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High Order Statistics From Lambert-Topp-Leone Distribution: Statistical Properties and Applications

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Abstract This paper investigates the statistical properties of high order statistics derived from the Lambert–Topp–Leone (LTL) distribution. The authors establish recurrence relations for both single and product moments of order statistics and explore their application in characterizing the LTL distribution. Comparative analysis is conducted with related distributions, namely the Power Inverted Topp–Leone (PITL) and Topp–Leone Lomax (TLL) distributions. Through simulations, the study examines the impact of parameter variations on the mean and variance of order statistics across different scenarios, highlighting the stability and flexibility of the LTL model. Theoretical results are validated with a real dataset on stress measurements in concrete, where the LTL distribution demonstrates superior goodness-of-fit compared to competing models. The findings underscore the robustness and adaptability of the LTL distribution for modeling lifetime and reliability data, and suggest directions for future research in statistical modeling and inference.

Keywords High Order Statistics,Lambert–Topp–Leone (LTL) Distribution , Recurrence Relations, Order Statistics ,Statistical Modeling, Lifetime and Reliability Analysis

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1. Introduction

The Lambert–Topp–Leone (LTL) distribution is a flexible probability model that builds on the classical Topp–Leone distribution by adding a Lambert W-based transformation. This change makes it more flexible in terms of form, which gives you more control over skewness and tail behavior. This makes it a good choice for modeling a wide range of lifetime and reliability data. Topp and Leone (1955) came up with the original Topp–Leone distribution to describe bounded random variables. It has since been used successfully in reliability analysis, hydrology, and quality control. But even while it is valuable, the original form may not always be flexible enough to capture complicated data patterns in the real world [26].

The Lambert transformation adds to the Topp–Leone family, giving the LTL distribution the same support and reliability as the original model while also making it better at modeling because it can fix problems with skewness and kurtosis. This improvement makes the LTL distribution a useful option for applications that deal with lifespan data that is limited, acceptance sampling strategies, and assessments of how well a model fits.

The LTL distribution can also be used as a starting point to build generalized forms like the Power Inverted Lambert–Topp–Leone (PILTL) distribution, which has more shape parameters for even more versatility. These kinds of generalizations are very helpful for things like statistical inference, simulation studies, and finding analytical features like moments, order statistics, and recurrence relations. A random variable X follows the Lambert-Topp- Leone (LTL) distribution, with parameters $\alpha \in (0, e)$ and $\beta, b > 0$, denoted as $X \sim LTL(\alpha, \beta, b)$,

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if its CDF and PDF are given, respectively, by [26].

$$G(x;\alpha,\beta,b) = 1 - \left[1-(1-y^2)^b\right]\alpha^{\left((1-y^2)^b\right)} \tag{1}$$

$$g(x;\alpha,\beta,b) = \frac{2\beta}{b}y(1-y^2)^{\beta-1}\alpha^{(1-y^2)^b}\{1-\log(\alpha)[1-(1-y^2)^b]\} \tag{2}$$

where $y = 1 - x/b$, $0 < x < b$, $0 < p < 1$, $b > 0$ and $0 < \alpha < e$. $\alpha \in (0,e)$ is an extra shape parameter, and $e \approx 2.718$ is Euler's number.

2. Maximum Likelihood Estimation

The maximum probability estimation for the LTL distribution is covered in this section, taking into account both known and unknown b scenarios. When fitting proportional data, the case where b is known if we assume $\theta = (\alpha,\beta)$ so the likelihood function can be written as [26].

$$\begin{aligned}\ell(\theta;x) = n\log(2) + n\log(\beta) - n\log(b)\\ +\textstyle\sum_{i=1}^n \log(y_i) + (\beta-1)\sum_{i=1}^n \log(1-y_i^2)\\ +\log(\alpha)\textstyle\sum_{i=1}^n (1-y_i^2)^\beta + \sum_{i=1}^n \log\{1-\log(\alpha)[1-(1-y_i^2)^\beta]\}\end{aligned} \tag{3}$$

The ML estimator $\hat{\theta} = (\hat{\alpha},\hat{\beta})$ of $\theta = (\alpha,\beta)$ can be derived by resolving the system of equations represented by the subsequent score equations:

$$\begin{aligned}0 = \tfrac{\partial\ell(\tilde{\alpha};x)}{\partial\alpha} = \textstyle\sum_{i=1}^n (1-y_i^2)^\beta - \sum_{i=1}^n \tfrac{1-(1-y_i^2)^\beta}{1-\log(\alpha)[1-(1-y_i^2)^\beta]}\\ 0 = \tfrac{\partial\ell(\tilde{\alpha};x)}{\partial\beta} = \tfrac{n}{\beta} + \textstyle\sum_{i=1}^n \log(1-y_i^2)\\ +\log(\alpha)\textstyle\sum_{i=1}^n \log(1-y_i^2)(1-y_i^2)^\beta + \log(\alpha)\sum_{i=1}^n \tfrac{\log(1-y_i^2)(1-y_i^2)^\beta}{1-\log(\alpha)[1-(1-y_i^2)^\beta]}.\end{aligned}$$

As the root of this system lacks a closed form, the maximum likelihood estimates for $\theta = (\alpha,\beta)$ must be derived by numerical methods.

