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Alpha Power One-Parameter Weibull Distribution: Its Properties, Simulations and Applications to Real-Life Data

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Abstract In this paper, we introduce a new lifetime distribution called alpha power one-parameter Weibull (APOPW) distribution based on the alpha power transformation method has been defined and studied. Various statistical properties of the newly proposed distribution including moments, moment generating function, quantile function, Rényi and Shannon entropy, stress-strength reliability, mean deviations, and extreme order statistics have been obtained.
Several estimation techniques are studied, including maximum likelihood estimation (MLE), Anderson–Darling (AD), least squares estimation (LSE), Cramér–von Mises (CVM), and maximum product of spacings (MPS). The estimators compared their efficiency based on average absolute bias (BIAS), mean squared error (MSE), and mean absolute relative error (MRE), identifying that MLE as the most robust method across various sample sizes increase.
The efficiency and flexibility of the new distribution are illustrated by analysing two real-live data sets, and compare its goodness-of-fit against several existing lifetime distributions.

Keywords Alpha power transformation, One-parameter Weibull distribution, Moment, Quantile, Estimation Methods, Simulation.

AMS 2010 subject classifications 60E05, 62E10.

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1. Introduction

In 2017, Mahdavi and Kundu [1] introduced a new method, called alpha power transformation (APT) to incorporate skewness to the baseline continuous distribution. The cumulative distribution function (CDF) for the APT family of distributions is defined as follows:

$$F_{APT}(t) = \begin{cases} \frac{\alpha^{F(t)}-1}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\ F(t) & \text{if } \alpha = 1 \end{cases} \quad (1)$$

The corresponding probability density function (PDF) for the APT family is given by:

$$f_{APT}(t) = \begin{cases} \frac{log\alpha}{\alpha-1}\alpha^{F(t)}f(t) & \text{if } \alpha > 0, \alpha \neq 1 \\ f(t) & \text{if } \alpha = 1 \end{cases} \quad (2)$$

The Weibull distribution (see Weibull 1951)[17] is a continuous probability distribution used widely in statistics and engineering. It is commonly used for analyzing biostatistics, medical, actuarial science and hydrological data sets, as one of the most popular distributions in analysing the lifetime data due to the wide variety of shapes it

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can be assumed by varying its shape parameter. The cumulative distribution function of the Weibull distribution is defined as:

$$F_W(t) = 1 - e^{-\beta t^{\lambda}}, \quad t, \lambda, \beta > 0. \tag{3}$$

Where $\lambda > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

In this paper we set β equal to 1 and thus the equation (3) becomes as follows:

$$F(t) = 1 - e^{-t^{\lambda}}, \quad t, \lambda > 0. \tag{4}$$

called the cumulative distribution function (CDF) of the One-Parameter Weibull distribution.

In this work, we introduce a new distribution called the **Alpha Power One-Parameter Weibull (APOPW)** distribution, which is derived based on the novel technique for creating new distributions, that of alpha power transformation (APT) introduced by Mahdavi and Kundu in [1]. The APOPW distribution is an extension of the classical Weibull distribution, adding a parameter for increased flexibility in modeling different hazard shapes, particularly non-monotonic ones, which the classic Weibull distribution cannot. Explore its fundamental properties. Particular attention is given to the cumulative distribution function (CDF), as well as the monotonic behavior of the probability density (PDF), Survival and hazard functions. Its statistical properties such as moments, mean, variance, moment generating function, skewness and kurtosis, quantile function, Rényi and Shannon entropy, stress-strength reliability, mean deviations, and extreme order statistics are discussed. Estimation of its parameters is made using five methods : maximum likelihood estimation (MLE), Anderson–Darling (AD), least squares estimation (LSE), Cramér–von Mises (CVM), and maximum product of spacings (MPS). Comprehensive Simulations of the model are conducted, and its applications to tow real data sets are also presented.

More recent, many generalizations of the alpha power transformation methods on modeling Weibull distributions have been attempted by various researchers, the alpha power Weibull distribution by Nassar et al. (2017) [18], a new extension of Weibull distribution called alpha power transformed Weibull distribution by Dey et al. (2017) [19], a new modified alpha power Weibull distribution by Seema et al. (2021) [30], and alpha power transformed generalized Weibull distribution by Merga et al. (2024) [21].

The paper is organized into several sections. It begins with an introduction in Section 1, introduces the new distribution by a presentation of the APOPW Model using the APT method in Section 2. Section 3 covers the derivation of survival and hazard functions for the new distribution, deriving and discussing its statistical properties. Section 4 elaborates on the several parameter estimation Methods. The later section include the presentation of a comprehensive simulation study in Section 5, an application around to model comparison in fitting to real data Sets in Section 6, and the conclusion in Section 7.

2. The APOPW Model

In this paper, we introduce the new distributions, called the Alpha-Power One-Parameter Weibull distribution (APOPWD for short). The probability function of APOPW with two parameters α, λ, where $\alpha > 0$ and $\lambda > 0$ are the shape parameters, are obtained by using the CDF and PDF of one-parameter Weibull distribution in α - power transformation. The corresponding density of the APOPW distribution of $t > 0$ is:

$$f_{APOPW}(t) = \begin{cases} \frac{log\alpha}{\alpha-1}\alpha^{1-e^{-t^{\lambda}}}\lambda t^{\lambda-1}e^{-t^{\lambda}} & \text{if } \alpha > 0, \alpha \neq 1 \\ \lambda t^{\lambda-1}e^{-t^{\lambda}} & \text{if } \alpha = 1 \end{cases} \tag{5}$$

Figure 2 shows the behavior of the density function under various parameter configurations. The results indicate that the PDF is unimodal. And the cumulative distribution function is given by :

$$F_{APOPW}(t) = \begin{cases} \frac{\alpha^{1-e^{-t^{\lambda}}}-1}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\ 1 - e^{-t^{\lambda}} & \text{if } \alpha = 1 \end{cases} \tag{6}$$

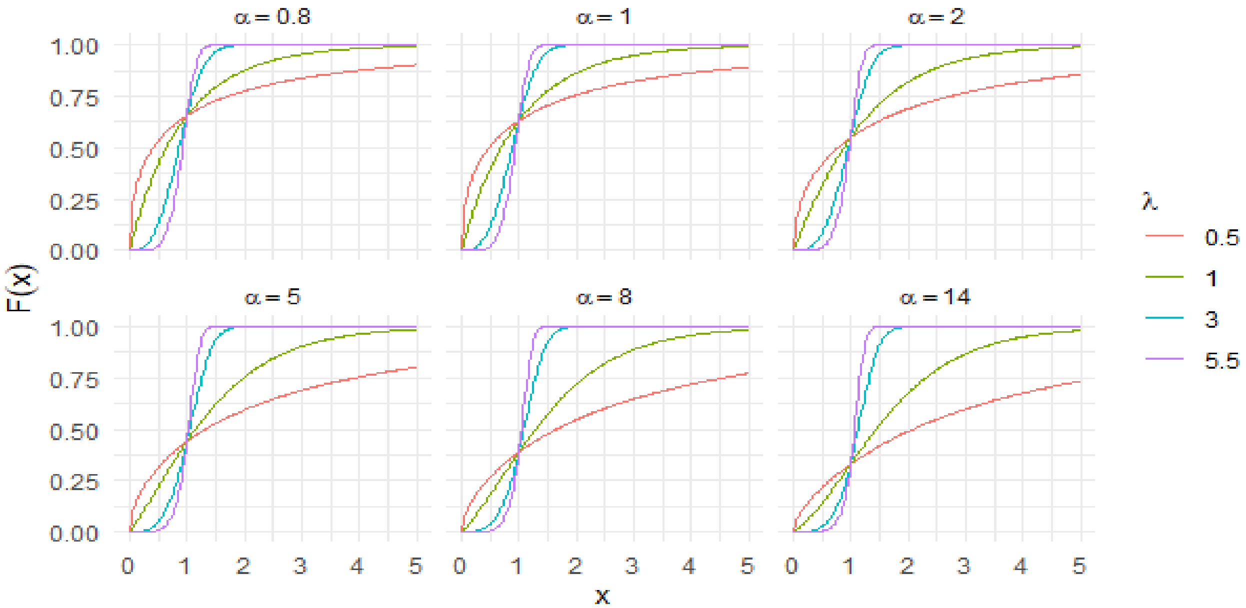

Figure 1. CDF $F_{APOPW}(x)$ for different α and λ values

As observed in Figure 1, the CDF increases to one at different rates depending on the parameter values.

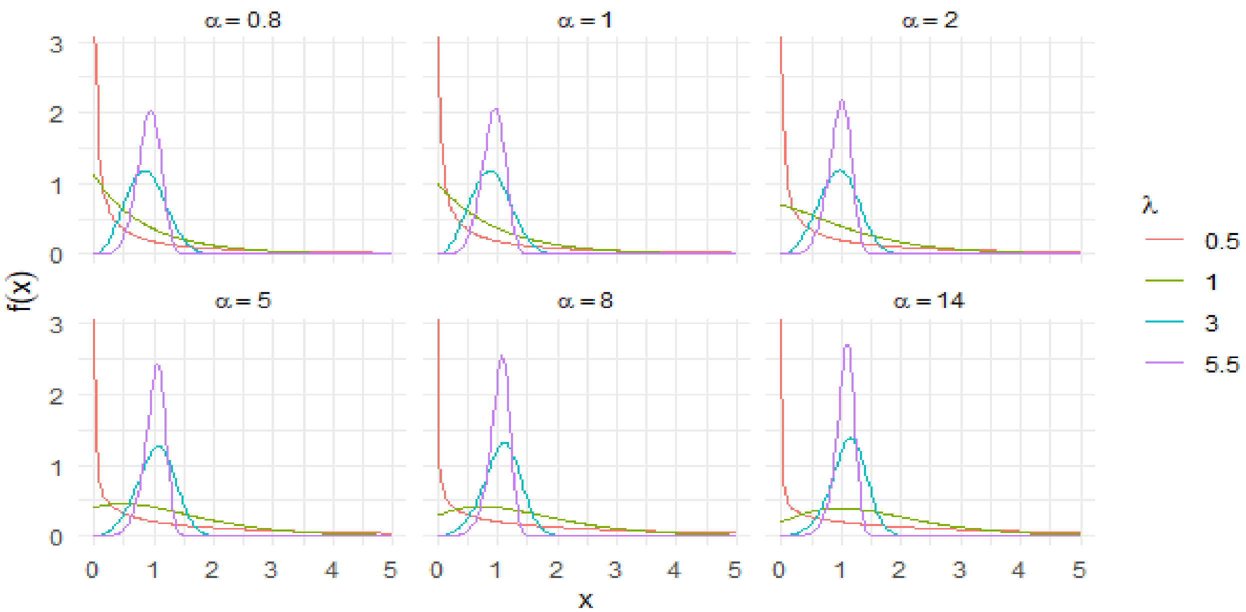

Figure 2. PDF $f_{APOPW}(x)$ for different α and λ values

Figure 2 Shows that for a fixed λ, increasing α shifts the mode of the PDF to the right and elongates the tail.

