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Restrained Domination Coalition Number of Paths and Cycles

A. H. Shola Nesam [1], S. Amutha [2,*], N. Anbazhagan [1]

[1] *Department of Mathematics, Alagappa University, India*
[2] *Ramanujan Centre for Higher Mathematics, Alagappa University, India*

Abstract A restrained domination coalition (or simply rd_c) consists of two disjoint subsets of vertices R_1 and R_2 of a graph G_h. Neither R_1 nor R_2, on its own, is a restrained dominating set (RD-set). However, when combined, they together form an RD-set for the graph. A restrained domination coalition partition (rd_{cp}) is a vertex partition $\pi_r = \{R_1, R_2, .., R_l\}$ where each element of $R_i \in \pi_r$ is either an RD-set consisting of a single vertex, or a non-RD-set that forms an rd_c with a set R_j in π_r. In this work, we initiated the concept of rd_c and rd_c-graph. We further proved the existence of rd_c for any simple graph. Moreover, we determine the exact value of this parameter in special graph families such as complete multipartite graphs, paths and cycles, while establishing the relation between rd_c-number and graph invariants like vertex degree. We further characterized the rd_c-graphs of paths. This study applies rd_c-partitioning to cybersecurity, structuring networks into collaborative security clusters that detect, contain, and neutralize threats in real time.

Keywords coalition, restrained domination coalition, dominating set, restrained dominating set

AMS 2010 subject classifications 05C69

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1. Introduction

Let $G_h = (V_h, E_h)$ be a finite, simple, and undirected graph, where V_h represents the set of vertices with n elements (order) and E_h represents the set of edges with m elements (size). For an element in the vertex set $r_h \in V_h$, the *open neighborhood* of r_h is the set $N(r_h) = \{s_h : r_h s_h \in E_h\}$ and the *closed neighborhood* is the set that contains both the set $N(r_h)$ and the vertex r_h. A vertex that is adjacent to all other vertices in the graph is referred to as a *full vertex*, while a vertex that has no connections to any other vertex is known as an *isolated vertex*. The smallest and largest degrees of a graph G_h are indicated by $\delta(G_h)$ and $\Delta(G_h)$, respectively. $G[M]$ represents the subgraph induced by a vertex subset $M \subseteq V_h$. A set $V_x \subset V_h$ is termed a *singleton vertex set* if V_x contains exactly one element. A subset V_d of the vertex set is called a *dominating set* when each vertex in $V_h \setminus V_d$ is adjacent to at least one vertex in V_d. Various new types of dominating sets and their properties have been explored in recent research [1, 2]. A subset $V_r \subseteq V_h$ is called a *restrained dominating set (RD-set)* if it satisfies two conditions. First, every vertex in the complement set $V_h \setminus V_r$ must be adjacent to at least one vertex within V_r, ensuring that the dominating property is maintained. Second, each vertex in $V_h \setminus V_r$ must also be adjacent to at least one other vertex within the same set $V_h \setminus V_r$, (i.e.) the graph $G[V_h \setminus V_r]$ contains no isolated vertices. This condition ensures that the remaining vertices are not completely disconnected but remain integrated within the graph's structure. The *restrained domination number*, denoted by $\gamma_r(G_h)$, is the minimum size of a restrained dominating set in the graph G_h. Restrained domination was first introduced in 1999 by Gayla S. Domke and others [3].

*Correspondence to: S. Amutha (Email: amuthas@alagappauniversity.ac.in). Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi-630 003, Tamilnadu, India.

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Hayes and others introduced the study of *coalition* and *coalition partition* in [4]. In a graph G_h, a coalition is formed by two distinct vertex subsets C_x and C_y. Individually, neither C_x nor C_y is a dominating set of G_h. However, when these two sets are combined, their union, $C_x \cup C_y$, satisfies the condition of a dominating set, ensuring that every vertex in the graph is either included in the union or adjacent to at least one of its vertices. A *coalition partition* in G_h refers to a partition of the vertex set into distinct subsets, denoted as $\psi = \{C_1, C_2, \ldots, C_k\}$, where each subset C_i satisfies one of two conditions. Either it is a dominating set on its own, which consists of only one vertex, ensuring that all other vertices in the graph are adjacent to it, or it is not a dominating set on its own but collaborates with another subset C_j within ψ to form a coalition that collectively dominates the graph. The *singleton vertex partition* of a graph G_h with vertex set $V_h = \{s_1, s_2, ..., s_n\}$, is the partition $\psi = \{C_1, C_2, ...C_k\}$ such that for $j \in \{1, 2, ..., n\}$, $C_j = \{s_j\}$.

