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Analysis of Queueing-Inventory System that Delivers an Item for Pre-Booked Orders

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Abstract This paper considers the delivery of an item for pre-booked orders in a queueing-inventory system. This system considers the maximum capacity of stock to be S units with two waiting platforms and two dedicated servers. The demands arrive according to the Poisson process and enter into the platform 1 (PL 1) of size M, including one at the service point. The server 1 (SR 1) is used to pick up orders from the client in PL 1, and the assumption is that orders will be picked even if the stock level is zero. Following order selection, the client joins the platform 2 (PL 2), which has a virtual waiting area of size N. Subsequently, server 2 (SR 2) fabricates the selected orders one by one and distributes them to the PL 2 client. Both service durations are distributed exponentially. The arriving clients are lost if the PL 1 is full. An external supplier replenishes the stock, which is carried out according to the (s, Q) reordering policy. The exponential distribution determines the lead time for reordering. Several numerical results for different parameters are given to clarify the system's key performance indicators. In addition, an investigation is conducted into the required numerical interpretations to improve the suggested model.

Keywords Delivery Server, (s, Q) Ordering Policy, Multi Server Service Facilities, Fabrication, Poisson Process.

AMS 2010 subject classifications 60K25, 90B05, 90B22

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1. Introduction

Several researchers have closely examined queuing-inventory models over the last three decades. The availability of stock in queueing-inventory system (QIS) models should also be considered a key component. In this paper, the researcher introduces that the orders will be picked even if the stock level is zero, and the picked orders are considered to be assembled one by one; that is, it involves actively working with raw materials to create pre-booked order (transforming raw materials into finished goods) and delivering them to the clients. These deliveries focus on improving customer satisfaction by offering personalised, eco-friendly, and real-time solutions tailored to evolving demands. In this QIS mainly deals with two types of business processes: a finished goods-based business and a raw materials-based business.

In 2007, Krishnamoorthy and Anbazhagan [1] analyzed service facilities with N policy and considered a finite buffered queue in an stochastic queueing-inventory system (SQIS). According to this operating policy,

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when the queue length is nil, the system turns off the server, and the service doesn't start until the client's level meets threshold level N. For many dynamic control issues, N-policy is the best operational strategy. In 2015, Amirthakodi and Sivakumar [2] examined an SQIS featuring a service facility and a finite orbit size, where clients arrive at service locations according to a Poisson process and the service time is exponentially distributed. Anbazhagan et al. [3] considered a queuing-inventory system incorporating a service facility and a finite buffered queue. They assumed that demand follows a Poisson process and is delivered to the client after being processed for a random duration. There are two service levels in the system. Jeganathan et al. [4] investigated an integrated and interconnected SQIS that handles fresh, returned, and refurbished items. This system provides a multi-type service facility to approach multi-class consumers through a dedicated channel. Anbazhagan et al. [5] developed a two-commodity Markovian stock system that includes complementary and retrial demand. Numerous research have explored various forms of service facilities in queueing-inventory systems, refer to [6, 7, 8, 9].

Yadavalli et al. [10] considered a continuous system for perishable stock within a finite population, featuring two heterogeneous servers, one of which is unreliable. This article examines a system with two heterogeneous servers designated server 1 and server 2 and a finite capacity waiting area. The service time distribution for each server operates independently. Jehoashan Kingsly et al. [11] developed a queuing-inventory system featuring two heterogeneous servers, a shared, finite buffer, and a flexible server. They assumed the presence of two classes of clients: high-priority and low-priority. The dedicated server exclusively serves high-priority clients, while the flexible server attends to both client types. Jeganathan et al. [12] analyzed a retrial queuing-inventory system characterized by a stock-dependent demand rate, comparing (s, S) and (s, Q) ordering policies.

He and Jewkes [13] addressed how to manage production in response to stochastic demand by proposing optimal and near-optimal inventory control rules for make-to-order systems. He et al. [14] looked more closely at inventory and backlog levels when examining performance metrics in these kinds of systems. Our study goes beyond their models' focus on reactive production by examining a queueing-inventory system that fulfils pre-booked requests, with a focus on timely delivery and anticipatory inventory control. Artalejo and Phung-Duc [15] examined the steady-state behaviour of Markovian retrial queues that feature two-way communication. This article focuses on a system where a single server receives incoming calls from clients, defining a busy period until the server initiates outgoing calls once idle. The client's retrial is also considered. Hiroyuki Sakurai and Phung [16] developed a model of two-way communication retrial queues involving multiple types of outgoing calls. We suppose that different outgoing call types exhibit unique exponential distributions. Two-way communication can be modified to improve performance, including Retrial Queues with Balanced Call Blending [17], retry queues with several kinds of outbound calls and unreliable servers [18]. Dey and Deepak [19] examined two-way communication with orbit queues with server vacation in queueing theory, where a single server is considered to perform M-type incoming calls and L-type outgoing calls. Each such type is exponentially distributed.