2.1. Shape behavior

The effect of λ:

- Controls the spread and tail behavior of the distribution.
- Appears in the exponent and power terms: t^{λ}, $t^{\lambda-1}$.
- $\lambda < 1$: Heavy right tail, high density near zero.
- $\lambda = 1$: Exponential-like decay.
- $\lambda > 1$: Peak shifts right, tail becomes thinner.

The effect of α:

- Modulates the skewness and modality of the distribution.
- Appears in the transformation $\alpha^{1-e^{-t^{\lambda}}}$.
- $\alpha < 1$: CDF grows slowly, PDF is flatter—more weight in the tail.
- $\alpha > 1$: CDF grows quickly, PDF is sharper—more weight near the mode.
- $\alpha = 1$: Distribution simplifies to the Weibull distribution.

The interactive effects of α and λ:
The interaction between α and λ creates nonlinear distortions in the shape of the distribution:

α	λ	PDF Behavior	CDF Behavior
$\alpha < 1$	$\lambda < 1$	Flat, heavy-tailed	Slow accumulation
$\alpha > 1$	$\lambda < 1$	Sharp near zero, fast decay	Rapid accumulation early
$\alpha < 1$	$\lambda > 1$	Broad peak, long tail	Gradual rise
$\alpha > 1$	$\lambda > 1$	Narrow peak, short tail	Steep rise then plateau

So,

- λ shapes the core decay and spread—it's the engine of the Weibull-like behavior.
- α acts as a distortion lens, warping the distribution to be more peaked or more spread out.
- When $\alpha = 1$, the distortion disappears and the distribution becomes Weibull.
- Together, α and λ allow the APOPW distribution to flexibly model a wide range of behaviors—from early failures to long lifetimes, from sharp peaks to heavy tails.

Theorem 1
The function of the new formula for the APOPW distribution is a PDF.

Proof
The function of the new formula for the APOPW distribution to be a PDF if $f_{APOPW}(t) > 0$ and $\int_0^{\infty} f_{APOPW}(t) = 1$. Then

$$\int_0^{\infty} f_{APOPW}(t)\,dt = \int_0^{\infty} \frac{\log\alpha}{\alpha-1}\alpha^{1-e^{-t^{\lambda}}}\lambda t^{\lambda-1}e^{-t^{\lambda}}\,dt \tag{7}$$

Let use $\delta = 1 - e^{-t^{\lambda}}$, then the derivatives give $\frac{d\delta}{dt} = \lambda t^{\lambda-1}e^{-t^{\lambda}}$ and $dt = \frac{d\delta}{\lambda t^{\lambda-1}e^{-t^{\lambda}}}$, so

$$\int_0^{\infty} f_{APOPW}(t)\,dt = \frac{\log\alpha}{\alpha-1}\int_0^{\infty} \frac{\alpha^{1-e^{-t^{\lambda}}}\lambda t^{\lambda-1}e^{-t^{\lambda}}}{\lambda t^{\lambda-1}e^{-t^{\lambda}}}d\delta = \frac{\log\alpha}{\alpha-1}\int_0^{\infty} \alpha^{\delta}d\delta \tag{8}$$

using exponential rule $\int \alpha^{\delta}d\delta = \frac{\alpha^{\delta}}{\log\alpha}$, then

$$\int_{0}^{\infty} f_{APOPW}\left(t\right)dt = \frac{\log\alpha}{\alpha-1}\int_{0}^{\infty}\alpha^{\delta}d\delta = \frac{\log\alpha}{\alpha-1}\frac{\alpha^{\delta}}{\log\alpha}\Big|_{0}^{\infty} = \frac{\alpha^{1-e^{-t^{\lambda}}}}{\alpha-1}\Big|_{0}^{\infty} = 1 \tag{9}$$

Hence, it indicates that $f_{APOPW}\left(t\right)$ satisfies the definition of PDF.

3. Main properties

3.1. Survival and hazard functions

The survival functions corresponding to the CDF defined in (6) are given by

$$S_{APOPW}(t) = \begin{cases} \frac{\alpha}{\alpha-1}(1-\alpha^{-e^{-t^{\lambda}}}) & \text{if } \alpha > 0, \alpha \neq 1 \\ e^{-t^{\lambda}} & \text{if } \alpha = 1 \end{cases} \tag{10}$$

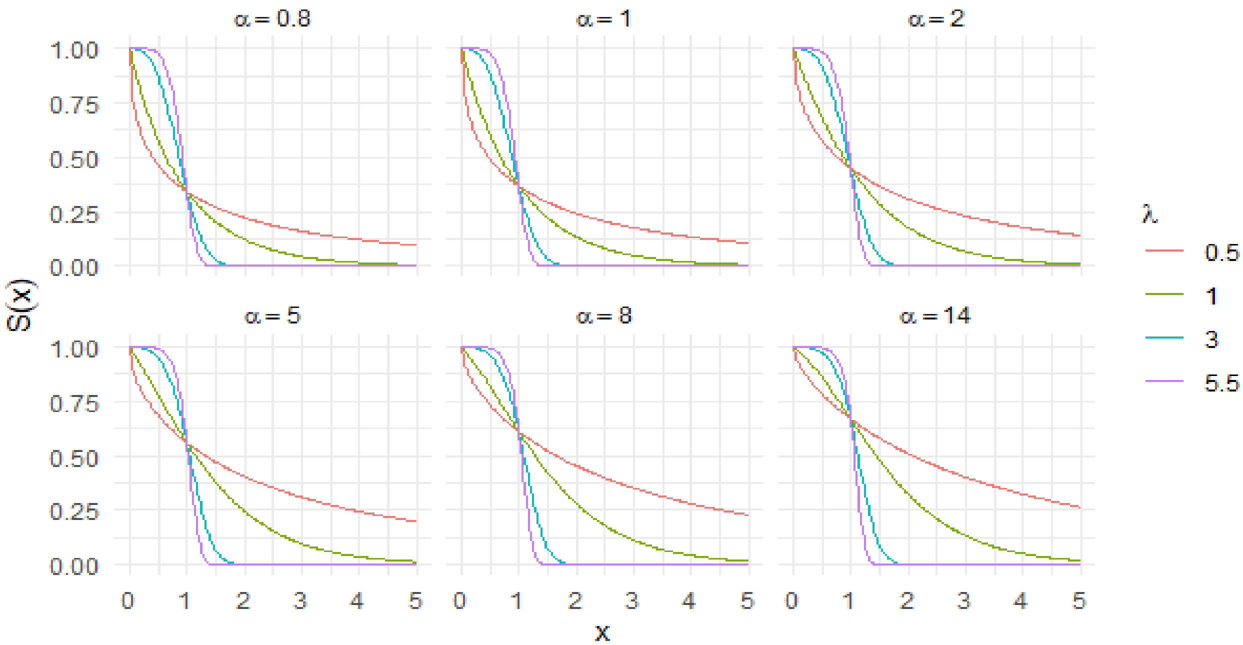

Figure 3. Survival curves for APOPWD for different α and λ values

Figure 3 Shows that for a fixed λ, increasing α shifts the mode of the survival function to the right and elongates the tail.

and the hazard functions are given by

$$h_{APOPW}(t) = \begin{cases} \frac{log(\alpha)\lambda t^{\lambda-1}e^{-t^{\lambda}}}{\alpha^{e^{-t^{\lambda}}}-1} & \text{if } \alpha > 0, \alpha \neq 1 \\ \lambda t^{\lambda-1} & \text{if } \alpha = 1 \end{cases} \tag{11}$$

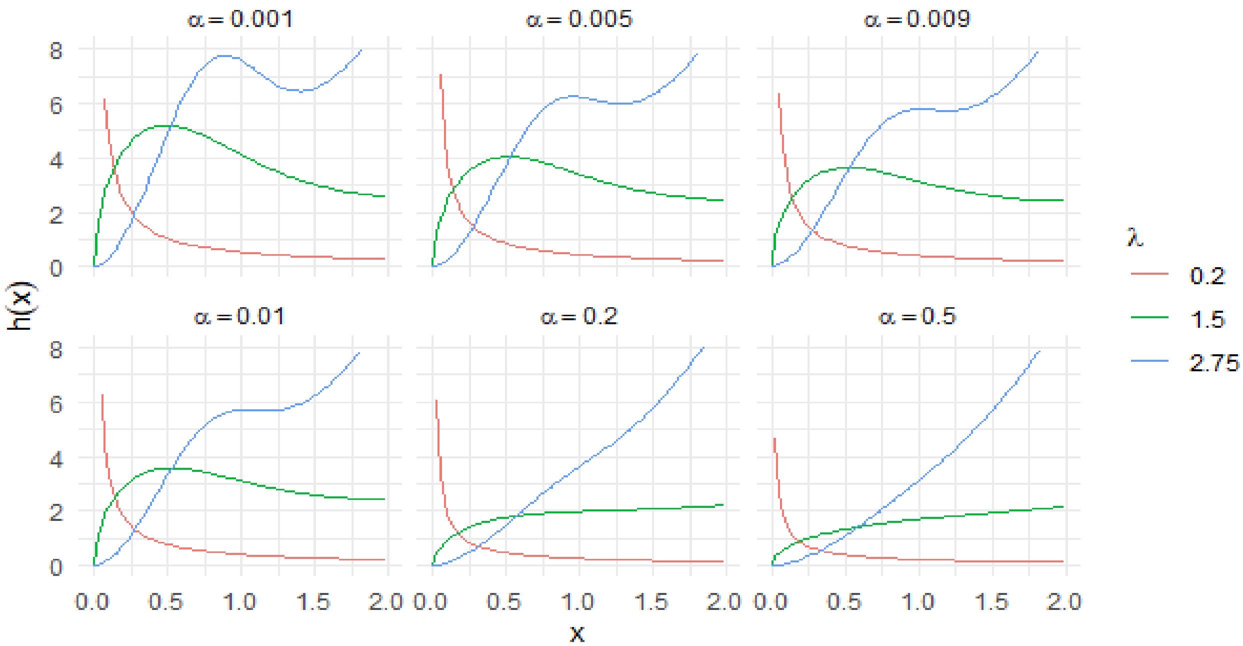

Figure 4. Hazard function $h_{APOPW}(t)$ for different α and λ values.

The hazard function exhibits diverse shapes depending on the parameters α and λ. For $\alpha < 1$, the hazard rate is typically decreasing, indicating early failures. For $\alpha > 1$, it becomes increasing, suggesting aging or wear-out mechanisms. In some cases, particularly when α is very small and λ is moderate, the hazard function may exhibit a *bathtub shape*, which is common in reliability studies. This flexibility makes the distribution suitable for modeling various lifetime behaviors.