A series of interrelated studies have significantly advanced the understanding of specialized coalition structures in graphs. The self-coalition graphs, introduced by Haynes [5], explored graph structures that are isomorphic to their own coalition graph under a singleton partition. Building on this, the same authors formalized the concept of coalition graphs $CG(G_h, \pi)$, where a coalition partition π of a graph G_h generates a new graph, and proved that every graph can be modeled as a coalition graph using constructive methods involving auxiliary graphs [6]. Extending the theory, Alikhani introduced total coalitions, a variant where all dominating sets must be total and applied the concept to trees and connected graphs to establish foundational properties [7]. Samadzadeh studied a variant of coalition called paired coalitions, focusing on the role of perfect matchings and providing structural characterizations of triangle-free and unicyclic graphs along with results for trees [8]. Meanwhile, Bakhshesh investigated singleton coalition graph chains, addressing open questions from previous work and establishing links between coalition number, graph order, and degree for graphs with low minimum degree [10]. Moreover, Bakhshesh and Henning introduced the minmin coalition number, defined as the minimum order among all minimal coalition partitions, and proposed polynomial-time algorithms and necessary conditions for its existence [11]. Several other variants of coalitions such as independent coalition [12], double coalition [13], perfect coalition [14], k-coalitions [15] and so on have also been defined.

To advance future study, a novel perspective on coalitions that deals with restrained dominating sets and their corresponding graphs has been defined and studied in this work. Section 2 introduces and studies the properties of restrained domination coalitions and restrained domination coalition graph. We focus on establishing the restrained coalition number of various graph classes like complete multipartite graphs, paths and cycles. Section 3 presents an application of restrained coalition partitioning (rd_c-partitioning) in cybersecurity, demonstrating how this approach enhances network resilience against cyber threats. Finally, Section 4 concludes the paper with some problems for further research.

2. Restrained coalition

In this section we focuses on the foundational aspects of restrained domination coalitions (rd_c), beginning with their existence and initial findings. We further define rd_c-graphs and determine the exact value of rd_c-numbers for various special graph classes.

2.1. Existence and preliminary results

We begin this section by defining restrained domination coalition and restrained domination coalition partition and assert the existence and initial outcomes of restrained domination coalitions in a graph.

Definition 2.1
(Restrained domination coalition) For a graph G_h, two distinct vertex subsets R_x and R_y are said to form a restrained domination coalition (or simply rd_c) if neither R_x nor R_y is an RD-set of G_h. However, $R_x \cup R_y$ is an RD-set of G_h.

Definition 2.2
(Restrained domination coalition partition) A restrained domination coalition partition (abbreviated as rd_{cp}), in

a given graph G_h is a specific type of vertex partition denoted as $\pi_r = \{R_1, R_2, \ldots, R_l\}$. Within this partition, each subset R_i falls into one of two categories: it is either an RD-set containing a single vertex, or a non-RD-set that forms a rd_c with another subset R_j from the same partition π_r. Consequently, R_i and R_j are referred to as rd_c-partners. The largest possible order of an rd_{cp} in a graph G_h is defined as the rd_c-number, denoted by $RD_c(G_h)$. An rd_{cp} that attains this maximum order is specifically referred to as an $RD_c(G_h)$-partition, which is represented by Π_r.

For example, consider the graph G_h depicted in fig. 1 and examine $\pi_r(G_h) = \{\{u_1, u_5\}, \{u_2\}, \{u_3\}, \{u_4\}\}$, where none of these sets are RD-sets of G_h. Note that $\{u_1, u_5\}$ is the rd_c-partner of $\{u_2\}, \{u_3\}$ and $\{u_4\}$. The partition $\pi_r(G_h)$ is the one with maximum cardinality and thus $RD_c(G_h) = 4$.

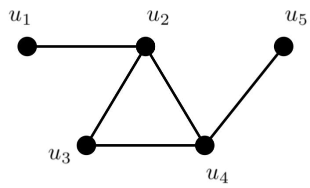

Figure 1. The graph G_h

Observation 2.3
Let $\pi_r(G_h) = \{R_1, R_2, ..., R_l\}$ be an rd_{cp} of a graph G_h. Then $|R_i \cup R_j| \geq \gamma_r(G)$ for any pair of rd_c partners in $\pi_r(G_h)$.

