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Cosets and Congruence Relations over Residuated Lattices in the SoftMultiset Framework

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Abstract This paper investigates cosets and congruence relations within the framework of residuated lattices over soft multisets. By combining the algebraic rigor of residuated lattices with the flexibility of soft multiset theory, it establishes a unified structure for handling graded uncertainty. The notions of soft multiset filters, equivalence, and congruence relations are defined and their interconnections are explored through formal theorems. The proposed framework generalizes classical lattice concepts and strengthens the algebraic basis for reasoning under uncertainty.

Keywords softmultiset; softmultiset filter; cosets of softmultiset filter; softmultiset equivalence relation; softmultiset congruence relation.

AMS 2010 subject classifications 08A72; 03G10.

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1. Introduction

The algebraic structure of residuated lattices has been regarded as a fundamental framework in the study of non-classical logics, algebraic logic, and soft computing systems. It has been recognized that residuated lattices provide a unifying foundation for the analysis of implication, conjunction, and order within many-valued and substructural reasoning [13, 14, 8]. Within these algebraic systems, the concept of *filters* has been considered as a cornerstone, since filters enable the characterization of deductive systems and provide essential tools for constructing quotient structures and analyzing the algebraic behavior of elements.

In the conventional setting, filters in residuated lattices have been employed to describe deductive systems and algebraic completeness [26, 22, 29]. The framework of filters has further been generalized to fuzzy and soft environments, in which uncertainty and degrees of membership are incorporated into algebraic reasoning [18, 19, 21, 25, 3]. Through these generalizations, several important properties such as comaximality, pseudo-irreducibility[20], and structural indecomposability have been studied, which allow for a more nuanced understanding of algebraic and logical structures in contexts where classical assumptions may not hold.

Alongside these algebraic advancements, the introduction of the *soft set* theory [17] and its variants, including fuzzy soft sets [1, 10], multiset soft sets [2, 7, 24], and rough soft sets, has been recognized as a powerful means for handling parameterized uncertainty. These theories have been designed to represent data and knowledge that involve multiple occurrences or uncertain attributes. When combined with the algebraic concepts of residuated lattices, soft set theory provides a robust framework for managing imprecise, redundant, or uncertain information while preserving the underlying algebraic structure.

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Within this background, the theory of multiset filters of residuated lattices [4, 5, 25, 30] has been formulated to merge the principles of multiset-based reasoning with algebraic operations. This unification has been utilized to analyze decision-making and approximate reasoning models where repetition and multiplicity of parameters occur simultaneously. The algebraic behavior of these structures has been explored to provide new insights into structural logic and soft computation [16, 15], enabling more effective approaches for modeling complex systems that require both multiplicity and order considerations.

Although residuated lattices and their filters have been examined extensively, a comprehensive theoretical framework for the study of cosets, equivalence relations, and congruence relations of *soft multiset filters* has not yet been established. This absence has restricted the development of quotient structures and congruence-based homomorphisms in the soft multiset setting. Consequently, the present study has been undertaken to extend the theoretical foundation of residuated lattices by defining cosets and congruence relations of soft multiset filters, thereby filling a significant gap in the current literature.

Accordingly, the objectives of this study are articulated as follows:

1. To rigorously define and examine the structural framework of **cosets of soft multiset filters** within residuated lattices, uncovering their intrinsic algebraic properties;
2. To formulate and explore **soft multiset congruence relations** generated by these filters, and to establish their coherence and compatibility with both lattice and monoidal operations;
3. To demonstrate that the ensemble of all cosets naturally constitutes a residuated lattice, and to establish an **isomorphism theorem** for this construction, extending the classical results to the soft multiset context; and
4. To highlight the novel contribution of this study by distinguishing the quotient structures over soft multiset filters from existing results in classical and multiset filter theory.

Through these developments, the algebraic foundation of soft multiset theory has been extended. This extension has been considered to strengthen the bridge between the theories of residuated lattices and soft algebraic structures[23], thereby enhancing their mathematical depth and applicability in approximate reasoning, decision analysis, and soft computing [11, 16, 15, 28].