3. Generalized Order Statistics

The concept of generalized order statistics (GOS) was introduced by [13]. This concept includes various models of order for random variables [22]. To simplify, let G represent an absolutely continuous distribution function with a density function f. The random variables $X(1,n,\widetilde{m},k),\ldots,X(n,n,\widetilde{m},k)$ are referred to as generalized order statistics (GOS) based on F if their joint probability density function (pdf)follows the structure [23]:

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{G}(x_i)\right]^{m_i} g(x_i)\right)\left[\bar{G}(x_n)\right]^{k-1} g(x_n)$$

the survival function which can be written as

$$\bar{G}(x;\alpha,\beta,b) = [1-(1-y^2)^b]\alpha^{((1-y^2)^b)} \tag{4}$$

for $G^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < G^{-1}(1)$ with parameters $n \in N, n \ge 2, k > 0, \widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}, M_r = \sum_{i=r}^{n-1} m_i$, such that $\alpha_r = k + n - r + M_r > 0$, for all $r \in \{1,2,\ldots,n-1\}$ for $\alpha_i \ne \alpha_j, i \ne j$ for all $i.j \,\epsilon\, (1,2,\ldots,n-1)$ the pdf of $X(r,n,\widetilde{m},k)$ is given by [8] in the following manner

$$g_{X(r,n,\widetilde{m},k)}\ (x) = K_{r-1} g\,(x) \sum_{i=1}^{r} a_i\,(r) \left[\bar{G}\,(x)\right]^{\alpha_i - 1} \tag{5}$$

The joint pdf of $X\,(r, n, \widetilde{m}, k)$ and $X\,(s, n, \widetilde{m}, k)\,, 1 \leq r < s \leq n$ is given as

$$g_{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}\ (x,t) = C_{s-1} \sum_{i=r+1}^{s} {a_i}^{(r)}\,(s) \left[\frac{\bar{G}\,(t)}{\bar{G}\,(x)}\right]^{\alpha_i} \left(\sum_{i=1}^{r} a_i\,(r) \left[\bar{G}\,(x)\right]^{\alpha_i}\right) \frac{g\,(x)}{\bar{G}\,(x)} \frac{g\,(t)}{\bar{G}\,(t)} \tag{6}$$

where $x < t$ and

$$a_i\,(r) = \prod_{\substack{j=1\\ j\neq i}}^{r} \frac{1}{\delta_j - \delta_i}, 1 \leq i \leq r \leq n,$$

$${a_i}^{(r)}\,(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s} \frac{1}{\delta_j - \delta_i}, r+1 \leq i \leq s \leq n.$$

It may be noted that for $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$,

$$a_i\,(r) = \frac{(-1)^{r-i}}{(m+1)^{r+1}\,(r-1)!} \binom{r-1}{r-i}, \tag{7}$$

and

$${a_i}^{(r)}\,(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}\,(s-r-1)!} \binom{s-r-1}{s-i}. \tag{8}$$

Therefore pdf $X\,(r, n, m, k)$ given in Eq.(5) reduces to [25]

$$g_{X(r,n,m,k)}\ (x) = \frac{K_{r-1}}{(r-1)!} \left[\bar{G}\,(x)\right]^{\alpha_r - 1} g\,(x)\, g_m^{r-1} \left[G\,(x)\right] \tag{9}$$

and joint pdf $X\,(r, n, m, k)$ and $X\,(s, n, m, k)$ given in Eq. (6) reduces to

$$\begin{aligned} g_{X(r,n,m,k),X(s,n,m,k)}\ (x,y) = \frac{K_{s-1}}{(r-1)!\,(s-r-1)!} \left[\bar{G}\,(x)\right]^{m} g\,(x)\, g_m^{r-1} \left[G\,(x)\right] \\ \left\{h_m \left[G\,(t)\right] - h_m \left[G\,(x)\right]\right\}^{s-r-1} \left[\bar{G}\,(t)\right]^{\alpha_s - 1} g\,(t)\,, x < t, \end{aligned} \tag{10}$$

where

$$K_{r-1} = \prod_{\substack{j=1\\ j\neq i}}^{r} \delta_j, \alpha_i = k + (n-i)\,(m+1)\,,$$

$$\begin{cases} K_{r-1} = \prod_{\substack{j=1\\ j\neq i}}^{r} \delta_j, & m \neq -1 \\ -ln\,(1-x)\,, & m = -1 \end{cases}$$

and

$$l_m\,(x) = h_m\,(x) - h_m\,(0)\,, x\epsilon\,[0,1)$$

We will also set $X\,(0, n, m, k) = 0$. If $m = 0, k = 1$, then $X\,(r, n, m, k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X\,(r, n, m, k)$ reduces to the k_{th} record values [20].

Elamir and Seheult in 2003 [10], proposed trimmed L-moments (TL-moments) as an alternative to standard L-moments. Compared to both conventional L-moments and traditional moments, TL-moments provide distinct advantages. The r_{th} generalized TL-moment, where c_1 and c_2 represent the counts of the smallest and largest values trimmed, respectively, is defined as follows:

$$L_r^{(c_1,c_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{r-k+t_1:r+c_1+c_2}\right); \quad c_1, c_2 \\ = 1, 2, \ldots and\, r = 1, 2, \ldots, \tag{11}$$

Here, $E\left(X_{r-i+t_1:r+c_1+c_2}\right)$ represents the expected value of the $(r-i+c_1+1)(r-i+c_2-1)$ ith order statistic from a random sample of size $(r+c_1+c_2)$ When $c_1 = c_2 = 0$, this reduces to the original L-moments as defined by [11].

This characterization has been used by numerous researchers, including [14, 15, 8, 4, 20, 2, 3, 16, 5, 17, 18, 1].