Proposition 2.4
For a complete graph G_k of k elements, $RD_c(G_k) = k$.

Proof
It is clear that the singleton vertex partition $\pi_r = \{\{u_1\}, \{u_2\}, ..., \{u_k\}\}$ is an rd_{cp} of G_k having the vertex set $V_h = \{u_1, u_2, ..., u_k\}$ where each $\{u_i\}$ for $i = \{1, 2, ..., k\}$ is a singleton RD-set. □

Note that from proposition 2.4 , for every k, there is a graph G_h of order k whose rd_c number is $RD_c(G_h) = k$. We now prove the existence of rd_{cp} for every simple graph.

Theorem 2.5
If G_h is a simple graph then G_h admits an rd_{cp}.

Proof
Consider any $x_h \in V_h$ that denotes a vertex in a graph G_h such that $\delta(G_h) = deg(x_h)$. Considering the following two cases,
Case 1: Let x_h be a full vertex of G_h, then G_h is a complete graph (since x_h is a full vertex having minimum degree). Then by proposition 2.4, G_h admits an rd_{cp}.
Case 2: Define the vertex partition $\pi_r = \{R_1, R_2\}$ where $R_1 = V_h - \{x_h\}$ and $R_2 = \{x_h\}$. As $G[V_h - R_1]$ has an isolated vertex (i.e. x_h) and since x_h is not a full vertex, neither R_1 nor R_2 is an RD-set. However, $R_1 \cup R_2 = V_h$ is an RD-set of G_h. Thus R_1 and R_2 are rd_c-partners, forming an rd_{cp} of G_h of order at least two. □

Corollary 2.6
$2 \leq RD_c(G_h) \leq r$ for any graph G_h with r vertices.

The bounds in the corollary 2.6 are sharp where the lower bound is attained by the path with two vertices and the upper bound is attained by the complete graph on r vertices.

Corollary 2.7
$RD_c(G_h) = 1$ if and only if $G_h = K_1$.

Since every RD-set in G_h is also a dominating set, an obvious question arises: are domination coalitions and rd_c structures equivalent. Consider the graph $G_h = P_2 \cup P_1$ as illustrated in the figure 2, where $V(G_h) = \{p_{1,1}, p_{2,1}, p_{2,2}\}$ and $\{p_{2,1}, p_{2,2}\} \in E(G_h)$.

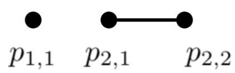

Figure 2. $P_1 \cup P_2$

Consider the vertex partitions of G_h, $\pi_1 = \{\{p_{1,1}\}, \{p_{2,1}\}, \{p_{2,2}\}\}, \pi_2 = \{\{p_{1,1}\}, \{p_{2,1}, p_{2,2}\}\}, \pi_3 = \{\{p_{1,1}, p_{2,2}\}, \{p_{2,1}\}\}$ and $\pi_3 = \{\{p_{1,1}, p_{2,1}\}, \{p_{2,2}\}\}$. Observe that π_1 is not an rd_{cp} of G_h but satisfies the condition of a coalition partition of G_h. Conversely, neither π_3 nor π_4 qualify as a coalition partition, yet both of them is an rd_{cp} of G_h. Also, π_2 leads to an intriguing observation that it satisfies both coalition and rd_c conditions. Since π_1 is a coalition partition but not an rd_{cp} of G_h, the two problems are not equivalent, highlighting the importance of studying the rd_{cp} problem separately. Moreover, these partitions also implies that for $G_h = P_2 \cup P_1$, we have $2 = RD_c(G_h) \leq C(G_h) = 3$ where $C(G_h)$ is a coalition number of G_h.

Additionally, theorem 2.5 is applied in constructing π_2, where the vertex of minimum degree is placed in its own distinct element.

The following lemma establishes that the set consists of the end vertices must be contained within the union of every pair of rd_c-partners (A, B) in any rd_{cp} of a graph G_h with $\delta(G_h) = 1$.

Lemma 2.8
If π_r is an rd_{cp} of a graph G_h with $\delta(G_h) = 1$ and $A, B \in \pi_r$, then $X_e \subseteq A \cup B$ for each pair of rd_c-partners (A, B), where X_e is the set of end vertices.