In recent years, significant attention has been devoted to the structural extensions and generalizations of residuated lattices. Advancements have been made in the study of uniform residuated lattices and their topological completions [27], as well as in the analysis of filter-based distance functions and continuity concepts [9]. Further developments have been reported in the context of residuated multilattices where soft set theory has been incorporated [6], indicating a growing interest in integrating soft computing principles with algebraic structures for enhanced theoretical and practical applications.

The present investigation has been structured in accordance with the progressive development of the theory. Section 2 has been devoted to the fundamental concepts and preliminary results on residuated lattices and soft multiset filters. Section 3 has included the formulation of cosets of soft multiset filters together with their algebraic characterizations. Section 4 has presented the notions of soft multiset equivalence and congruence relations. The paper has been concluded with remarks emphasizing the implications of the results for further research in algebraic logic and soft computing.

2. Preliminaries

Definition 2.1

[26] A residuated lattice of the type (2,2,2,2,0,0) is an ordered algebraic structure $(\mathcal{A}, \curlywedge, \curlyvee, \odot, \rightharpoonup, 0, 1)$ that satisfies the following conditions:

$\mathscr{X}_1$: $(\mathcal{A}, \curlywedge, \curlyvee)$ is a bounded lattice

$\mathscr{X}_2$: $(\mathcal{A}, \odot, 1)$ is a commutative monoid

$\mathscr{X}_3$: $e_1 \odot e_2 \preceq e_3 \;\Leftrightarrow\; e_1 \preceq e_2 \rightharpoonup e_3$ for any $e_1, e_2, e_3 \in \mathcal{A}$ where $e_i \rightharpoonup e_j = \max\{e_k \in \mathcal{A} : e_i \odot e_k \preceq e_j\}$

Proposition 2.2

[26] In any residuated lattice $\mathcal{A}$, the following conditions hold for each $e_1, e_2, e_3, e_4 \in \mathcal{A}$

(r_1) : $0 \rightharpoonup e_1 = 1 = e_1 \rightharpoonup e_1,\ 1 \rightharpoonup e_1 = e_1$
(r_2) : $e_1 \preceq (e_1 \rightharpoonup e_2) \rightharpoonup e_2\ ;\ e_1 \odot (e_1 \rightharpoonup e_2) \preceq e_2$ and $e_1 \rightharpoonup (e_2 \rightharpoonup e_1) = 1$
(r_3) : $e_1 \rightharpoonup (e_2 \rightharpoonup e_3) = (e_1 \odot e_2) \rightharpoonup e_3 = e_2 \rightharpoonup (e_1 \rightharpoonup e_3)$
(r_4) : $e_1 \rightharpoonup e_2 \preceq (e_2 \odot e_3) \rightharpoonup (e_2 \odot e_3)$
(r_5) : $e_1 \preceq e_2 \Rightarrow e_1 \odot e_3 \preceq e_2 \odot e_3$
(r_6) : $e_1 \odot e_2 \preceq e_1 \curlywedge e_2$
(r_7) : $e_1 \odot (e_2 \curlyvee e_3) = (e_1 \odot e_2) \curlyvee (e_1 \odot e_3)$
(r_8) : $e_1 \rightharpoonup e_2 \preceq (e_3 \rightharpoonup e_1) \rightharpoonup (e_3 \rightharpoonup e_2)$ and $e_1 \rightharpoonup e_2 \preceq (e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3)$
(r_9) : $e_1 \preceq e_2 \Rightarrow\ e_2 \rightharpoonup e_3 \preceq e_1 \rightharpoonup e_3$ and $e_3 \rightharpoonup e_2 \preceq e_3 \rightharpoonup e_2$
(r_{10}) : $e_1 \preceq e_2\ \Leftrightarrow\ e_1 \rightharpoonup e_2 = 1$
(r_{11}) : $(e_1 \rightharpoonup e_2) \odot (e_3 \rightharpoonup e_4) \preceq (e_1 \curlywedge e_3) \rightharpoonup (e_2 \curlywedge e_4)$
(r_{12}) : $(e_1 \rightharpoonup e_2) \odot (e_3 \rightharpoonup e_4) \preceq (e_1 \curlyvee e_3) \rightharpoonup (e_2 \curlyvee e_4)$
(r_{13}) : $(\bigvee_{i \in \blacksquare} e_i) \rightharpoonup e = \bigwedge_{i \in \blacksquare} (e_i \rightharpoonup e)$
(r_{14}) : $e_1 \rightharpoonup (e_2 \curlyvee e_3) = (e_1 \rightharpoonup e_2) \curlyvee (e_1 \rightharpoonup e_3)$

