STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–7.
Published online in International Academic Press (www.IAPress.org)

Fractional Medium Domination Number of Graphs

G. Uma [1], S. Amutha [2,*], N. Anbazhagan [1]

[1]*Department of Mathematics, Alagappa University, karaikudi-630 003, Tamilnadu, India*
[2]*Ramanujan Centre for Higher Mathematics, Alagappa University, karaikudi-630 003, Tamilnadu, India*

Abstract This work develops the framework of the fractional medium domination number($MD_f(G)$), focusing on connected, undirected graphs without loops. The ($MD_f(G)$) is defined as the ratio of the fractional total domination value ($T_{DV_f}(G)$) to the total number of unordered pairs of distinct vertices in a graph. This new parameter expands on traditional domination concepts by incorporating fractional values, providing a more refined measure of domination in graphs. The fractional domination value between vertices is computed as the sum of fractional contributions from their common neighbors, where each contribution is inversely proportional to the degree of the respective vertex. The paper explores bounds for the fractional medium domination number across various graph families and presents computational methods for determining $MD_f(G)$ using Python programming. Practical applications, such as network optimization and disaster relief, are also discussed to illustrate the significance of this parameter.

Keywords Fractional Medium Domination Number, Fractional Domination Number, Network Optimization.

AMS 2010 subject classifications 05C69

DOI: 10.19139/soic-2310-5070-2961

1. Introduction

Extensive studies have explored various aspects of domination in graph theory, leading to the introduction of several variants that enhance our understanding and expand their practical applications [1, 2, 3]. Among these variants, fractional domination has emerged as a prominent topic, contributing to both theoretical advancements and practical uses. Its continued development and utilization are widely acknowledged, prompting further research and exploration [4, 5]. Devvrit et al.[6] proposed the concept of fractional eternal domination, combining fractional principles with eternal domination to model secure and continuous resource distribution in dynamic networks. Vijayalakshmi et al.[7] introduced the fractional restrained domination parameter, which adds specific constraints to fractional dominating functions, making it suitable for applications requiring selective control. Meenakshi and Pankajam[8] presented the total regular fractional domination number, integrating total regular domination with fractional constraints to study domination in uniformly connected graphs. Ebin Raja Merly and Saranya[9] introduced medium domination decomposition, merging geodetic decomposition with the medium domination number to analyze how domination values are distributed across different graph segments. In this article, G refers to a graph that is undirected, connected, and free of loops. A set $S \subseteq V(G)$ is called a dominating set if every vertex outside S is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the size of the smallest dominating set [10]. A fractional dominating function $f : V(G) \rightarrow [0, 1]$ assigns a weight to each vertex in the graph such that for every vertex v, the sum of the weights of vertices in the closed neighborhood $N[v]$ is no

*Correspondence to: S. Amutha (Email: amuthas@alagappauniversity.ac.in). Ramanujan Centre for Higher Mathematics, Alagappa University, karaikudi-630 003, Tamilnadu, India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

less than 1. The fractional domination number $\gamma_f(G)$ represents the minimum total weight of f over all vertices in the graph [11, 12]. The medium domination number ($MD(G)$) of a connected, undirected, loopless graph G of order n is defined as the ratio of the total domination value ($T_{DV}(G)$) to the total number of vertex pairs in G, i.e., $MD(G) = \frac{T_{DV}(G)}{\binom{n}{2}}$, where $T_{DV}(G) = \sum_{u,v \in V(G)} \mathrm{dom}(u, v)$ is the sum of the domination values for unordered pairs of distinct vertices u, $v \in V(G)$ [13]. Extending this framework, we define a new parameter called the fractional medium domination number, denoted as $MD_f(G)$. It is defined as the ratio of the fractional total domination value ($T_{DV_f}(G)$) to the total number of unordered vertex pairs.