3.2. The Moments and Moment Generating Function

The kth Moments for the APOPWD distribution are defined as follows:

$$E\left(T^x\right) = \int_0^\infty t^x f_{APOPW}(t)dt$$
$$= \int_0^\infty t^x \frac{log\alpha}{\alpha - 1}\alpha^{1-e^{-t^\lambda}} \lambda t^{\lambda-1} e^{-t^\lambda} dt$$

using the following string representation:

$$\alpha^{-z} = \sum_{k=0}^{\infty} \frac{(-log\alpha)^k z^k}{k!} \tag{12}$$

expanding $\alpha^{-e^{-t^\lambda}}$ in PDF and letting$z = e^{-t^\lambda}$, then the density of APOPW distribution can be written as follows:

$$f_{APOPW}(t) = \frac{\alpha \log \alpha}{\alpha - 1}\lambda t^{\lambda-1} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^k}{k!} \left(e^{-t^\lambda}\right)^k e^{-t^\lambda} = \frac{\alpha}{1-\alpha}\lambda t^{\lambda-1} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^{k+1}}{k!} e^{-(k+1)t^\lambda} \tag{13}$$

Now, we can obtain the moments :

$$\begin{aligned} E\left(T^{x}\right) &= \int_{0}^{\infty} t^{x} \frac{\alpha}{1-\alpha} \lambda t^{\lambda-1} \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!} e^{-(k+1)t^{\lambda}} \\ &= \frac{\alpha}{1-\alpha} \lambda \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!} \int_{0}^{\infty} t^{x+\lambda-1} e^{-(k+1)t^{\lambda}} dt \end{aligned} \tag{14}$$

Let use $u = (k+1)\, t^{\lambda}$, then the derivatives of z with respect to t give $\frac{du}{dt} = (k+1)\, \lambda t^{\lambda-1}$, so

$$\begin{aligned} E(T^{x}) &= \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)} \int_{0}^{\infty} \left(\left(\frac{u}{(k+1)}\right)^{\frac{1}{\lambda}}\right)^{x} e^{-u} du \\ &= \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)\,(k+1)^{\frac{x}{\lambda}}} \int_{0}^{\infty} u^{\frac{x}{\lambda}} e^{-u} du \\ E(T^{x}) &= \frac{\alpha}{1-\alpha} \Gamma(\frac{x}{\lambda}+1) \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)^{\frac{x}{\lambda}+1}} \end{aligned} \tag{15}$$

The moment generating function $M_T(x)$ of the APOPW distribution For a continuous random variable T, is given by

$$M_T(\nu) = E\left(e^{\nu T}\right) = \int_{0}^{\infty} e^{\nu t} f_{APOPW}(t) dt \tag{16}$$

Using series representation $e^{\nu t} = \sum\limits_{x=0}^{\infty} \frac{(\nu t)^x}{x!}$, so

$$\begin{aligned} M_T(\nu) &= \int_{0}^{\infty} \sum_{x=0}^{\infty} \frac{(\nu t)^{x}}{x!} f_{APOPW}(t)\, dt = \sum_{x=0}^{\infty} \frac{(\nu)^{x}}{x!} \int_{0}^{\infty} t^{x} f_{APOPW}(t)\, dt = \sum_{x=0}^{\infty} \frac{(\nu)^{x}}{x!} E\left(T^{x}\right) \\ &= \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \sum_{x=0}^{\infty} \frac{(-\log\alpha)^{k+1} (\nu)^{x}\, \Gamma\left(\frac{x}{\lambda}+1\right)}{x!k!\,(k+1)^{\frac{x}{\lambda}+1}} \end{aligned} \tag{17}$$

The mean

$$\mu = E(T) = \frac{\alpha}{1-\alpha} \Gamma(\frac{1}{\lambda}+1) \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)^{\frac{1}{\lambda}+1}} \tag{18}$$

The variance of the APOPW distribution is

$$Var(T) = \frac{\alpha}{1-\alpha} \Gamma(\frac{2}{\lambda}+1) \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)^{\frac{2}{\lambda}+1}} - \left(\frac{\alpha}{1-\alpha} \Gamma(\frac{1}{\lambda}+1) \sum_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!\,(k+1)^{\frac{1}{\lambda}+1}}\right)^{2} \tag{19}$$

We obtained now the coefficient of variation λ, skewness and kurtosis of the APOPW distribution

$$C.V = \frac{\sqrt{\frac{\alpha}{1-\alpha} \Gamma(\frac{2}{\lambda}+1) \sum\limits_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{2}{\lambda}+1}} - \left(\frac{\alpha}{1-\alpha} \Gamma(\frac{1}{\lambda}+1) \sum\limits_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{1}{\lambda}+1}}\right)^{2}}}{\frac{\alpha}{1-\alpha} \Gamma(\frac{1}{\lambda}+1) \sum\limits_{k=0}^{\infty} \frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{1}{\lambda}+1}}}$$

$$skewness = \frac{\frac{\alpha}{1-\alpha}\Gamma(\frac{3}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{3}{\lambda}+1}}}{\left\{\frac{\alpha}{1-\alpha}\Gamma(\frac{2}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{2}{\lambda}+1}}-\left(\frac{\alpha}{1-\alpha}\Gamma(\frac{1}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{1}{\lambda}+1}}\right)^2\right\}^{\frac{3}{2}}}$$

$$kurtosis = \frac{\frac{\alpha}{1-\alpha}\Gamma(\frac{4}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{4}{\lambda}+1}}}{\left\{\frac{\alpha}{1-\alpha}\Gamma(\frac{2}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{2}{\lambda}+1}}-\left(\frac{\alpha}{1-\alpha}\Gamma(\frac{1}{\lambda}+1)\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!(k+1)^{\frac{1}{\lambda}+1}}\right)^2\right\}^{2}}$$

3.3. *Quantile function*

The quantile function T of the APOPW distribution is given by

$$t_\beta = \left\{-\log\left(\frac{\log\left(\alpha/\left((\alpha-1)\beta+1\right)\right)}{\log\alpha}\right)\right\}^{\frac{1}{\lambda}} \tag{20}$$

where $F_{APOPW}(t) = \beta = \frac{\alpha^{1-e^{-t^\lambda}}-1}{\alpha-1}$ and then $t = F^{-1}_{APOPW}(\beta)$. For more details see [2].
The median of the APOPW distribution can be obtained as

$$M = t_{\frac{1}{2}} = \left\{-\log\left(\frac{\log\left(2\alpha/\left(\alpha+1\right)\right)}{\log\alpha}\right)\right\}^{\frac{1}{\lambda}} \tag{21}$$

3.4. *The Rényi and Shannon Entropy*

The entropy of a random variable T is a measure of variation of uncertainty (see, Rényi, 1961) [3], that of the Alpha Power one-parameter distribution is given by

$$I_{RE}(s) = \frac{1}{1-s}\log\left(\int_{-\infty}^{+\infty} f_{APOPW}(t)^s\,dt\right) \tag{22}$$

where s (integer) > 0 and $s \neq 0$. Using the PDF in (13) we have:

$$\begin{aligned} I_{RE}(s) &= \frac{1}{1-s}\log\left(\int_0^{+\infty}\left\{\frac{\alpha}{1-\alpha}\lambda t^{\lambda-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}e^{-(k+1)t^\lambda}\right\}^s dt\right) \\ &= \frac{1}{1-s}\log\left(\left\{\frac{\alpha}{1-\alpha}\lambda\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\right\}^s\int_0^{+\infty} t^{s(\lambda-1)}e^{-s(k+1)t^\lambda}\,dt\right) \end{aligned} \tag{23}$$

Let $v = s(k+1)t^\lambda \implies \frac{dv}{dt} = \lambda s(k+1)t^{\lambda-1}$, the Rényi entropy reduces to

$$\begin{aligned} I_{RE}(s) &= \frac{1}{1-s}\log\left(\left\{\frac{\alpha}{1-\alpha}\lambda\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\right\}^s\int_0^{+\infty}\left(\frac{v}{s(k+1)}\right)^{\frac{(\lambda-1)(s-1)}{\lambda}}\frac{e^{-v}}{\lambda s(k+1)}dv\right) \\ &= \frac{s}{1-s}\log\left\{\frac{\alpha}{1-\alpha}\lambda\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\right\} + \frac{1}{1-s}\log\left\{\frac{\Gamma\left(\frac{(\lambda-1)(s-1)}{\lambda}+1\right)}{\lambda\left\{s(k+1)\right\}^{\frac{(\lambda-1)(s-1)}{\lambda}+1}}\right\} \end{aligned} \tag{24}$$

The Shannon entropy SE_T from a random variable T is defined as

$$SE_T = -\int_{-\infty}^{+\infty} f_{APOPW}(t) log f_{APOPW}(t) dt \tag{25}$$

Using the PDF in (13), SE_T is

$$SE_T = \log\left(\frac{\alpha-1}{\alpha\lambda\log\alpha}\right) + \frac{\alpha}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\left(\frac{\log\alpha}{k+2} - \frac{1}{(k+1)^2} + (\lambda-1)\lambda I\right) \tag{26}$$

where

$$I = \int_0^{\infty} t^{\lambda-1} e^{-(k+1)t^{\lambda}} log t dt$$

Now by applying the transformation $t^{\lambda} = y$ and

$$\int_0^{\infty} e^{-\vartheta t}\log t dt = -\frac{1}{\vartheta}(c + \log\vartheta)$$

where c is the Euler constant. Then the Shannon entropy SE_T is derived as :

$$SE_T = \log\left(\frac{\alpha-1}{\alpha\lambda\log\alpha}\right) + \frac{\alpha}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\log\alpha)^{k+1}}{k!}\left(\frac{\log\alpha}{k+2} - \frac{1}{(k+1)^2} + \frac{(1-\lambda)(c+\log(k+1))}{(k+1)\lambda}\right) \tag{27}$$

3.5. Stress-Strength Reliability

The measure of reliability has many applications, especially in the area of engineering. The component fails at the instant that the random stress T_2 applied to it exceeds the random strength T_1, and the component will function satisfactorily whenever $T_1 > T_2$. Hence, $R = P[T_2 < T_1]$ is a measure of component reliability. We derive the reliability R when T_1 and T_2 have independent $APOPW(\alpha_1, \lambda)$ and $APOPW(\alpha_2, \lambda)$ distributions. The reliability is defined by

$$\begin{aligned} R = P[T_2 < T_1] &= \int_0^{\infty} f_1(t;\alpha_1,\lambda) F_2(t;\alpha_2,\lambda) dt \\ &= \int_0^{\infty} \frac{log\alpha_1}{\alpha_1 - 1}\alpha_1^{1-e^{-t^{\lambda}}}\lambda t^{\lambda-1}e^{-t^{\lambda}}\left(\frac{\alpha_2^{1-e^{-t^{\lambda}}}-1}{\alpha_2-1}\right) dt \end{aligned} \tag{28}$$

Using the string representation in equation (12), R can be written as :

$$R = \frac{\lambda\alpha_1\alpha_2\log\alpha_1}{(\alpha_1-1)(\alpha_2-1)}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\log\alpha_1)^k(-\log\alpha_2)^m}{k!m!}\int_0^{+\infty} t^{\lambda-1}e^{-(k+m+1)t^{\lambda}} dt - \frac{1}{\alpha_2-1} \tag{29}$$

By using the transformation $y = (k+m+1)t^{\lambda}$, the stress-strength reliability reduces to

$$R = \frac{\lambda \alpha_1 \alpha_2 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\log \alpha_1)^k (-\log \alpha_2)^m}{k! m!} \int_0^{+\infty} \left(\frac{y}{(k+m+1)} \right)^{\frac{\lambda-1}{\lambda}}$$
$$\times \frac{e^{-y}}{\lambda (k+m+1) \left(\frac{y}{(k+m+1)\beta} \right)^{\frac{\lambda-1}{\lambda}}} dy - \frac{1}{\alpha_2 - 1} \tag{30}$$
$$R = \frac{1}{\alpha_2 - 1} \left(\frac{\alpha_1 \alpha_2}{(1-\alpha_1)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\log \alpha_1)^{k+1} (-\log \alpha_2)^m}{k! m! (k+m+1)} - 1 \right)$$