Proof
For the purpose of contradiction, let $X_e \nsubseteq A \cup B$, then there exists a vertex say x_h in X_e such that x_h is not in the union of A and B. Let y_h be the support vertex to x_h. If y_h is not in the union of A and B then x_h is not dominated by $A \cup B$. Thus $y_h \in A \cup B$. Now x_h has no adjacent vertex in $G[V_h - (A \cup B)]$, contradicting the fact that (A, B) are rd_c-partners. □

Lemma 2.9
If Π_r is an $RD_c(G_h)$-partition of order $RD_c(G_h) \geq 3$ of a graph G_h with $\delta(G_h) = 1$, then there exists an element in Π_r that includes every end vertex of G_h.

Proof
Let X_e contain all the end vertices of G_h. If X_e consists of a single element, then the result is trivial. Let $|X_e| \geq 2$. For the purpose of contradiction, let x_h and y_h such that $x_h, y_h \in X_e$ that are contained in different members of Π_r. Let $x_h \in V_{x_h}$ and $y_h \in V_{y_h}$, where $\{V_{x_h}, V_{y_h}\} \in \Pi_r$. Further, let V_{z_h} be a set in Π_r, such that $V_{z_h} \notin \{V_{x_h}, V_{y_h}\}$. By lemma 2.8, V_{z_h} has no rd_c-partner, a contradiction. □

Combining lemma 2.8 and 2.9, we have the next result.

Theorem 2.10
If Π_r is an $RD_c(G_h)$-partition of order $RD_c(G_h) \geq 3$ of a graph G_h with $\delta(G_h) = 1$ and let V_e be the member of Π_r containing the end vertices of G. Then for any paired (X, Y) of RD_c-partners in G_h, $V_e \in \{X, Y\}$.

The following theorem establishes a bound on the quantity of rd_c-partners of an element in an rd_{cp} of the graph G_h.

Theorem 2.11
If π_r is a rd_{cp} and A_h be an element of π_r, then A_h has at most $\Delta(G_h) + 1$ rd_c-partners in π_r and this bound is exact.

Proof
Assume that π_r is any rd_{cp} of G_h and $A_h \in \pi_r$. If A_h is an RD-set then A_h is a singleton set and A_h cannot be an rd_c-partner of any other sets. Therefore, assume that A_h is not an RD-set of G_h. If A_h is a non-RD-set then by the definition of an RD-set, for a vertex $w \notin A_h$, we have :
Case 1: When $N(w) \notin A_h$. Then any set in π_r that forms an rd_c with A_h must dominate w by including at least a single vertex from $N[w]$, such sets has a upper bound of $|N[w]| = deg(w) + 1$ which is at most $\Delta(G) + 1$.
Case 2: When $N(w) \in A_h$ and $N(w) \notin V_h - A_h$. That is, w is dominated by A_h and no edges exist between w and any other vertices in $G[V_h - A_h]$, as $N(w) \subseteq A_h$. Then A_h has exactly one rd_c-partner, say B_h and B_h should contain w.
Combining both the cases, we get that if $A_h \in \pi_r$, then A_h has at most $\Delta(G_h) + 1$, rd_c-partners. To prove the sharpness, consider the graph $H = K_p \cup K_1$, $p \geq 3$ with the vertex set $V(H) = \{w_1, w_2, ..., w_p\} \cup \{x\}$. The singleton vertex partition is an $RD_c(G)-$partiton of H, where the set $\{x\}$ has $\Delta + 1 = p$, rd_c-partners. That is for $1 \leq j \leq p$, the sets $\{w_j\}$ is an rd_c-partner of $\{x\}$. □

Using the above theorem, we prove the following.

Theorem 2.12
If $RD_c(G_h)$ is an rd_c-number of a graph G_h with $\delta(G_h) = 1$, then $RD_c(G_h) \leq \Delta(G_h) + 2$.

Proof
Let $w \in V_h$ and $deg(w) = 1$. Let π_r be an rd_{cp} of G_h and let W be an element of π_r such that $w \in W$. By lemma 2.8 for any two sets $U, X \in \pi_r$ that form an rd_c, $U = W$ or $X = W$. Therefore, the sets of π_r other than W form an rd_c with W. By theorem 2.11, W has at most $\Delta(G_h) + 1$, rd_c-partners. Hence, $RD_c(G_h) \leq \Delta(G_h) + 2$. □

Next we define the rd_c-graph for the corresponding rd_{cp}.

Definition 2.13
(Restrained domination coalition graph) Given an rd_{cp}, $\pi_r = \{R_1, R_2, \ldots, R_l\}$ of order l for a graph G_h, the rd_c-graph, denoted as $RD_cG(G_h, \pi_r)$, is defined as follows: The vertex set of $RD_cG(G_h, \pi_r)$ consists of l vertices, each uniquely associated with an element of π_r. An edge exists between two vertices R_i and R_j in $RD_cG(G_h, \pi_r)$ if and only if the R_i and R_j together form an rd_c in G_h.

