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Multivariate Cubic Transmuted Family of Distribution with Applications

Hayfa Abdul Jawad Saieed [1,*], Khalida Ahmed Mohammed [2], Mhasen Saleh Altalib [3]

[1,2,3] *Department of Statistics and Informatics, College of Computer science and Mathematics, University of Mosul, Mosul, Iraq*

Abstract The need to construct multivariate probability distributions is important in modeling dependent variables. Developing flexible multivariate distribution that can model skewness and complex dependence is a challenging task. There are different forms to the same distribution are available. For this reason, the research is ongoing into ways to construct multivariate families from univariate margins. In this paper, we introduce a multivariate cubic transmuted (MCT) family.The marginal cumulative distribution function of each variable belongs to a univariate cubic transmuted family. This new family applied to (p) baseline Weibull variables named a multivariate cubic transmuted Weibull distribution (CT_PW). Statistical properties of (CT_PW) have been studied, and the parameters have been estimated by maximum likelihood (ML) method. A real data set for bone density test by photon absorption in the peripheral bones of olderly women fitted by (CT_PW), trivariate transmuted Weibull (T_3W) and $FGMW$ distributions. The important theoretical conclusions are: the marginal distributions belong to multivariate cubic family with dimension less than p, joint moments of any order depend on raw moments of each baseline variable and moments of the largest order statistics of random samples of sizes two and three drawn from each baseline distribution. In real application, the (CT_3W) is a better fit to bone density data.

Keywords Bone density test, Multivariate Cubic transmuted family, New class multivariate family, Univariate Cubic transmuted family.

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1. Introduction

Univariate distributions are not adequate to explain many ties in real data sets. For instance, variables of some phenomena must be studied together for example: in economics, financial, environmental pollutions, engineering fields, the climatic variables such as drought, wind speed and rainfalls, tests to measure calcium and minerals in different places of bone segment of spine, hyper forearm of osteoporosis patients must be measured together.Another example is risk factors of serious diseases such as cancers, kidney failure, heart attacks and others. All these cases above must be analyzed by multivariate modeling.

With the development technology, the construction of continuous multivariate distributions depending on baseline variables whose density functions are the same univariate distribution with different values of their parameters become an important. The easier procedure is a bivariate Gumbel family introduced by [6] this family used to generate many continuous bivariate distributions. A generalization of univariate $T-X$ family to bivariate $T-X$ family proposed by [5] which is used for developing different forms of bivariate generaters and G-class. The joint cumulative distribution function $(c.d.f.)$ of this family is:

*Correspondence to: Hayfa Abdul Jawad Saieed (Email: hayfaajwad.65@uomosul.edu.iq). Department of Statistics and Informatics, College of Computer science and Mathematics, University of Mosul, Mosul, Iraq.

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$$F(x,y) = \int_{a_2}^{W_2\left(G_2(y)\right)} \int_{a_1}^{W_1\left(G_1(x)\right)} r(u_1,u_2)\,du_1\,du_2 \tag{1}$$

Whereas $a_i < u_i < b_i \quad$ for $i = 1,2$,
$r(u_1,u_2)$ is any joint density function and $W_1[\cdot], W_2[\cdot]$ are absolutely continuous functions of $G_1(\cdot)$ and $G_2(\cdot)$ respectively. The simpler form of $r(u_1,u_2)$ in (1) was given by [2] defining the support of $r(u_1,u_2)$ on $[0,1]\times[0,1]$ and $W_1[\cdot] = G_1(\cdot)$ and $W_2[\cdot] = G_2(\cdot)$. in this case, the joint $c.d.f.$ defined as:

$$F(x,y) = \int_{0}^{G_2(y)} \int_{0}^{G_1(x)} \Big(1+\lambda_1(1-2u_1)+\lambda_2(1-2u_2)+2\lambda_3(1-u_1-u_2)\Big)\,du_1\,du_2 \tag{2}$$

The family given in (2) named as a bivariate transmuted family (T_2) family.
It is a generalization of univariate quadratic transmuted family [11]. A new bivariate family of distribution based on bivariate density in $T-X$ family has been proposed by [3] they defined r(-,-) as :

$$r(u_1,u_2) = 1+\lambda_1(1-2u_1)+\lambda_2(1-2u_2)+\lambda_3(1-4u_1u_2) \tag{3}$$

Whereas $\mathbf{u}\in[0,1]^2$, the transmutation parameters λ_i for $i=1,2,3$ satisfy the following conditions: $\lambda_i \in [-1,1]\ (i=1,2,3),\ \sum_{i=1}^{3}\lambda_i \geq -1,\ \lambda_1+\lambda_2+3\lambda_3 \leq 1,\ -1 \leq \lambda_1+\lambda_3 \leq 1,\ -1 \leq \lambda_2+\lambda_3 \leq 1.$ Also, they introduced a multivariate family and applied it on Weibull baseline variables.

A generalization of [2] has been done by [5], which is a p-variate transmuted family of distributions where the joint $c.d.f.$ of a random vector $\underline{X} = (X_1, X_2, \ldots, X_p)'$ be:

$$F(\underline{X}) = \prod_{i=1}^{p} G_i(X_i)\left(1+\sum_{i=1}^{p}(\lambda_i+\lambda_{p+1})(1-G_i(X_i))\right), \quad \underline{X}\in\mathbb{R}^p \tag{4}$$

whereas $G_i(X_i)$ is a $c.d.f.$ of the i-th baseline distribution. The transmutation parameters $\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}$ satisfying following inequalities:
$-1 \leq \lambda_i, \lambda_{p+1} \leq 1$, $-1 \leq \lambda_i + \lambda_{p+1} \leq 1$ for $i = 1,2,\ldots,p$, and $-1 \leq \sum_{i=1}^{p}\lambda_i + p\lambda_{p+1} \leq 1$.

The marginal c.d.f. of a single variable of $\mathbf{X}$ in (4) belongs to the k-transmuted family [10, 11] at $k=1$, which is called a univariate quadratic transmuted family. Reference [4] introduced a new class of bivariate and multivariate (NCM) families based on the multivariate $T-X$ family where the form of $r(u_1,u_2)$ is defined as

$$r(\mathbf{u}) = 1+\xi\left(1-\prod_{i=1}^{p}\delta_i\, u_i^{\xi_{i-1}}\right)$$

where $-1 \leq \xi \leq 2m\ln(\delta_i^{-1})$, $\delta_i \geq 1$, and $\xi_i \geq 1$ for $i = 1,2,\ldots,p$. Reference [14] proposed a bivariate family of distributions whose conditional distribution is a transmuted distribution; their family depends on three baseline independent continuous distributions and three dependence parameters, and they studied specific bivariate models. Reference [1] introduced a bivariate cubic transmuted family, which is an extension of a univariate cubic transmuted family. Their family is characterized by providing flexible dependence between variables, and non-linear relationships are integrated through complex relations.

The aim of this paper is to introduce new multivariate family which is a generalization of [4]. This family, named a multivariate cubic transmuted (MCT). It can be used for any baseline continuous distribution that has a closed-form differentiable $c.d.f.$ The family is a novel extension of a univariate cubic transmuted family or an extension of a bivariate quadratic transmuted family; the MCT_p can represent wider shapes of margins including asymmetric, heavy, and light tails. In this family, two transmutation parameters are added to each baseline distribution. These parameters allow for modeling dependence, skewness, and tail behavior.