Due to the growing popularity of electronic commerce, parcel delivery services are now widely available. The queuing system was implemented to provide a home delivery service from the courier office for clients' homes. Hideyama et al. [20] executed parcel delivery services with parcel lockers. Due to the receiver's absence and the courier's need to resend the package later, the re-delivery issue has also worsened. A locker facility would be highly beneficial to delivering the packages. Additionally, it lowers the additional transit expenses. Van Duin [21] modified parcel delivery service also uses lockers. Schnieder et al. [22] discussed using the staff-manned collection and delivery points with parcel lockers. Unnikrishnan and Figliozzi [23] explored the impact of the COVID-19 pandemic on home delivery purchases and expenditures. Yoon-Joo Park [24] examined delivery services in online grocery markets, analyzing variations in delivery times and packaging types. Ekramol et al. [25] have analyzed serving eligible clients with reworks in a stochastic stock system. Additionally, home service is available for returning items, with service times following an exponential distribution. Further related works can be found in [26, 27, 28, 29], while diverse models of delivery service facilities in queueing-inventory systems

1.1. Stimulation

A practical experience motivated the authors to develop this mathematical model. The author recently went to purchase brand-new eyewear from an optical store. While there, the author made fascinating observations regarding how that exhibit room operated. The showroom only picks up orders from the clients, and when the order is placed, it will be produced by a server inside the showroom and delivered to the clients. This shop inspired the author's mathematical model because of how effectively it functioned and seemed. Similar procedures are used in other contexts, such as bakeries that provide bespoke cakes, where customers place counter orders, have them produced in the kitchen, then have them delivered. Additionally, there are tailor shops where clients select materials or designs, and the tailor creates the clothing before it is delivered. The authors were motivated to develop the suggested model by these real world systems effective order processing and servicing methods.

1.2. Research Gap

We have identified the following assumptions as research gaps.

- The majority of current queueing-inventory models place little emphasis on pre-scheduled (pre-booked) client orders and instead concentrate on walk-in or real-time demand.
- No one has examined how orders will be picked even when the stock level is zero, despite the fact that numerous researchers have examined how to take orders and provide delivery methods in SQIS.

1.3. The Paper's Structure

The document is structured as follows: Section 2 describes the model. Section 3 analyses the model's transitions and discusses the main results. Section 4 focuses on the steady-state analysis. In Section 5, the model's critical metrics are outlined. Section 6 interprets the numerical illustration. Finally, Section 7 presents the conclusion and provides suggestions.

Notation

- $[Q]_{st}$: The submatrix element at $(s,t)^{th}$ position of Q.
- **0** : Zero matrix.
- **I** : Identity matrix.
- **e** : Column vector of 1's with appropriate order.
- **FCFS** : First Come First Serve.
- **SQIS** : Stochastic Queueing-Inventory System.
- **QIS** : Queueing-Inventory System.
- **SR 1** : server 1.
- **SR 2** : server 2.
- **PL 1** : Platform 1.
- **PL 2** : Platform 2.

2. Model Description

This study explains the delivers an item for pre-booked orders in queueing-inventory system and the assumption that orders will be picked even if the stock level is zero. This system considers the maximum stock level to be S units under a continuous review (s, Q) ordering policy is accepted. Upon the stock level falling to s, an order of Q items (where $Q(=(S-s)>s+1)$) is placed. The stock reorder level is set at s. The lead time for restocking comes after an exponential distribution with parameter β. And considers two waiting platforms, namely platform

1 (PL 1) and platform 2 (PL 2), and two dedicated servers, namely server 1 (SR 1) and server 2 (SR 2). All arriving clients follow a Poisson process and pass into the PL 1 with a parameter λ of size M, including one at the service point. The arriving clients are lost if the PL 1 is full. The SR 1 is only used to pick up orders from the client in PL 1, and following order selection, the client joins into the PL 2, it has a virtual waiting area of size N $(M > N)$. When the platform 2 reaches its capacity N, server 1 stops service temporarily. Once space becomes available in platform 2, server 1 resumes service. Subsequently, SR 2 assemble the pre-booked orders one by one (transforming raw materials into finished goods) and using delivery technique for distributes them to the PL 2 client. Both service durations are distributed exponentially using parameters μ and α, respectively $(\alpha > \mu)$. First-Come, First-Served (FCFS) is the method by which the service is run.

3. Examination of the Model

With the on-hand stock level, the amount of clients in the PL 2 and the amount of clients in the PL 1 represented by the variables $\eta_1(t)$, $\eta_2(t)$ and $\eta_3(t)$, respectively, at time t. It can be demonstrated that the triplet $\{(\eta_1(t), \eta_2(t), \eta_3(t)); t \geq 0\}$ forms a continuous-time markov chain with the following state space,

E=$\{(\tau_1,\tau_2,\tau_3)/\tau_1$=0,1,2,...,S; τ_2=0,1,2,...,N;τ_3=0,1,2,...,M$\}$

In this process, we utilize the following arguments,

Transition

- **Transitions resulting from arriving customers:**
  $* (\tau_1,\tau_2,\tau_3) \rightarrow (\tau_1, \tau_2, \tau_3 + 1)$: the rate is λ,
  τ_1=0,1,2,...,S; τ_2=0,1,2,...,N;τ_3=0,1,2,...,M-1
- **Transitions resulting from order picking:**
  $* (\tau_1,\tau_2,\tau_3) \rightarrow (\tau_1, \tau_2 + 1, \tau_3 - 1)$: the rate is μ,
  τ_1=0,1,2,...,S; τ_2=0,1,2,...,N-1;τ_3=1,2,...,M
- **Transitions resulting from delivering the picked order:**
  $* (\tau_1,\tau_2,\tau_3) \rightarrow (\tau_1 - 1, \tau_2 - 1, \tau_3)$: the rate is α,
  τ_1=1,2,...,S; τ_2=1,2,...,N;τ_3=0,1,2,...,M
- **Transitions resulting from replenishment:**
  $* (\tau_1,\tau_2,\tau_3) \rightarrow (\tau_1 + Q, \tau_2, \tau_3)$: the rate is β,
  τ_1=0,1,2,...,s; τ_2=0,1,2,...,N;τ_3=0,1,2,...,M