$$MD_f(G) = \frac{T_{DV_f}(G)}{\binom{n}{2}} = \frac{T_{DV_f}(G)}{\frac{n(n-1)}{2}},$$

where $T_{DV_f}(G) = \sum_{u,v \in V(G)} \mathrm{dom}_f(u, v)$ is the sum of the fractional domination values for unordered pairs of distinct vertices u, $v \in V(G)$, expressed as

$$\mathrm{dom}_f(u, v) = \sum_{w \in N[u] \cap N[v]} f(w),$$

where $f(w) = \frac{1}{\deg(w)}$ represents the fractional value assigned to vertex w, based on its degree ($\deg(w)$). The fractional medium domination number normalizes the total fractional domination value by the number of vertex pairs, providing a measure of the overall fractional domination efficiency in the graph. This number is useful for evaluating the connectedness and influence of vertex pairs across the entire graph in fractional terms. In the realm of disaster relief, the ability to allocate resources efficiently often determines the outcome, highlighting the importance of strategic planning and distribution method. The fractional medium domination number offers a cutting-edge solution to optimize these networks. By modeling relief centers as vertices and their connections as edges in a graph, we can assign each center a fractional value $f(w) = \frac{1}{\deg(w)}$, reflecting its capacity to assist others based on its connectivity. The fractional domination between each pair of centers is calculated, showing how effectively resources are shared, whether directly connected or not. By dividing the fractional total domination influence by the number of possible center pairs, we arrive at the fractional medium domination number $MD_f(G)$. A value equal to 1 represents a perfectly balanced distribution network, where the resource allocation or domination efficiency is optimal across the entire graph. A value near 1 indicates that the network is close to being optimal, with minor imbalances or slight variations in domination efficiency. Conversely, lower values (significantly less than 1) highlight critical gaps in coverage, pointing to areas of inefficiency or weaker connectivity in the network. However, when the fractional medium domination number $MD_f(G)$ exceeds 1, it indicates a highly robust and interconnected network. In such cases, the graph exhibits significant redundancy, ensuring that even in scenarios of multiple node failures, the overall system maintains its functionality. This powerful tool not only helps planners identify bottlenecks but also allows them to strategically prioritize upgrades and improve overall resource allocation. With its wide-reaching applications in disaster management, public health logistics, and transportation networks, the fractional medium domination number is a game-changer in optimizing real-world systems that depend on timely, equitable resource distribution. In Fig. 1, for the graph

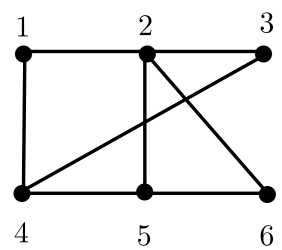

Figure 1. Graph G

G, the fractional domination values are as follows: $\mathrm{dom}_f(1, 2) = \frac{1}{4} + \frac{1}{2}, \mathrm{dom}_f(1, 3) = \frac{1}{4} + \frac{1}{3}, \mathrm{dom}_f(1, 4) =$

$\frac{1}{2}+\frac{1}{3}, \text{dom}_f(1,5)=\frac{1}{4}+\frac{1}{3}, \text{dom}_f(1,6)=\frac{1}{4}, \text{dom}_f(2,3)=\frac{1}{4}+\frac{1}{2}, \text{dom}_f(2,4)=\frac{1}{2}+\frac{1}{2}+\frac{1}{3}, \text{dom}_f(2,5)=\frac{1}{4}+\frac{1}{3}+\frac{1}{2}, \text{dom}_f(2,6)=\frac{1}{4}+\frac{1}{3}+\frac{1}{2}, \text{dom}_f(3,4)=\frac{1}{2}+\frac{1}{3}, \text{dom}_f(3,5)=\frac{1}{4}+\frac{1}{3}, \text{dom}_f(3,6)=\frac{1}{4}, \text{dom}_f(4,5)=\frac{1}{3}+\frac{1}{3}, \text{dom}_f(4,6)=\frac{1}{3}, \text{dom}_f(5,6)=\frac{1}{4}+\frac{1}{3}+\frac{1}{2}$. Thus, $T_{DV_f}(G)=11, \quad MD_f(G)=\frac{T_{DV_f}(G)}{\binom{6}{2}}=\frac{11}{15}$.

