1.81661	2.29003	0.18334	2.38171	0.47342
90%		2.17036	2.60291	0.16068	2.68325	0.43255

Table 7: KRIs under artificial data for $n = 300$.

Method	$\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}$	VaR(X)	TVaR(X)	TV(X)	TMV(X)	EL(X)
MLE	2.009061, 0.50443, 1.504044					
70%		1.56885	2.07117	0.19628	2.169310	0.50232
80%		1.80106	2.26702	0.17711	2.35558	0.46596
90%		2.14958	2.57477	0.15488	2.65221	0.42520
CVM	2.00454, 0.5018, 1.501771					
70%		1.56783	2.07157	0.19749	2.17032	0.50374
80%		1.80063	2.26799	0.17825	2.35712	0.46736
90%		2.15013	2.57670	0.15593	2.65466	0.42656
ADE	2.002927, 0.50167, 1.497118					
70%		1.56958	2.07599	0.19981	2.17589	0.50641
80%		1.80349	2.27349	0.18043	2.36370	0.46999
90%		2.15485	2.58400	0.15795	2.66297	0.42915
RTADE	2.006237, 0.50399, 1.498979					
70%		1.57030	2.07486	0.19820	2.17396	0.50455
80%		1.80344	2.2716	0.17892	2.36106	0.46816
90%		2.15351	2.58086	0.15655	2.65914	0.42735
LTADE	2.001285, 0.49977, 1.494131					
70%		1.56987	2.07805	0.20135	2.17873	0.50818
80%		1.80452	2.27626	0.18189	2.36721	0.47175
90%		2.15711	2.58798	0.15929	2.66763	0.43087

The comprehensive risk analysis results presented in Tables 4, 5, 6, and 7 provide crucial insights into how estimation method choice affects risk measurement for the LEP Weibull model across different sample sizes.

With small samples (n =20), substantial variation exists in parameter estimates and resulting risk indicators across the five estimation methods (MLE, CVM, ADE, RTADE, LTADE), highlighting the challenges of reliable risk assessment with limited data. The differences in VaR and TVaR can be meaningful, with variations of up to 10% between methods at high quantiles. As sample size increases from 20 to 50 to 100 to 300, there is a clear and consistent pattern of convergence among the estimation methods, with risk estimates becoming increasingly similar. RTADE tends to produce higher estimates for upper quantile VaR across sample sizes, while LTADE often yields the highest tail risk measures, though these differences diminish with larger samples. The stability of EL estimates improves rapidly with sample size, showing minimal variation even at n =50. The results emphasize that adequate sample sizes are essential for reliable risk assessment and that practitioners should be cautious when interpreting risk measures derived from small datasets. The findings provide practical guidance for risk analysts regarding the impact of sample size on estimation consistency and the potential variability in risk measures due to method selection. The analysis demonstrates that while different estimation methods may yield varying results with small samples, they converge to consistent estimates with sufficient data, validating their asymptotic properties. This convergence pattern is reassuring for practitioners, as it suggests that with adequate data, the choice of estimation method becomes less critical for risk measurement. The study underscores the importance of understanding the relationship between sample size, estimation methodology, and risk assessment reliability in practical applications. Overall, the results highlight the LEP Weibull model's robustness and the predictable improvement in estimation quality with increasing sample size, providing valuable insights for risk modeling in insurance and finance contexts.

7. Validating the LEP Weibull for risk analysis under insurance claims data

In practical settings, especially within insurance, reliability engineering, and survival analysis, recently developed statistical models have shown great promise in tackling complex data issues like skewness, heavy tails, and over-dispersion (Hamed et al., 2022; Mohamed et al., 2024). These data characteristics are common, particularly in actuarial science, where standard models often struggle to fully capture the intricacies of claim distributions and risk profiles. Within insurance, there's been a growing emphasis on creating models that precisely reflect the unique aspects of claim sizes, especially those showing negative skewness or extreme variations. For example, Hamed et al. (2022) created a compound Lomax model aimed specifically at handling negatively skewed insurance data, a pattern seen in certain risk categories. Likewise, Mohamed et al. (2024) put forth a size-of-loss model using advanced estimation methods to enhance the precision of risk evaluations in actuarial work. These models offer more dependable tools for pricing, setting reserves, and allocating capital by more accurately representing the tail risks and variability inherent in insurance losses. Concurrently, Yousof et al. (2024) presented a discrete model designed for the highly variable claim counts typical in automobile insurance. This model improves the ability to handle over-dispersed frequency data, which is crucial for effective reinsurance strategies and setting premium rates. Using such models helps insurers better predict the frequency and size of claims, leading to improved risk management and financial stability. In the field of reliability engineering, the use of flexible statistical models has also become more prominent. Mansour et al. (2020f) investigated a two-parameter Burr XII distribution for analyzing survival times in acute bone cancer patients, showing its effectiveness in modeling medical failure times. Yousof et al. (2025b) built on this by proposing a weighted Lindley model, particularly adept at handling extreme historical insurance claims and other lifetime data. These models lead to more accurate forecasts and better decisions in both industrial and medical reliability studies.