Subsequently, the system's activities show that the triplet stochastic process (η_1, η_2, η_3) = $\{(\eta_1(t), \eta_2(t), \eta_3(t)); t \geq 0\}$ is a Markov chain operating in continuous time with a finite discrete state space equivalent to E. This process's infinitesimal generator is

$$P = (a((\tau_1,\tau_2,\tau_3), (\tilde{\tau_1},\tilde{\tau_2},\tilde{\tau_3}))), (\tau_1,\tau_2,\tau_3), (\tilde{\tau_1},\tilde{\tau_2},\tilde{\tau_3}) \epsilon E$$

The process's infinitesimal generator P is generated by

$$
P = \begin{array}{c|cccccccccccc}
 & S & S-1 & S-2 & \dots & Q+1 & Q & \dots & s & s-1 & \dots & 1 & 0 \\
\hline
S & Z_0 & X & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
S-1 & \mathbf{0} & Z_0 & X & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
S-2 & \mathbf{0} & \mathbf{0} & Z_0 & \ddots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Q+1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Z_0 & X & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
Q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & Z_0 & \ddots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s & Y & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & Z_1 & X & \dots & \mathbf{0} & \mathbf{0} \\
s-1 & \mathbf{0} & Y & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & Z_1 & \ddots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Y & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & Z_1 & X \\
0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & Y & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & Z_2
\end{array}
$$

where,

$$
X = \begin{array}{c|cccccc}
 & 0 & 1 & 2 & \dots & N-1 & N \\
\hline
0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
1 & A & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
2 & \mathbf{0} & A & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
N-1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\
N & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A & \mathbf{0}
\end{array}
$$

With,

$$
A = \begin{array}{c|cccccc}
 & 0 & 1 & 2 & \dots & M-1 & M \\
\hline
0 & \alpha & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
1 & \mathbf{0} & \alpha & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
2 & \mathbf{0} & \mathbf{0} & \alpha & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M-1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \alpha & \mathbf{0} \\
M & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \alpha
\end{array}
$$

Also,

$$Y = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} D & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & D \end{pmatrix} \end{array}$$

With,

$$D = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} \beta & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \beta & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \beta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \beta \end{pmatrix} \end{array}$$

Further,

$$Z_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} B_0 & C & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_1 & C & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_1 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & B_1 & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & B_2 \end{pmatrix} \end{array}$$

With,

$$C = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mu & \mathbf{0} \end{pmatrix} \end{array}$$

$$B_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -\lambda & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\mu+\lambda) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mu+\lambda) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\mu+\lambda) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\mu \end{pmatrix} \end{array}$$

$$B_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\alpha+\lambda) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\alpha+\mu+\lambda) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\alpha+\mu+\lambda) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\alpha+\mu+\lambda) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -(\alpha+\mu) \end{pmatrix} \end{array}$$

$$B_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\alpha+\lambda) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\alpha+\lambda) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\alpha+\lambda) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\alpha \end{pmatrix} \end{array}$$

Further more,

$$Z_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} E_0 & C & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_1 & C & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & E_1 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & E_1 & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & E_2 \end{pmatrix} \end{array}$$

With,

$$E_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\lambda+\beta) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\mu+\lambda+\beta) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mu+\lambda+\beta) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\mu+\lambda+\beta) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -(\mu+\beta) \end{pmatrix} \end{array}$$

$$E_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\alpha+\lambda+\beta) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\alpha+\mu+\lambda+\beta) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\alpha+\mu+\lambda+\beta) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\alpha+\mu+\lambda+\beta) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -(\alpha+\mu+\beta) \end{pmatrix} \end{array}$$

$$E_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\alpha+\lambda+\beta) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\alpha+\lambda+\beta) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\alpha+\lambda+\beta) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\alpha+\lambda+\beta) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -(\alpha+\beta) \end{pmatrix} \end{array}$$

And,

$$Z_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} E_0 & C & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_0 & C & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & E_0 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & E_0 & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & F_0 \end{pmatrix} \end{array}$$

With,

$$F_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & M-1 & M \end{array} \\ \begin{pmatrix} -(\lambda+\beta) & \lambda & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\lambda+\beta) & \lambda & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -(\lambda+\beta) & \lambda \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\beta \end{pmatrix} \end{array}$$

The matrices A, B_0, B_1, B_2, C, D, E_0, E_1, E_2, and F_0 are square matrices of order $(M+1)$. It is also noticed that the matrices Z_0, Z_1, Z_2, X, and Y are square matrices of order $(N+1)(M+1)$.