2. Bounds of fractional medium domination number of graphs

In this section, we explore the bounds of the fractional medium domination number, $MD_f(G)$, for certain graph classes. This parameter reflects the average fractional domination value across all vertex pairs in a graph. By analyzing different graph families, we derive bounds that highlight the structural influence of each class on $MD_f(G)$.

Result 2.1
$0 \leq \text{dom}_f(u',v') \leq n$.

Theorem 2.2
For a path graph P_n with $n \geq 2$ vertices, $MD_f(P_n)=\frac{3n-2}{n(n-1)}$.

Proof
Let the vertices of P_n be labeled as $v_1, v_2, \ldots, v_n$, where v_1 and v_n are the endpoints. Therefore, the fractional domination value depends on the number of common neighbors and their degrees. For $n=2$, the only pair is (v_1,v_2). Since both vertices have degree 1, $\text{dom}_f(1,2)=1+1=2$. Thus, $T_{DV_f}(P_2)=2$ and $MD_f(P_2)=\frac{2}{\binom{2}{2}}=2$. For $n=3$, the unordered pairs are $\{(1,2),\ (1,3),\ (2,3)\}$ and $\text{dom}_f(1,2)=1+\frac{1}{2}=\frac{3}{2}$, $\text{dom}_f(1,3)=\frac{1}{2}$, $\text{dom}_f(2,3)=1+\frac{1}{2}=\frac{3}{2}$. Hence, $T_{DV_f}(P_3)=\frac{3}{2}+\frac{1}{2}+\frac{3}{2}=\frac{7}{2}$, and $MD_f(P_3)=\frac{\frac{7}{2}}{\binom{3}{2}}=\frac{7}{6}$. For $n=4$, the unordered pairs are $\{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$ and $\text{dom}_f(1,2)=1+\frac{1}{2}=\frac{3}{2}$, $\text{dom}_f(1,3)=\frac{1}{2}$, $\text{dom}_f(1,4)=0$, $\text{dom}_f(2,3)=\frac{1}{2}+\frac{1}{2}=1$, $\text{dom}_f(2,4)=\frac{1}{2}$, $\text{dom}_f(3,4)=\frac{1}{2}+1=\frac{3}{2}$. Thus $T_{DV_f}(P_4)=\frac{3}{2}+\frac{1}{2}+0+1+\frac{1}{2}+\frac{3}{2}=5$, and $MD_f(P_4)=\frac{5}{\binom{4}{2}}=\frac{5}{6}$. In general for $n \geq 3$, nonzero contributions come only from pairs at distance 1 or 2. For adjacent pairs, there are $n-1$ edges and each endpoint edge contributes $\frac{3}{2}$ also each of the $n-3$ interior edges contributes 1. Hence, total from adjacent pairs $=2\cdot\frac{3}{2}+(n-3)\cdot 1=n$. For distance-2 pairs, there are $n-2$ such pairs. Each is dominated only by the middle vertex of degree 2, contributing $\frac{1}{2}$. Hence, total from distance-2 pairs $=(n-2)\cdot\frac{1}{2}=\frac{n-2}{2}$. Adding, $T_{DV_f}(P_n)=n+\frac{n-2}{2}=\frac{3n-2}{2}$. The number of unordered pairs is $\binom{n}{2}=\frac{n(n-1)}{2}$. Therefore, $MD_f(P_n)=\frac{T_{DV_f}(P_n)}{\binom{n}{2}}=\frac{\frac{3n-2}{2}}{\frac{n(n-1)}{2}}=\frac{3n-2}{n(n-1)}$. □

Theorem 2.3
For a cycle graph C_n with $n \geq 3$ vertices, $MD_f(C_n)=\dfrac{3}{n-1}$.