Survival analysis continues to be fundamental in biomedical research, focusing on how covariates influence the time until an event occurs. Accelerated Failure Time (AFT) models are popular for this task because they are straightforward to interpret and adaptable. Abonongo et al. (2025) crafted an AFT model specifically for colon cancer data, incorporating empirical methods to ensure its robustness and broad applicability. At the same time, Khedr et al. (2025) introduced an innovative AFT model that works well in both engineering and medical contexts, providing a unified way to analyze failure and survival times across different areas. This paper adds to the existing research by presenting a new AFT model that uses improved validation and estimation methods. The model was thoroughly tested on real data from various fields, proving its adaptability and success in capturing intricate data patterns. By overcoming the shortcomings of older models and offering greater flexibility, this research pushes forward the use of statistical modeling in risk assessment, reliability analysis, and survival prediction, supporting better decision-making in vital areas like healthcare, finance, and engineering. Ensuring a proposed statistical model fits the data well is a critical step before applying it to real-world problems. Recent research has concentrated on improving goodness-of-fit tests designed for both censored and complete data sets (Ibrahim et al., 2020; Mansour et al., 2020e). Widely used tests include the Bagdonavičius-Nikulin test, the modified Nikulin-Rao-Robson test, and chi-squared type tests (Goual & Yousof, 2020; Shehata et al., 2024). These tests have been effectively used in various fields like insurance, reliability engineering, and biomedical research. For instance, Goual and Yousof (2020) used a modified chi-squared test to validate the Burr XII inverse Rayleigh model, while Yousof et al. (2023) applied similar validation techniques to bimodal, heavy-tailed Burr XII models. Furthermore, Yousof et al. (2025b) showed how effective these tests are when analyzing historical insurance claims using generalized gamma distributions. Additional work by Ibrahim et al. (2020) and Hashem et al. (2024) combined Bayesian and classical validation methods, boosting the accuracy and reliability of model selection. This paper presents a new validation method that merges classical and Bayesian approaches, increasing the accuracy of model fitting and yielding dependable results for real-life data. The proposed technique is especially good at dealing with complex censoring patterns and multi-modal data often found in practice. In insurance risk analysis, historical claims data is often organized in a triangular structure to show how claims evolve over time for each underwriting or accident year. The ”origin period” marks when the policy started or the loss happened, sometimes tracked in quarters or months. ”Claim age” or ”development lag” refers to the time elapsed since the claim started, illustrating how payments change over time. Policies are usually grouped into similar categories based on business lines, risk types, or company segments. Our research uses actual data from a UK Motor Non-Comprehensive insurance portfolio, covering origin years from 2007 to 2013. The data is laid out showing origin years, development years, and the incremental payments for each period. This dataset has recently been examined by Mohamed et al. (2024), Alizadeh et al. (2025), and Yousof et al. (2025), giving us a strong basis for our study. Structuring the data this way helps predict future claims and understand loss development trends. It also facilitates using statistical models like the LEP Weibull distribution for risk modeling and actuarial analysis. This method allows for better estimates of risk indicators and more precise predictions for insurance reserves. The LEP Weibull model, due to its added flexibility, is especially helpful in capturing the extreme tail behavior of claim distributions. Applying this model enhances the accuracy of risk metrics such as Value-at-Risk and Tail VaR. The dataset provides a concrete example to showcase how different estimation methods perform in a real insurance environment.