4. Analysis in the steady state

The homogeneous Markov process is evident from P's structure. This is an irreducible, aperiodic, persistent non-null function $\{(\eta_1(t), \eta_2(t), \eta_3(t)); t \geq 0\}$ on the finite state space E. The limiting distribution, thus,

$$\boldsymbol{\phi}^{(\tau_1,\tau_2,\tau_3)} = \lim_{t\to\infty} Pr\big(\eta_1(t) = \tau_1, \eta_2(t) = \tau_2, \eta_3(t) = \tau_3 | \eta_1(0), \eta_2(0), \eta_3(0)\big)$$

exists and is independent of the initial state . Let

$\boldsymbol{\Phi} = \big(\boldsymbol{\phi}^{(0)}, \boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, ..., \boldsymbol{\phi}^{(S)}\big)$

$$\boldsymbol{\phi}^{(\tau_1)} = \big(\boldsymbol{\phi}^{(\tau_1,0)}, \boldsymbol{\phi}^{(\tau_1,1)}, \boldsymbol{\phi}^{(\tau_1,2)}, ..., \boldsymbol{\phi}^{(\tau_1,N)}\big), \tau_1 = 0, 1, 2, ..., S$$

$$\boldsymbol{\phi}^{(\tau_1,\tau_2)} = \big(\phi^{(\tau_1,\tau_2,0)}, \phi^{(\tau_1,\tau_2,1)}, \phi^{(\tau_1,\tau_2,2)}, ..., \phi^{(\tau_1,\tau_2,M)}\big), \tau_1 = 0, 1, 2, ..., S;\ \tau_2 = 0, 1, 2, ..., N,$$

where $\phi^{(\tau_1,\tau_2,\tau_3)}$ denotes the steady state probability for the state (τ_1,τ_2,τ_3) of the process exists and is given by

$$\mathbf{\Phi}P = 0 \ \ and \ \ \mathbf{\Phi}e = 1 \tag{1}$$

The equation $\mathbf{\Phi}P$=0 yields the following set of equation.

$$\begin{aligned}
\phi^{(c+1)}X + \phi^{(c)}Z_2 &= \mathbf{0}, \ c = 0 && (2)\\
\phi^{(c+1)}X + \phi^{(c)}Z_1 &= \mathbf{0}, \ c = 1, 2, ..., s && (3)\\
\phi^{(c+1)}X + \phi^{(c)}Z_0 &= \mathbf{0}, \ c = s+1, s+2, ..., Q-1 && (4)\\
\phi^{(c+1)}X + \phi^{(c)}Z_0 + \phi^{(c-Q)}Y &= \mathbf{0}, \ c = Q && (5)\\
\phi^{(c+1)}X + \phi^{(c)}Z_0 + \phi^{(c-Q)}Y &= \mathbf{0}, \ c = Q+1, Q+2, ..., S-1 && (6)\\
\phi^{(c)}Z_0 + \phi^{(c-Q)}Y &= \mathbf{0}, \ c = S && (7)
\end{aligned}$$

After lengthy simplifications, the above equation, (except (5)), yield

$$\phi^{c} = \begin{cases}
(-1)^{(Q-c)}\phi^{(Q)}(XZ_0^{-1})^{(Q-s-1))}(XZ_1^{-1})^{(s)}(XZ_2^{-1}), & \text{c=0,}\\
(-1)^{(Q-c)}\phi^{(Q)}(XZ_0^{-1})^{(Q-s-1)}(XZ_1^{-1})^{(s+1-c)}, & \text{c=1,2,...,s,}\\
(-1)^{(Q-c)}\phi^{(Q)}(XZ_0^{-1})^{(Q-c)}, & \text{c=s+1, s+2, ..., Q-1,}\\
I, & \text{c=Q,}\\
\sum\limits_{d=0}^{S-c}(-1)^{(2Q+1-c)}\phi^{(Q)}(XZ_0^{-1})^{(S+s-c-d)}(XZ_1^{-1})^{(d+1)}(YZ_0^{-1}), & \text{c=Q+1, Q+2,...,S,}
\end{cases}$$

where $\phi^{(Q)}$ can be obtained by solving equations (5) and $\mathbf{\Phi}e = 1$. That is,

$$\phi^{(Q)}\Bigg\{\sum_{d=0}^{s-1}\{(-1)^{(Q)}(XZ_0^{-1})^{(2s-1-d)}(XZ_1^{-1})^{(d+1)}(YZ_0^{-1})\}X + Z_0 + \\ \{(-1)^{(Q)}(XZ_0^{-1})^{(Q-s-1)}(XZ_1^{-1})^{(s)}(XZ_2^{-1})\}Y\Bigg\} = 0$$

and

$$\begin{aligned}
&\phi^{Q}\Bigg\{(-1)^{Q}(XZ_0^{-1})^{(Q-s-1))}(XZ_1^{-1})^{s}(XZ_2^{-1})\\
&+\sum_{c=1}^{s}\{(-1)^{(Q-c)}(XZ_0^{-1})^{(Q-s-1))}(XZ_1^{-1})^{(s+1-c)}\}\\
&+\sum_{c=s+1}^{Q-1}\{(-1)^{(Q-c)}(XZ_0^{-1})^{(Q-c)}\} + I\\
&+\sum_{c=Q+1}^{S}(-1)^{(2Q+1-c)}\sum_{d=0}^{S-c}\{(XZ_0^{-1})^{(S+s-c-d)}(XZ_1^{-1})^{(d+1)}(YZ_0^{-1})\}\Bigg\}e = 1.
\end{aligned}$$

5. Assessment of Expected System Performance Measures

To calculate the Expected Total Cost (ETC) rate, we compute a number of system performance metrics in its steady state in this section.