2. Multivariate Cubic Transmuted Family of Distributions.

It is known that the multivariate $T-X$ family be

$$F(\underline{X}) = \int_0^{G_p(x_p)} \int_0^{G_{p-1}(x_{p-1})} \cdots \int_0^{G_2(x_2)} \int_0^{G_1(x_1)} r(\underline{u})\, d\underline{u} \tag{5}$$

where $\underline{u} = (u_1, u_2, \ldots, u_p)'$ is defined on support $[0,1]^p$, and $G_i(X_i)$ is the c.d.f. of X_i of the i-th baseline population for $i = 1, 2, \ldots, p$. There are many kinds of multivariate distributions depending on the mathematical form of $r(\underline{u})$. We introduce a new form of this function:

$$r(\underline{u}) = 1 + \sum_{i=1}^{p} \lambda_{i1}(1-2u_i) + \lambda_{p+1}\left(p - 2\sum_{i=1}^{p} u_i\right) + 2^{p-1}\sum_{i=1}^{p} \lambda_{i2}\left(\prod_{j=1}^{p} u_j\right)(2-3u_i) \tag{6}$$

whereas

$$-1 \leq \lambda_{ij}, \lambda_{p+1} \leq 1, -1 \leq \lambda_{i1} + \lambda_{p+1} \leq 1, -1 \leq \sum_{i=1}^{p} \lambda_{i1} + p\lambda_{p+1} \leq 1, -2 \leq \lambda_{i2} \leq 1, \text{for } i = 1, 2, \ldots, p, j = 1, 2.$$

The above constraints are placed because the marginal pdf for each variable u_i belongs to a univariate cubic transmuted family, so that these constraints must be the same as the constraints of the univariate cubic family of distributions [11, 13]. The marginal joint pdf of u_i, u_j can be found from (6) as:

$$r(u_i, u_j) = \int_{u_1} \cdots \int_{u_{i-1}} \int_{u_{i+1}} \cdots \int_{u_{j-1}} \int_{u_{j+1}} \cdots \int_{u_p} r(\mathbf{u})\, du_p \cdots du_{j+1}\, du_{j-1} \cdots du_{i+1}\, du_{i-1} \cdots du_1$$

After mathematical simplification of the above integral, $r(u_i, u_j)$ becomes:

$$r(u_i, u_j) = 1 + \delta_1(1-2u_i) + \delta_2(1-2u_j) + 2\lambda_{i2}u_iu_j(2-3u_i) + 2\lambda_{j2}u_iu_j(2-3u_j)$$

whereas

$$\delta_1 = \lambda_{i1} + \lambda_{p+1}, \quad \delta_2 = \lambda_{j1} + \lambda_{p+1}, \quad -1 \leq \delta_1, \delta_2 \leq 1, \quad -1 \leq \delta_1 + \delta_2 \leq 1, \quad -2 \leq \lambda_{i2}, \lambda_{j2} \leq 1.$$

The graph of $r(u_i, u_j)$ has been done at different values of transmutation parameters which are represented in Figure(1). We note from Figure(1) below the sensitivity of $r(u_1, u_2)$ to the values of the transmutation parameters. Parameters δ_1 and δ_2 determine the slope of the surface, λ_{12} shows the strength of the positive relationship between u_1 and u_2, while λ_{22} shows the strength of the negative relationship between u_1 and u_2. The negative value of λ_{22} makes the surface more downwardly curved at high values of u_2, so the surface appears clearly concave. If δ_1 is negative, the surface is steeper and the difference between the maximum and minimum values becomes wider.

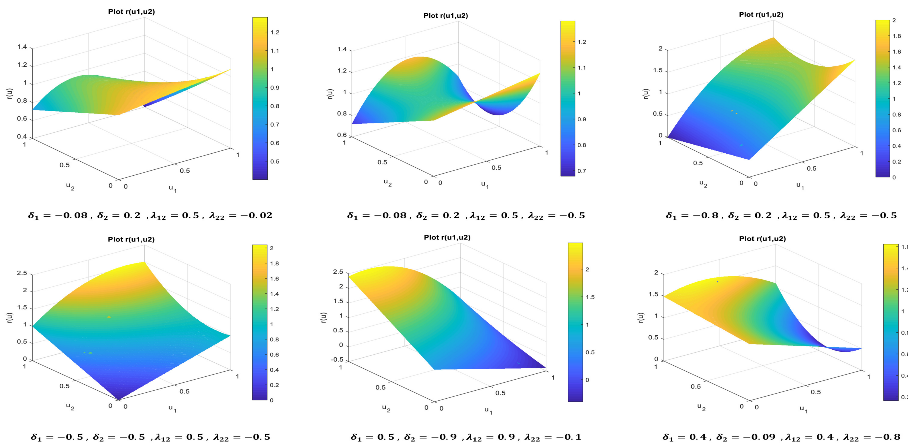

Figure 1. $r(u_i, u_j)$ curves at different values of transmutation parameters.

Substituting (6) into (5), the joint $c.d.f.$ of $\underline{X}$ is

$$F(\underline{X}) = \prod_{j=1}^{p} G_j(X_j) \left[\left(1 + \sum_{i=1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - G_i(X_i)) \right) + \sum_{i=1}^{p} \lambda_{i2} \left(\prod_{j=1}^{p} G_j(X_j) \right) (1 - G_i(X_i)) \right] \quad (7)$$

Taking the p-th partial derivatives of $F(\underline{X})$ with respect to $X_1, X_2, \ldots, X_p$, the joint probability density function $(p.d.f.)$ of $\underline{X}$ is:

$$f(\underline{X}) = \frac{\partial^p F(\underline{X})}{\partial X_1 \partial X_2 \ldots \partial X_p} = \prod_{j=1}^{p} g_j(X_j) \left[\left(1 + \sum_{i=1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) \right) \right.$$
$$\left. + 2^{p-1} \sum_{i=1}^{p} \lambda_{i2} \left(\prod_{j=1}^{p} G_j(X_j) \right) (2 - 3G_i(X_i)) \right] \quad (8)$$

whereas $\underline{X} \in \mathbb{R}^p$ and $G_i(X_i)$, $g_j(X_j)$ are the $c.d.f.$ and $p.d.f.$ of the j-th baseline distribution. The random vector $\underline{X}$ whose joint $p.d.f.$, $c.d.f.$ defined in (7) and (8) is a (CTp) family denoted by $\underline{X} \sim \text{CTp}(\underline{\lambda}, \underline{\theta})$, where $\underline{\theta}$ contains the parameters of all baseline distributions also

$$\underline{\lambda} = (\lambda_{11}, \lambda_{21}, \ldots, \lambda_{p1}, \lambda_{p+1}, \lambda_{12}, \lambda_{22}, \ldots, \lambda_{p2})'.$$

The $CT_p(\underline{\lambda}, \underline{\theta})$ family reduces to a bivariate transmuted family by [2], if $p = 2$ and $\lambda_{i2} = 0$ for all $i = 1, 2, \ldots, p$. It reduces to the multivariate transmuted family introduced by [5] when $\lambda_{i2} = 0$ for all $i = 1, 2, \ldots, p$. The CTp$(\underline{\lambda}, \underline{\theta})$ family reduces to a univariate cubic transmuted family introduced by [11] if $p = 1$, it reduces to the univariate quadratic transmuted family if $\lambda_{i2} = 0$ for all $i = 1, 2, \ldots, p$ and $p = 1$.