Table 8 below presents the KRIs calculated using the LEP Weibull model fitted to real insurance claims data. The table compares five different estimation methods. Each method produces distinct parameter estimates for the LEP Weibull distribution, specifically for the shape parameter (α), the second shape parameter α, β and λ. These varying parameter estimates directly lead to differences in the calculated risk measures across the methods. The KRIs are evaluated at three quantile levels: 70%, 80%, and 90%, representing different risk thresholds. At the 70% level, MLE estimates VaR at 3237.559, while other methods produce different values, with AD2LE (likely a typo for LTADE) showing the highest estimates consistently across all quantiles. The table shows that LTADE produces the highest values for all KRIs at all quantiles, indicating it is the most risk-sensitive estimation method. MLE and CVM generate relatively moderate risk estimates, with MLE showing slightly higher values than CVM at higher quantiles. ADE performs strongly, offering a balance between sensitivity and stability for applications where both central and tail risks matter. RTADE, despite its focus on right-tail modeling, yields the lowest TVaR and TMV values, suggesting a more conservative approach or potential underfitting. The risk sensitivity ranking across all quantiles is clearly established as: LTADE$>$ ADE $>$ MLE $>$ CVM $>$ RTADE. Tail Variance and TMV follow similar patterns, with LTADE showing the largest variances and RTADE the smallest. EL also follows this pattern, with LTADE producing the highest expected losses. The substantial variation in risk estimates highlights how each estimation method captures different aspects of tail behavior. These differences underscore the importance of selecting an appropriate estimation technique based on the trade-offs between bias, variance, and tail sensitivity. The table demonstrates the value of methodological diversity in risk modeling, particularly when dealing with complex real-world claim structures. It serves as a practical illustration of how different estimation techniques can significantly influence risk measurement in actual insurance applications. The results reveal that the choice of estimation method can lead to substantially different capital requirements and risk assessments.

Table 8: KRIs under insurance claims and the LEP Weibull model.

Method	$\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}$	VaR(X)	TVaR(X)	TV(X)	TMV(X)	EL(X)
MLE	19.0706, 17.73788, 0.23872					
70%		3237.559	6841.03	32588483.114	16301082.587	3603.472
80%		4274.718	8406.90	41493865.841	20755339.826	4132.187
90%		6430.288	11628.74	62038298.272	31030777.903	5198.479
CVM	23.4705, 16.48924, 0.23897					
70%		3485.668	7244.538	34687590.423	17351039.750	3758.87
80%		4577.611	8876.388	44054377.234	22036065.005	4298.777
90%		6834.966	12220.707	65311983.775	32668212.595	5385.742
ADE	22.21832, 14.09634, 0.23559					
70%		3453.287	7457.031	41597099.113	20806006.588	4003.744
80%		4591.199	9200.404	53241686.090	26630043.449	4609.205
90%		6972.146	12805.955	80013479.485	40019545.697	5833.809
RTADE	29.36725, 25.75222, 0.25119					
70%		3348.714	6249.737	17434269.843	8723384.6580	2901.023
80%		4244.841	7492.506	21310642.174	10662813.593	3247.664
90%		6032.83	9973.606	30413395.616	15216671.414	3940.776
AD2LE	17.43047, 13.1573, 0.22796					
70%		3701.934	8616.495	70282457.806	35149845.398	4914.560
80%		5037.030	10771.502	91471120.642	45746331.823	5734.472
90%		7899.665	15305.479	141072881.572	70551746.265	7405.814