3.6. Mean deviations

The deviation from the mean and the median are used to measure the dispersion and spread in a population from the centre. The mean deviation from the mean and median can be written as

$$D(\mu) = \int_0^{\infty} |t - \mu| f_{APOPW}(t) dt = 2\mu F_{APOPW}(\mu) - 2 \int_0^{\mu} t f_{APOPW}(t) dt, \tag{31}$$

$$D(M) = \int_0^{\infty} |t - M| f_{APOPW}(t) dt = \mu - 2 \int_0^{M} t f_{APOPW}(t) dt. \tag{32}$$

where $D(\mu)$ and $D(M)$ is the mean deviation from the mean and median respectively.
Using the PDF in (13) and the transformation $u = (k+1) t^{\lambda}$, the required integral becomes :

$$\begin{aligned} \int_0^b t f_{APOP}(t) dt &= \frac{\alpha}{1-\alpha} \lambda \sum_{k=0}^{\infty} \frac{(-\log \alpha)^{k+1}}{k!} \int_0^b t^{\lambda} e^{-(k+1)t^{\lambda}} dt \\ &= \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^{k+1}}{k! (k+1)^{\frac{1}{\lambda}+1}} \gamma \left(\frac{1}{\lambda} + 1, (k+1) b^{\lambda} \right) \end{aligned}$$

where γ is the lower incomplete gamma function.

we obtain,

$$\begin{aligned} D(\mu) &= 2\mu F_{APOPW}(\mu) - 2 \int_0^{\mu} t f_{APOPW}(t) dt, \\ &= 2\mu F_{APOPW}(\mu) - \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^{k+1}}{k! (k+1)^{\frac{1}{\lambda}+1}} \gamma \left(\frac{1}{\lambda} + 1, (k+1) \mu^{\lambda} \right). \end{aligned} \tag{33}$$

and

$$\begin{aligned} D(M) &= \mu - 2 \int_0^{M} t f_{APOPW}(t) dt, \\ &= \mu - \frac{\alpha}{1-\alpha} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^{k+1}}{k! (k+1)^{\frac{1}{\lambda}+1}} \gamma \left(\frac{1}{\lambda} + 1, (k+1) M^{\lambda} \right). \end{aligned} \tag{34}$$

3.7. Extreme order statistics of APOPW

Let $T_1, T_2, ..., T_n$ be a sample of n random variables that follows the APOPW distribution. For α = 1, the APOPW reduces to one-parameter Weibull distribution. The asymptotic distribution of the sample maximum is of extreme type. According to Arnold et al. (2008) [22], we find that the asymptotic distribution of the sample minimum of $T_{\overline{1,n}}$ is Weibull type with shape parameter $\lambda > 0$:

$$\lim_{x\to 0} \frac{F_{APOPW}\,(xt)}{F_{APOPW}\,(x)} = t$$

and

$$\lim_{x\to \infty} \frac{f_{APOPW}\,(xt)}{f_{APOPW}\,(x)} = t^{\lambda-1}.$$

4. Estimation Methods

Maximum likelihood estimation

Let $T_1, T_2, \ldots, T_n$ be a random sample of size n taken from the APOPW distribution. Then the log-likelihood function is given by

$$\begin{aligned}
\ln\ L\,(\alpha,\lambda) &= \ln\left\{\prod_{i=1}^{n} f_{APOP}\,(t_i;\alpha,\lambda)\right\} \\
&= \ln\ \left\{\prod_{i=1}^{n} \frac{\log}{\alpha-1}\alpha^{1-e^{-t_i^{\lambda}}}\lambda t_i^{\lambda-1}e^{-t_i^{\lambda}}\right\} \\
&= \ln\left\{\left(\frac{\lambda\alpha\log\alpha}{\alpha-1}\right)^{n}\prod_{i=1}^{n}\alpha^{-e^{-t_i^{\lambda}}}t_i^{\lambda-1}e^{-t_i^{\lambda}}\right\} \\
&= n\ln\left(\frac{\log\alpha}{\alpha-1}\right)+n\ln\lambda\alpha+(\lambda-1)\sum_{i=1}^{n}\ln t_i-\ln\alpha\sum_{i=1}^{n}e^{-t_i^{\lambda}}-\sum_{i=1}^{n}t_i^{\lambda}
\end{aligned} \tag{35}$$

Determining the maximum likelihood estimators for the parameters α and λ requires the use of numerical methods, such as the Newton-Raphson or BFGS algorithms, because the likelihood function is nonlinear.

Least Squares Estimators

Gauss is credited with developing the least squares approach, which is a kind of estimation. It is typically applied to the linear model's parameter estimate. Swain, Venkatraman, and Wilson (1988) [16] estimated the parameters of beta distributions using least squares estimates and weighted least squares estimates. Also see Alkasasbeh and Raqab (2009) [4], Gupta and Kundu (2001) [8], and Kundu and Raqab (2005) [9].

Let $T_1, T_2, \ldots, T_n$ be order statistics from a random sample of size n from a distribution function $F(\cdot)$ and suppose $T_{(i)}$ denotes the ordered sample. The proposed method uses the distribution of $F(T_{(i)}),\ i = 1, 2, \ldots, n$. For a sample of size n, we have:

$$\mathbb{E}[F(T_{(i)})] = \frac{i}{n+1}, \quad \mathrm{Var}[F(T_{(i)})] = \frac{i(n-i+1)}{(n+1)^2(n+2)},$$

$$\mathrm{Cov}(F(T_{(i)}), F(T_{(j)})) = \frac{i(n-j+1)}{(n+1)^2(n+2)}, \quad \text{for } i < j.$$

The least squares estimates can be obtained by minimizing:

$$\sum_{i=1}^{n}\left(F(T_{(i)})-\frac{i}{n+1}\right)^2$$

with respect to the unknown parameters.

Methods of Anderson-Darling

In 1952, Anderson and Darling [12] created the Anderson-Darling test to identify sample distributions that deviate from normal. This test served as an alternative to existing statistical tests in this regard. According to Anderson and Darling (1954) [13], Pettitt (1976) [7], and Stephens (2013) [14], the AD test specifically converges very rapidly towards the asymptote.

The Anderson-Darling estimators of parameters are obtained by minimizing the function:

$$AD=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log(F(t_{(i)}))+\log(S(t_{(i)}))\right]$$

The Cramér–von Mises Estimation

Macdonald (1971) [10] provides empirical support for the selection of Cramér–von Mises type minimum distance estimators, demonstrating that the estimator's bias is lower than that of other minimum distance estimators.

The Cramér–von Mises estimate of the parameters is obtained by minimizing:

$$C=\frac{1}{12n}\sum_{i=1}^{n}\left(F(t_{(i)})-\frac{2i-1}{2n}\right)^2$$

Method of Maximum Product of Spacings

The maximum product of spacings (MPS) approach was presented by Cheng and Amin (1979 [5], 1983 [6]) as a substitute for MLE in the estimation of parameters for continuous univariate distributions. A similar technique was independently developed as an estimate for the Kullback–Leibler measure of information by Ranneby (1984) [11].

The estimation of parameters is carried out by maximizing the following equation:

$$X=\left(\prod_{i=1}^{n+1}D_i\right)^{1/(n+1)}, \quad \text{or, equivalently,}$$

$$\log(X)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log(D_i), \quad \text{where } D_i=F(t_{(i)})-F(t_{(i-1)})$$

5. Simulation

Using a simulation study, we generate samples (with increasing sample sizes) from the new APOPW distribution with given parameter values and estimate them back using the Maximum Likelihood Estimation (MLE), Anderson–Darling (AD), Cramér–von Mises (CVM), Maximum Product of Spacings (MPS), and Ordinary Least Squares (OLS) methods. To generate random samples, we employed the inverse transform sampling technique: independent uniform random numbers $U_1, \ldots, U_n \sim \text{Uniform}(0,1)$ were generated, and each was transformed via the quantile function $Q_F(u;\theta)$. Since the quantile function does not admit a closed form, it was obtained numerically by solving

$$F_X(x;\theta) = u, \quad 0 < u < 1,$$

for x using a root-finding routine. This ensures that all simulated data are exact realizations from the specified distribution. The performance of the estimation methods is then measured using the mean square error (MSE), bias, and mean relative error (MRE) for each case. Different sample sizes are considered in the experiments, ranging from $n = 25$ to 1000, and various values of the parameters α and λ are used. These methods of estimation asymptotically converge in probability to the actual parameter values, and the solution is unique.

The results in Tables 1 to 4 and the figures from 5 to 8 show that all methods of the estimation approach the true value as the sample size increases, but the MLE is the best. The results show that as the sample size increases, the parameter estimates approach the true values, with the respective MSE, bias, and MRE computed values all approaching zero. This is also shown to be true for each case of the four simulation studies. The evaluation of estimation method efficacy in this investigation is based on the utilization of Three specific metrics, delineated as follows:

1. The average of absolute bias (BIAS):

$$\left|\text{bias}\left(\hat{\theta}\right)\right| = \frac{1}{N}\sum_{i=1}^{N}\left|\left(\hat{\theta}_i - \theta_0\right)\right|. \tag{36}$$

2. The mean squared error (MSE):

$$\text{MSE}\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_0\right)^2. \tag{37}$$

3. The mean absolute relative error (MRE):

$$\text{MRE}\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|\hat{\theta}_i - \theta_0\right|}{\theta_0}. \tag{38}$$

Where $\hat{\theta} = \left(\hat{\alpha}, \hat{\lambda}\right)$.

Table 1. Simulation values of Estimates, Bias, MSE, and MRE for $(\alpha = 0.25, \lambda = 0.75)$.