4. Moments of Order Statistics

This paper provides a characterization of the LTL distribution for single and product moments of generalized order statistics (GOS) [24]. The J_th moment of order statistics can be written as:

$$\begin{aligned} E\left[X^j(r,n,m,k)\right] &= \frac{K_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{G}(x)\right]^{\alpha_r-1} g(x)\, g_m^{r-1}\left[G(x)\right]dx \\ &= \frac{K_{r-1}}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{G}(x)\right]^{\alpha_r-1} g(x)\left[1-\bar{G}(x)^{m+1}\right]^{r-1}dx \\ &= \frac{K_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{G}(x)\right]^{\alpha_r+w(m+1)-1} g(x)\,dx \\ &= \frac{jK_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!\left[\alpha_r+w(m+1)\right]}\int_0^\infty x^{j-1}\left[\bar{G}(x)\right]^{\alpha_r+w(m+1)}dx \end{aligned}$$

using Eq. (1), we get

$$E\left[X^j(r,n,m,k)\right] = \frac{jK_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!\left[\alpha_r+w(m+1)\right]}\int_0^\infty x^{j-1}\left[\left[1-(1-y^2)^b\right]\alpha^{\left((1-y^2)^b\right)}\right]^{\alpha_r+w(m+1)}dx \tag{12}$$

since $y = 1 - \frac{x}{b}$ and let $R = \alpha_r + w(m+1)$

$$E\left[X^j(r,n,m,k)\right] = \frac{jK_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!\left[\alpha_r+w(m+1)\right]}\int_0^b x^{j-1}\left[\left[1-(1-(1-x/b)^2)^b\right]\alpha^{\left((1-(1-x/b)^2)^b\right)}\right]^R dx \tag{13}$$

So, [19] we solve the following integral I_1 to represent the formula for the individual moments of generalized order statistics (GOS) derived from the LTL distribution.

$$I1 = \int_0^b x^{j-1} \cdot \left(1 - \left[1 - \left(1 - \frac{x}{b}\right)^2\right]^b \cdot \alpha^{\left(1-\left(1-\frac{x}{b}\right)^2\right)^b}\right)^R dx$$

Let $u(x) = 1 - \left(1 - \frac{x}{b}\right)^2$. Then $u(x) = x(2b - x)/b^2$. The integrand can be written as :

$$\left(\left(1 - u(x)^b\right) \cdot \alpha^{u(x)^b}\right)^R$$

Parameter ranges assumed: $b > 0$, $j > 0$ (so x^{j-1} is integrable near 0), $\alpha > 0$ for exponential manipulations; R can be any real number, but special simplifications occur when R is a nonnegative integer.
substitution: $u = 1 - \left(1 - \frac{x}{b}\right)^2$ Compute u in terms of x: $u = 1 - \left(1 - \frac{x}{b}\right)^2 = x(2b - x)/b^2$. Solve for x in terms of u and $x = b \cdot (1 - \sqrt{1-u})$. Differentiate to get dx in terms of du: $du/dx = (2/b) \cdot (1 - x/b) = (2/b) \cdot \sqrt{1-u}$. Therefore $dx = (b/2) \cdot du/\sqrt{1-u}$.
When $x = 0$, $u = 0$. When $x = b$, $u = 1$. So u integrates from 0 to 1. and by expressing x^{j-1} and dx in u we have $x^{j-1} = b^{j-1} \cdot (1 - \sqrt{1-u})^{j-1}$. $dx = (b/2) \cdot du/\sqrt{1-u}$. and by Substituting into original integral:

$$I1 = \left(\frac{b^j}{2}\right) \cdot \int_{u=0}^1 \frac{(1 - \sqrt{1-u})^{j-1}}{\sqrt{1-u}} \cdot (1 - u^b)^R \cdot \alpha^{Ru^b} du. \tag{14}$$

This is an exact algebraic transformation. It simplifies the rational-power structure, but still contains non-elementary factors, in general.
and by applying the second substitution to remove the square root: $t = \sqrt{1-u}$. Then $u = 1 - t^2$, $du = -2tdt$. so the limits change to: $u = 0 \to t = 1$; $u = 1 \to t = 0$. The minus sign reverses limits and cancels with the -2 factor; after simplification we obtain a convenient forward-integral from 0 to 1. Compute transformed factors: $1 - \sqrt{1-u} = 1 - t$. $(1 - u^b) = 1 - (1 - t^2)^b$. $\alpha^{Ru^b} = \alpha^{R(1-t^2)^b}$. dx and prefactors combine to give the final transformed integral

$$I1 = b^j \cdot \int_{t=0}^1 (1 - t)^{j-1} \cdot \left[1 - (1 - t^2)^b\right]^R \cdot \alpha^{R(1-t^2)^b} dt \tag{15}$$

This is the main simplified representation we will use for analytic manipulations or numerical evaluation. For arbitrary parameters it does not admit a simple elementary anti-derivative. The following sections provide useful special-case closed forms and series expansions that allow exact or approximate evaluation. since α arbitrary ($\alpha > 0$) expand $\left[1 - (1 - t^2)^b\right]^R$ by binomial theorem as before. The integral becomes:

$$I1 = b^j \cdot \sum_{k=0}^R \binom{R}{k} \cdot (-1)^k \cdot \int_0^1 (1 - t)^{j-1}(1 - t^2)^{bk} \cdot \alpha^{R(1-t^2)^b} dt. \tag{16}$$

The remaining factor $\alpha^{R(1-t^2)^b}$ is not polynomial in t. Write $\alpha^{R(1-t^2)^b} = \exp\left(R(1 - t^2)^b \cdot \ln \alpha\right)$ expand the exponential as a power series:

$$\alpha^{R(1-t^2)^b} = \sum_{n=0}^\infty \frac{(R \ln \alpha)^n}{n!} \cdot (1 - t^2)^{bn}.$$

Substitute into $I1$ (and swap sum and integral under appropriate convergence conditions):

$$I1 = b^j \cdot \sum_{k=0}^R \binom{R}{k} \cdot (-1)^k \cdot \sum_{n=0}^\infty \frac{(R \ln \alpha)^n}{n!} \cdot \int_0^1 (1 - t)^{j-1}(1 - t^2)^{b(k+n)} dt. \tag{17}$$

Each inner integral is of the same Beta-type form and can be reduced similarly. Thus, I_1 admits a double series representation (finite sum over k and infinite sum over n) with Beta-function coefficients. This is a convergent series for all finite $\alpha > 0$. Series expansion using the exponential for arbitrary R However, we can split as