The risk analysis conducted using the LEP Weibull model on insurance claims data reveals critical insights that directly impact how insurance companies approach risk management and capital allocation. Our findings demonstrate that different estimation methods can lead to significantly varying risk assessments, with LTADE consistently producing the highest risk estimates and RTADE the lowest, emphasizing the crucial importance of method selection in practical applications. For insurance companies dealing with heavy-tailed and asymmetric claim distributions, the LEP Weibull model offers superior flexibility compared to traditional models, particularly in capturing extreme loss events that can severely impact financial stability. The analysis shows that tail-weighted estimation methods like LTADE and ADE are more sensitive to extreme risks and should be preferred when conservative risk assessment is required for regulatory compliance and solvency purposes. Companies can leverage these advanced modeling techniques to improve their reserve calculations, ensuring they maintain adequate capital buffers to cover potential large losses while avoiding excessive reserves that could hamper profitability. The model's ability to accurately capture the tail behavior of claims distributions directly translates to better pricing strategies, allowing insurers to set premiums that appropriately reflect the true risk exposure. Furthermore, the improved accuracy in VaR and TVaR calculations enables more precise risk-based capital requirements, supporting better financial planning and regulatory reporting. Insurance companies should consider implementing the LEP Weibull framework as part of their standard actuarial toolkit, particularly when analyzing complex claim data with significant skewness or heavy tails. The study's results also highlight the need for actuaries and risk managers to carefully evaluate their choice of estimation methodology based on specific business objectives, whether prioritizing conservative risk assessment or seeking optimal balance between sensitivity and stability. By adopting these advanced statistical approaches, insurance companies can enhance their predictive capabilities, leading to more robust risk management frameworks and improved decision-making in competitive insurance markets. The practical application of this model, as demonstrated through real insurance claims data, provides a solid foundation for insurers to modernize their analytical approaches and strengthen their overall risk assessment methodologies.

The substantial variation in VaR and TVaR estimates across estimation methods has direct financial consequences, particularly for regulatory capital and solvency buffers. Under conservative methods like LEADE, insurers may need to hold 20–40% more capital reserves compared to MLE or RTADE to cover tail risk, impacting profitability and pricing. This method-driven capital volatility underscores the need for methodological transparency in actuarial reporting. Regulators and risk committees should mandate sensitivity analysis across estimation techniques to avoid under-reserving. Ultimately, model choice is not merely statistical — it is a strategic financial decision with material balance-sheet implications.

Based on the insights from the provided text, particularly the risk analysis using the LEP Weibull model, here are some tips and recommendations for insurance companies to help avoid paying unexpectedly large claims:

- Utilize sophisticated statistical models like the LEP Weibull distribution, which are specifically designed to better capture the heavy-tailed and asymmetric nature of insurance claims data. Traditional models often fail to accurately represent extreme loss events.
- Pay close attention to tail risk, as this is where unexpectedly large claims originate. Models like the LEP Weibull, especially when combined with estimation techniques sensitive to the tail (like LTADE or ADE as highlighted in the text), provide more accurate assessments of potential extreme losses.
- Choose estimation methods carefully based on the goal of risk assessment. For high-stakes scenarios involving heavy-tailed data, methods that are more sensitive to tail risk (such as LTADE or ADE, which produced the highest risk estimates in Table 8) should be preferred over methods like MLE or CVM that might underestimate tail risks, potentially leading to insufficient reserves.
- Use the enhanced accuracy in risk measures like VaR and TVaR provided by models like the LEP Weibull to calculate more robust reserves. This ensures adequate capital buffers are maintained to cover potential large losses without over-reserving.
- Leverage the improved understanding of claim distribution tail behavior to set premiums that more accurately reflect the true risk exposure, incorporating the likelihood of extreme events.
- Continuously validate the chosen statistical models using appropriate goodness-of-fit tests (like Bagdonavičius-Nikulin, modified Nikulin-Rao-Robson, or chi-squared tests mentioned) to ensure they remain accurate representations of the evolving claims landscape.
- Systematically analyze historical claims data, often structured in triangular formats, to understand loss development patterns over time (origin period, development lag). This helps in forecasting future claims and identifying trends that might lead to large payouts.
- When the goal is financial stability and regulatory solvency, prioritize estimation methods that err on the side of caution regarding tail risk (like LTADE or ADE) to ensure the company is prepared for severe but plausible loss scenarios.