2.1. Marginal and conditional distributions:

Consider a random vector $\underline{X}$ partitioned into two vectors:
$\underline{X} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix}$, where $\underline{X}^{(1)}$ is a $(q \times 1)$ vector of any q variables belonging to $\underline{X}$, and $\underline{X}^{(2)}$ is a $((p - q) \times 1)$ vector containing the remaining variables of $\underline{X}$. Consider that $\underline{X}^{(1)} = (X_1, X_2, \ldots, X_q)'$ and

$\underline{X}^{(2)} = (X_{q+1}, X_{q+2}, \ldots, X_p)'$. The marginal c.d.f. for each group of variables $\underline{X}^{(1)}$, $\underline{X}^{(2)}$ are:

$$\begin{aligned} F(\underline{X}^{(1)}) &= \lim_{\underline{X}^{(2)} \to \infty} F(\underline{X}) \\ &= \prod_{j=1}^{q} G_j(X_j) \Bigg[\Big(1 + \sum_{i=1}^{q} (\lambda_{i1} + \lambda_{p+1})(1 - G_i(X_i)) \Big) + \sum_{i=1}^{q} \lambda_{i2} \Big(\prod_{j=1}^{q} G_j(X_j) \Big) (1 - G_i(X_i)) \Bigg] \end{aligned} \tag{9}$$

$$\begin{aligned} F(\underline{X}^{(2)}) = \lim_{\underline{X}^{(1)} \to \infty} F(\underline{X}) = \prod_{j=q+1}^{p} G_j(X_j) \Bigg[\Big(1 + \sum_{i=q+1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - G_i(X_i)) \Big) \\ + \sum_{i=q+1}^{p} \lambda_{i2} \Big(\prod_{j=q+1}^{p} G_j(X_j) \Big) (1 - G_i(X_i)) \Bigg] \end{aligned} \tag{10}$$

joint marginal $p.d.f.$ of $\underline{X}^{(1)}$ is partial derivative of $F(\underline{X}^{(1)})$ q times relative to all variables of $\underline{X}^{(1)}$ be:

$$\begin{aligned} f(\underline{X}^{(1)}) = \frac{\partial^q F(\underline{X}^{(1)})}{\partial X_1 \partial X_2 \ldots \partial X_q} = \prod_{j=1}^{q} g_j(X_j) \Bigg[\left(1 + \sum_{i=1}^{q} (\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) \right) \\ + \sum_{i=1}^{q} \lambda_{i2} \left(\prod_{j=1}^{q} G_j(X_j) \right) (2 - 3G_i(X_i)) \Bigg] \end{aligned} \tag{11}$$

In the same way the joint marginal $(p.d.f.)$ of $\underline{X}^{(2)}$ be.

$$\begin{aligned} f(\underline{X}^{(2)}) = \frac{\partial^{(p-q)} F(\underline{X}^{(2)})}{\partial X_{q+1} \partial X_{q+2} \ldots \partial X_p} = \prod_{j=q+1}^{p} g_j(X_j) \Bigg[\left(1 + \sum_{i=q+1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) \right) \\ + 2^{p-1} \sum_{i=q+1}^{p} \lambda_{i2} \left(\prod_{j=q+1}^{p} G_j(X_j) \right) (2 - 3G_i(X_i)) \Bigg] \end{aligned} \tag{12}$$

The joint conditional $p.d.f.$ of $\underline{X}^{(1)} \mid \underline{X}^{(2)}$ is:

$$f(\underline{X}^{(1)} \mid \underline{X}^{(2)}) = \frac{f(\underline{X})}{f(\underline{X}^{(2)})} = \prod_{j=1}^{q} g_j(X_j) \; \Delta_2(\underline{X}) \; C_2(\underline{X}^{(2)}) \tag{13}$$

where

$$\Delta_2(\underline{X}) = 1 + \sum_{i=1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) + 2^{p-1} \sum_{i=1}^{p} \lambda_{i2} \left(\prod_{j=1}^{p} G_j(X_j) \right) (2 - 3G_i(X_i)),$$

and

$$C_2(\underline{X}^{(2)}) = \left[1 + \sum_{i=q+1}^{p} (\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) + 2^{p-q-1} \sum_{i=q+1}^{p} \lambda_{i2} \left(\prod_{j=q+1}^{p} G_j(X_j) \right) (2 - 3G_i(X_i)) \right]^{-1}$$

In the Same way, the joint conditional p.d.f. of $\underline{X}^{(2)} \mid \underline{X}^{(1)}$ is:

$$f(\underline{X}^{(2)} \mid \underline{X}^{(1)}) = \frac{f(\underline{X})}{f(\underline{X}^{(1)})} = \prod_{j=q+1}^{p} g_j(X_j) \; \Delta_2(\underline{X}) \; C_1(\underline{X}^{(1)}) \tag{14}$$

where

$$C_1(\underline{X}^{(1)}) = \left[1 + \sum_{i=1}^{q}(\lambda_{i1} + \lambda_{p+1})(1 - 2G_i(X_i)) + 2^{q-1}\sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}G_j(X_j)\right)(2 - 3G_i(X_i))\right]^{-1}$$

joint conditional $c.d.f.$ of $\underline{X}^{(1)} \mid \underline{X}^{(2)}$ is:

$$F(\underline{X}^{(1)} \mid \underline{X}^{(2)}) = \int_{\underline{X}^{(1)}} f(\underline{X}^{(1)} \mid \underline{X}^{(2)})\, d\underline{X}^{(1)} \tag{15}$$

Put (13) into (15), then:

$$F(\underline{X}^{(1)} \mid \underline{X}^{(2)}) = \left[\left(\prod_{j=1}^{q}G_j(X_j)\right)\left(1 + \sum_{i=1}^{q}(\lambda_{i1} + \lambda_{p+1})(1 - G_i(X_i))\right) + \sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}G_j(X_j)\right)(1 - G_i(X_i))\right] C_2(\underline{X}^{(2)}). \tag{16}$$

The joint conditional $c.d.f.$ of $\underline{X}^{(2)} \mid \underline{X}^{(1)}$ is:

$$F(\underline{X}^{(2)} \mid \underline{X}^{(1)}) = \left[\left(\prod_{j=q+1}^{p}G_j(X_j)\right)\left(1 + \sum_{i=q+1}^{p}(\lambda_{i1} + \lambda_{p+1})(1 - G_i(X_i))\right) + \sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}G_j(X_j)\right)(1 - G_i(X_i))\right] C_1(\underline{X}^{(1)}). \tag{17}$$

2.2. Joint moments:

1. The joint moments between all variables of $(\underline{X})$ of order $(r_1, r_2, \ldots, r_p)$ is:

$$E\left(\prod_{j=1}^{p}X_j^{r_j}\right) = \int_{\underline{X}}\left(\prod_{j=1}^{p}X_j^{r_j}\right) f(\underline{X})\, d\underline{X} \tag{18}$$