Consider the graph G_h in figure 3 and consider the rd_{cp} of G_h, $\pi_r(G_h) = \{\{u_1, u_5\}, \{u_2\}, \{u_3\}, \{u_4\}\}$. In this partition, $\{u_1, u_5\}$ serves as an rd_c-partner of the sets $\{u_2\}, \{u_3\}$ and $\{u_4\}$. Consequently, the vertex corresponding to $\{u_1, u_5\}$ is adjacent to the vertices corresponding to $\{u_2\}, \{u_3\}$ and $\{u_4\}$ in $RD_cG(G_h, \pi_r)$. However, since any union combination of the sets $\{u_2\}, \{u_3\}$ and $\{u_4\}$ does not form an RD-set, the vertices corresponding to $\{u_2\}, \{u_3\}$ and $\{u_4\}$ are not adjacent in $RD_cG(G_h, \pi_r)$.

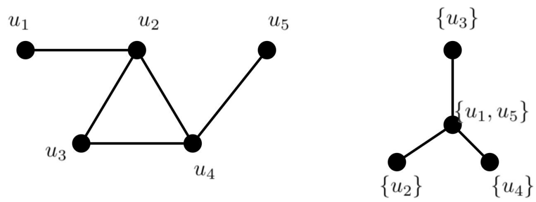

Figure 3. A $rd_c(G_h)$-partition π_r and $RD_cG(G_h, \pi_r) \approxeq K_{1,3}$

Note that the complete graph K_n has precisely one $RD_c(G)$-partition, which is its singleton partition given in proposition 2.4, and hence $RD_cG(K_n, \Pi_r) = \overline{K_n}$.

Remark 2.14
Given that every graph G_h has at least one rd_{cp}, each graph also has a minimum of one associated rd_c-graph, $H = RD_cG(G_h, \pi_r)$ and based on the number of rd_{cp} in the graph, it may have multiple associated rd_c-graphs. By using the rd_{cp} in theorem 2.5, we can say that every graph other than complete graph has at least one complete graph (for example, K_2) as their rd_c-graph.

The following theorem establishes a bound on the largest possible degree of $RD_cG(G_h, \pi_r)$.

Lemma 2.15
If Δ_r be the maximum degree of $RD_cG(G_h, \pi_r)$ then $\Delta_r(RD_cG(G_h, \pi_r)) \leq \Delta(G_h) + 1$.

Proof
Let $RD_cG(G_h, \pi_r)$ be the corresponding rd_c-graph of the rd_{cp}, π_r. Let $U_h \in \pi_r$ such that $|U_h| \geq 2$ and $deg_{RD_cG(G_h,\pi_r)}(U_h) = \Delta_r$. Since $U_h \in \pi_r$, by theorem 2.11, U_h has at most $\Delta(G_h) + 1$, rd_c-partners. Consequently, U_h can be adjacent to at most $\Delta(G_h) + 1$ vertices in $RD_cG(G_h, \pi_r)$. Then $\Delta_r(RD_cG(G_h, \pi_r)) = deg_{RD_cG(G_h,\pi_r)}(U_h) \leq \Delta(G_h) + 1$. □

Note that the smallest possible degree, $\delta_r(RD_cG(G_h, \pi_r)) = 0$ is attained whenever G_h has a full vertex v_f which is not a support vertex. Futhermore, the existence of v_f increases the number of components in the rd_c-graph of G_h.

2.2. *Restrained domination coalition number for special graphs*

We now find the exact value of $RD_c(G_h)$ for some special classes of graphs like complete multipartite, paths and cycles.

Proposition 2.16
For any star $K_{1,n}$ and $n \geq 2$, $RD_c(K_{1,n}) = 2$.

Proof
Consider a star graph $K_{1,n}$ with the vertex set $V(K_{1,n}) = \{s_1, s_2, ..., s_{n-1}, s_n\}$ where $deg(s_n) = n - 1$ and $deg(s_l) = 1$ for $l = 1, 2, ..., n$. For $K_{1,n}$, $\pi_r = \{\{s_1, s_2, ..., s_{n-1}\}, \{s_n\}\}$ is an rd_{cp}. Thus, $RD_c(K_{1,n}) \geq 2$. If $RD_c(K_{1,n}) > 2$, then by theorem 2.9, there exists a set in the rd_{cp} which contains $s_1, s_2, ..., s_{n-1}$. Hence, it follows that $RD_c(K_{1,n}) = 2$, which is a contradiction. Hence, $RD_c(K_{1,n}) = 2$. □

Proposition 2.17
Let G_h be a complete multipartite graph of order m, then $RD_c(G_h) = m$.