5.1. Expected stock Level [E_I]:

Let E_I represent the average stock level in the steady state.

$$E_I = \sum_{\tau_1=1}^{S} \sum_{\tau_2=0}^{N} \sum_{\tau_3=0}^{M} \tau_1 \phi^{(\tau_1,\tau_2,\tau_3)}$$

5.2. Expected Rate of Reorder [E_R]:

Let E_R represent the expected rate of reorder in the steady state.

$$E_R = \sum_{\tau_2=1}^{N} \sum_{\tau_3=0}^{M} \alpha \phi^{(s+1,\tau_2,\tau_3)}$$

5.3. Expected Number of clients in the PL 1 [E_W]:

Let E_W represent the expected number of clients in the waiting hall in the steady state.

$$E_W = \sum_{\tau_1=0}^{S} \sum_{\tau_2=0}^{N} \sum_{\tau_3=1}^{M} \tau_3 \phi^{(\tau_1,\tau_2,\tau_3)}$$

5.4. Expected Number of clients in the PL 2 [E_P]:

Let E_P represent the expected number of clients in the pool in the steady state.

$$E_P = \sum_{\tau_1=0}^{S} \sum_{\tau_2=1}^{N} \sum_{\tau_3=0}^{M} \tau_2 \phi^{(\tau_1,\tau_2,\tau_3)}$$

6. Numerical Experiments

A set of exemplary results is presented. The outcomes of numerical experiments conducted for various examples are considered, revealing that the ETC is minimized.

6.1. Formulation of the Cost Function

The three-dimensional stochastic process's ETC value is calculated by

$$ETC(S, s) = c_h E_I + c_{sc} E_R + c_{wcq} E_W + c_{wcp} E_P.$$

Where,
c_h : The cost of stock holding per unit item.
c_{sc} : The setup cost per order.

c_{wcq} : The cost incurred by client for waiting in the PL 1 per unit time.
c_{wcp} : The cost incurred by client for waiting in the PL 2 per unit time.

6.2. Results and Discussions

Table 1 and Figure 1 indicates the consequence of S $\in$ [35, 41] and s $\in$ [2, 6] on the optimal expected total cost value, while keeping all other parameters fixed. In **Table 1**, we note that each row and column has the minimum possible value for ETC. The minimum possible ETC rate is indicated by the underlined values in each row, while the bold values in each column represent the minimum possible ETC rate. The value with both bold and underlined formatting specifies the least cost rate of the function ETC* (S*,s*). It gives ETC*(S*,s*) = 1.307220 is achieved at S* = 38, s* = 4 and $\beta = 5.1$. The fixed parameters and cost values as follows λ = 4.6, μ = 13.2, α = 14.5, N = 10, M = 12, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 2 indicates the consequence of S $\in$ [36, 48], s $\in$ [1, 9] and β = 3.3, 4.01, 5.1 on the optimal expected total cost value with all other parameters held constant.. Resupply will be completed more quickly, as well as fabrication and delivery service, when the lead time rate rises. If the β increases, then the optimal ETC*(S*,s*) decreases. And assumed the parameters and cost values as follows λ = 4.6, μ = 13.2, α = 14.5, N = 10, M = 12, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 3 illustrate the impact of S $\in$ [39, 45], s $\in$ [1, 9] and λ = 4.4, 4.6, 4.8 on the optimal expected total cost value, assuming all other parameters remain constant. As the influx of clients into the system rises, there is a concurrent increase in the volume of items being purchased. If the λ increases, the optimal ETC*(S*,s*) also increases. The following fixed parameters and cost levels are assumed β = 4.01, μ = 13.2, α = 14.5, N = 10, M = 12, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 4 indicates the consequence of S $\in$ [39, 45], s $\in$ [1, 9] and μ = 12, 14, 16 on an ideal projected total cost when every other parameter is set. If the μ increases, then the optimal ETC*(S*,s*) decreases. The assumed fixed parameters and cost values are as follows β = 4.01, α = 14.5, λ = 4.6, N = 10, M = 12, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 5 illustrate the consequence of S $\in$ [39, 45], s $\in$ [1, 9] and α = 12, 14, 16 on the optimal expected total cost value, with all other parameters held constant. The stock level drops if production and delivery service are finished promptly. If the α increases, the optimal ETC*(S*,s*) decreases. The following fixed parameters and cost levels are assumed β = 4.01, μ = 13.2, λ = 4.6, N = 10, M = 12, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 6 demonstrates that an increase in the service rate α, results in a decrease in the expected stock level. Conversely, an increase in the lead time rate β leads to an increase in the expected stock level. With the values S = 42, s = 5, N = 10, M = 12, λ = 4.6, μ = 13.2, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Figure 7 shown that if increases the service rate α, then the ETC decreases. Similarly, if an arrival client λ increases, then the ETC increases. With the values S = 42, s = 5, N = 10, M = 12, μ = 13.2, β = 4.01, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

In **Table 2** gives the optimal expected total cost, considering both setup costs (c_{sc}) and holding costs (c_h). If both setup cost and holding cost increase, it suggests that the optimal expected total cost also increase. With the values are N = 10, M = 12, λ = 4.6, μ = 13.2, α = 14.5, β = 4.01, c_{wcq} = 0.1, c_{wcp} = 0.02.