Definition 2.3
[30] A soft multiset $(\mathcal{F}, \mathcal{A})$ over $(\mathcal{U}, \mathcal{E})$ is called a soft multiset filter of $\mathcal{A}$ if it satisfies the following:
$\mathbb{S}_1$: $e_1 \preceq e_2 \Rightarrow \mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{e_2}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2 \in \mathcal{A}$
$\mathbb{S}_2$: $\min\{\mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_2}_{(\mathcal{F},\mathcal{A})}(u)\} \leqslant \mathcal{C}^{e_1 \odot e_2}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2 \in \mathcal{A}$

Proposition 2.4
[30] Let $(\mathcal{F}, \mathcal{A})$ be a softmultiset over $(\mathcal{U}, \mathcal{E})$. Then $(\mathcal{F}, \mathcal{A})$ satisfies the following :
$\mathbb{S}_3$: $\mathcal{C}^{e}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u).\ \forall\ e \in A$
$\mathbb{S}_4$: $\min\{\mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\} \leqslant \mathcal{C}^{e_2}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2 \in \mathcal{A}$
$\mathbb{S}_5$: $e_1 \preceq e_2 \rightharpoonup e_3 \Rightarrow \min\{\mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{e_2}_{(\mathcal{F},\mathcal{A})}(u)\} \leqslant \mathcal{C}^{e_3}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2, e_3 \in \mathcal{A}$
$\mathbb{S}_6$: $\mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) \Rightarrow \mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{e_2}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2 \in \mathcal{A}$
$\mathbb{S}_7$: $\mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \min\{\mathcal{C}^{(e_1 \rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_3 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\}$ for any $u \in \mathcal{U},\ e_1, e_2, e_3 \in \mathcal{A}$
$\mathbb{S}_8$: $\mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{((e_1 \odot e_3) \rightharpoonup (e_2 \odot e_3))}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2, e_3 \in \mathcal{A}$
$\mathbb{S}_9$: $\mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{((e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3))}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2, e_3 \in \mathcal{A}$
$\mathbb{S}_{10}$: $\mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{((e_3 \rightharpoonup e_1) \rightharpoonup (e_3 \rightharpoonup e_2))}_{(\mathcal{F},\mathcal{A})}(u)$ for any $u \in \mathcal{U},\ e_1, e_2, e_3 \in \mathcal{A}$

Example 2.5
Let $U = \{x_1, x_2\},\ E = \{e_1 \preceq e_2 \preceq e_3\}$, with $e_i \odot e_j = \min(e_i, e_j)$. Define soft multiset $(\mathcal{F}, \mathcal{A})$ with:

$$C^{e_1}_{(F,A)}(x_1) = 1, \quad C^{e_1}_{(F,A)}(x_2) = 0$$
$$C^{e_2}_{(F,A)}(x_1) = 2, \quad C^{e_2}_{(F,A)}(x_2) = 1$$
$$C^{e_3}_{(F,A)}(x_1) = 3, \quad C^{e_3}_{(F,A)}(x_2) = 2$$

Both filter conditions hold, hence (F, A) is a soft multiset filter.

2.1. Notation

The following notations are employed throughout this paper, with their respective meanings, unless stated otherwise:

- $\mathcal{A} = (\mathcal{A}, \curlywedge, \curlyvee, \preceq, \odot, \rightharpoonup, 0, 1)$ denotes a *residuated lattice*, where $a \curlywedge b$ represents the greatest lower bound (g.l.b) of a, b and $a \curlyvee b$ represents the least upper bound (l.u.b) of a, b. Here, 0 and 1 denote the g.l.b and l.u.b of A, respectively.