Proof
Consider the complete multipartite graph G_h of order m with k partite sets.The singleton vertex partition is the required rd_{cp} of maximum cardinality. This is because each vertex belonging to one partite set forms an rd_c together with all the vertices in the remaining $k - 1$ partite sets. □

We now proceed to establish the value of $RD_c(P_k)$ for the paths P_k.

Theorem 2.18
For any path P_k, $RD_c(P_k) = \begin{cases} 2 & \text{if } 2 \leq k \leq 5 \\ 3 & \text{if } k \geq 6. \end{cases}$

Proof
Consider the path P_2 with the vertex set $V = \{v_1, v_2\}$. since the singleton vertex partition is the only rd_{cp}, $RD_c(P_2) = 2$. Now, consider the path P_3 with the vertex set $V = \{v_1, v_2, v_3\}$. It is clear that $RD_c(P_3) \neq 3$, since $\gamma_r(P_3) = 3$. Thus $RD_c(P_3) \leq 2$. The rd_{cp} of P_3 is $\{\{v_1\}, \{v_2, v_3\}\}$ and hence $RD_c(P_3) = 2$. Let P_4 be a path having the vertex set $\{v_1, v_2, v_3, v_4\}$, then $RD_c(P_4) = 2$ (since $\gamma_r(P_4) = 4$). If $RD_c(P_4) \geq 3$, then by theorem 2.10, there exists an $RD_c(G)-$partiton π_r such that $V_e \in \pi_r$ where V_e contains the end vertices of P_4 (i.e. $v_1, v_4 \in V_e$). Since V_e is not an RD-set either v_2 or v_3 (not both) should be in V_e. If suppose $v_3 \in V_e$ then π_r contains two elements, which is a contradiction. Therefore, $RD_c(P_4) = 2$. Similarly, $RD_c(P_5) = 2$.
For path P_k of order $k \geq 6$ with $V(P_k) = \{v_1, v_2, ..., v_k\}$, the partition $\{\{v_1, v_6, v_7, ..., v_k\}, \{v_2, v_3\}, \{v_4, v_5\}\}$ is an rd_{cp}. Thus $RD_c(P_k) \geq 3$. To prove $RD_c(P_k) \leq 3$, let π_r be an $RD_c(G)$-partition (i.e., rd_{cp} of maximum cardinality) of P_k. By lemma 2.9, there is an element, say V_e of π_r that contains the end vertices of P_k which is $\{v_1, v_n\} \subseteq V_e$. To show that V_e forms an rd_c with a maximum of two other sets, assume the contrary, that there

are three sets in π_r, say V_1, V_2, V_3 that form an rd_c with V_e. Let V_e fail to dominate the vertex v_i. Assume that $v_{i-1} \in V_1, v_i \in V_2$, and $v_{i+1} \in V_3$ (as shown in the fig.4).

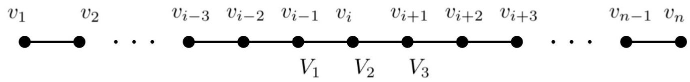

Figure 4. Path

As $V_e \cup V_1$ is an RD-set, $v_{i-2} \in V_e \cup V_1$ and similarly $V_e \cup V_3$ is an RD-set, $v_{i+2} \in V_e \cup V_3$. Suppose $v_{i-2} \in V_e$ then $G[V \setminus (V_e \cup V_2)]$ has an isolated vertex v_{i-1}. Thus $v_{i-2} \in V_1$. Suppose $v_{i+2} \in V_e$ then $G[V \setminus (V_e \cup V_2)]$ has an isolated vertex v_{i+1}. Thus $v_{i+2} \in V_3$. Now, since $v_{i-2} \in V_1$ and $v_{i+2} \in V_3$ we arrive at a contradiction that $V_1 \cup V_e$ does not dominate v_{i+1} and similarly $V_3 \cup V_e$ does not dominate v_{i-1}.

Hence, V_e forms an rd_c with a maximum of two other sets. Now theorem 2.10 implies that $RD_c(P_k) \leq 3$. □

We next characterize the rd_c-graphs of paths (i.e. RD_cP-graphs).