In **Table 3** the effects of the service rates μ and α were investigated. The ETC decreases as both service rates increase.. With the values are S = 42, s = 5, N = 10, M = 12, λ = 4.6, β = 4.01, c_h = 0.05, c_{sc} = 4.95, c_{wcq} = 0.1, c_{wcp} = 0.02.

In **Table 4** examined the consequence of the service rate μ and the arriving clients rate λ. The ETC increases with the number of arriving clients. Conversely, the ETC decreases as the service rate rises. With the values are S = 42, s = 5, N = 10, M = 12, α =14.5; β = 4.01, c_h = 0.05, c_{sc} = 4.95, c_{wcq} = 0.1, c_{wcp} = 0.02.

In **Table 5** examined the effects of the service rate α and the lead time rate β. The ETC decreases with an increase in the service rate. Similarly, the ETC decreases as the lead time rate rises. With the values are N = 10, M = 12, λ = 4.6, μ = 13.2, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

In **Table 6**, the service rate α and the number of clients in the PL 2 (N) were investigated. As the service rate is increased, the ETC is observed to fall. As the number of clients is increased, the ETC is observed to rise. With the values are S = 42, s = 5, M = 12, λ = 4.6, μ = 13.2, β = 4.01, c_h = 0.05, c_{sc} = 5, c_{wcq} = 0.1, c_{wcp} = 0.02.

Table 1. Expected total cost ETC(S,s).

S\s	2	3	4	5	6
35	1.390882	1.336856	1.332944	1.352229	1.375106
36	**1.389256**	1.327942	1.319535	1.337952	1.364690
37	1.390711	1.323523	1.311088	1.327706	1.356212
38	1.394807	**1.323023**	**1.307220**	1.321781	1.350496
39	1.401136	1.325844	1.307349	**1.319976**	**1.348040**
40	1.409343	1.331432	1.310843	1.321812	1.348893
41	1.419124	1.339300	1.317110	1.326716	1.352777

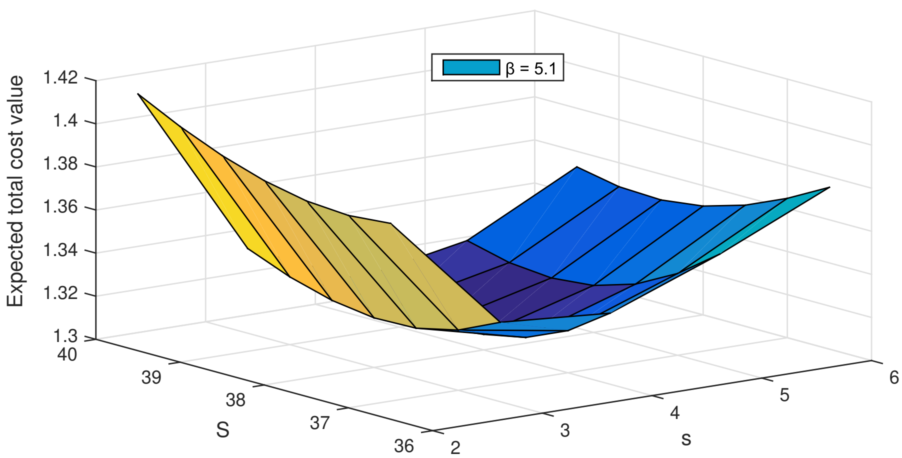

Figure 1. Effect of S ∈ [35, 41] and s ∈ [2, 6] on the optimal expected total cost value.

Table 2. Optimal expected total cost on setup cost (c_{sc}) vs holding cost (c_h).

c_{sc} \ c_h	0.03	0.04	0.05	0.06	0.07	0.08
4	44 5 0.938250	43 5 1.159872	41 5 1.374408	39 4 1.575944	39 4 1.774205	39 4 1.972465
5	45 6 0.957321	43 5 1.184310	42 5 1.402933	41 5 1.616701	39 4 1.825825	39 4 2.024086
6	45 6 0.971152	45 6 1.204539	43 5 1.428749	42 5 1.645980	41 5 1.858994	40 5 2.068340
7	45 6 0.984983	45 6 1.218370	44 6 1.449958	43 5 1.673188	42 5 1.889028	41 5 2.101287
8	45 6 0.998815	45 6 1.232201	45 6 1.465588	44 6 1.694813	42 5 1.916323	42 5 2.132075

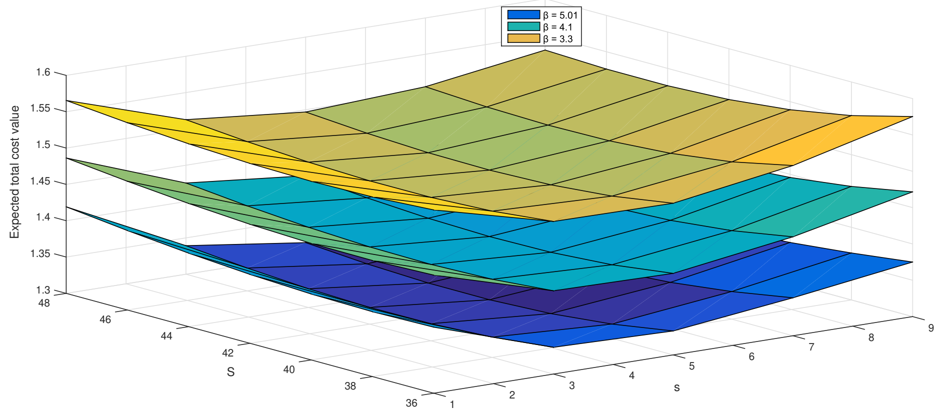

Figure 2. Effect of S $\in$ [36, 48], s $\in$ [1, 9] and β = 3.3, 4.01, 5.1 on the optimal expected total cost value.