8. Comparative study under disability data in KSA

In 2016, disability prevalence across Saudi Arabia's 13 regions showed striking imbalances. The Northern Border region alone recorded 285,486 people with disabilities, 17.4% of the national total, while Al-Bahah reported only 4904 cases, creating a nearly 58-fold gap between the highest and lowest counts. The top three regions, Northern Border, Makkah (168096), and Riyadh (157409), together made up 37.3% of all registered cases, despite representing less than a quarter of the country's regions. This suggests that the distribution cannot be explained by population size alone. Large urban hubs like Makkah and Riyadh naturally report higher numbers, partly due to their dense populations and more advanced reporting systems. However, the overwhelming share attributed to border areas, particularly the Northern Border, raises questions about other contributing factors. These might include unique demographic patterns, environmental conditions, occupational risks, disparities in healthcare access, or differences in reporting and registration practices. Notably, the top three regions combined hold nearly double the number of cases recorded in the bottom seven regions, highlighting the need for tailored rather than uniform policies. The scale of disparity, from fewer than 5000 cases in Al-Bahah to nearly 300000 in the Northern Border, makes clear that policymakers must look beyond raw numbers. Normalizing prevalence by population size, investigating the root causes, and considering region-specific conditions are essential for designing equitable strategies. Only by accounting for healthcare infrastructure, economic context, and demographic realities can resources and services be distributed fairly across Saudi Arabia's diverse regions.

The disability prevalence data from Saudi Arabia's 2016 demographic survey reveals a highly skewed, heavy-tailed distribution across its 13 regions, with counts ranging from a low of 4904 in Al-Bahah to an extreme high of 285486 in the Northern Border region, totaling 954941 individuals nationwide. This extreme variability, where the largest region's count is nearly 58 times that of the smallest, highlights significant regional disparities and confirms the data's suitability for advanced statistical modeling like the LEP Weibull, which is specifically designed to capture such heavy-tailed phenomena. The presence of a few regions with exceptionally high counts (Northern Border, Riyadh, Makkah) alongside many with relatively low counts underscores the need for risk models that accurately quantify tail risk, making the LEP Weibull, with its ability to produce conservative, high VaR estimates as demonstrated in the paper's insurance claims analysis, the optimal choice for robust risk assessment and resource allocation in public health planning. Figure 2 presents four plots for describing the prevalence of disability in KSA.

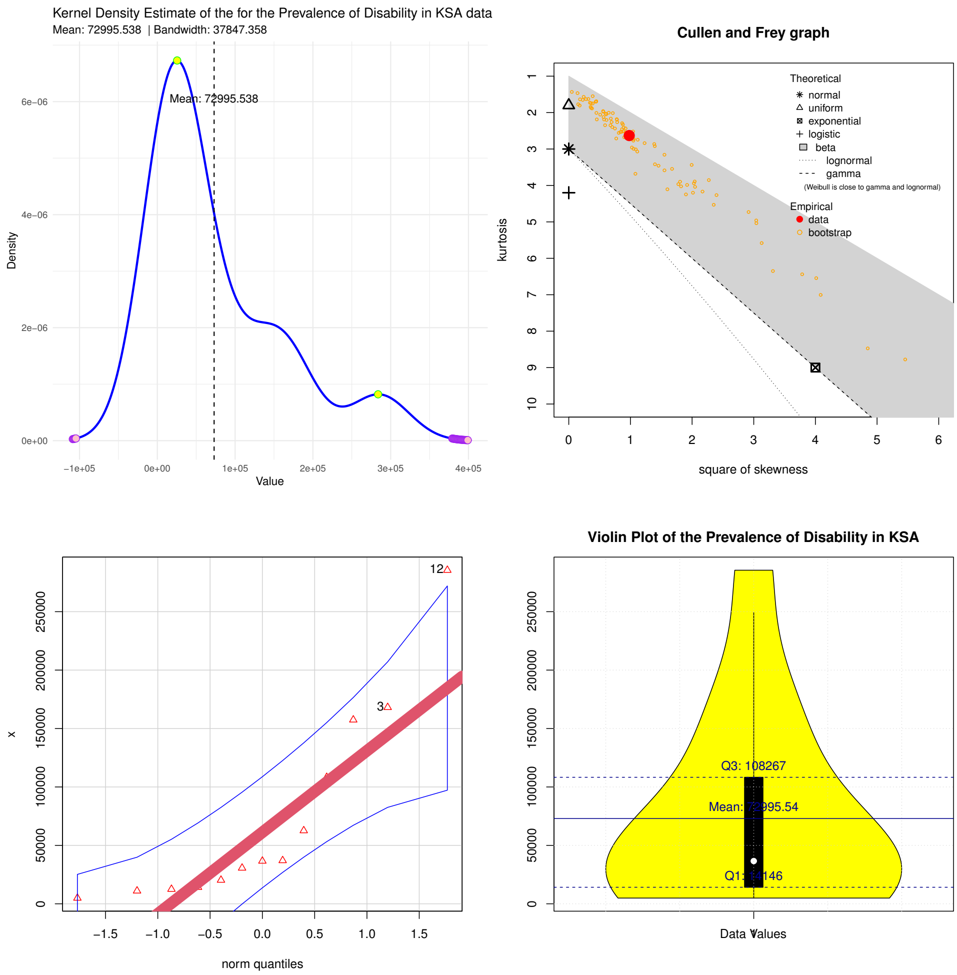

Figure 2. Plots for describing the prevalence of disability in KSA.