Sample Size	Method	$\hat{\alpha}$	$\hat{\lambda}$	Bias($\hat{\alpha}$)	Bias($\hat{\lambda}$)	MSE($\hat{\alpha}$)	MSE($\hat{\lambda}$)	MRE($\hat{\alpha}$)	MRE($\hat{\lambda}$)
25	MLE	0.324369	0.813353	0.179711	0.108679	0.076437	0.019392	0.718845	0.144905
50	MLE	0.271795	0.771895	0.105164	0.053250	0.026644	0.005048	0.420656	0.071000
100	MLE	0.267370	0.770592	0.089287	0.048144	0.012593	0.004043	0.357147	0.064192
150	MLE	0.258696	0.761456	0.072934	0.041724	0.008692	0.002739	0.291737	0.055632
300	MLE	0.250777	0.758737	0.044352	0.026779	0.003045	0.001197	0.177407	0.035705
500	MLE	0.257296	0.754542	0.033648	0.022357	0.001826	0.000732	0.134593	0.029809
1000	MLE	0.249704	0.750975	0.023988	0.014207	0.000837	0.000306	0.095952	0.018942
25	AD	0.396285	0.741796	0.231119	0.103433	0.200720	0.016522	0.924475	0.137910
50	AD	0.303949	0.752035	0.120141	0.076031	0.026041	0.009526	0.480562	0.101374
100	AD	0.245611	0.754589	0.073828	0.043996	0.007876	0.003138	0.295310	0.058662
150	AD	0.255527	0.756381	0.067221	0.042240	0.007268	0.003011	0.268884	0.056320
300	AD	0.249384	0.750655	0.046625	0.030872	0.003050	0.001492	0.186500	0.041162
500	AD	0.254486	0.748928	0.032611	0.022270	0.001876	0.000806	0.130442	0.029693
1000	AD	0.245710	0.752525	0.022046	0.017482	0.000744	0.000458	0.088184	0.023310
25	CVM	0.298819	0.792968	0.191086	0.130043	0.079134	0.025123	0.764342	0.173391
50	CVM	0.281107	0.768103	0.125054	0.087496	0.028446	0.014256	0.500216	0.116661
100	CVM	0.264641	0.757663	0.094035	0.068163	0.016109	0.007417	0.376140	0.090885
150	CVM	0.265220	0.752102	0.067040	0.047200	0.008889	0.003557	0.268159	0.062933
300	CVM	0.248879	0.757505	0.042238	0.036216	0.002827	0.002163	0.168954	0.048288
500	CVM	0.254776	0.753045	0.035079	0.028135	0.002009	0.001278	0.140315	0.037513
1000	CVM	0.253803	0.751619	0.028523	0.019872	0.001191	0.000584	0.114093	0.026496
25	MPS	0.359585	0.717946	0.189287	0.101523	0.107988	0.014917	0.757149	0.135364
50	MPS	0.316639	0.713727	0.131460	0.076779	0.049231	0.008451	0.525840	0.102373
100	MPS	0.288796	0.719775	0.086983	0.045951	0.013935	0.003244	0.347933	0.061269
150	MPS	0.267216	0.728722	0.065284	0.041768	0.006981	0.002664	0.261134	0.055691
300	MPS	0.261940	0.740038	0.042325	0.027451	0.003190	0.001164	0.169301	0.036602
500	MPS	0.253688	0.745564	0.031612	0.021930	0.001729	0.000767	0.126446	0.029240
1000	MPS	0.257929	0.746072	0.026261	0.015204	0.001116	0.000349	0.105043	0.020273
25	OLS	0.354414	0.740086	0.201054	0.107783	0.080182	0.020045	0.804215	0.143711
50	OLS	0.309470	0.731532	0.133111	0.082354	0.038424	0.009795	0.532443	0.109806
100	OLS	0.262265	0.750616	0.083670	0.054020	0.011016	0.004548	0.334682	0.072026
150	OLS	0.259752	0.749154	0.067524	0.050141	0.007154	0.004143	0.270096	0.066855
300	OLS	0.265572	0.741730	0.045741	0.033617	0.003368	0.001833	0.182964	0.044823
500	OLS	0.249839	0.745994	0.037211	0.027198	0.002394	0.001072	0.148843	0.036263
1000	OLS	0.254805	0.750165	0.026729	0.020412	0.001095	0.000650	0.106914	0.027217

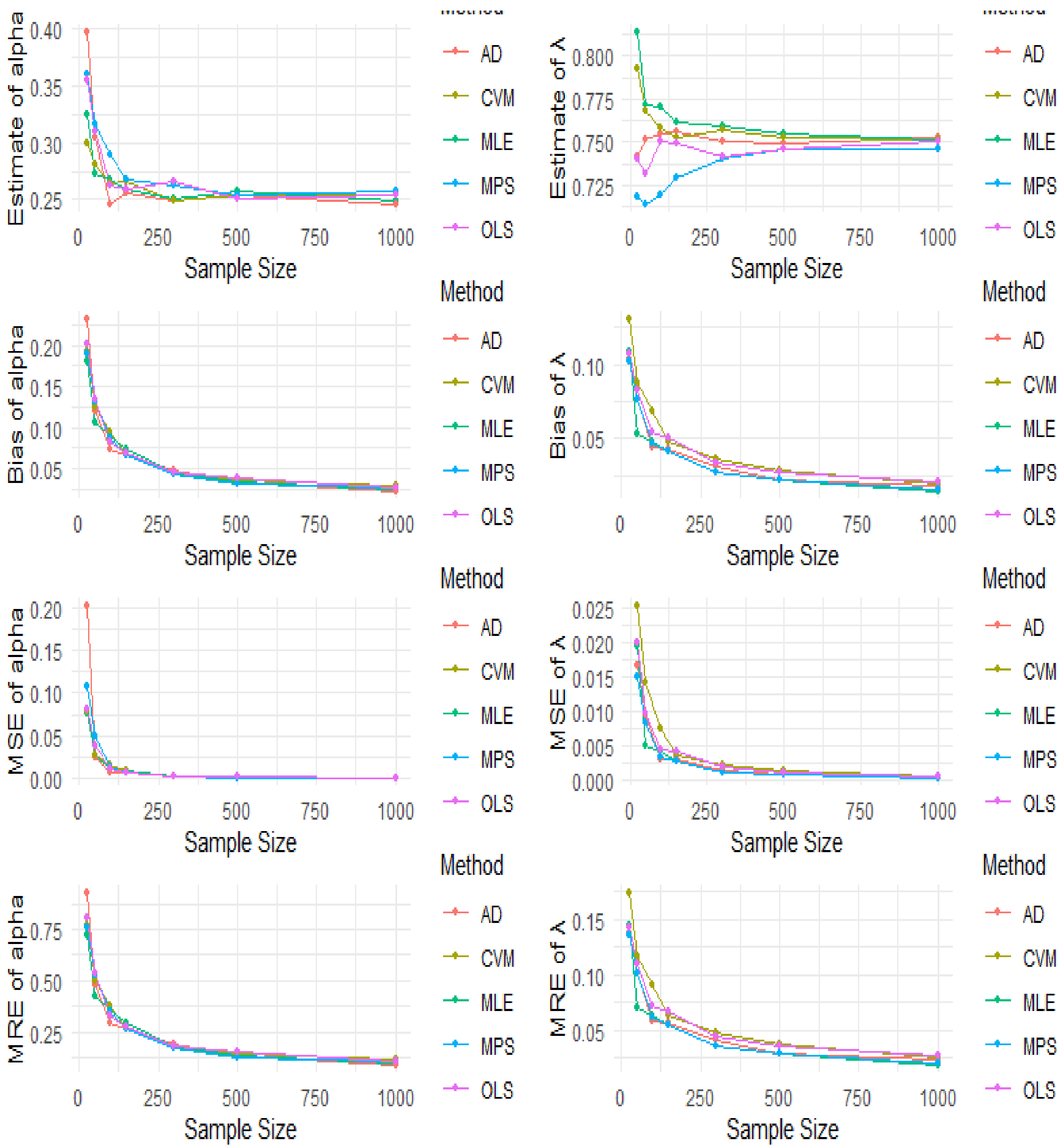

Figure 5. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 1.

Table 2. Simulation values of Estimates, Bias, MSE, and MRE for ($\alpha = 0.5, \lambda = 1$).

Sample Size	Method	$\hat{\alpha}$	$\hat{\lambda}$	Bias($\hat{\alpha}$)	Bias($\hat{\lambda}$)	MSE($\hat{\alpha}$)	MSE($\hat{\lambda}$)	MRE($\hat{\alpha}$)	MRE($\hat{\lambda}$)
25	MLE	0.604622	1.047836	0.335612	0.145728	0.217046	0.037464	0.671225	0.145728
50	MLE	0.517619	1.010105	0.193513	0.087307	0.059491	0.012565	0.387026	0.087307
100	MLE	0.498266	1.014357	0.130619	0.063215	0.026686	0.006716	0.261238	0.063215
150	MLE	0.516519	1.011581	0.128424	0.048852	0.026613	0.004187	0.256848	0.048852
300	MLE	0.511985	1.001961	0.089815	0.034152	0.012456	0.002008	0.179629	0.034152
500	MLE	0.496278	1.008507	0.062265	0.026687	0.006331	0.001036	0.124529	0.026687
1000	MLE	0.506416	1.004580	0.051366	0.019808	0.003855	0.000635	0.102731	0.019808
25	AD	0.575676	1.019750	0.301122	0.128744	0.211167	0.035588	0.602245	0.128744
50	AD	0.556715	0.997130	0.224822	0.100479	0.112401	0.015585	0.449643	0.100479
100	AD	0.542661	1.016364	0.144104	0.065705	0.036198	0.007568	0.288207	0.065705
150	AD	0.514910	1.001622	0.126531	0.052092	0.028367	0.004634	0.253062	0.052092
300	AD	0.492253	0.999875	0.091754	0.041477	0.013160	0.002558	0.183509	0.041477
500	AD	0.502007	1.003336	0.057953	0.032656	0.005686	0.001677	0.115905	0.032656
1000	AD	0.499201	0.998304	0.046245	0.019017	0.003448	0.000543	0.092490	0.019017
25	CVM	0.583454	1.055700	0.346805	0.159331	0.224253	0.049371	0.693609	0.159331
50	CVM	0.560392	1.001347	0.244876	0.116635	0.101998	0.020250	0.489753	0.116635
100	CVM	0.553924	0.992614	0.162703	0.072389	0.048181	0.007816	0.325405	0.072389
150	CVM	0.515043	1.011700	0.119995	0.068832	0.023563	0.009268	0.239991	0.068832
300	CVM	0.499455	1.007186	0.088295	0.045690	0.011263	0.003284	0.176590	0.045690
500	CVM	0.493539	1.001203	0.073037	0.036136	0.008245	0.002000	0.146074	0.036136
1000	CVM	0.497856	1.004157	0.048515	0.026809	0.003700	0.001189	0.097030	0.026809
25	MPS	0.603761	0.925469	0.275039	0.132997	0.137672	0.027679	0.550078	0.132997
50	MPS	0.565887	0.933412	0.193693	0.096719	0.068649	0.013239	0.387386	0.096719
100	MPS	0.537141	0.980598	0.142307	0.061653	0.030917	0.005822	0.284613	0.061653
150	MPS	0.565756	0.979002	0.135245	0.056873	0.033685	0.005091	0.270491	0.056873
300	MPS	0.535503	0.980922	0.093439	0.035997	0.014105	0.001853	0.186879	0.035997
500	MPS	0.524481	0.991853	0.068327	0.029910	0.007343	0.001412	0.136654	0.029910
1000	MPS	0.494172	0.997925	0.042785	0.020478	0.003033	0.000690	0.085570	0.020478
25	OLS	0.666928	1.036401	0.397914	0.182625	0.297665	0.079671	0.795828	0.182625
50	OLS	0.576001	0.994268	0.212287	0.099645	0.094956	0.016035	0.424575	0.099645
100	OLS	0.552238	0.981959	0.154626	0.086677	0.047188	0.010492	0.309252	0.086677
150	OLS	0.508533	0.998313	0.113951	0.069449	0.021360	0.006893	0.227903	0.069449
300	OLS	0.510027	0.996797	0.080990	0.049710	0.010379	0.003839	0.161979	0.049710
500	OLS	0.510507	1.004018	0.066430	0.030823	0.007011	0.001563	0.132860	0.030823
1000	OLS	0.508137	0.998619	0.053165	0.026633	0.004298	0.001096	0.106330	0.026633