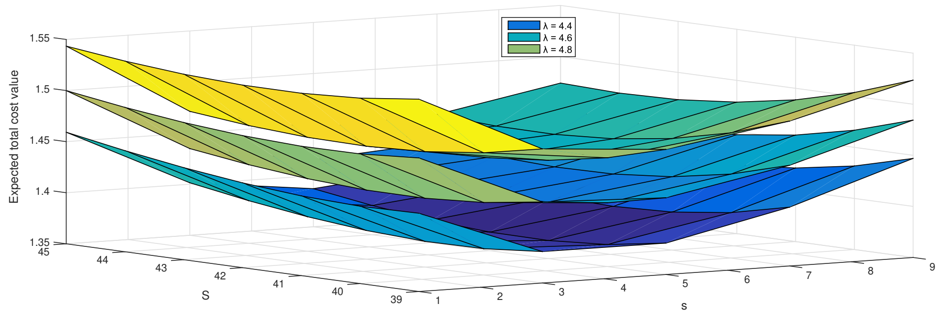

Figure 3. Effect of S $\in$ [39, 45], s $\in$ [1, 9] and λ = 4.4, 4.6, 4.8 on the optimal expected total cost value.

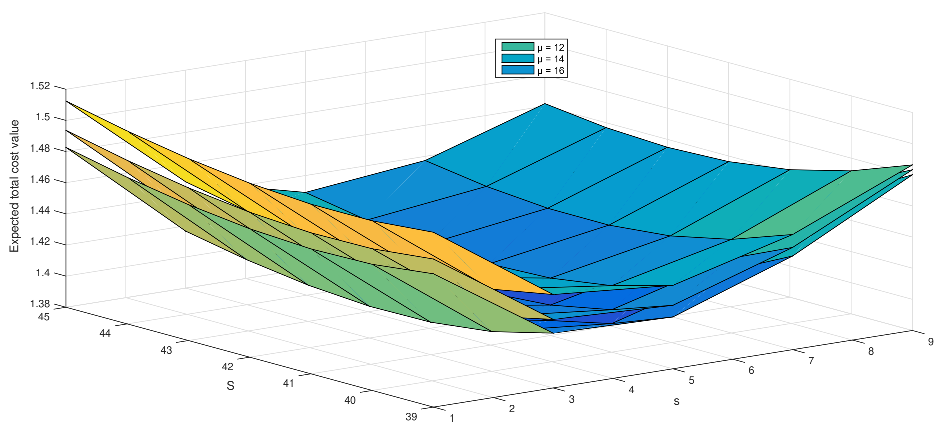

Figure 4. Effect of S $\in$ [39, 45], s $\in$ [1, 9] and μ = 12, 14, 16 on the optimal expected total cost value.

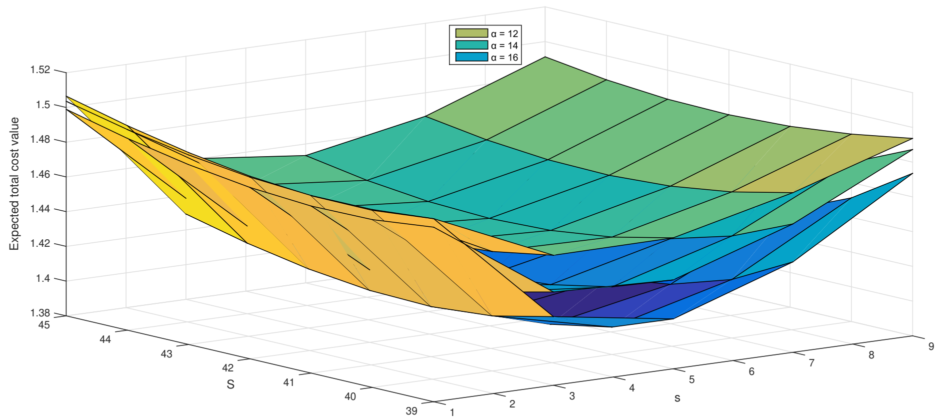

Figure 5. Effect of S ∈ [39, 45], s ∈ [1, 9] and α = 12, 14, 16 on the optimal expected total cost value.

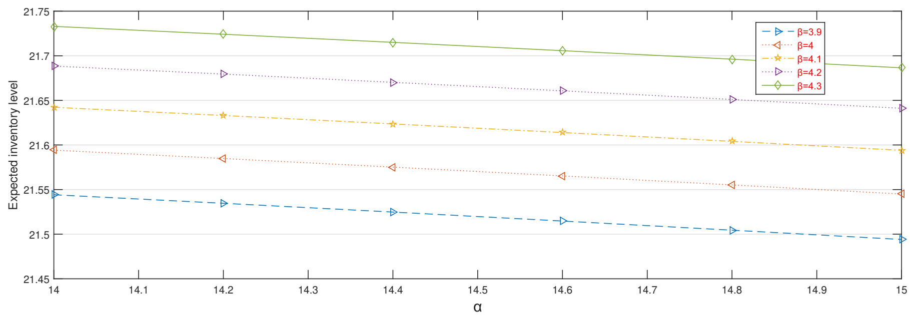

Figure 6. Expected inventory level on (β vs α).