Put (8) into (18), the joint moment of $(\underline{X})$ of order $(r_1, r_2, \ldots, r_p)$ is:

$$E\left(\prod_{j=1}^{p}X_j^{r_j}\right) = \prod_{j=1}^{p}\tilde{E}X_j^{r_j}\left[1 + \sum_{i=1}^{p}(\lambda_{i1} + \lambda_{p+1})\right] - \sum_{i=1}^{p}(\lambda_{i1} + \lambda_{p+1})\tilde{E}X_{i(2,2)}^{r_i}\left(\prod_{\substack{j=1\\ j\neq i}}^{p}\tilde{E}X_j^{r_j}\right) - \sum_{i=1}^{p}\lambda_{i2}\tilde{E}X_{i(3,3)}^{r_i}\left(\prod_{\substack{j=1\\ j\neq i}}^{p}\tilde{E}X_{j(2,2)}^{r_j}\right) + \sum_{i=1}^{p}\lambda_{i2}\left(\prod_{j=1}^{p}\tilde{E}X_{j(2,2)}^{r_j}\right) \tag{19}$$

Where $\tilde{E}X_j^{r_j}$ is the r_j-th moment around zero of X_j on the baseline density $g_j(X_j)$ for $j = 1, 2, \ldots, p$, and $\left(\tilde{E}X_{j(2,2)}^{r_j}, \tilde{E}X_{j(3,3)}^{r_j}\right)$ be the r_j^{th} moments around zero of X_j based on greatest order statistic of random samples of size 2 and 3 taken from the baseline density $g_j(X_j)$ for $j = 1, 2, \ldots, p$, respectively.

2-The marginal joint moments of $\underline{X}^{(1)}$ be:

$$E\left(\prod_{i=1}^{q} X_i^{r_i}\right) = \prod_{j=1}^{q} \tilde{E}X_j^{r_j}\left[1+\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\right] - \sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\tilde{E}X_{i(2,2)}^{r_i}\left(\prod_{\substack{j=1\\j\neq i}}^{q}\tilde{E}X_j^{r_j}\right)$$
$$-\sum_{i=1}^{q}\lambda_{i2}\tilde{E}X_{i(3,3)}^{r_i}\left(\prod_{\substack{j=1\\j\neq i}}^{q}\tilde{E}X_{j(2,2)}^{r_j}\right)+\sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}\tilde{E}X_{j(2,2)}^{r_j}\right). \tag{20}$$

and marginal joint raw moments of $\underline{X}^{(2)}$ be:

$$E\left(\prod_{j=q+1}^{p} X_j^{r_j}\right) = \prod_{j=q+1}^{p} \tilde{E}X_j^{r_j}\left[1+\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\right] - \sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\tilde{E}X_{i(2,2)}^{r_i}\left(\prod_{\substack{j=q+1\\j\neq i}}^{p}\tilde{E}X_j^{r_j}\right)$$
$$-\sum_{i=q+1}^{p}\lambda_{i2}\tilde{E}X_{i(3,3)}^{r_i}\left(\prod_{\substack{j=q+1\\j\neq i}}^{p}\tilde{E}X_{j(2,2)}^{r_j}\right)+\sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\tilde{E}X_{j(2,2)}^{r_j}\right). \tag{21}$$

2.3. Conditional moments.

The conditional moments of $\left(\prod_{i=1}^{q} X_i^{r_i}\,\middle|\,\underline{X}^{(2)}\right)$ be:

$$E\left(\prod_{i=1}^{q} X_i^{r_i}\,\middle|\,\underline{X}^{(2)}\right) = \int_{\underline{X}^{(1)}}\left(\prod_{i=1}^{q} X_i^{r_i}\right) f\left(\underline{X}^{(1)}\mid\underline{X}^{(2)}\right)\, d\underline{X}^{(1)} \tag{22}$$

Put (13) into (22), then

$$E\left(\prod_{i=1}^{q} X_i^{r_i}\,\middle|\,\bar{X}^{(2)}\right) = \Delta_2(\bar{X})+C_2\left(\bar{X}^{(2)}\right)\left[\left(\prod_{j=1}^{q}\tilde{E}X_j^{r_j}\right)\left(1+\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\right)\right.$$
$$-\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\,\tilde{E}X_{i(2,2)}^{r_i}\left(\prod_{\substack{j=1\\j\neq i}}^{q}\tilde{E}X_j^{r_j}\right)-\sum_{i=1}^{q}\lambda_{i2}\,\tilde{E}X_{i(3,3)}^{r_i}\left(\prod_{\substack{j=1\\j\neq i}}^{q}\tilde{E}X_{j(2,2)}^{r_j}\right)$$
$$\left.+\sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}\tilde{E}X_j^{r_j}\right)\right] \tag{23}$$

and $E\left(\prod_{i=q+1}^{p} X_i^{r_i} \,\middle|\, \underline{X}^{(1)}\right)$ is:

$$E\left(\prod_{i=q+1}^{p} X_i^{r_i} \,\middle|\, \underline{X}^{(1)}\right) = \Delta_2(\underline{X}) + C_1(\underline{X}^{(1)}) \Bigg[\prod_{j=q+1}^{p} \tilde{E}X_j^{r_j}\Big(1 + \sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\Big) - \sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\tilde{E}X_{i(2,2)}^{r_i} \prod_{\substack{j=q+1 \\ j\neq i}}^{p} \tilde{E}X_j^{r_j} - \sum_{i=q+1}^{p}\lambda_{i2}\tilde{E}X_{i(3,3)}^{r_i}\left(\prod_{\substack{j=q+1 \\ j\neq i}}^{p} \tilde{E}X_{j(2,2)}^{r_j}\right) + \sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\tilde{E}X_j^{r_j}\right)\Bigg] \tag{24}$$

The benefit of using joint, marginal and conditional moments of CTp family lies in the ability to provide a deeper understanding, a more flexible and comprehensive distributional view of the structure of the data distribution especially for extreme values, the shape and behavior of the tail and outliers. The modeling is more robust in non-normal and heavy-tailed cases, for example: in the field of finance, engineering and insurance. The extreme values are important in modeling rainfall, maximum losses, ...etc.

2.4. Dependence measures.

Measure of dependence are important for the association among variables. Some of these popular measures are obtained for proposed CT_P family:

1- Kendall's Tau coefficient The τ coefficient measures the dependence between two random variables $X_j, X_{j'}$ where $j \neq j'$ and $j = 1, 2, \ldots, p$ defined as [4]:

$$\tau = 4\int_{\mathbf{X}^{(1)}} F(\mathbf{X}^{(1)})\, f(\mathbf{X}^{(1)})\, \partial\mathbf{X}^{(1)} - 1$$

Where $\mathbf{X}^{(1)} = (X_j\ X_{j'})'$ and $F(\mathbf{X}^{(1)})$, $f(\mathbf{X}^{(1)})$ defined in (10), (11) respectively. Putting (10), (11) in the formula of τ above and defining one to one transformations:

$$u_j = G_j(X_j), \quad u_{j'} = G_{j'}(X_{j'}) \quad \text{so that} \quad \partial u_j = g_j(X_j)\,\partial X_j, \quad \partial u_{j'} = g_{j'}(X_{j'})\,\partial X_{j'} \quad \text{where } 0 < u_j, u_{j'} < 1$$

And making mathematical simplifications τ as:

$$\tau = \frac{1}{45}\left[\lambda_{j2}\,\delta_{j1} + \lambda_{j'2}\,\delta_{j'1}\right] - \frac{1}{9}\left[\lambda_{j2} + \lambda_{j'2} + 2\,\delta_{j1}\,\delta_{j'1} + \lambda_{j'2}\,\delta_{j1} + \lambda_{j2}\,\delta_{j'1}\right] + \frac{\lambda_{j2}^2}{3} - \frac{2\,\lambda_{j2}\,\lambda_{j'2}}{25}$$

Where

$$\delta_{j1} = \lambda_{j2} + \lambda_{p+1}, \quad \delta_{j'1} = \lambda_{j'2} + \lambda_{p+1}, \quad -1 \leq \delta_{j1}, \delta_{j'1} \leq 1, \quad -1 \leq \delta_{j1} + \delta_{j'1} \leq 1, \quad -2 \leq \lambda_{j2}, \lambda_{j'2} \leq 1.$$

The τ coefficient was determined at different values of parameters, these values shown in Table(1).