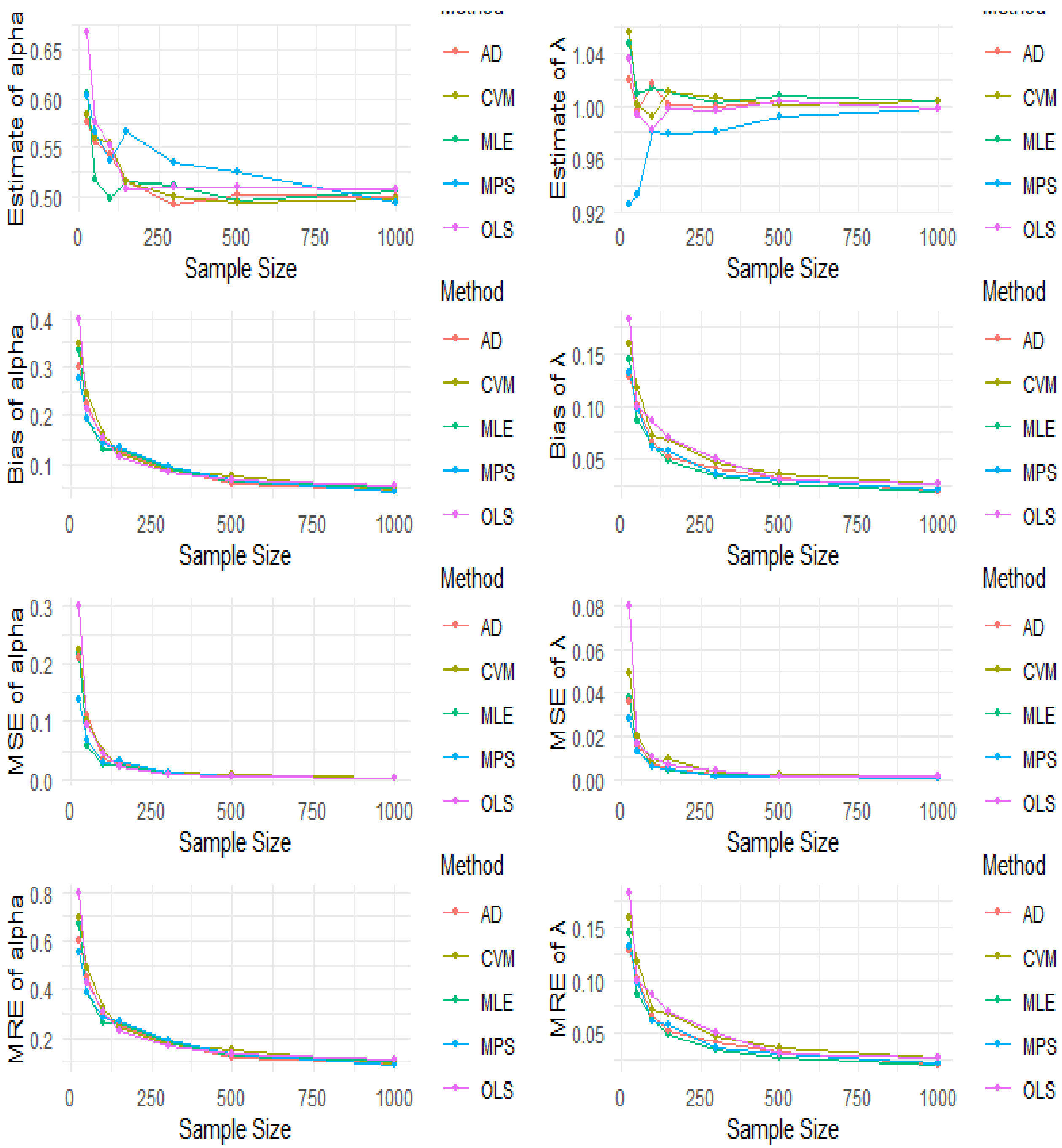

Figure 6. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 2.

Table 3. Simulation values of Estimates, Bias, MSE, and MRE for $(\alpha = 2, \lambda = 1)$.

Sample Size	Method	$\hat{\alpha}$	$\hat{\lambda}$	Bias($\hat{\alpha}$)	Bias($\hat{\lambda}$)	MSE($\hat{\alpha}$)	MSE($\hat{\lambda}$)	MRE($\hat{\alpha}$)	MRE($\hat{\lambda}$)
25	MLE	3.212831	1.067780	2.008607	0.125458	11.573667	0.032344	1.004304	0.125458
50	MLE	2.233042	1.028175	0.887122	0.076538	1.346299	0.011634	0.443561	0.076538
100	MLE	2.182920	1.005188	0.623708	0.053182	0.860550	0.004656	0.311854	0.053182
150	MLE	1.971027	0.996391	0.395466	0.041949	0.247083	0.003001	0.197733	0.041949
300	MLE	2.122692	1.011020	0.425895	0.033407	0.319862	0.001742	0.212948	0.033407
500	MLE	2.020594	1.005905	0.255106	0.029985	0.105357	0.001272	0.127553	0.029985
1000	MLE	2.012578	0.998449	0.162673	0.016597	0.047159	0.000440	0.081337	0.016597
25	AD	2.933367	1.007198	1.506997	0.120514	11.956806	0.021521	0.753499	0.120514
50	AD	2.165274	1.006941	0.812413	0.083725	1.359413	0.012374	0.406207	0.083725
100	AD	2.112530	0.998841	0.531703	0.060039	0.529811	0.005870	0.265851	0.060039
150	AD	2.208226	1.009680	0.553967	0.049818	0.554768	0.004523	0.276983	0.049818
300	AD	1.987243	0.994139	0.306564	0.041227	0.142356	0.002721	0.153282	0.041227
500	AD	2.049418	1.000130	0.232564	0.026023	0.090476	0.001150	0.116282	0.026023
1000	AD	2.009585	0.997505	0.177858	0.019207	0.052310	0.000597	0.088929	0.019207
25	CVM	2.562685	1.023676	1.344952	0.158900	6.203513	0.042634	0.672476	0.158900
50	CVM	2.198450	1.015783	0.808844	0.112450	1.225484	0.023045	0.404422	0.112450
100	CVM	2.157844	0.998602	0.522876	0.072332	0.452549	0.008609	0.261438	0.072332
150	CVM	1.982215	1.003727	0.459847	0.060209	0.395603	0.005596	0.229924	0.060209
300	CVM	2.083594	1.002607	0.372878	0.042888	0.217785	0.002876	0.186439	0.042888
500	CVM	2.089927	0.996862	0.271897	0.030191	0.138053	0.001304	0.135949	0.030191
1000	CVM	2.019047	1.003084	0.185343	0.023690	0.050599	0.000928	0.092671	0.023690
25	MPS	2.322445	0.906181	1.122302	0.136755	2.811999	0.026985	0.561151	0.136755
50	MPS	2.385805	0.961383	0.773570	0.085553	1.292548	0.010449	0.386785	0.085553
100	MPS	2.189917	0.974577	0.597207	0.059745	0.569064	0.005025	0.298603	0.059745
150	MPS	2.199184	0.988174	0.455814	0.053204	0.524846	0.004061	0.227907	0.053204
300	MPS	2.102561	0.989501	0.290804	0.035801	0.156821	0.001922	0.145402	0.035801
500	MPS	2.034230	0.987667	0.229667	0.025709	0.094513	0.001027	0.114833	0.025709
1000	MPS	2.025788	0.992832	0.163091	0.020662	0.061937	0.000654	0.081545	0.020662
25	OLS	2.794893	1.010620	1.481708	0.173504	6.340959	0.051214	0.740854	0.173504
50	OLS	2.141161	1.020272	0.745891	0.106810	1.115792	0.017786	0.372945	0.106810
100	OLS	2.186301	0.991772	0.686903	0.069867	0.867224	0.008199	0.343451	0.069867
150	OLS	2.089332	0.997897	0.444889	0.061191	0.315059	0.005769	0.222444	0.061191
300	OLS	2.030200	0.997201	0.303155	0.042940	0.140567	0.002951	0.151578	0.042940
500	OLS	2.027001	0.996773	0.267153	0.033483	0.110639	0.001629	0.133577	0.033483
1000	OLS	2.003389	1.005798	0.157795	0.026513	0.042861	0.001047	0.078898	0.026513

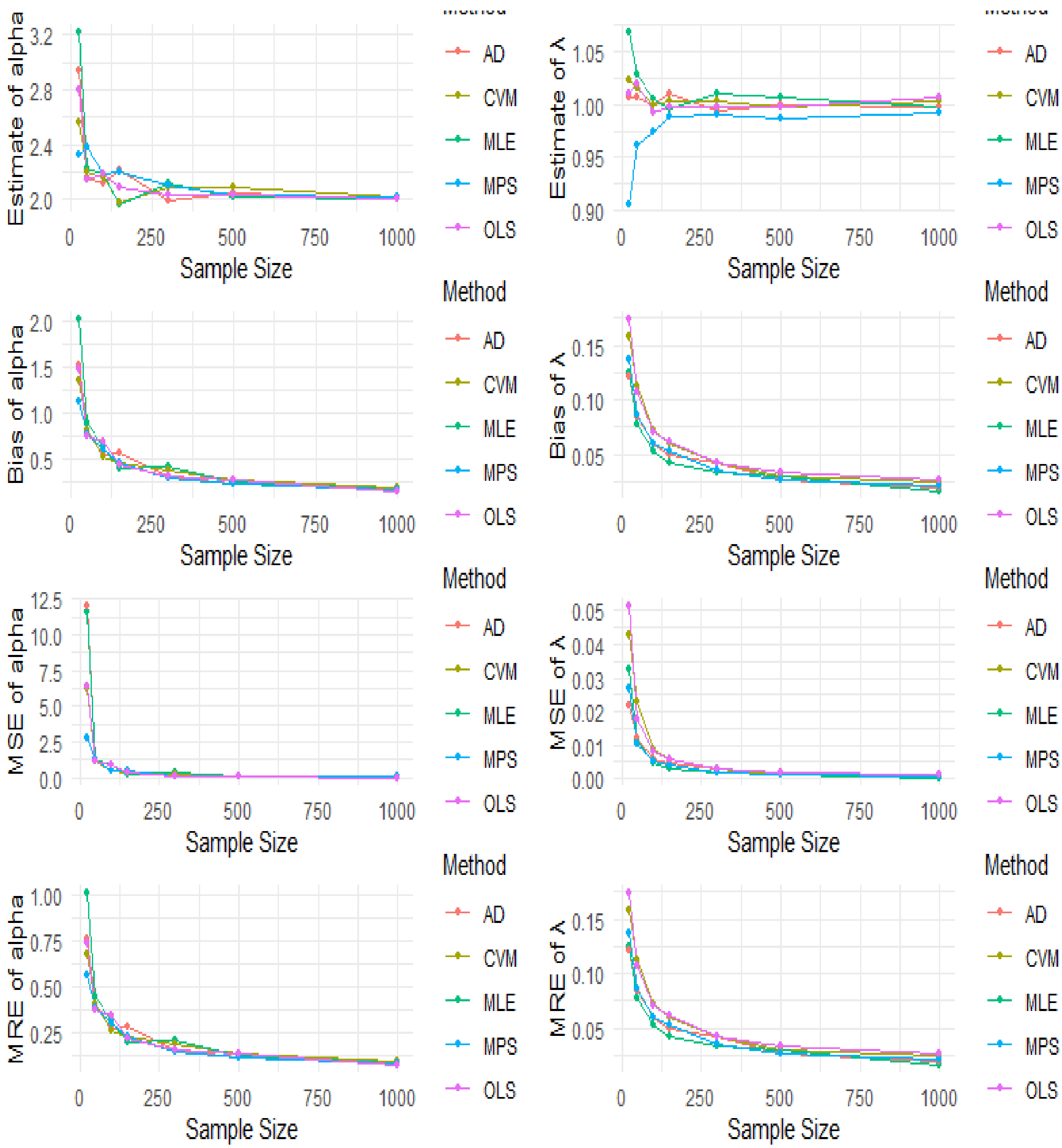

Figure 7. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 3.