- $\mathbb{N}$ represents the set of all natural numbers including zero.
- $(\mathbb{N}, \min, \max)$ forms a lattice with respect to the usual order relation $\leqslant$, i.e., for $a, b \in \mathbb{N}$, $a \leqslant b \Leftrightarrow \min\{a, b\} = a$ and $\max\{a, b\} = b$.
- $\mathcal{A}$ denotes the parameter set of a *soft multiset filter* $(\mathcal{F}, \mathcal{A})$, where $\mathcal{A}$ is a residuated lattice.
- $\mathcal{U}$ denotes a universal multiset, and $\mathcal{P}(\mathcal{U})$ represents the powerset of the universal multiset $\mathcal{U}$.
- If $\mathcal{A}$ is the parameter set of a soft multiset filter $(\mathcal{F}, \mathcal{A})$ forming a residuated lattice, then the quotient structure $\mathcal{A}/\mathcal{F}$ is well-defined.
- $(\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A}) = \{e_1 \in \mathcal{A} \;:\; C^{e_1}_{(\mathcal{F},\mathcal{A})}(u) = C^{1}_{(\mathcal{F},\mathcal{A})}(u)\}$
  i.e., $e \in \mathcal{F}_{\mathcal{F}(1)} \Leftrightarrow C^{e}_{(\mathcal{F},\mathcal{A})}(u) = C^{1}_{(\mathcal{F},\mathcal{A})}(u)$
- $\xi(e_i, e_j) = \{u/C^{(e_i,e_j)}_{(\xi,\mathcal{A}\times\mathcal{A})}(u) \mid u \in \mathcal{U} \text{ and } e_i, e_j \in \mathcal{A}\}$ where $C^{(e_i,e_j)}_{(\xi,\mathcal{A}\times\mathcal{A})}(u) = \min\{C^{e_i \rightharpoonup e_j}_{(\mathcal{F},\mathcal{A})}(u),\; C^{e_j \rightharpoonup e_i}_{(\mathcal{F},\mathcal{A})}(u)\}$
- $\xi_{\mathcal{F}}(e_i, e_j) = \{u/C^{(e_i,e_j)}_{(\xi_{\mathcal{F}},\mathcal{A}\times\mathcal{A})}(u) \mid u \in \mathcal{U} \text{ and } e_i, e_j \in \mathcal{A}\}$,
  where $C^{(e_i,e_j)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) = \min\{C^{(e_i \rightharpoonup e_j)}_{(\mathcal{F},\mathcal{A})}(u),\; C^{(e_j \rightharpoonup e_i)}_{(\mathcal{F},\mathcal{A})}(u)\}$ for any $u \in \mathcal{U},\; e_i, e_j \in \mathcal{A}$

3. Cosets of a softmultiset filter

Definition 3.1
Let $(\mathcal{F}, \mathcal{A})$ be a softmultiset filter over $(\mathcal{U}, \mathcal{E})$. Then the mapping $e_i F : A \to P(U)$ for some 'i' is said to be a coset of a softmultiset filter (F, A) if it is defined for any $e_j \in A$ as $C^{e_j}_{(e_i F, A)}(\mathrm{u}) = \min\{C^{(e_i \rightharpoonup e_j)}_{(F,A)}(\mathrm{u}),\; C^{(e_j \rightharpoonup e_i)}_{(F,A)}(\mathrm{u})\}$.

Note that for any $e \in \mathcal{A},\; u \in \mathcal{U},\; C^{e}_{(1_{\mathcal{F}},\mathcal{A})}(u) = C^{e}_{(\mathcal{F},\mathcal{A})}(u)$

Example 3.2
Consider the parameter lattice $A = \{e_1, e_2, e_3, e_4\}$ defined by

$$e_1 \preceq e_2, \quad e_1 \preceq e_3, \quad e_2, e_3 \preceq e_4,$$

where $e_2 \curlywedge e_3 = e_1$ and $e_2 \curlyvee e_3 = e_4$. Define the residuum on A is given by

$$e_i \rightharpoonup e_j = \begin{cases} e_4, & e_i \preceq e_j, \\ e_j, & \text{otherwise.} \end{cases}$$