Table 1. Kendall's Tau coefficient of CT_2 family at different values of transmutation parameters

$\delta_j = 0.5, \delta_{j'} = 0.5, \lambda_{j'2} = -1.9$	.	$\delta_j = 1, \delta_{j'} = -1, \lambda_{j'2} = -1.9$	.
τ	λ_{j2}	τ	λ_{j2}
0.807	-1.3	0.958	-1.2
0.577	-1	0.898	-1.1
0.406	-0.7	0.846	-1
0.269	0.3	0.800	-0.9
0.322	0.5	0.728	-0.7
0.401	0.7	0.683	-0.5
0.507	0.9	0.673	-0.1
		0.687	0
		0.707	0.1
		0.857	0.5
		0.872	0.7

From Table(1) above, we observe that there is a positive strong hierarchical association between $X_j, X_{j'}$ when $\delta_j = 1$, $\delta_{j'} = -1$, $\lambda_{j'2} = -1.9$. While the association is between weak and strong positive when $\delta_j = \delta_{j'} = 0.5$ and $\lambda_{j'2} = -1.9$ with different values of λ_{j2}. It is seen that the increasing values of λ_{j2} from -1.2 up to -0.9 increases the strength of the positive relationship between two variables, and if the values of $\lambda_{j2} > -1.2$ the strength of association decreases. On the other hand, the increasing values of δ_j and decreasing values of $\delta_{j'}$ and another parameters are constant make the association stronger.

2- Spearman's ρ coefficient

The Spearman's rho (ρ) coefficient is another popular measure for dependence among variables. This coefficient is defined as [14]:

$$\rho = 12 \int_0^1 \int_0^1 R(u_j, u_{j'})\, du_j\, du_{j'} - 3$$

where $R(u_j, u_{j'})$ is a bivariate cubic transmuted cdf obtained from (6). After some mathematical simplifications the ρ coefficient becomes as:

$$\rho = \delta_j + \delta_{j'} + \tfrac{1}{3}\left(\lambda_{j2} + \lambda_{j'2}\right)$$

$$-1 \leq \delta_j + \delta_{j'} \leq 1.$$

For the value of ρ to fall within the interval $[0, 1]$, with condition

$$-3(1 + \delta_j + \delta_{j'}) \;\leq\; \lambda_{j2} + \lambda_{j'2} \;\leq\; 3(1 - \delta_j - \delta_{j'})$$

must be met. The ρ coefficient was determined at different values of CT_2 transmutation parameters, the results are presented in Table(2).

Table 2. Spearman's rho coefficient at different values of CT_2 transmutation parameters

$\delta'_j = 0.08,\ \delta_j = 0.2,$		$\delta'_j = \delta_j = 0.2$		$\delta'_j = 0.5, \delta_j = -0.9$		$\delta'_j = 0.5, \delta_j = -0.9$	
$\lambda'_{j2} = -2$		$\lambda'_{j2} = 1$		$\lambda'_{j2} = -2$		$\lambda'_{j2} = -1.8$	
λ_{j2}	ρ	λ_{j2}	ρ	λ_{j2}	ρ	λ_{j2}	ρ
-1.3	-0.94	-1	0.4	0.2	-1	0	-1
-1.1	-0.84	-0.8	0.4666	0.4	-1	0.2	-0.9333
-0.9	-0.806	-0.6	0.5333	0.6	-0.93333	0.4	-0.866
-0.6	-0.706	-0.4	0.6	0.7	-0.86667	0.6	-0.8
-0.3	-0.60667	-0.2	0.6666	0.8	-0.8	0.8	-0.733
-0.1	-0.54	0.2	0.8	1	-0.73333	1	-0.666
0.4	-0.373	0.4	0.8666				
0.6	-0.306	0.6	0.9333				
0.8	-0.24	0.8	1				
1	-0.253	1	1				

It is noted from Table(2) above that the values of the transmutation parameters have an effect on the Spearman correlation values. It is noted that all values of Spearman correlations are negative and lies between very strong and medium strength when $\lambda_{j'2} = -2$, while the correlation is positive, it lies between moderate and highly relationship. This result indicates that MCT_p can be used for modeling variables with positive and negative correlations [12, 15].

2.5. Multivariate Cubic Transmuted Weibull Distribution.

In this section, we use the CT_p family on Weibull baseline distributions to construct a new multivariate cubic transmuted Weibull distribution $(CTpW)$. The baseline distributions for each $X_1, X_2, \ldots, X_p$ are .
$X_j \sim W(\alpha_j, \beta_j)$ for $j = 1, 2, \ldots, p$, where $(p.d.f.'s)$, $(c.d.f.'s)$ are:

$$g_j(X_j) = \frac{\alpha_j}{\beta_j^{\alpha_j}} X_j^{\alpha_j - 1} e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}, \quad G_j(X_j) = 1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}, \quad \text{for } j = 1, 2, \ldots, p. \tag{25}$$

where $X_j > 0$; $\alpha_j, \beta_j > 0$, for $j = 1, 2, \ldots, p$.
Putting (25) into (7) and (8), the $(c.d.f.)$, $(p.d.f.)$ of CT_pW are:

$$F(\underline{X}) = \prod_{j=1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right) \times \left[\left(1 + \sum_{i=1}^{p}(\lambda_{i1} + \lambda_{p+1})e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right) + \sum_{i=1}^{p}\lambda_{i2}\left(\prod_{j=1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right] \tag{26}$$

$$f(\underline{X}) = \prod_{j=1}^{p}\left[\frac{\alpha_j}{\beta_j^{\alpha_j}} X_j^{\alpha_j - 1} e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right] \times \left[\left(1 + \sum_{i=1}^{p}(\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right) + 2^{p-1}\sum_{i=1}^{p}\lambda_{i2}\left(\prod_{j=1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right] \tag{27}$$

where the transmutation parameters $\lambda_{i1}, \lambda_{i2}$ for $i = 1, 2, \ldots, p$ and $\lambda_{(p+1)}$ satisfy all conditions mentioned in section (2). The marginal $(c.d.f.)$ for each group $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are:

$$F(\underline{X}^{(1)}) = \prod_{j=1}^{q}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right) \times \left[\left(1 + \sum_{i=1}^{q}(\lambda_{i1} + \lambda_{p+1})e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right) + \sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right] \tag{28}$$

$$F(\underline{X}^{(2)}) = \prod_{j=q+1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right) \times \left[\left(1 + \sum_{i=q+1}^{p}(\lambda_{i1} + \lambda_{p+1})e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right) + \sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}\right] \tag{29}$$