$\exp(R\ln(1-(1-t^2)^b)) * \exp(R(1-t^2)^b \ln\alpha)$. The second exponential can be expanded as before. The first factor $\exp(R\ln(1-(1-t^2)^b))$ can be expanded using a generalized binomial series when $|(1-t^2)^b| < 1$ (which holds for t in $(0,1)$ if $b > 0$). Use the series:

$$(1-z)^R = \sum_{m=0}^{\infty} \binom{R}{m} (-1)^m z^m$$

where $\binom{R}{m} = \frac{R(R-1)\cdots(R-m+1)}{m!}$. Taking $z = (1-t^2)^b$ yields:

$$[1-(1-t^2)^b]^R = \sum_{m=0}^{\infty} \binom{R}{m} (-1)^m (1-t^2)^{bm}.$$

This yields an infinite series representation (valid when the series converges absolutely, typically for t in $[0,1)$ and $\text{Re}(R) > -1$):

$$I_1 = b^j \cdot \sum_{m=0}^{\infty} \binom{R}{m} (-1)^m \cdot \int_0^1 (1-t)^{j-1} (1-t^2)^{bm} \cdot \alpha^{R(1-t^2)^b}\, dt. \quad (18)$$

Again, expand $\alpha^{R(1-t^2)^b}$ as an exponential series and combine to obtain a double infinite series with Beta-function integrals. Under typical conditions, the series converges and provides a practical way to compute I_1 to arbitrary precision

Practical Numeric Evaluation Recommendation: For given numeric parameters (b, j, α, R), evaluate the transformed integral:

$$I_1 = b^j \cdot \int_0^1 (1-t)^{j-1} [1-(1-t^2)^b]^R \alpha^{R(1-t^2)^b}\, dt$$

using high-accuracy numerical quadrature (adaptive Gauss-Kronrod or similar).and use the finite-sum Beta-function expressions to get exact algebraic evaluations. we have:

$$I_1 = \frac{b^j}{2} \cdot \sum_{k=0}^{m} \left[\binom{m}{k} \cdot (-1)^k \cdot \sum_{r=0}^{m} \left[\binom{m}{r} \cdot (-1)^r \cdot \beta\left(\frac{r+1}{2}, bk+1\right) \right] \right] \quad (19)$$

Then the moment of order statistics will be as:

$$E\left[X^j\,(r,n,m,k)\right] = \frac{jK_{r-1} \sum_{w=0}^{r-1} \binom{r-1}{w} (-1)^w}{(m+1)^{r-1} (r-1)!\, [\alpha_r + w\,(m+1)]} \frac{b^j}{2} \cdot \sum_{k=0}^{m} \left[\binom{m}{k} \cdot (-1)^k \cdot \sum_{r=0}^{m} \left[\binom{m}{r} \cdot (-1)^r \cdot \beta\left(\frac{r+1}{2}, bk+1\right) \right] \right] \quad (20)$$

the single moments of order statistics (OS) for the LTL distribution are obtained by taking $m = 0, k = 1$

5. Some Other Related Models

We will briefly review some statistical distributions related to to make comparison among them.

5.1. Power Inverted Topp-Leone Distribution PITL

A random sample X is deemed to conform to a PITL distribution if its probability density function (PDF) is articulated as [25]:

$$g\ (x;\ \alpha, \beta) = 2\alpha\beta x^{2\beta-1} \left(1+x^\beta\right)^{-2\alpha-1} \left(1+2x^\beta\right)^{\alpha-1}; x, \alpha, \beta > 0. \quad (21)$$

The cumulative distribution function (CDF) is the following:

$$G\ (x;\ \alpha, \beta) = 1 - \frac{\left(1 + 2x^{\beta}\right)^{\alpha}}{\left(1 + x^{\beta}\right)^{2\alpha}} \tag{22}$$

The single moments of order statistics for the PITL distribution are obtained as follows.

$$E\left[X_{r:n}^{j}\right] = \frac{jn!}{(r-1)!\alpha\left[n+r-1+w\right]}\frac{1}{\beta}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^{w}\sum_{i=0}^{\infty}\binom{\alpha\left[n+r-1+w\right]}{i}\beta\left(\frac{j}{\beta}+i, \frac{-j}{\beta}+R\right) \tag{23}$$

5.2. *Topp-Leone Lomax distribution TLL*

The TLL distribution is one of these flexible and capable generalized distributions, it is a sub model of Topp-Leone family distribution. The TLL distribution was introduced by [29]. A random variable X is said to have a TLL distribution, if its probability density function (pdf) is of the form [27]

$$g\ (x;\ \alpha, \beta) = 2\alpha\beta(1+x)^{-2\beta-1}[1-(1+x)^{-2\beta}]^{\alpha-1}, \quad x \geq 0, \alpha > 0, \beta > 0, \tag{24}$$

and the corresponding distribution function (CDF) is

$$G\ (x;\ \alpha, \beta) = [1-(1+x)^{-2\beta}]^{\alpha} \tag{25}$$

The individual moments of order statistics for the TLL distribution are derived as follows.

$$E\left[X_{r:n}^{j}\right] = \alpha\binom{n}{r}\sum_{j=0}^{n-r}\sum_{i=0}^{k}(-1)^{k+j-i}\binom{n-r}{j}\binom{k}{i}\beta\left(r+j, 1-\frac{i}{2}\right), \tag{26}$$

6. Simulation

The Lambert–Topp–Leone (LTL), Power Inverted Topp–Leone (PITL), and Topp–Leone Lomax (TLL) distributions are the three competing distributions that are investigated in this study. The simulation technique is used to analyze the behavior of order statistics from these distributions. An examination of the mean and variance of order statistics was carried out under a number of different variable combinations. In particular, the shape parameters (α, β) were assigned values of (2, 3), (3, 2), and (4, 3), and the scaling parameter b was varied between 2 and 3. These parameter selections capture differences in skewness and tail heaviness, which enables a detailed evaluation of the flexibility of the distributions and their sensitivity to changes in the parameters.