Proof
The $\text{dom}_f(u',v')$ depends on the number of common neighbors and degrees it share. Since each vertex has exactly two neighbors, the intersection $N[u]\cap N[v]$ contains at most two vertex when u and v are adjacent, and is empty otherwise. For $n=3$, the cycle C_3 is a triangle. The unordered pairs are $\{(1,2),(1,3),(2,3)\}$. Each pair is adjacent and dominated by all three vertices, each of degree 2. Thus, each pair contributes $\frac{3}{2}$. So, $T_{DV_f}(C_3)=3\cdot\frac{3}{2}=\frac{9}{2}$. The number of pairs is $\binom{3}{2}=3$. Hence, $MD_f(C_3)=\dfrac{9/2}{3}=\dfrac{3}{2}$, which agrees with the formula $3/(3-1)=3/2$. For $n=4$, The cycle C_4 has unordered pairs $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. The adjacent pairs are $(1,2),(2,3),(3,4),(4,1)$, each dominated by two vertices, contributing 1 each. and non adjacent pairs are $(1,3),(2,4)$, each dominated by two vertices of degree 2, contributing 1 each. Thus, $T_{DV_f}(C_4)=6$. The number of pairs is $\binom{4}{2}=6$. So, $MD_f(C_4)=\frac{6}{6}=1$, which matches the formula $3/(4-1)=1$. For $n\geq 5$, every vertex has degree 2. Nonzero contributions come only from pairs at distance 1 or 2 and for adjacent pairs, there are n such

pairs, each dominated by its two endpoints, contributing 1 each and for distance-2 pairs, There are n such pairs, each dominated by exactly one middle vertex, contributing $\frac{1}{2}$ each. So the total is $T_{DV_f}(C_n) = n + n/2 = \frac{3n}{2}$. The number of pairs is $\binom{n}{2} = \frac{n(n-1)}{2}$. Therefore, $MD_f(C_n) = \frac{\frac{3n}{2}}{\frac{n(n-1)}{2}} = \frac{3}{n-1}$.. □

Theorem 2.4
For a complete graph K_n with $n \geq 2$ vertices, $MD_f(K_n) = \frac{n}{n-1}$.

Proof
In K_n, every vertex has degree $n-1$. For any two distinct vertices u, v, every vertex of K_n dominates both u and v. Each vertex contributes $\frac{1}{n-1}$, and there are n such vertices. Thus $\text{dom}_f(u, v) = n \cdot \frac{1}{n-1} = \frac{n}{n-1}$. For $n = 2$, the only pair (v_1, v_2) has $T_{DV_f}(K_2) = 2$, so $MD_f(K_2) = 2$. For $n = 3$, each pair contributes $\frac{3}{2}$, has $T_{DV_f}(K_2) = \frac{9}{2}$ so $MD_f(K_3) = \frac{3}{2}$. For $n = 4$, each pair contributes $\frac{4}{3}$, has $T_{DV_f}(K_2) = \frac{24}{3} = 8$ so $MD_f(K_4) = \frac{4}{3}$. Similarly, since this value is the same for every unordered pair, the total fractional domination value for any K_n is $T_{DV_f}(K_n) = \binom{n}{2} \cdot \frac{n}{n-1}$. Dividing by the number of pairs gives $MD_f(K_n) = \frac{T_{DV_f}(K_n)}{\binom{n}{2}} = \frac{n}{n-1}$. Hence, the proof. □

Theorem 2.5
For a star graph $K_{1,n}$ with $n \geq 1$ leaves, $MD_f(K_{1,n}) = \frac{3n+1}{n(n+1)}$.