Theorem 2.19
A graph H is an RD_cP-graph if and only if $H \in \{P_2, P_3\}$

Proof

Consider any rd_{cp} π_r of paths P_2 through P_5, then π_r contains only two elements and the corresponding rd_c-graph is, $RD_cG(P_i, \pi_r) \cong P_2$ for $2 \leq i \leq 5$. For $k \geq 6$, consider the rd_cp $\pi_r = \{\{v_1, v_6, v_7, ..., v_k\}, \{v_2, v_3\}, \{v_4, v_5\}\}$ where $V(P_k) = \{v_1, v_2, ..., v_k\}$, then $RD_cG(P_k, \pi_r) \cong P_3$. Thus P_2 and P_3 are RD_cP-graphs.

Conversely, suppose H is an RD_cP-graph of n_h vertices. By theorem 2.18, we have $2 \leq n_h \leq 3$. For $n_h = 2$, the graphs containing two vertices are either P_2 or $\overline{P}_2$ and since n_h is the number of vertices of RD_cP-graph, $RD_cP \ncong \overline{P}_2$. For $n_h = 3$, applying theorem 2.10, we conclude that the element in π_r containing the end vertices is the only rd_c-partner for the remaining sets when $\delta(G) = 1$ (i.e.) $RD_cG(G, \pi_r) \cong K_{l-1,1}$ where $|\pi_r| = l$. Hence, we get $H \in \{P_2, P_3\}$.

□

Theorem 2.20

For any cycle C_l, $RD_c(C_l) = \begin{cases} l & \text{if } l = 3, 4 \\ 4 & \text{if } l \equiv 0 \mod 4 \\ 6 & \text{if } l \equiv 0 \mod 3 \\ 3 & \text{otherwise.} \end{cases}$

Proof
Since the singleton vertex partition gives the rd_{cp} for $l = 3$ and 4, $RD_c(C_l) = l$. Consider the cycle C_l of order $l \geq 5$. Let $V(C_l) = \{v_1, v_2, ..., v_l\}$.

Case 1: $l \equiv 0 \mod 3$. Consider the sets, $R_1^1 = \{v_1\}, R_2^1 = \{v_2\}, R_3^1 = \{v_3\}, R_4^1 = \{v_4, v_7, v_{10}..., v_{4+3(\lfloor(l-4)/3\rfloor)}\}, R_5^1 = \{v_5, v_8, v_{11}, ..., v_{5+3(\lfloor(l-5)/3\rfloor}\}, R_6^1 = \{v_6, v_9, v_{12}, ..., v_{6+3(\lfloor(l-6)/3\rfloor}\}$. Then $\pi_{r_1} = \{R_1^1, R_2^1, R_3^1, R_4^1, R_5^1, R_6^1\}$ is an rd_{cp} of C_l when $l \equiv 0 \mod 3$ such that R_1^1 and R_4^1 are rd_c-partners, R_2^1 and R_5^1 are rd_c-partners and R_3^1 and R_6^1 are rd_c-partners. Thus $RD_c(C_l) = 6$.

Case 2: $l \not\equiv 0 \mod 3$.

subcase 2.1: $l \equiv 0 \mod 4$. Consider the sets $R_1 = \{v_1, v_5, ...v_{1+4\lfloor(l-1)/4\rfloor}\}$, $R_2 = \{v_2, v_6, ...v_{2+4\lfloor(l-1)/4\rfloor}\}$, $R_3 = \{v_3, v_7, ...v_{3+4\lfloor(l-1)/4\rfloor}\}$ and $R_4 = \{v_4, v_8, ...v_{4+4\lfloor(l-1)/4\rfloor}\}$. Then $\pi_{r_2} = \{R_1, R_2, R_3, R_4\}$ is an is an rd_{cp} of C_l when $l \not\equiv 0 \mod 3$ and $l \equiv 0 \mod 4$ such that R_1 is an rd_c-partner of R_4 and R_2 is an rd_c-partner of R_3. Thus $RD_c(C_l) = 4$.

subcase 2.2: $l \not\equiv 0 \mod 4$. Consider the vertex partition $\pi_{r_3} = \{\{v_1, v_7, ..., v_l\}, \{v_2, v_3\}, \{v_4, v_5\}\}$. The partition π_{r_2} forms an rd_{cp}, since R_1 is an rd_c-partner of R_4 and R_2 is an rd_c-partner of R_3. Hence, $RD_c(C_l) = 3$ when $l \not\equiv 0 \mod 3$ and $l \not\equiv 0 \mod 4$. □