The study illustrates the impact of parameter selections on model stability, skewness, and the behavior of extreme values by systematically comparing outcomes across several scenarios. A number of different scenarios were used in the study. The findings of the simulation offer insights into the susceptibility of each distribution to changes in form and scale. They also demonstrate how LTL constantly maintains stable statistical features even when the parameter choices are extremely extreme. On the other hand, PITL and TLL display a greater degree of variability, particularly in terms of tail behavior. This phenomenon may have an impact on the degree to which these models are suitable for modeling heavily skewed or heavily tailing data.

In addition to this, the research investigates higher-order moments and the impact of parameter combinations on the convergence of order statistics. This provides valuable insights into the modeling of dependability and risk assessment. The findings highlight the significance of selecting proper distributions for applications in engineering, finance, and data science, which are fields in which an understanding of extreme occurrences and variability is essential. The purpose of this work is to validate existing models by validating theoretical predictions through simulation. Additionally, this study gives direction for future applications that require probabilistic modeling that is robust, adaptable, and accurate.

The findings, taken as a whole, highlight the fact that the selection of distribution and parameter configuration has a considerable influence in effectively capturing data features. This is especially true in situations where tail behavior and skewness have a substantial impact on decision-making processes. The comparative study that is presented here makes a contribution to the expanding body of work on advanced order statistics and the practical applications of these statistics in stochastic modeling.

Table 1. Mean of Order Statistics , *b = 2*

		Alpha = 2 , Beta = 3			***Alpha = 3 , Beta = 4***			***Alpha = 4 , Beta = 3***		
n	***r***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***
8	1	0.0122	0.0246	0.0341	0.1122	0.1249	0.1346	0.0185	0.0330	0.0440
	2	0.0213	0.0433	0.0461	0.1215	0.1442	0.1471	0.0291	0.0551	0.0583
	3	0.0373	0.0643	0.0521	0.1384	0.1664	0.1534	0.0484	0.0810	0.0651
	4	0.0455	0.0861	0.0661	0.1551	0.1899	0.1683	0.0677	0.1091	0.0829
	5	0.0645	0.0991	0.0762	0.1666	0.2041	0.1791	0.0812	0.1264	0.0961
	6	0.0722	0.1205	0.0862	0.1757	0.2280	0.1900	0.0921	0.1560	0.1092
	7	0.0835	0.1462	0.1021	0.1870	0.2574	0.2074	0.1056	0.1935	0.1304
	8	0.1016	0.1765	0.1231	0.2069	0.2930	0.2309	0.1298	0.2404	0.1597
9	1	0.0120	0.0249	0.0346	0.1120	0.1252	0.1352	0.0185	0.0333	0.0447
	2	0.0212	0.0442	0.0471	0.1214	0.1451	0.1482	0.0290	0.0561	0.0597
	3	0.0376	0.0664	0.0534	0.1383	0.1686	0.1548	0.0483	0.0836	0.0674
	4	0.0460	0.0899	0.0683	0.1470	0.1940	0.1706	0.0583	0.1140	0.0860
	5	0.0660	0.1041	0.0791	0.1682	0.2097	0.1823	0.0831	0.1333	0.0999
	6	0.0746	0.1280	0.0900	0.1774	0.2365	0.1941	0.0941	0.1668	0.1142
	7	0.0865	0.1574	0.1074	0.1903	0.2704	0.213	0.1096	0.2104	0.1373
	8	0.1062	0.1930	0.1309	0.2120	0.3128	0.2398	0.1361	0.2672	0.1709
	9	0.1165	0.2103	0.1765	0.2235	0.3340	0.2930	0.1504	0.2964	0.2404
10	1	0.0120	0.0252	0.0352	0.1120	0.1255	0.1358	0.0185	0.0337	0.0454
	2	0.0214	0.0451	0.0482	0.1216	0.1461	0.1493	0.0293	0.0573	0.0610
	3	0.0383	0.0686	0.0548	0.1390	0.1710	0.1563	0.0491	0.0864	0.0691
	4	0.0470	0.0940	0.0706	0.1481	0.1985	0.1731	0.0596	0.1195	0.0889
	5	0.0682	0.1097	0.0823	0.1705	0.2159	0.1857	0.0858	0.1409	0.1040
	6	0.0774	0.1365	0.0941	0.1804	0.2462	0.1986	0.0976	0.1791	0.1196
	7	0.0903	0.1704	0.1139	0.1945	0.2857	0.2206	0.1147	0.2306	0.1468
	8	0.1120	0.2128	0.1398	0.2185	0.3371	0.2500	0.1442	0.3034	0.1840
	9	0.1235	0.2340	0.1930	0.2314	0.3636	0.3128	0.1603	0.3384	0.2672
	10	0.1279	0.2551	0.2387	0.2364	0.3905	0.3695	0.1666	0.3777	0.3470

This table presents a comparative assessment of the mean values of order statistics derived from three distinct probability distributions: the Lambert–Topp–Leone (LTL), the Inverted Power Topp–Leone (IPTL), and the Topp–Leone Lomax (TLL), assessed across various parameter configurations. The results unequivocally demonstrate that as the order r increases, the mean values of the order statistics increase consistently across all three models. This pattern illustrates the expected theoretical characteristic of order statistics, in which higher-order positions correlate with higher expected values. Moreover, comparing the distributions reveals significant disparities in the central tendency. The IPTL distribution generally yields higher mean values compared to the LTL distribution, whereas the TLL distribution often occupies an intermediate position, serving as a transitional instance. These findings highlight that even with equal sample sizes and parameter settings, the fundamental distributional structure can significantly impact the positioning of order statistics, thereby influencing subsequent inferential or reliability assessments.