Proof
Let $K_{1,n}$ consist of a central vertex v_0 and n leaf vertices $v_1, v_2, \ldots, v_n$. It consists of $n+1$ vertices. In a star graph, the central vertex v_0 is adjacent to all leaf vertices and leaf vertices are only adjacent to v_0 and not to each other. Consider, for example $K_{1,3}$, the unordered pairs are $\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$ and $\text{dom}_f(1,2) = \frac{1}{3} + 1 = \frac{4}{3}$, $\text{dom}_f(1,3) = \frac{1}{3} + 1 = \frac{4}{3}$, $\text{dom}_f(1,4) = \frac{1}{3} + 1 = \frac{4}{3}$, $\text{dom}_f(2,3) = \frac{1}{3}$, $\text{dom}_f(2,4) = \frac{1}{3}$, $\text{dom}_f(3,4) = \frac{1}{3}$. Thus $T_{DV_f}(K_{1,3}) = 3 \times \frac{4}{3} + 3 \times \frac{1}{3} = \frac{15}{3} = 5$, and $MD_f(K_{1,3}) = \frac{\frac{15}{3}}{\binom{4}{2}} = \frac{5}{6}$. Similarly, the total fractional domination value $T_{DV_f}(K_{1,n})$ is the sum of the fractional domination contributions over all unordered vertex pairs. By calculation, this total is $T_{DV_f}(K_{1,n}) = \frac{3n+1}{2}$. The total number of vertex pairs in $K_{1,n}$ is $\binom{n+1}{2} = \frac{n(n+1)}{2}$. Thus, the fractional medium domination number is $MD_f(K_{1,n}) = \frac{T_{DV_f}(K_{1,n})}{\binom{n+1}{2}} = \frac{\frac{3n+1}{2}}{\frac{n(n+1)}{2}} = \frac{3n+1}{n(n+1)}$. □

The results on the relations between $MD_f(G)$ and the degrees of vertices are as follows.

Result 2.6
$MD_f(G) \geq \frac{\sum_{j=1}^{n} \deg(v_j)}{n(n-1)}$, referring to $\deg(v_j)$ is the degree of v_j.

Proof
Assign to each vertex v_j a fractional weight $f(v_j) = 1/\deg(v_j)$ where $1 \leq j \leq n$. Consider selecting an unordered pair of vertices uniformly at random. The expected contribution of vertex v_j to $\text{dom}_f(u, v)$ is at least $\deg(v'_j)/\binom{n}{2}$, since it participates in all pairs involving its neighbors. Summing over all vertices, $MD_f(G) = \frac{T_{DV_f}(G)}{\binom{n}{2}} \geq \frac{\sum_{j=1}^{n} \deg(v_j)}{n(n-1)}$. □

Result 2.7
$MD_f(G) \geq \frac{d(G)}{n-1}$, referring to $d(G)$ indicates the mean degree of the vertices in the graph.

Proof
In Result 2.6, apply the definition of the mean degree as $d(G) = \frac{1}{n} \sum_{j=1}^{n} \deg(v_j)$. Hence, the result follows. □

Result 2.8
$MD_f(G) \geq \frac{\delta(G)}{n-1}$, referring to $\delta(G)$ signifies the smallest degree among all vertices in the graph.

Proof
Let $\delta(G)$ be the minimum degree in G. For any pair (u, v), the $\mathrm{dom}_f(u, v)$ is the sum of contributions from all vertices in $N[u] \cap N[v]$, where each vertex w contributes at least $f(w) = 1/\deg(w) \geq 1/\delta(G)$. Since each vertex has at least $\delta'(G)$ neighbors, its total contribution over all pairs is at least $\delta(G) \cdot \frac{1}{\deg(w)} \geq 1$. Summing over all n vertices gives $T_{DV_f}(G) \geq \frac{n\delta(G)}{2}$. Dividing by $\binom{n}{2} = \frac{n(n-1)}{2}$, we have $MD_f(G) = \frac{T_{DV_f}(G)}{\binom{n}{2}} \geq \frac{n\delta(G)/2}{n(n-1)/2} = \frac{\delta(G)}{n-1}$. □

We propose the following bounds of $MD_f(G)$, derived from the addition or deletion of vertices and edges in the graph based on case analysis of standard graph families (paths, cycles, and complete graphs). A general result is left open for future work.