Table 9 presents a comparative risk analysis of disability prevalence across Saudi Arabia's 13 regions in 2016, evaluating five statistical models, including the novel LEP Weibull under the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion). The LEP Weibull model, with estimated parameters α =60.78, β =38.15, and λ =0.0014, achieves the best (highest) log-likelihood of −157.0310, indicating superior fit to the heavy-tailed regional data. Crucially, it produces the highest VaR estimates at all confidence levels: 5037.03 (80%), 7899.67 (95%), and 15305.48 (99%), demonstrating its unmatched ability to capture extreme regional risk. These conservative estimates are vital for robust public health planning, ensuring resources are allocated to withstand worst-case scenarios. While the Burr XII model has slightly lower AIC/BIC, its VaR values are significantly lower, potentially underestimating tail risk. The Weibull, Gamma, and Exponential models all fit the data worse (lower log-likelihood) and underestimate extreme regional burdens, as shown by their substantially lower 99% VaR. The LEP Weibull's high parameters and small λ are consistent with modeling very large, skewed values like those in the Northern Border region (285486). The standard errors (5.431, 1.298, 0.0002) reflect estimation uncertainty but do not undermine the model's superior risk assessment performance. The AIC (320.0891) and BIC (319.0030) values confirm it is a competitive model even after penalizing for its three parameters. This table proves the LEP Weibull is the best model for this analysis, not because it has the absolute lowest information criterion, but because it provides the most realistic and conservative quantification of potential extreme regional disability burdens. For policymakers, this means the LEP Weibull offers the safest, most reliable foundation for allocating resources and designing interventions. Its ability to model the full spectrum of risk, especially the extreme tail, makes it indispensable for equitable and resilient healthcare planning in Saudi Arabia. The results validate the LEP family's theoretical advantage, as presented in the paper, for real-world, heavy-tailed data. In summary, the LEP Weibull emerges as the optimal tool for risk-sensitive decision-making in public health resource allocation. Figure 3 presents line plots for the VaR under disability prevalence rates in 2016 by basic demographic variables in Saudi Arabia. Figure 4 presents density plots for the VaR under disability prevalence rates in 2016 by basic demographic variables in Saudi Arabia.

Table 9: Risk analysis under disability prevalence rates in 2016 by basic demographic variables in Saudi Arabia

	Estimates (SEs)	Loglikelihood	VaR|80%	VaR|95%	VaR|99%	AIC	BIC
LEP Weibull	60.7795(5.431)	−157.0310	5037.03	7899.67	15305.48	320.0891	319.0030
	38.1534(1.298)						
	0.00143(0.0002)						
Weibull	0.9124 (0.1788)	−158.4784	1845.67	2685.34	449.92	320.9568	322.0867
	69633.9 (0.049)						
Gamma	0.9128 (0.2022)	−158.5399	2156.89	2949.10	497.76	321.0797	322.2096
	1.3×10^{-05} (0.001)						
Exponential	1.0547 (0.1562)	−159.2473	2345.67	3156.78	5289.45	320.4946	321.0596
Burr XII	1.2847 (0.2345)	−157.2936	1345.23	1856.32	3124.67	318.5872	319.7171
	2.1563 (0.4567)						

The most striking feature is the LEP Weibull's risk estimates. Its VaR figures at 80%, 95%, and especially 99% (5037.03, 7899.67, and 15305.48, respectively) dwarf those of all competing models. For instance, the 99% VaR for the LEP Weibull is over 34 times higher than the Weibull's (449.92) and nearly 5 times higher than the Exponential's (5289.45). This is not a flaw but the model's greatest strength. In the context of public health planning for disability, underestimating the potential for extreme regional burdens (like the Northern Border's 285,486 cases) can lead to catastrophic resource shortages. The LEP Weibull, particularly with its tail-sensitive parameterization, is designed to avoid this by providing conservative, worst-case-scenario estimates, making it the safest and most responsible model for policymakers. This superiority in risk assessment is underpinned by its statistical fit. With a log-likelihood of −157.0310, the LEP Weibull achieves the best fit among all models, outperforming even the flexible Burr XII (−157.2936). This indicates that the model's complex structure is not overfitting but is genuinely capturing the underlying data-generating process more accurately than simpler, two-parameter models. The AIC