Table 2. Variance of Order Statistics, $b = 2$

		Alpha = 2 , Beta = 3			***Alpha = 3 , Beta = 4***			***Alpha = 4 , Beta = 3***		
n	***r***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***
8	1	0.3342	0.4520	0.3886	0.4213	0.5717	0.4907	0.5161	0.7056	0.6031
	2	0.3675	0.4978	0.4294	0.4637	0.6303	0.5427	0.5692	0.7804	0.6689
	3	0.3954	0.5363	0.4639	0.4993	0.6798	0.5868	0.6140	0.8438	0.7249
	4	0.4132	0.5609	0.4860	0.5220	0.7114	0.6152	0.6427	0.8846	0.7611
	5	0.4865	0.6626	0.5784	0.6158	0.8427	0.7340	0.7619	1.0553	0.9138
	6	0.5565	0.7603	0.6686	0.7058	0.9697	0.8505	0.8773	1.2228	1.0656
	7	0.7011	0.9644	0.8613	0.8927	1.2370	1.1016	1.1209	1.5840	1.3996
	8	1.0012	1.3974	1.2920	1.2856	1.8155	1.6732	1.6509	2.4085	2.1998
9	1	0.3968	0.5382	0.4656	0.5011	0.6822	0.5891	0.6163	0.8470	0.7277
	2	0.4442	0.6038	0.5248	0.5617	0.7667	0.6650	0.6929	0.9562	0.8249
	3	0.4849	0.6603	0.5764	0.6138	0.8398	0.7314	0.7593	1.0515	0.9104
	4	0.5116	0.6975	0.6105	0.6480	0.8880	0.7754	0.8031	1.1149	0.9675
	5	0.6266	0.8589	0.7609	0.7962	1.0984	0.9705	0.9945	1.3953	1.2239
	6	0.7445	1.0262	0.9209	0.9491	1.3186	1.1798	1.1955	1.6966	1.5057
	7	1.0159	1.4189	1.3142	1.3050	1.8447	1.7032	1.6778	2.4518	2.2434
	8	1.7215	2.4966	2.5466	2.2593	3.3632	3.4368	3.0864	4.9809	5.1193
	9	2.0982	3.1107	3.3759	2.7879	4.2872	4.7015	3.9533	6.8476	7.7842
10	1	0.4870	0.6633	0.5791	0.6165	0.8436	0.7348	0.7627	1.0565	0.9148
	2	0.5591	0.7640	0.6720	0.7091	0.9744	0.8549	0.8816	1.2291	1.0713
	3	0.6241	0.8554	0.7576	0.7930	1.0938	0.9661	0.9903	1.3890	1.2181
	4	0.6680	0.9174	0.8164	0.8498	1.1752	1.0428	1.0645	1.4994	1.3205
	5	0.8743	1.2126	1.1044	1.1186	1.5668	1.4224	1.4226	2.0463	1.8416
	6	1.1055	1.5511	1.4526	1.4238	2.0248	1.8904	1.8435	2.7228	2.5199
	7	1.7620	2.5612	2.6290	2.3155	3.4582	3.5584	3.1753	5.1600	5.3518
	8	2.2479	3.3634	3.7501	3.0020	4.6818	5.3037	4.3249	5.7382	6.2711
	9	2.6631	4.0926	4.9590	3.6091	5.8746	7.4137	5.4500	7.8320	9.8940
	10	3.7861	6.3187	7.3766	5.3628	7.1721	9.6992	6.4257	8.8739	10.352

The variances of the order statistics are shown here for a variety of different models. There is a correlation between higher order r and an increase in the variance, which is to be expected given that bigger order statistics often have a greater dispersion. It is often the case that the IPTL distribution has the highest variance values, which indicates that it has heavier tails and a greater degree of variability. In the majority of instances, LTL generates lower variances, which indicates a more concentrated distribution around its center values. It has been demonstrated through additional observations that the difference in variance between the models becomes more evident as the value of n increases. This highlights the influence that the sample size has on the dispersion. To be more specific, IPTL exhibits a rapid development in variance for higher-order statistics, which makes it less dependable for modeling stable data. However, in comparison to LTL, TLL still has a tendency to exaggerate variability, despite the fact that it retains a moderate variance pattern. The consistent growth of LTL variances represents stability without overfitting severe oscillations, as indicated by the gradual increase. These findings lend credence to the notion that LTL offers a more equitable compromise between the degree of variability and the degree of precision when modeling order statistics..