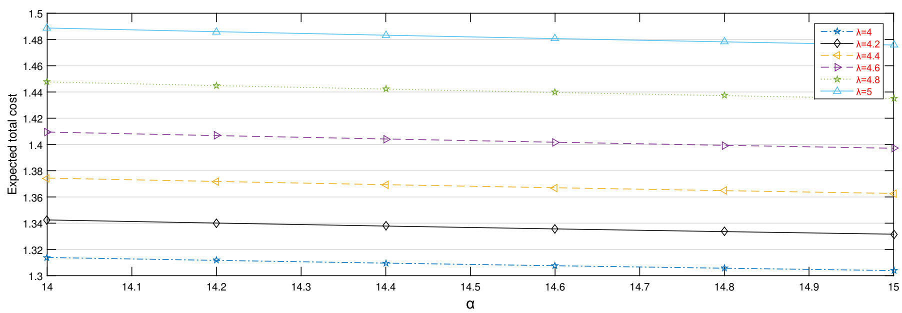

Figure 7. Expected total cost on (α vs λ).

Table 3. Expected total cost on (μ vs α)

μ / α	14.3	14.4	14.5	14.6	14.7	14.8
13.0	1.405828	1.404576	1.403351	1.402155	1.400986	1.399845
13.1	1.404946	1.403683	1.402447	1.401239	1.400059	1.398907
13.2	1.404090	1.402815	1.401568	1.400349	1.399158	1.397994
13.3	1.403257	1.401972	1.400714	1.399484	1.398282	1.397107
13.4	1.402447	1.401152	1.399884	1.398643	1.397431	1.396245
13.5	1.401661	1.400355	1.399077	1.397826	1.396603	1.395407

Table 4. Expected total cost on (λ vs μ)

λ/μ	13	13.2	13.4	13.6	13.8	14
2	1.173857	1.173369	1.172902	1.172453	1.172023	1.171610
3	1.213582	1.212631	1.211724	1.210859	1.210033	1.209243
4	1.309371	1.307851	1.306410	1.305044	1.303748	1.302517
5	1.481879	1.480026	1.478277	1.476627	1.475068	1.473596
6	1.722449	1.720922	1.719481	1.718121	1.716837	1.715625

Table 5. Optimal expected total cost on (α vs β)

α /β	**3.9**	**4.0**	**4.1**	**4.2**	**4.3**	**4.4**
	42 5	42 5	42 5	42 5	41 5	41 5
14.2						
	1.418561	1.407781	1.398025	1.389183	1.380260	1.371947
	42 5	42 5	42 5	42 5	41 5	41 5
14.3						
	1.417263	1.406477	1.396720	1.387879	1.378941	1.370635
	42 5	42 5	42 5	42 5	41 5	41 5
14.4						
	1.415992	1.405202	1.395443	1.386603	1.377643	1.369344
	42 5	42 5	42 5	42 5	41 5	41 5
14.5						
	1.414751	1.403954	1.394192	1.385352	1.376369	1.368075
	42 5	42 5	42 5	42 5	41 5	41 5
14.6						
	1.413538	1.402734	1.392969	1.384128	1.375117	1.366829
	42 5	42 5	42 5	42 5	41 5	41 5
14.7						
	1.412354	1.401542	1.391773	1.382931	1.373888	1.365605

Table 6. Expected total cost (α vs N)

α / N	8	9	10	11	12	13
14	1.359312	1.381584	1.409452	1.443885	1.485544	1.534183
14.2	1.357349	1.379256	1.406761	1.440930	1.482581	1.531676
14.4	1.355506	1.377051	1.404181	1.438053	1.479637	1.529104
14.6	1.353780	1.374965	1.401713	1.435258	1.476720	1.526479
14.8	1.352167	1.372997	1.399356	1.432550	1.473838	1.523813
15	1.350661	1.371143	1.397109	1.429931	1.471000	1.521115

7. Conclusion and Suggestions

The paper proposed a novel approach for taking order from the clients in the SQIS, even in instances where there is no stock available in the system with adaptive delivery techniques. By incorporating contemporary delivery methodologies and optimising multi-server configurations, these systems efficiently reduce delays and improve service quality. The work developed a finite generator transition matrix based on the model description. Additionally, the study calculated the models unique stationary probability, which enhanced understanding of system dynamics and optimised order fulfilment tactics. Various system performance metrics were calculated by the study using the resulting vector, and the ETC for the suggested model was determined. The effectiveness and financial feasibility of the suggested strategy within the SQIS framework were both clarified by this analysis. The ETC is optimised as a function of S and s for each variation in the parameters λ, β, μ, and α. This demonstrates how varying parameter values result in different impacts on the optimal ETC when all other parameters are held constant.

This study's work can be expanded in the future with careful thought towards multi-servers and multi-suppliers. Then improve this model by the Markovian arrival process and phase-type distributed service for convenient delivery.

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