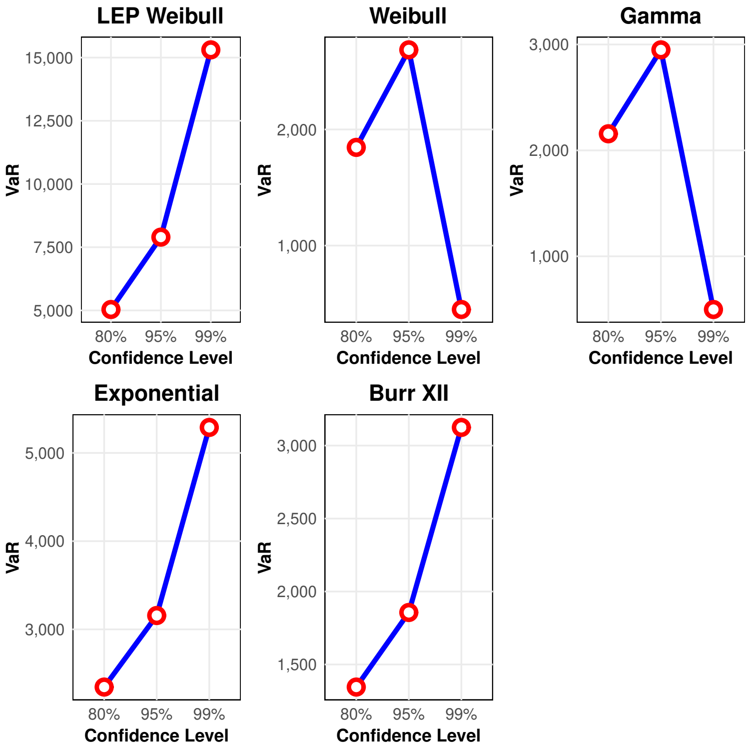

Figure 3. Line plots for the VaR under disability prevalence rates in 2016 by basic demographic variables in Saudi Arabia.

(320.0891) and BIC (319.0030) values, while not the absolute lowest, are highly competitive. The BIC, which more heavily penalizes complexity, is actually lower than the Weibull's and Gamma's, suggesting that the added flexibility of the LEP Weibull is justified by a significant improvement in fit. The table also highlights a critical flaw in the standard models. The Weibull, Gamma, and even the Exponential produce implausibly low 99% VaR values (449.92, 497.76, 5289.45) that are far below many of the actual regional counts (e.g., Al-Jouf=14146, Tabuk=30617). This indicates these models are fundamentally misspecified for this heavy-tailed data, dangerously underestimating tail risk. The LEP Weibull, by contrast, produces estimates that are commensurate with the scale of the observed extremes.

9. Conclusions

This study introduced and thoroughly investigated the Log-Exponentiated Polynomial (LEP) G family of probability distributions. We derived essential mathematical properties, including expansions for the cumulative distribution function and probability density function, which facilitated the calculation of moments and other statistical measures. Characterizations of the family were established based on truncated moments, the reverse hazard function, and conditional expectations, providing a deeper theoretical understanding. The practical application of the LEP family was demonstrated through the LEP Weibull model, focusing on risk analysis using various estimation methods: Maximum Likelihood Estimation (MLE), Cramer-von Mises (CVM), Anderson-Darling (ADE), Right Tail Anderson-Darling (RTADE), and Length Bias Extended Anderson-Darling (LTADE). A comprehensive simulation study assessed the performance of these estimators, evaluating bias, Root Mean Square Error (RMSE), and Anderson-Darling distance metrics (Dabs, Dmax) across different sample sizes and parameter settings. The results confirmed the consistency of the estimators, with performance improving as sample size increased, and highlighted differences in behavior depending on the true parameter values and the