Table 3. Mean of Order Statistics, b = 3

		Alpha = 2 , Beta = 3			Alpha = 3 , Beta = 4			Alpha = 4 , Beta = 3		
n	***r***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***
8	1	0.0128	0.0258	0.0358	0.1178	0.1311	0.1413	0.0194	0.0347	0.0462
	2	0.0224	0.0455	0.0484	0.1276	0.1514	0.1545	0.0306	0.0579	0.0612
	3	0.0392	0.0675	0.0547	0.1453	0.1747	0.1611	0.0508	0.0851	0.0684
	4	0.0478	0.0904	0.0694	0.1629	0.1994	0.1767	0.0711	0.1146	0.0870
	5	0.0677	0.1041	0.0800	0.1749	0.2143	0.1881	0.0853	0.1327	0.1009
	6	0.0758	0.1265	0.0905	0.1845	0.2394	0.1995	0.0967	0.1638	0.1147
	7	0.0877	0.1535	0.1072	0.1964	0.2703	0.2178	0.1109	0.2032	0.1369
	8	0.1067	0.1853	0.1293	0.2172	0.3077	0.2424	0.1363	0.2524	0.1677
9	1	0.0128	0.0266	0.0370	0.1198	0.1340	0.1447	0.0198	0.0356	0.0478
	2	0.0227	0.0473	0.0504	0.1299	0.1553	0.1586	0.0310	0.0600	0.0639
	3	0.0402	0.0710	0.0571	0.1480	0.1804	0.1656	0.0517	0.0895	0.0721
	4	0.0492	0.0962	0.0731	0.1573	0.2076	0.1825	0.0624	0.1220	0.0920
	5	0.0706	0.1114	0.0846	0.1800	0.2244	0.1951	0.0889	0.1426	0.1069
	6	0.0798	0.1370	0.0963	0.1898	0.2531	0.2077	0.1007	0.1785	0.1222
	7	0.0926	0.1684	0.1149	0.2036	0.2893	0.2279	0.1173	0.2251	0.1469
	8	0.1136	0.2065	0.1401	0.2268	0.3347	0.2566	0.1456	0.2859	0.1829
	9	0.1247	0.2250	0.1889	0.2391	0.3574	0.3135	0.1609	0.3171	0.2572
10	1	0.0131	0.0275	0.0384	0.1221	0.1368	0.1480	0.0202	0.0367	0.0495
	2	0.0233	0.0492	0.0525	0.1325	0.1592	0.1627	0.0319	0.0625	0.0665
	3	0.0417	0.0748	0.0597	0.1515	0.1864	0.1704	0.0535	0.0942	0.0753
	4	0.0512	0.1025	0.0770	0.1614	0.2164	0.1887	0.0650	0.1303	0.0969
	5	0.0743	0.1196	0.0897	0.1858	0.2353	0.2024	0.0935	0.1536	0.1134
	6	0.0844	0.1488	0.1026	0.1966	0.2684	0.2165	0.1064	0.1952	0.1304
	7	0.0984	0.1857	0.1242	0.2120	0.3114	0.2405	0.1250	0.2514	0.1600
	8	0.1221	0.2320	0.1524	0.2382	0.3674	0.2725	0.1572	0.3307	0.2006
	9	0.1346	0.2551	0.2104	0.2522	0.3963	0.3410	0.1747	0.3689	0.2912
	10	0.1394	0.2781	0.2602	0.2577	0.4256	0.4028	0.1816	0.4117	0.3782

This table expands the mean order statistics analysis for $b = 3$. The mean goes up with r, just like in Table 1, which shows that the results are consistent across changes in the parameters. The values are a little higher than those for $b = 2$, which is because the shape parameter scales the data. IPTL once again has the highest averages, which demonstrates that it is more sensitive to higher-order values than LTL and TLL. When you look more closely, the LTL distribution shows the smoothest change in mean values, which means its estimation is more stable. IPTL, on the other hand, shows sharper increases, which could make the effect of high values in the data seem stronger than it really is. TLL usually lies somewhere in the middle, although it tends to be more like IPTL at higher orders, which means that growth patterns are comparable when r is higher. The elevated mean levels across all models for b=3 further substantiate the impact of the shape parameter in elongating the distribution. These data highlight that the selection of parameters directly influences the interpretation of order statistics, and LTL consistently provides a balanced response across many settings.

This is a summary of the variance results for the value of b equal to three. The variances increase with the value of r, as seen in Table 2, which is evidence that higher order statistics carry a greater amount of variability. When comparing different distributions, IPTL demonstrates a greater degree of dispersion, but LTL retains a degree of variance that is substantially smaller. In general, the variances at $b = 3$ are higher than those at $b = 2$; this highlights the importance that the shape parameter has in distribution spread. The effect of raising b is evident, as the variances between the two values are generally higher.After further investigation, it is discovered that the variances of IPTL increase at a considerably faster rate than those of LTL or TLL. This is a reflection of the fact

Table 4. Variance of Order Statistics, b = 3

		Alpha = 2 , Beta = 3			***Alpha = 3 , Beta = 4***			***Alpha = 4 , Beta = 3***		
n	***r***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***	***LTL***	***IPTL***	***TLL***
8	1	0.3676	0.4972	0.4275	0.4634	0.6289	0.5398	0.5677	0.7762	0.6634
	2	0.4043	0.5476	0.4723	0.5101	0.6933	0.5970	0.6261	0.8584	0.7358
	3	0.4349	0.5899	0.5103	0.5492	0.7478	0.6455	0.6754	0.9282	0.7974
	4	0.4545	0.6170	0.5346	0.5742	0.7825	0.6767	0.7070	0.9731	0.8372
	5	0.5352	0.7289	0.6362	0.6774	0.9270	0.8074	0.8381	1.1608	1.0052
	6	0.6122	0.8363	0.7355	0.7764	1.0667	0.9356	0.9650	1.3451	1.1722
	7	0.7712	1.0608	0.9474	0.9820	1.3607	1.2118	1.2330	1.7424	1.5396
	8	1.1013	1.5371	1.4212	1.4142	1.9971	1.8405	1.8160	2.6494	2.4198
9	1	0.4762	0.6458	0.5587	0.6013	0.8186	0.7069	0.7396	1.0164	0.8732
	2	0.5330	0.7246	0.6298	0.6740	0.9200	0.7980	0.8315	1.1474	0.9899
	3	0.5819	0.7924	0.6917	0.7366	1.0078	0.8777	0.9112	1.2618	1.0925
	4	0.6139	0.8370	0.7326	0.7776	1.0656	0.9305	0.9637	1.3379	1.1610
	5	0.7519	1.0307	0.9131	0.9554	1.3181	1.1646	1.1934	1.6744	1.4687
	6	0.8934	1.2314	1.1051	1.1389	1.5823	1.4158	1.4346	2.0359	1.8068
	7	1.2191	1.7027	1.5770	1.5660	2.2136	2.0438	2.0134	2.9422	2.6921
	8	2.0658	2.9959	3.0559	2.7112	4.0358	4.1242	3.7037	5.9771	6.1432
	9	2.5178	3.7328	4.0511	3.3455	5.1446	5.6418	4.7440	8.2171	9.3410
10	1	0.6331	0.8623	0.7528	0.8015	1.0967	0.9552	0.9915	1.3735	1.1892
	2	0.7268	0.9932	0.8736	0.9218	1.2667	1.1114	1.1461	1.5978	1.3927
	3	0.8113	1.1120	0.9849	1.0309	1.4219	1.2559	1.2874	1.8057	1.5835
	4	0.8684	1.1926	1.0613	1.1047	1.5278	1.3556	1.3839	1.9492	1.7167
	5	1.1366	1.5764	1.4357	1.4542	2.0368	1.8491	1.8494	2.6602	2.3941
	6	1.4372	2.0164	1.8884	1.8509	2.6322	2.4575	2.3966	3.5396	3.2759
	7	2.2906	3.3296	3.4177	3.0102	4.4957	4.6259	4.1279	6.7080	6.9573
	8	2.9223	4.3724	4.8751	3.9026	6.0863	6.8948	5.6224	7.4597	8.1524
	9	3.4620	5.3204	6.4467	4.6918	7.6370	9.6378	7.0850	10.1816	12.862
	10	4.9219	8.2143	9.5896	6.9716	9.3237	12.609	8.3534	11.536	13.457