Observation 2.9
For a graph G, where G is a path, cycle, or complete graph with n vertices, the following bounds hold:
(i)$MD_f(G - v) < MD_f(G), \quad \forall v \in V(G)$, where $MD_f(G - v)$ denotes the fractional medium domination number of the graph obtained by deleting a vertex v.
(ii)$MD_f(G + v) < MD_f(G), \quad \forall v \in V(G)$, where $MD_f(G + v)$ denotes the fractional medium domination number of the graph obtained by adding a new vertex adjacent to some vertex in G.
(iii) $MD_f(G - e) < MD_f(G), \quad \forall e \in E(G)$, where $MD_f(G - e)$ denotes the fractional medium domination number of the graph obtained by deleting an edge e.

We utilized Python to implement our theoretical findings on the bounds of the fractional medium domination number $MD_f(G)$ for specific graph classes, including path graphs, cycle graphs, complete graphs, and star graphs. This focused approach offers a clearer understanding of the parameter'n behavior across structurally diverse graphs. The implementation is based on analytically derived expressions and is designed for efficiency. Notably, the program operates with a time complexity of $O(1)$, performing calculations in constant time regardless of the input size. This ensures the tool is effective for both validating theoretical results and supporting practical applications.

Python Program to Calculate $MD_f(G)$ of special classes of graphs

```
import math

def MD_f():
    print("Choose the graph type to calculate MD_f(G):")
    print("1. Path graph (P_n)")
    print("2. Cycle graph (C_n)")
    print("3. Complete graph (K_n)")
    print("4. Star graph (K_{1,n})")
    print("0. Exit")

    while True:
        try:
    choice = int(input("Enter your choice (1, 2, 3, 4, or 0 to exit): "))

    if choice in [1, 2, 3, 4]:
    while True:
    try:
    n = int(input("Enter the number of vertices (n): "))
```

```
# Validate input based on graph type
if choice == 1 and n < 2:
print("Invalid input: n must be at least 2 for P_n.")
continue
elif choice == 2 and n < 3:
print("Invalid input: n must be at least 3 for C_n.")
continue
elif choice == 3 and n < 2:
print("Invalid input: n must be at least 2 for K_n.")
continue
elif choice == 4 and n < 1:
print("Invalid input: n must be at least 1 for K_{1,n}.")
continue

# Calculate MD_f()
if choice == 1:
# Path graph P_n
md_f_ps = (3 * n - 2) / (n * (n - 1))
print(f"MD_f(P_{n}): {md_f_ps:.6f}")

elif choice == 2:
# Cycle graph C_n
md_f_cs = 3 / (n - 1)
print(f"MD_f(C_{n}): {md_f_cs:.6f}")

elif choice == 3:
# Complete graph K_n
md_f_ks = n / (n - 1)
print(f"MD_f(K_{n}): {md_f_ks:.6f}")

elif choice == 4:
# Star graph K_{1,n}
md_f_k1n = (3 * n + 1) / (n * (n + 1))
print(f"MD_f(K_{{1,{n}}}): {md_f_k1n:.6f}")

break  # exit inner loop once valid input is processed

except ValueError:
print("Invalid input: Please enter an integer value for n.")

elif choice == 0:
print("Exiting program.")
break
else:

print("Invalid choice. Please enter 1, 2, 3, 4, or 0 to exit.")

except ValueError:
print("Invalid input. Please enter a valid choice.")
```

```
if __name__ == "__main__":
    MD_f()
```

3. Conclusion

In this work, we defined the $MD_f(G)$ for connected, undirected graphs that are free from loops. This new parameter extends traditional domination theory by incorporating fractional values, providing a more detailed measure of domination in graphs. By defining the fractional domination value between vertex pairs and calculating it based on their common neighbors, we have developed a method that offers greater flexibility compared to classical approaches. We also derived bounds for the fractional medium domination number in different graph families, further expanding the understanding of its properties. Additionally, we demonstrated the practical computation of $MD_f(G)$ using Python programming, which makes this approach useful for real-world applications such as network optimization, resource allocation, and disaster relief operations. Future work will focus on determining $MD_f(G)$ for special graph classes such as grid graphs, trees, and regular graphs. A central open problem is to establish sharp upper and lower bounds for $MD_f(G)$ in terms of classical invariants like average degree, connectivity, or girth. Another promising direction is the design of efficient algorithms for computing $MD_f(G)$ in large-scale networks, which would broaden its potential applications in optimization and network analysis.