After putting (25) into (11) and (12), the marginal probability density functions $(p.d.f.)$ for each $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are:

$$f(\underline{X}^{(1)}) = \prod_{j=1}^{q}\left[\frac{\alpha_j}{\beta_j^{\alpha_j}}X_j^{\alpha_j-1}e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right] \times \left[\left(1 + \sum_{i=1}^{q}(\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right) + 2^{q-1}\sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right] \tag{30}$$

$$f(\underline{X}^{(2)}) = \prod_{j=q+1}^{p}\left[\frac{\alpha_j}{\beta_j^{\alpha_j}}X_j^{\alpha_j-1}e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right] \times \left[\left(1 + \sum_{i=q+1}^{p}(\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right) + 2^{p-q-1}\sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right] \tag{31}$$

Putting (25) into (13) and (14), the conditional probability density functions $(p.d.f.)$ of $\underline{X}^{(1)} \mid \underline{X}^{(2)}$, $\underline{X}^{(2)} \mid \underline{X}^{(1)}$ respectively are:

$$\begin{aligned} f(\underline{X}^{(1)} \mid \underline{X}^{(2)}) = \prod_{j=1}^{q}\left[\frac{\alpha_j}{\beta_j^{\alpha_j}}X_j^{\alpha_j-1}e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right] &\left[1 + \sum_{i=1}^{p}(\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right. \\ &\left. + 2^{p-1}\sum_{i=1}^{p}\lambda_{i2}\left(\prod_{j=1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right] \\ &\times \left[\left(1 + \sum_{i=q+1}^{p}(\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right)\right. \\ &\left. + 2^{p-q-1}\sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1)\right]^{-1} \end{aligned} \tag{32}$$

$$f(\underline{X}^{(2)} \mid \underline{X}^{(1)}) = \prod_{j=q+1}^{p} \left[\frac{\alpha_j}{\beta_j^{\alpha_j}} X_j^{\alpha_j - 1} e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right] \left[1 + \sum_{i=1}^{p} (\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right.$$
$$\left. + 2^{p-1} \sum_{i=1}^{p} \lambda_{i2} \left(\prod_{j=1}^{p} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) (3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right]$$
$$\times \left[\left(1 + \sum_{i=1}^{q} (\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right) \right.$$
$$\left. + 2^{q-1} \sum_{i=1}^{q} \lambda_{i2} \left(\prod_{j=1}^{q} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) (3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right]^{-1} \tag{33}$$

Putting (25) into (16) and (17), the conditional cumulative distribution functions $(c.d.f.)$ of $\underline{X}^{(1)} \mid \underline{X}^{(2)}$ and $\underline{X}^{(2)} \mid \underline{X}^{(1)}$ are, respectively:

$$F(\underline{X}^{(1)} \mid \underline{X}^{(2)}) = \prod_{j=1}^{q} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \left[1 + \sum_{i=1}^{q} (\lambda_{i1} + \lambda_{p+1}) e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} \right.$$
$$\left. + \sum_{i=1}^{q} \lambda_{i2} \left(\prod_{j=1}^{q} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} \right]$$
$$\times \left[\left(1 + \sum_{i=q+1}^{p} (\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right) \right.$$
$$\left. + 2^{p-q-1} \sum_{i=q+1}^{p} \lambda_{i2} \left(\prod_{j=q+1}^{p} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) (3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right]^{-1} \tag{34}$$

$$F(\underline{X}^{(2)} \mid \underline{X}^{(1)}) = \prod_{j=q+1}^{p} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \left[1 + \sum_{i=q+1}^{p} (\lambda_{i1} + \lambda_{p+1}) e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} \right.$$
$$\left. + \sum_{i=q+1}^{p} \lambda_{i2} \left(\prod_{j=q+1}^{p} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} \right]$$
$$\times \left[\left(1 + \sum_{i=1}^{q} (\lambda_{i1} + \lambda_{p+1})(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right) \right.$$
$$\left. + 2^{q-1} \sum_{i=1}^{q} \lambda_{i2} \left(\prod_{j=1}^{q} \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right) \right) (3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}} - 1) \right]^{-1} \tag{35}$$

2.6. Moments of CT_pW distribution

The r_j^{th} moment about zero of the baseline Weibull distribution is:

$$E^* X_j^{r_j} = \beta_j^{r_j} \, \Gamma \left(1 + \frac{r_j}{\alpha_j} \right), \quad \text{for } j = 1, 2, \ldots, p \tag{36}$$

The r_j^{th} moment around zero of largest order statistic of random samples of sizes (2) and (3) taken from baseline Weibull distribution are:

$$E^* X_{j(2,2)}^{r_j} = \beta_j^{r_j} \, \Gamma \left(1 + \frac{r_j}{\alpha_j} \right) \left(1 - 2^{-\frac{r_j}{\alpha_j}} \right) \tag{37}$$

$$E^{*}X_{j(3,3)}^{r_j}=3\beta_j^{r_j}\,\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}+3^{-\left(\frac{r_j}{\alpha_j}+1\right)}\right) \tag{38}$$

Substituting (36), (37), (38), into (19), (20), (21), (23) and (24), the joint moments between variables of $\underline{X}$, $\underline{X}^{(1)}$,$\underline{X}^{(2)}$ of some orders are:

$$\begin{aligned}E\left(\prod_{j=1}^{p}X_j^{r_j}\right)&=\left(\prod_{j=1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right)\left[1+\sum_{i=1}^{p}(\lambda_{i1}+\lambda_{p+1})\right]\\&-\sum_{i=1}^{p}(\lambda_{i1}+\lambda_{p+1})\left(\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}\right)\right)\prod_{\substack{j=1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\\&-\sum_{i=1}^{p}\lambda_{i2}\left(3\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}+3^{-\left(\frac{r_i}{\alpha_i}+1\right)}\right)\right)\\&\times\prod_{\substack{j=1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)+\sum_{i=1}^{p}\lambda_{i2}\prod_{j=1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)\end{aligned} \tag{39}$$

$$\begin{aligned}E\left(\prod_{j=1}^{q}X_j^{r_j}\right)&=\left(\prod_{j=1}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right)\left[1+\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\right]\\&-\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\left(\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}\right)\right)\prod_{\substack{j=1\\j\neq i}}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\\&-\sum_{i=1}^{q}\lambda_{i2}\left(3\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}+3^{-\left(\frac{r_i}{\alpha_i}+1\right)}\right)\right)\\&\times\prod_{\substack{j=1\\j\neq i}}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)+\sum_{i=1}^{q}\lambda_{i2}\prod_{j=1}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)\end{aligned} \tag{40}$$

$$\begin{aligned}E\left(\prod_{j=q+1}^{p}X_j^{r_j}\right)&=\left(\prod_{j=q+1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right)\left[1+\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\right]\\&-\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\left(\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}\right)\right)\prod_{\substack{j=q+1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\\&-\sum_{i=q+1}^{p}\lambda_{i2}\left(3\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}+3^{-\left(\frac{r_i}{\alpha_i}+1\right)}\right)\right)\\&\times\prod_{\substack{j=q+1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)+\sum_{i=q+1}^{p}\lambda_{i2}\prod_{j=q+1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)\end{aligned} \tag{41}$$