Table 4. Simulation values of Estimates, Bias, MSE, and MRE for $(\alpha = 1, \lambda = 1)$.

Sample Size	Method	$\hat{\alpha}$	$\hat{\lambda}$	Bias($\hat{\alpha}$)	Bias($\hat{\lambda}$)	MSE($\hat{\alpha}$)	MSE($\hat{\lambda}$)	MRE($\hat{\alpha}$)	MRE($\hat{\lambda}$)
25	MLE	1.000000	1.043379	0.000000	0.138504	0.000000	0.034957	0.000000	0.138504
50	MLE	1.000000	1.013742	0.000000	0.084444	0.000000	0.011718	0.000000	0.084444
100	MLE	1.000000	1.001318	0.000000	0.055014	0.000000	0.005235	0.000000	0.055014
150	MLE	1.000000	1.014566	0.000000	0.044767	0.000000	0.003222	0.000000	0.044767
300	MLE	1.000000	1.001807	0.000000	0.035574	0.000000	0.001927	0.000000	0.035574
500	MLE	1.000000	0.998352	0.000000	0.024632	0.000000	0.000948	0.000000	0.024632
1000	MLE	1.000000	1.000111	0.000000	0.016885	0.000000	0.000487	0.000000	0.016885
25	AD	1.000000	0.950257	0.000000	0.134100	0.000000	0.028746	0.000000	0.134100
50	AD	1.000000	0.982530	0.000000	0.100758	0.000000	0.015885	0.000000	0.100758
100	AD	1.000000	0.992657	0.000000	0.063703	0.000000	0.006305	0.000000	0.063703
150	AD	1.000000	0.986089	0.000000	0.056962	0.000000	0.005152	0.000000	0.056962
300	AD	1.000000	0.991485	0.000000	0.039224	0.000000	0.002617	0.000000	0.039224
500	AD	1.000000	1.005577	0.000000	0.034094	0.000000	0.001850	0.000000	0.034094
1000	AD	1.000000	0.998423	0.000000	0.021410	0.000000	0.000773	0.000000	0.021410
25	CVM	1.000000	1.009650	0.000000	0.205481	0.000000	0.079776	0.000000	0.205481
50	CVM	1.000000	1.016141	0.000000	0.131566	0.000000	0.030086	0.000000	0.131566
100	CVM	1.000000	0.993484	0.000000	0.094587	0.000000	0.013029	0.000000	0.094587
150	CVM	1.000000	0.992549	0.000000	0.076261	0.000000	0.008857	0.000000	0.076261
300	CVM	1.000000	1.007258	0.000000	0.053902	0.000000	0.004705	0.000000	0.053902
500	CVM	1.000000	0.996156	0.000000	0.040162	0.000000	0.002374	0.000000	0.040162
1000	CVM	1.000000	0.994240	0.000000	0.027664	0.000000	0.001222	0.000000	0.027664
25	MPS	1.000000	0.922500	0.000000	0.127447	0.000000	0.025582	0.000000	0.127447
50	MPS	1.000000	0.941602	0.000000	0.096777	0.000000	0.013453	0.000000	0.096777
100	MPS	1.000000	0.973487	0.000000	0.062208	0.000000	0.006019	0.000000	0.062208
150	MPS	1.000000	0.975185	0.000000	0.054526	0.000000	0.004502	0.000000	0.054526
300	MPS	1.000000	0.988498	0.000000	0.035935	0.000000	0.001886	0.000000	0.035935
500	MPS	1.000000	0.990280	0.000000	0.028205	0.000000	0.001246	0.000000	0.028205
1000	MPS	1.000000	0.991401	0.000000	0.019585	0.000000	0.000559	0.000000	0.019585
25	OLS	1.000000	0.921417	0.000000	0.197258	0.000000	0.066185	0.000000	0.197258
50	OLS	1.000000	0.958158	0.000000	0.115767	0.000000	0.020936	0.000000	0.115767
100	OLS	1.000000	0.978956	0.000000	0.087494	0.000000	0.012557	0.000000	0.087494
150	OLS	1.000000	0.983878	0.000000	0.069138	0.000000	0.007375	0.000000	0.069138
300	OLS	1.000000	0.993597	0.000000	0.050315	0.000000	0.003943	0.000000	0.050315
500	OLS	1.000000	0.998652	0.000000	0.035044	0.000000	0.001961	0.000000	0.035044
1000	OLS	1.000000	0.996512	0.000000	0.030609	0.000000	0.001492	0.000000	0.030609

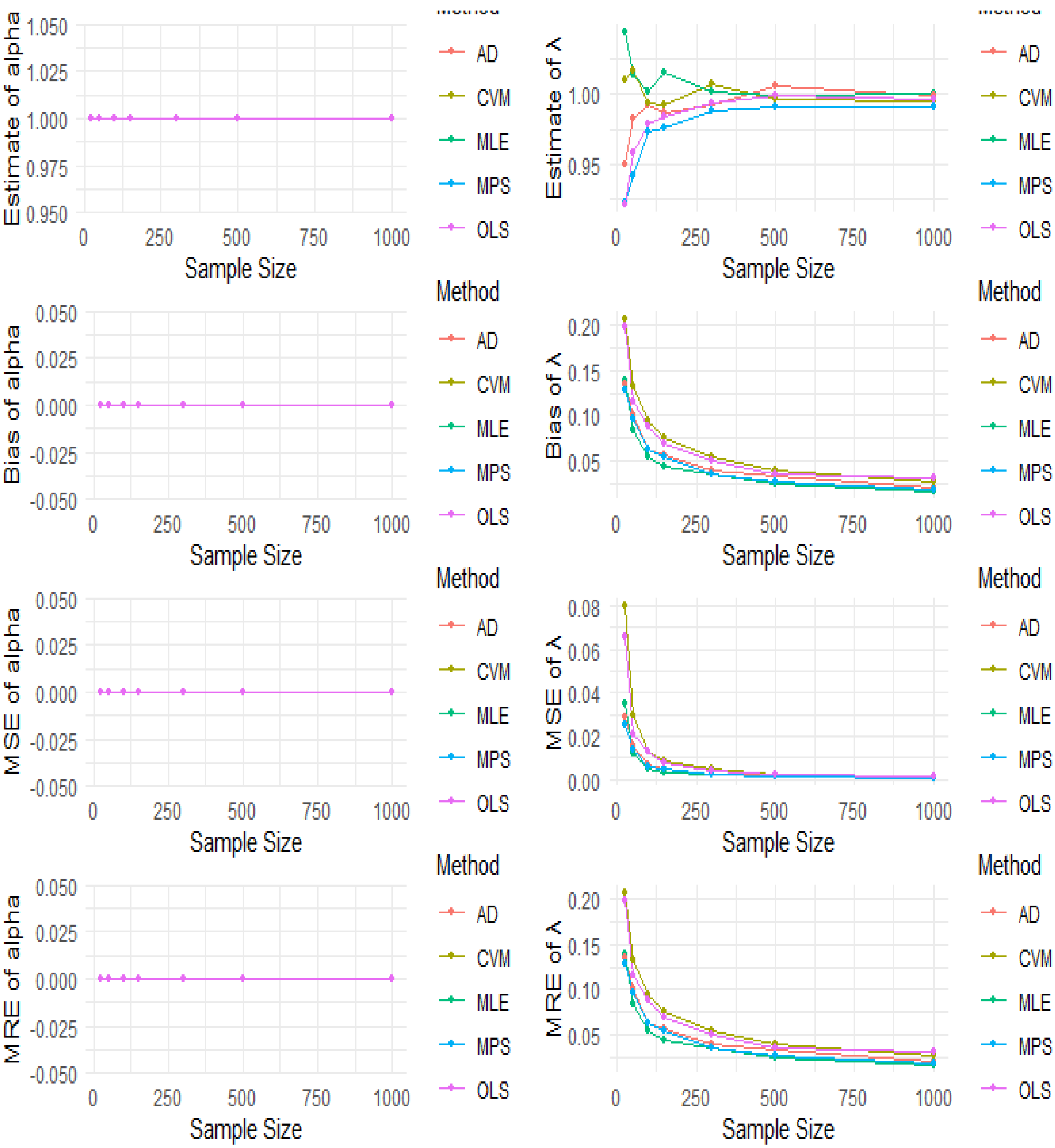

Figure 8. Graphical representation of Estimate, Bias, MSE, and MRE values presented in Table 4.

6. Model Comparison in Fitting to Real Data Sets

Two real datasets were analyzed to illustrate the applicability of the proposed distribution. The first dataset comprises the remission times of 128 bladder cancer patients (Aldeni et al., 2017)[23], which naturally aligns with the survival/lifetime framework. The second dataset included 24 determinations of copper concentration (in ppm) in wholemeal flour (Analytical Methods Committee, 1989)[26]. Although this dataset is not a lifetime dataset, it was selected to test the model's flexibility in fitting positive and skewed data, highlighting its potential utility beyond traditional survival analysis. This section compares the APOPW distribution to other comparable lifetime distributions, such as the Power XLindley [27], Gamma Lindley [24], Gamma, Quasi Lindley [20], New Quasi Lindley [15], NTPQED [29], Two parameter Lindley2 [28]. The maximum likelihood estimates were obtained numerically by minimizing the negative log-likelihood function using the BFGS quasi-Newton method implemented in the `optim()` function in R. Convergence was declared when the relative change in both the parameter estimates and the objective function was below 10^{-8}, with a maximum of 1000 iterations. Initial guesses for (α, β) were set close to the true parameter values in the simulation study, while for real data they were chosen based on simple moment-based heuristics. To avoid convergence to local optima, several perturbed starting values were also tested, and the solution yielding the largest likelihood was retained. And to compare several distributions, we consider criteria like AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2LL (-2Log-Likelihood), and CAIC (Consistent Akaike Information Criterion), HQIC (Hannan-Quinn information criterion) for the data set. The better distribution corresponds to smaller AIC, BIC, -2LL, CAIC and HQIC values.

The first data set consists of the remission time of 128 bladder cancer patients. The data were taken from Aldeni et al. (2017) [23].