Acknowledgement

This article has been written with the joint financial support of the RUSA Phase $2.0(F24-51/2014-U)$, DST-FIST(SR/FIST/MS-I/2018/17), DST-PURSE 2nd Phase programme (SR/PURSE Phase 2/38), Govt. of India. Alagappa University Research Fund (AURF) Seed Money 2024 [grant sanctioned vide Letter No. AU/SO(P&D)/AURF Seed Money/2024 Alagappa University, Karaikudi, Tamil Nadu, India, Date 11th December 2024]

REFERENCES

1. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, “Domination in Graphs Advanced Topics”, Marcel Dekker Inc., 1998. https://doi.org/10.1201/9781315141428
2. N. Sridharan, S. Amutha and Ramesh Babu, “Bounds for λ-domination number $\gamma_\lambda(G)$ of a graph”, Journal of Computational Mathematica, vol. 1, no. 2, pp. 79-90, 2017. https://doi.org/10.26524/cm18
3. T. W. Haynes, S. T. Hedetniemi and P. J. Slater,”Fundamentals of domination in Graphs”, Marcel Dekker Inc., 1998. https://doi.org/10.1201/9781482246582
4. P. Shanthi, S. Amutha, N. Anbazhagan, G. Uma, Gyanendra Prasad Joshi and Woong Cho, “Characterization of Fractional Mixed Domination Number of Paths and Cycles”, Journal of Mathematics, Hindawi, vol. 2024, pp. 1-11, 2024. https://doi.org/10.1155/2024/6619654.
5. D. K. Thakkar, S. N. Badiyani, “Fractional Domination in graphs”, International Journal of Scientific and Innovative Mathematical Research, vol. 4, no. 4, 2016. https://www.arcjournals.org/pdfs/ijsimr/v4-i4/6.pdf
6. F. Devvrit, A. Krim-Yee, N. Kumar, G. MacGillivray, B. Seamone, V. Virgile, A. Xu, ”Fractional eternal domination: securely distributing resources across a network”, Discussiones Mathematicae Graph Theory, vol. 44, no. 4, 2024. https://doi.org/10.7151/dmgt.2503
7. P. Vijayalakshmi, K. Karuppasamy, and Tiji Thomas, “Fractional Restrained Domination”, Discrete Mathematics, Algorithms and Applications, vol. 17, no. 1, pp. 1–11, 2025. https://doi.org/10.1142/S1793830924500046
8. N. Meenakshi, V. Pankajam,“Fractal Graph of Total Regular Fractional Domination Number”, Advances in Nonlinear Variational Inequalities, vol. 28, no. 3n, 2025. https://doi.org/10.52783/anvi.v28.3125
9. E. Ebin Raja Merly, J. Saranya, “Medium Domination Decomposition of Graphs”, Ratio Mathematica, vol. 44, pp. 59–64, 2022. 10.23755/rm.v44i0.890
10. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, “Fundamentals of domination in Graphs”, Marcel Dekker Inc., 1998. https://doi.org/10.1201/9781482246582

11. G. Uma, S. Amutha, N. Anbazhagan, B. Koushick, “Exploring Prism Graphs with Fractional Domination Parameters”, Mathematics and Statistics, vol. 12, no. 5, pp. 448-454, 2024. 10.13189/ms.2024.120506.
12. G. Uma, S. Amutha, N. Anbazhagan, P. Shanthi, “Fractional Domination number of the fractal graphs”, Indian Journal of science and technology, vol. 17, no. 1, pp. 34-39, 2024. https://doi.org/10.17485/IJST/v17sp1.110
13. Vargor, D., & Dundar, P., ”The medium domination number of a graph”, International Journal of pure and applied mathematics, vol. 70, no. 3, pp. 297-306, 2011.