$$E\left(\prod_{j=1}^{q} X_j^{r_j} \mid \underline{X}^{(2)}\right) = \Delta_2(\underline{X}) + C_2\left(\underline{X}^{(2)}\right)\left(\prod_{j=1}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right)\left[1+\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\right.$$
$$-\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\left(\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}\right)\right)\prod_{\substack{j=1\\j\neq i}}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)$$
$$-\sum_{i=1}^{q}\lambda_{i2}\left(3\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}+3^{-\left(\frac{r_i}{\alpha_i}+1\right)}\right)\right)$$
$$\left.\times\prod_{\substack{j=1\\j\neq i}}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)+\sum_{i=1}^{q}\lambda_{i2}\prod_{j=1}^{q}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right] \quad (42)$$

where

$$\Delta_2(\underline{X}) = 1+\sum_{i=1}^{p}(\lambda_{i1}+\lambda_{p+1})\left(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)+2^{p-1}\sum_{i=1}^{p}\lambda_{i2}\left(\prod_{j=1}^{p}\left(1-e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)\left(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)$$

$$C_2\left(\underline{X}^{(2)}\right) = \left[1+\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\left(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)\right.$$
$$\left.+2^{p-q-1}\sum_{i=q+1}^{p}\lambda_{i2}\left(\prod_{j=q+1}^{p}\left(1-e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)\left(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)\right]^{-1}$$

$$E\left(\prod_{j=q+1}^{p} X_j^{r_j} \mid \underline{X}^{(1)}\right) = \Delta_2(\underline{X}) + C_1\left(\underline{X}^{(1)}\right)\left(\prod_{j=q+1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right)\left[1+\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\right.$$
$$-\sum_{i=q+1}^{p}(\lambda_{i1}+\lambda_{p+1})\left(\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}\right)\right)\prod_{\substack{j=q+1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)$$
$$-\sum_{i=q+1}^{p}\lambda_{i2}\left(3\beta_i^{r_i}\Gamma\left(1+\frac{r_i}{\alpha_i}\right)\left(1-2^{-\frac{r_i}{\alpha_i}}+3^{-\left(\frac{r_i}{\alpha_i}+1\right)}\right)\right)$$
$$\left.\times\prod_{\substack{j=q+1\\j\neq i}}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\left(1-2^{-\frac{r_j}{\alpha_j}}\right)+\sum_{i=q+1}^{p}\lambda_{i2}\prod_{j=q+1}^{p}\beta_j^{r_j}\Gamma\left(1+\frac{r_j}{\alpha_j}\right)\right] \quad (43)$$

where

$$C_1(\underline{X}^{(1)}) = \left[\left(1+\sum_{i=1}^{q}(\lambda_{i1}+\lambda_{p+1})\left(2e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)\right)\right.$$
$$\left.+2(p-q-1)\sum_{i=1}^{q}\lambda_{i2}\left(\prod_{j=1}^{q}\left(1-e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}}\right)\right)\left(3e^{-\left(\frac{X_i}{\beta_i}\right)^{\alpha_i}}-1\right)\right]^{-1} \quad (44)$$

3. Parameters Estimation

Assuming parameters of the CT_pW distribution are unknown. By providing a complete sample of size n, the (ML) estimators for all parameters of CT_pW under two-sided inequality constraints on the vector of transmutation parameters. $\underline{\lambda} = (\lambda_{i1}, \lambda_{i2}, \text{ for } i = 1, 2, \ldots, p, \lambda_{p+1})'$. The likelihood function is:

$$L(\underline{\theta}) = \prod_{k=1}^{n} \left\{ \prod_{j=1}^{p} \frac{\alpha_j}{\beta_j^{\alpha_j}} X_{kj}^{\alpha_j - 1} \exp\left(-\left(\frac{X_{kj}}{\beta_j} \right)^{\alpha_j} \right) \times \left[1 + \sum_{i=1}^{p} (\lambda_{i1} + \lambda_{p+1}) \left(2e^{-\left(\frac{X_{ki}}{\beta_i}\right)^{\alpha_i}} - 1 \right) \right. \right.$$
$$\left. \left. + 2^{p-1} \sum_{i=1}^{p} \lambda_{i2} \left(3e^{-\left(\frac{X_{ki}}{\beta_i}\right)^{\alpha_i}} - 1 \right) \left(\prod_{j=1}^{p} \left(1 - e^{-\left(\frac{X_{ki}}{\beta_i}\right)^{\alpha_i}} \right) \right) \right] \right\} \tag{45}$$

Where

$$\underline{\theta} = (\underline{\alpha} \quad \underline{\beta} \quad \underline{\lambda}_1 \quad \underline{\lambda}_2 \quad \lambda_{p+1})', \quad \underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)', \quad \underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)',$$

$$\underline{\lambda}_1 = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1p})', \quad \underline{\lambda}_2 = (\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2p})'.$$

Taking $\log_e$ to both sides of eq.(45), the vector $\boldsymbol{\theta}$ which maximizes $L(\boldsymbol{\theta})$ or $\log_e \big(L(\boldsymbol{\theta})\big)$ under two-sided linear inequality constraints on all transmutation parameters is the same as the vector $\boldsymbol{\theta}$ which minimizes $[-L(\boldsymbol{\theta})]$ or $[-\log_e \big(L(\boldsymbol{\theta})\big)]$.

The optimum solution for $\boldsymbol{\theta}$ can be found by using constrained non-linear optimization via `fmincon` with the interior-point algorithm [7, 9].

$\hat{\boldsymbol{\theta}}$ is the maximum likelihood (ML) estimator for $\boldsymbol{\theta}$ under the condition that the information matrix is negative definite. For a large sample size, $\hat{\boldsymbol{\theta}}$ is approximately distributed as:

$$\hat{\boldsymbol{\theta}} \sim N_{13}\left(\boldsymbol{\theta}, -\left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}'} \bigg|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}} \right)^{-1} \right)$$

Taking $\log_e$ to both sides of eq.(45), the vector $\boldsymbol{\theta}$ which maximizes $L(\boldsymbol{\theta})$ or $\log_e \big(L(\boldsymbol{\theta})\big)$ under two sided linear inequality constraints on all transmutation parameters is the same of a vector $\boldsymbol{\theta}$ which minimizes $[-L(\boldsymbol{\theta})]$ or $[-\log_e \big(L(\boldsymbol{\theta})\big)]$. The optimum solution for $\boldsymbol{\theta}$ can be found by using the constrained non-linear optimization with using `fmincon` with the interior-point-algorithm [**?**].

$\hat{\boldsymbol{\theta}}$ is the ML estimator for $\boldsymbol{\theta}$ with condition that the Fisher information matrix is a negative definite matrix. A sample size is large, $\hat{\boldsymbol{\theta}}$ is approximately distributed as:

$$\hat{\boldsymbol{\theta}} \sim N_{13}\left(\boldsymbol{\theta}, \ -\left(\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}'} \bigg|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}} \right)^{-1} \right)$$

4. Real Application

In this section a real data set for a bone density test is taken from [8]. This test determines whether a person has osteoporosis. The test usually uses X-rays to measure grams of calcium and other bone minerals in a slice of bone. It is known that women were more susceptible to osteoporosis than men, so we used the measures of bone density of older women. This data contains the measures of bone density by photon absorption in the peripheral bones of (25) older women. The measurements were taken of three dominant bones: forearms (X_1), humerus (X_2), and ulna (X_3). The bone density for older women dataset were displayed in Table(3).