that IPTL is sensitive to changes in parameters and has a tendency to become unstable at higher-order values. Even though TLL yields moderate variations, it still tends to be more similar to IPTL in terms of variability, particularly when bigger samples are included. LTL, on the other hand, exhibits a slow and controlled development in variance, which makes it more ideal for practical modeling situations where stability is necessary. It is clear that parameter b is responsible for governing the dispersion structure of these distributions, as evidenced by the large rise in all variances when compared to b that is equal to two. LTL is further validated as the most reliable method for modeling consistent order statistics under a variety of scenarios as a result of these findings.

7. Real Data

A representation of the stress over concrete bars can be found below. A random experiment was chosen to apply pressure (kg/M) on a concrete block, and the data was acquired from a laboratory that specializes in civil engineering.

Table 5. Stress Concrete Data

0.0041	0.0068	0.0077	0.0083
0.0098	0.0099	0.0099	0.0099
0.0090	0.0090	0.0090	0.0090

Table 6. Goodness of Fit for the Real Data

Distribution	***K-S***	***AND***	***CVM***
LTL	0.2032	0.4206	0.0712
IPTL	1.5255	1.8041	0.8519
TLL	0.2486	0.8143	0.0939

Using K-S, Anderson-Darling, and Cramér–von Mises statistics, Table 6 checks how well three possible distributions (LTL, IPTL, and TLL) fit the real stress solid data. The best total fit is shown by LTL, which has the lowest test statistics across all measures. On the other hand, IPTL gets much higher values, which suggests that it is not a good fit for this dataset. The TLL model fits the data pretty well, but the LTL model, which is the best fit for the data, clearly does better.

Table 7. Mean of Order Statistics or Real Data, $b = 2$

r	1	2	3	4	5	6	7	8	9	10	11	12
LLT	0.013	0.023	0.042	0.051	0.075	0.085	0.099	0.123	0.135	0.140	0.165	0.186
IPTL	0.027	0.049	0.075	0.103	0.120	0.150	0.187	0.234	0.257	0.280	0.305	0.339
TLL	0.038	0.053	0.060	0.077	0.090	0.103	0.125	0.153	0.212	0.262	0.319	0.329

This table reports the expected means of order statistics for the fitted models using the real data. LTL consistently provides lower mean values compared to IPTL and TLL, which aligns with its better fit observed in Table 6. IPTL again produces noticeably higher means, showing its tendency toward inflated estimates. The differences across the three distributions highlight how model choice significantly influences predicted order statistic behavior.

Table 8. Variance of Order Statistics for Real Data, $b = 2$

r	1	2	3	4	5	6	7	8	9	10	11	12
LLT	0.535	0.615	0.686	0.734	0.961	1.216	1.938	2.472	2.929	4.164	5.224	7.761
IPTL	0.729	0.840	0.940	1.009	1.333	1.703	2.817	3.699	4.501	6.950	8.743	10.76
TLL	0.637	0.739	0.833	0.898	1.214	1.597	2.891	4.125	5.454	8.114	10.65	12.98

This table displays the variance of order statistics for the actual dataset. Consistent with previous findings, IPTL exhibits the greatest variations, signifying elevated dispersion and less stability in forecasts. LTL exhibits reduced variances while effectively encapsulating the fundamental traits of the data, hence affirming its appropriateness as the optimal distribution fit. TLL exhibits greater variability than LTL, hence affirming LTL's advantage in modeling this dataset.

Upon closer examination, the variances of IPTL increase significantly for higher orders, indicating that it overestimates variability and may be impractical for stable modeling. TLL, albeit being moderate, exhibits a significant increase in subsequent orders, hence diminishing its reliability. LTL variances increase more uniformly and proportionately, illustrating its equitable depiction of data distribution. This regulated expansion renders LTL both statistically efficient and more interpretable for practical applications.

8. Conclusion

Recurrence relations for the moments of order statistics derived from the Lambert–Topp–Leone (LTL) distribution were established and examined in this paper. These relations were then compared with similar models such as the Power Inverted Topp–Leone (PITL) and Topp–Leone Lomax (TLL). A comparison was made between these models and the LTL distribution. The results of the theoretical investigation indicated that the LTL distribution offers flexible expressions for both single moments and product moments. The results of the simulation demonstrated that LTL consistently produces more stable means and variances across a wide range of parameter values. On the other hand, PITL frequently displayed a larger degree of variability, whereas TLL displayed behavior that was intermediate. An additional confirmation of the efficiency of LTL was provided by its application to actual concrete stress data. The results of goodness-of-fit tests, such as K-S, Anderson–Darling, and Cramér–von Mises, showed that LTL performed better than PITL and TLL. Furthermore, the order statistics that were produced via LTL aligned more closely with empirical data, both in terms of the mean and the variance. The conclusion that can be drawn from these findings is that LTL is reliable when modeling lifetime and reliability data. The findings, taken as a whole, lend credence to the belief that LTL is a robust and versatile model for order statistics analysis. The Bayesian estimating method and its applications to multivariate instances could be investigated in subsequent work.

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