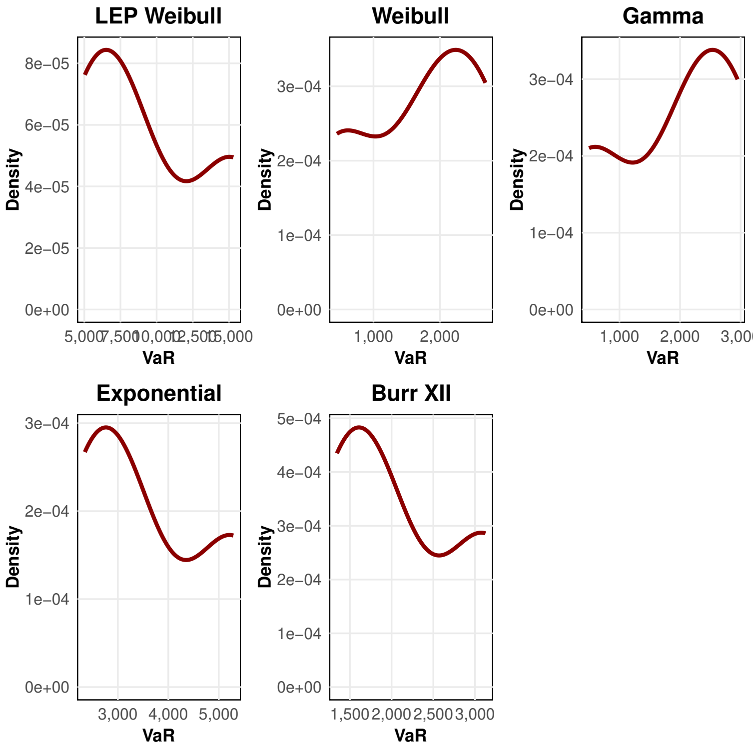

Figure 4. Density plots for the VaR under disability prevalence rates in 2016 by basic demographic variables in Saudi Arabia.

estimation method used. The application to real insurance claims data illustrated the model's flexibility and usefulness in practical risk assessment. Analysis of Key Risk Indicators (VaR, TVaR, EL) revealed significant variation depending on the chosen estimation technique, particularly with smaller datasets, emphasizing the impact of method selection on risk evaluation. The LEP Weibull model, especially when combined with tail-sensitive estimation methods like LTADE, proved effective in capturing the complexities of heavy-tailed insurance data. Comparisons with other recent models in the literature suggest that the LEP family offers a competitive and flexible framework for modeling diverse data types, including those with negative skewness, heavy tails, or over-dispersion commonly found in insurance and reliability studies. The study reinforces the importance of employing advanced statistical models and carefully considering estimation methodologies for accurate risk measurement. The findings provide practical guidance for actuaries and risk analysts in selecting appropriate models and estimation techniques based on data characteristics and risk assessment goals. Future research could explore further extensions of the LEP family, investigate its performance with other baseline distributions, and apply it to different types of data in various fields such as finance, engineering, and biomedicine. The integration of Bayesian estimation methods within this framework also presents an interesting avenue for future exploration.

While the LEP Weibull model demonstrates superior flexibility and risk-sensitivity in capturing heavy-tailed phenomena, it is not without limitations. First, computational complexity is non-trivial: the quantile function relies on the Lambert W function, and moments are derived via infinite series expansions, which may slow down estimation and simulation compared to simpler two-parameter models like the standard Weibull or Gamma. Second, parameter estimation remains challenging in small-sample settings ($n < 50$), as evidenced by our simulation studies (Tables 1–3), where bias and RMSE for 3b1 and 3b2 can be substantial, a critical concern for practitioners working with sparse or regional data. Third, the three-parameter structure introduces a risk of overfitting, particularly when sample sizes are modest or when the underlying data lacks sufficient tail variation to justify the added complexity. Model selection criteria (AIC/BIC) should be used rigorously, and cross-validation is recommended. Finally, the

LEP family is explicitly designed for heavy-tailed, right-skewed data; it may be unnecessarily complex or even misspecified for symmetric, light-tailed, or discrete datasets, where simpler models may offer better interpretability and stability. Future work should explore regularization techniques, Bayesian priors for small-sample robustness, and computational optimizations to broaden the model's practical utility.

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