Table 3. Bone density measurements for 25 older women (forearms X_1, humerus X_2, ulna X_3).

No.	X_1	X_2	X_3	No.	X_1	X_2	X_3	No.	X_1	X_2	X_3
1	1.103	2.139	0.873	10	0.921	1.954	0.823	19	0.856	2.028	0.578
2	0.842	1.873	0.590	11	0.792	1.624	0.686	20	0.890	2.187	0.758
3	0.925	1.887	0.767	12	0.815	2.204	0.678	21	0.688	1.650	0.533
4	0.857	1.739	0.706	13	0.755	1.508	0.662	22	0.940	2.334	0.757
5	0.795	1.734	0.549	14	0.880	1.786	0.810	23	0.493	1.037	0.546
6	0.787	1.509	0.782	15	0.900	1.902	0.723	24	0.835	1.509	0.618
7	0.933	1.695	0.737	16	0.764	1.743	0.586	25	0.915	1.971	0.869
8	0.799	1.740	0.618	17	0.733	1.863	0.672	-	-	-	-
9	0.945	1.811	0.853	18	0.932	2.028	0.836	-	-	-	-

Some descriptive statistics were measured and given in table(4).

Table 4. Summery important measures of bone variables

Measures	X_1	X_2	X_3
Minimum	0.493	1.037	0.533
Maximum	1.103	2.334	0.873
Mean	0.8438	1.8102	0.7044
Q_1	0.7895	1.6725	0.604
Median	0.856	1.811	0.706
Q_3	0.923	1.9995	0.796
Skewness	-0.84	-0.61	-0.02
Kurtosis	3.22	1.59	-1.20

The measures in Table (4) indicate that X_1, X_2, X_3 are negatively skewed.This means the values of the variables accumulate near the maximum values. This is clear because the median is smaller than the mean for each variable. It is seen that the kurtosis of X_1 and X_2 is greater than zero, which means the distributions of these variables are heavy-tailed and more peaked central area, while kurtosis of X_3 is negative, which means the distribution of X_3 has a lighter tail and a flatter central area. The positive kurtosis indicates that the sample data for X_1 and X_2 have outlier observations from the left, while the sample data of X_3 do not have outliers.

The estimated Spearman rho correlation coefficients.

$$\rho(X_1, X_2) = 0.628, \quad \rho(X_1, X_3) = 0.766, \quad \rho(X_2, X_3) = 0.459.$$

$$p - value = 0.0009937, \quad 0.00001575, \quad 0.0337$$

It is seen that all correlations are significant because all $p < 0.05$.
Fits of data has been done by our proposed CT_3W density defined in (24) and the trivariate families:

1-Trivariate transmuted Weibull (T_3W) distribution where the joint $p.d.f.$ of this family is:

$$f(\mathbf{X}) = \prod_{j=1}^{3} \frac{\alpha_j}{\beta_j^{\alpha_j}} X_j^{\alpha_j - 1} \exp\left(-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}\right) \left[1 + \sum_{i=1}^{3} (\lambda_{i1} + \lambda_4)\Big(1 - 2\big(1 - \exp\Big(-\Big(\frac{X_i}{\beta_i}\Big)^{\alpha_i}\Big)\big)\Big)\right]. \tag{46}$$

where

$$-1 \le \lambda_{i1},\, \lambda_4 \le 1, \qquad -1 \le \lambda_{i1} + \lambda_4 \le 1, \qquad -1 \le \sum_{i=1}^{3} (\lambda_{i1} + 3\lambda_4) \le 1, \text{for} \quad i = 1, 2, 3.$$

2- A new class of multivariate (NC_3) family of distributions: The (NC_3) family proposed by [4] depending on the baseline pdf and cdf Weibull distribution defined in (23), the new class trivariate Weibull (NC_3W) pdf defined as:

$$f(\underline{X}) = \prod_{j=1}^{3} \left[\frac{\alpha_j}{\beta_j^{\alpha_j}} X_j^{\alpha_j - 1} e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right] \left[(1+\xi) - \xi \prod_{j=1}^{3} \psi_j \left(1 - e^{-\left(\frac{X_j}{\beta_j}\right)^{\alpha_j}} \right)^{\psi_j - 1} \right] \tag{47}$$

Where

$$-1 \leq \xi \leq 2 \min\left(\psi_j^{-1}\right), \quad \psi_j \geq 1$$

The calculations of the constrained nonlinear optimization method using the fmincon algorithm with interior-point were performed in $MATLAB$ software. The starting points for the CT_3W, T_3W, and NC_3W parameters were randomly selected, provided that all constraints imposed on them were satisfied in each model. By calling fmincon, which is to minimize the negative of the likelihood function for each model. The optimum solutions that satisfy the objective function represent the ML estimators. These estimators, Akaike's and Bayesian information criterion (AIC), (BIC) were put in Table(5).

Table 5. Maximum likelihood estimation and goodness of fit measures for three trivariate Weibull distributions

Distribution	Parameter	Estimator	AIC	BIC
CT3W	α_1	7.9150	-162.0649	-146.2196
	α_2	8.6351		
	α_3	7.6780		
	β_1	0.8547		
	β_2	0.8902		
	β_3	0.8714		
	λ_{11}	-0.1270		
	λ_{21}	-0.9380		
	λ_{31}	0.9066		
	λ_4	-0.00005		
	λ_{12}	-1.7354		
	λ_{22}	-1.9336		
	λ_{32}	0.9228		
T3W	α_1	8.6143	-146.159	-133.970
	α_2	8.6050		
	α_3	9.3385		
	β_1	0.8906		
	β_2	0.8898		
	β_3	0.9051		
	λ_{11}	-0.8480		
	λ_{21}	-0.9918		
	λ_{31}	0.9910		
	λ_4	-0.0037		
NC3W	α_1	6.859	-151.401	-141.212
	α_2	8.616		
	α_3	8.588		
	β_1	0.844		
	β_2	0.889		
	β_3	0.890		
	ψ_1	1.608		
	ψ_2	3.000		
	ψ_3	1.000		
	ξ	-1.000		

From Table(5), it is show CT_3W is the most suitable for a bone density test data, set which has the smaller AIC and BIC criteria. It is also observed that the estimators of transmutation parameters λ_{i1} for $i = 1, 2, 3$ and λ_4 in CT_3W and T_3W have the same signs.

5. Contributions

In this paper, a new multivariate family of distribution was introduced. This family based on a generalization of univariate cubic transmuted family to a multivariate family, Different statistical properties of new family have been studied. It is concluded in theoretical analysis that the marginal distributions belong to transmutation family of order three, the joint and marginal moments are functions of moments of the largest order statistics of random samples of sizes two and three taken from baseline densities. In application, the CT_3W is the best fit than the T_3W and NC_3W distributions, so that CT_3W is a good fit for bone density studies.

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