The dataset 1 are: 0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180,4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

Table 5. MLEs of the parameters estimated for real dataset 1

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{b}$
Power XLindley	1.388593		0.2612941	
Gamma Lindley			0.1069747	0.09678821
Gamma		0.1251788	1.172319	
Quasi Lindley		0.3160336	0.04183527	
New Quasi Lindley		0.1140924	0.0005250499	
NTPQED			0.1885692	88.3213
Two parameter Lindley2	0.1077796		0.000843786	
Two parameter Lindley1	0.1069879		4722.193	
Weibull		1.047849	9.561373	
APOPWD	**434.8434**	**0.4671386**		

The second dataset is a numeric vector of 24 determinations of copper in wholemeal flour, expressed in parts per million. Data were taken from the Analytical Methods Committee (1989) [26].

Table 6. Analytical measures of fit criteria for survival time data of cancer (dataset 1)

Distribution	**AIC**	**BIC**	**-2LL**	**CAIC**	**HQIC**
Power XLindley	1127.514	1133.218	1123.514	1127.610	1129.831
Gamma Lindley	832.6839	838.3879	828.6839	832.7799	835.0015
Gamma	830.7356	836.4396	826.7356	830.8316	833.0531
Quasi Lindley	1196.427	1202.131	1192.427	1196.523	1198.745
New Quasi Lindley	832.8294	838.5334	828.8294	832.9254	835.147
NTPQED	841.4313	847.1353	837.4313	841.5273	843.7489
Two parameter Lindley2	832.686	838.3901	828.686	832.782	835.0036
Two parameter Lindley1	832.6839	838.388	828.6839	832.7799	835.0015
Weibull	832.1738	837.8778	828.1738	832.2698	834.4913
APOPWD	**830.2344**	**835.9385**	**826.2344**	**830.3304**	**832.552**

The dataset 2 are: 2.90, 3.10, 3.40, 3.40, 3.70, 3.70, 2.80, 2.50, 2.40, 2.40, 2.70, 2.20, 5.28, 3.37, 3.03, 3.03, 28.95, 3.77, 3.40, 2.20, 3.50, 3.60, 3.70, 3.70

Table 7. MLEs of the parameters estimated for real dataset 2

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{b}$
Power XLindley	1.484577		0.33913	
Gamma Lindley			0.4643355	35.07353
Gamma		0.574331	2.458373	
Quasi Lindley		0.7888356	0.00041516	
New Quasi Lindley		0.4639805	29.20064	
NTPQED			0.6523102	0.00062837
Two parameter Lindley2	0.4634209		39.34863	
Two parameter Lindley1	0.4762804		0.4750378	
Weibull		1.230793	4.673114	
APOPWD	**126.2734**	**0.6975994**		

Table 8. Analytical measures of fit criteria for survival time data of copper in wholemeal flour (dataset 2)

Distribution	**AIC**	**BIC**	**-2LL**	**CAIC**	**HQIC**
Power XLindley	177.3466	179.7027	173.3466	177.918	177.9716
Gamma Lindley	113.8102	116.1663	109.8102	114.3816	114.4353
Gamma	113.1218	115.4779	109.1218	113.6932	113.7469
Quasi Lindley	140.8624	143.2185	136.8624	141.4338	141.4875
New Quasi Lindley	113.7834	116.1395	109.7834	114.3549	114.4085
NTPQED	114.4569	116.8130	110.4569	115.0284	115.0820
Two parameter Lindley2	113.8517	116.2078	109.8517	114.4231	114.4767
Two parameter Lindley1	116.8540	119.2101	112.8540	117.4255	117.4791
Weibull	119.4160	121.7721	115.4160	119.9874	120.0411
APOPWD	**112.1528**	**114.5089**	**108.1528**	**112.7242**	**112.7779**

According to Tables 6,8, we can observe that APOPW distribution provide smallest AIC, BIC, -2LL, CAIC, HQIC values as compared to Power XLindley, gamma Lindley, gamma, quasi Lindley, new quasi Lindley, NTPQED, two-parameter L2, two parameter L1, and Weibull distributions, and hence best fits the data among all the models considered.

7. Conclusion

In this paper, we proposed a new lifetime distribution, the Alpha Power One-Parameter Weibull (APOPW) distribution with two shape parameters, which extends the one-parameter Weibull distribution. The proposed distribution exhibits desirable properties such as unimodality, light tails, and flexible hazard rate shapes. These features make it particularly suitable for modeling lifetime and reliability data. We explored its key the mathematical and statistical properties, including the cumulative distribution function, probability density function, moments (including mean, variance), moment-generating function, quantile function, mean deviation, Rényi and Shannon entropy, stress-strength reliability, mean deviations, and extreme order statistics.

Multiple parameter estimation methods were considered, and a comprehensive simulation study was conducted to evaluate estimation accuracy of five different methods for estimating the unknown parameters. The comprehensive simulation results, based on metrics such as average absolute bias (BAIS), mean squared error (MSE), and mean absolute relative error (MRE), indicate that the maximum likelihood estimation (MLE) method is the most efficient and reliable approach.

Furthermore, applications to two real datasets confirmed the superiority of the APOPW over several well-known distributions, such as the Power XLindley, Gamma Lindley, Gamma, Quasi Lindley, New Quasi Lindley, TPQED, Two parameter Lindley2, Two parameter Lindley1, and Weibull. Based on model selection criteria such as AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2LL (-2Log-Likelihood), and CAIC (Consistent Akaike Information Criterion), HQIC (Hannan-quinn information criterion), the APOPW distribution consistently provided the best fit to the data.

Overall, the Alpha Power One-Parameter Weibull (APOPW) distribution offers significant flexibility and modeling power under uncertainty, making it a valuable addition to the family of lifetime distributions, and the second dataset (copper in flour) is not a lifetime dataset, it was included to demonstrate the flexibility of the proposed model in handling different types of positive and skewed data, highlighting its potential utility beyond traditional survival analysis.

REFERENCES

1. Mahdavi, A., Kundu, D. *A new method for generating distributions with an application to exponential distribution*, Communications in Statistics-Theory and Methods, 46, pp. 6543–6557, 2017.
2. Karr, A.F. *Probability* , Springer Texts in Statistics, 1993.
3. Rényi, A. *On measures of entropy and information*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1, pp. 547-561, 1961.
4. Alkasasbeh, M.R. Raqab, M.Z. *Estimation of the generalized logistic distribution parameters*, Comparative study. Statistical Methodology, 6(3), 262–279. 2009.
5. Cheng, R.C.H., Amin, N.A.K. *Maximum product-of-spacings estimation with applications to the lognormal distribution*, Math Report, 791. 1979.
6. Cheng, R.C.H., Amin, N.A.K. *Estimating parameters in continuous univariate distributions with a shifted origin*, Journal of the Royal Statistical Society: Series B (Methodological), 45(3), 394–403. 1983.
7. Finsterwalder, C.E. *Collaborative Study of an Extension of the Mills et al. Method for the Determination of Pesticide Residues in Foods*, Journal of the Association of Official Analytical Chemists, 59(1), 169–171. 1976.
8. Gupta, R.D., Kundu, D. *Generalised exponential distribution: different methods of estimations*, Journal of Statistical Computation and Simulation, 69(4), 315–337. 2001.
9. Kundu, D., Raqab, M.Z. *Generalized Rayleigh distribution: different methods of estimations*, Computational Statistics and Data Analysis, 49(1), 187–200. 2005.
10. MACDONALD, P.D. *Estimation procedure for mixtures of distributions*, Journal of The Royal Statistical Society Series B-Statistical Methodology, 33(2), 326-+. 1971.

11. Ranneby, B. *The maximum spacing method. An estimation method related to the maximum likelihood method*, Scandinavian Journal of Statistics, 93–112. 1984.
12. Anderson, T.W., Darling, D.A. *Asymptotic theory of certain" goodness of fit" criteria based on stochastic processes*, The Annals of Mathematical Statistics, 193–212. 1952.
13. Anderson, T.W., Darling, D.A. *A test of goodness of fit*, Journal of the American Statistical Association, 49(268), 765–769. 1954.
14. Stephens, M. A. *EDF statistics for goodness of fit and some comparisons*, Journal of the American Statistical Association, 69(347), 730–737. 1974.
15. R. Shanker, A.H Ghebretsadik. *A new quasi Lindley distribution*, Internationl Journal of Statistics and Systems, 8, 143–156. 2013.
16. Swain, J. J., Venkatraman, S., Wilson, J. R. *Least-squares estimation of distribution functions in Johnson's translation system*, Journal of Statistical Computation and Simulation, 29(4), 271–297. 1988.
17. Weibull, W. *A statistical distribution functions of wide applicability*, J Appl Mech, 51, 293–297. 1951.
18. Nassar ,M., Alzaatreh ,A., Mead ,M., Abo-Kasem, O. *Alpha power Weibull distribution: properties and applications*, Communication in Statistics-Theory and Methods, 46(20), 10236–10252. 2017.
19. Dey ,S., Sharma, VK., Mesfioui ,M. *A new extension of Weibull distribution with application to lifetime data*, Ann Data Sci, 4(1), 31–61. 2017.
20. R. Shanker, A. Mishra. *A quasi Lindley distribution*, African Journal of Mathematics and Computer Science Research, 4, 64-71. 2013.
21. Merga ,A.A., Ayele, T.G. *Alpha power transformed generalized Weibull distribution with its detailed properties and applications*, Journal of probability and statistics,(1). 2024.
22. Arnold, B.C., Balakrishnan, N., Nagarajah, H.N. *A First Course in Order Statistics*, New York: John Wiley. 2008.
23. Aldeni, M., Lee, C., Famoye, F. *Families of distributions arising from the quantile of generalized lambda distribution*, Journal of Statistical Distributions and Applications, 4(1). 2017.
24. H. Zeghdoudi, S. Nedjar. *Gamma Lindley distribution and its application*, Journal of Applied Probability and Statistics, 11, 129-138. 2016.
25. R. Shanker, A. Mishra. *A two-parameter Lindley distribution*, Statistics in Transition new series, 14, 45-56. 2013.
26. Analytical Methods Committee. *Robust statistics – how not to reject outliers* , The Analyst, 114, 1693–1702. 1989.
27. M. Bouhadjar, Ahmed M. Gemeay, E.M. Almetwally, H. Zeghdoudi, E. Alshawarbeh, T.T Alanazi Talal, M.M. Abd El-Raouf, E. Hussam. *The Power XLindley Distribution: Statistical Inference, Fuzzy Reliability, and COVID-19 Application*, Journal of Function Spaces, (1). 2022.
28. R. Shanker, S. Sharma, R. Shanker. *A two-parameter Lindley distribution for modeling waiting and survival times data*, Applied Mathematics, 4, 363-368. 2013.
29. Fatma, Z.B., Halim, Z., Laxmi, P.S., Yusra, A.T., Bakr, M.E., Anoop, k., Ahmed, M.G. *Novel two-parameter quadratic exponential distribution: Properties, simulation, and applications*, Heliyon, 10. 2024.
30. Seema, C., Bhanita, D., Subrata, C. *A new Modified Alpha Power Weibull distribution: Properties, Parameter Estimation and applications*, Journal of the Indian for Probability and Statistics. 2021.