To illustrate the above theorem and the rd_c-graph of cycle, consider the cycle C_6 with vertices labelled as in the figure 5. The rd_{cp} of C_6 using the theorem 2.20 is $\pi_r = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}\}$. In this partition, $\{v_1\}$ serves as an rd_c-partner only for the set $\{v_4\}$. Consequently, the vertex corresponding to $\{v_1\}$ is adjacent to the vertex corresponding to $\{v_4\}$ in $RD_cG(C_6, \pi_r)$. Similarly, $\{v_2\}$ is adjacent to $\{v_5\}$ and $\{v_3\}$ is adjacent to $\{v_5\}$ in $RD_cG(C_6, \pi_r)$. Since no other combination forms an rd_c, there are no additional adjacency in $RD_cG(C_6, \pi_r)$.

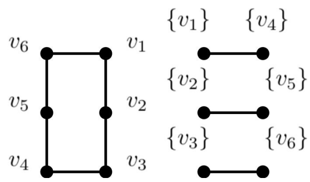

Figure 5. A $rd_c(C_6)$-partition π_r and $RD_cG(C_6, \pi_r) \approxeq 3P_2$

3. Application

Consider the network as a graph, where each node represents the devices or servers, and each edge represents the communication links. Now, consider the cybersecurity as an alliance rather than a single line of defense. In this approach, we use rd_c-partitioning to divide the network into security clusters (vertex partition). These clusters are non-restrained dominating sets, each responsible for monitoring specific threats, such as traffic anomalies, malware activity, or suspicious login attempts. But instead of working in isolation, they collaborate. If one cluster detects a potential threat, say a trojan on a user's device, its rd_c-partner (another cluster) immediately steps in, restricts the access, blocks suspicious activity, or isolates the compromised system to prevent the threat from spreading. For instance, consider the case modeled using the cycle C_6. The corresponding rd_{cp} and the associated rd_c-graph are illustrated in figure 5. In this setting, when the subset $\{v_1\}$ is exposed to a threat, its rd_c-partner $\{v_4\}$ promptly intervenes by limiting access, mitigating suspicious activity, or isolating the compromised component to prevent escalation. Similarly, for $\{v_2\}$ and $\{v_3\}$, their respective partners $\{v_5\}$ and $\{v_6\}$ undertake analogous protective actions to ensure system integrity. This teamwork stops vulnerabilities from being exploited and keeps threats from slipping through the cracks. This layered security model also ensures that every device outside the clusters is still connected to at least one of them (domination condition), and the real strength lies in the network's design that the devices not directly in these clusters are still monitored by other non-cluster devices (restrained condition). That way, even stealthy threats like stealth viruses or rootkits cannot slip through unnoticed. By maintaining this interconnected structure, we prevent any system from becoming an easy target, strengthening the network against cyberattacks while ensuring smooth, secure operations.

4. Conclusion and future works

In this study, we introduced and systematically investigated the concepts of restrained domination coalition rd_c and rd_c-graphs. We established the existence of rd_c for any simple graph and determined the exact value of the rd_c-number for several important graph families, including star graph, complete multipartite graphs, paths and cycles. Furthermore, we characterized the rd_c-graphs of paths. Additionally, we explored the relationship between the rd_c-number and fundamental graph invariants such as vertex degree. To explore new families of graphs through graph operations, establish bounds, identify structural properties, and investigate resilience to changes in real networks, future work will focus on addressing the following directions:

1. What is the value of $RD_c(G_h)$ of graph operations, such as corona, cartesian product, join, lexicographic, and so on?. These operations are particularly useful since they generate new families of graphs from well-known ones.

2. Determining Nordhaus and Gaddum bounds on the rd_c-number of a graph and its complement as it helps to measure the extremal behavior of the parameter and provides benchmarks for comparisons.
3. Studying which family of graphs have small and large $RD_c(G_h)$ may reveal on structural properties that influence rd_c.
4. Real networks often undergo changes (adding/removing nodes or links) This has practical implications, e.g., in fault tolerance, cybersecurity, or communication networks, where resilience to changes is crucial. A natural question arises how does the value of $RD_c(G_h)$ change when G_h is altered through vertex or edge operations? (see eg.[9])

Additionally, our study demonstrates the practical application of rd_c-partitioning in cybersecurity, where networks can be structured into collaborative security clusters that detect, contain, and neutralize threats.

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