Let $U = \{x_1, x_2\}$ and define the soft multiset (F, A) by the multiplicity table

	x_1	x_2
$C^{e_1}_{(F,A)}$	1	1
$C^{e_2}_{(F,A)}$	2	1
$C^{e_3}_{(F,A)}$	1	2
$C^{e_4}_{(F,A)}$	3	2

It is readily verified that (F, A) satisfies (S1) for all comparable pairs and (S2) for representative parameter pairs, and hence forms a soft multiset filter.
According to Definition 3.1, the coset of (F, A) with respect to $e_i \in A$ is defined by

$$C_{e_j}(e_i F, A)(x) = \min\big\{\, C^{e_i \rightharpoonup e_j}_{(F,A)}(x),\; C^{e_j \rightharpoonup e_i}_{(F,A)}(x) \,\big\}, \qquad \forall e_j \in A,\; x \in U.$$

Applying this definition yields the following cosets:
$e_1 F(e_1) = \{3/x_1, 2/x_2\}$; $e_2 F(e_1) = \{1/x_1, 1/x_2\}$; $e_3 F(e_1) = \{1/x_1, 1/x_2\}$; $e_4 F(e_1) = \{1/x_1, 1/x_2\}$.
$e_1 F(e_2) = \{1/x_1, 1/x_2\}$; $e_2 F(e_2) = \{3/x_1, 2/x_2\}$; $e_3 F(e_2) = \{2/x_1, 1/x_2\}$; $e_4 F(e_2) = \{2/x_1, 1/x_2\}$.
$e_1 F(e_3) = \{1/x_1, 1/x_2\}$; $e_2 F(e_3) = \{2/x_1, 1/x_2\}$; $e_3 F(e_3) = \{3/x_1, 2/x_2\}$; $e_4 F(e_3) = \{1/x_1, 2/x_2\}$.
$e_1 F(e_4) = \{1/x_1, 1/x_2\}$; $e_2 F(e_4) = \{2/x_1, 1/x_2\}$; $e_3 F(e_4) = \{1/x_1, 2/x_2\}$; $e_4 F(e_4) = \{3/x_1, 2/x_2\}$.

Lemma 3.3
If $(\mathcal{F},\mathcal{A})$ be a softmultiset filter over $(\mathcal{U},\mathcal{E})$, then $e_1\mathcal{F} = e_2\mathcal{F}$ if and only if $\mathcal{C}^{e_1\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{e_2\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u)$

Proof
Suppose that $(\mathcal{F},\mathcal{A})$ is a softmultiset filter over $(\mathcal{U},\mathcal{E})$.
let $e_1, e_2 \in \mathcal{A},\ u \in \mathcal{U}$ be arbitrary.

$$\begin{array}{lrcll} & e_1\mathcal{F} & = & e_2\mathcal{F} & \\ \Leftrightarrow & \mathcal{C}^{e_1}_{(e_1\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{e_1}_{(e_2\mathcal{F},\mathcal{A})}(u) & \text{for any } e_i \in e_1\mathcal{F}(= e_2\mathcal{F}),\ u \in \mathcal{U} \\ \Leftrightarrow & \mathcal{C}^{e_1\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) & = & \min\{\mathcal{C}^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\} & \\ \Leftrightarrow & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & = & \min\{\mathcal{C}^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\} & \\ \Leftrightarrow & \mathcal{C}^{e_1\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \text{and} \\ & \mathcal{C}^{e_2\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \text{by } \mathbb{S}_3 \end{array}$$

Corollary 3.4
Let $(\mathcal{F},\mathcal{A})$ be a softmultiset filter over $(\mathcal{U},\mathcal{E})$. Then $e_1\mathcal{F} = e_2\mathcal{F}$ if and only if $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_2$ where $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_2$ if and only if $e_1 \rightharpoonup e_2 \in (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})$ and $e_2 \rightharpoonup e_1 \in (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})$.

Proof
Suppose $(\mathcal{F},\mathcal{A})$ is a soft multiset filter over $(\mathcal{U},\mathcal{E})$ and let $e_1, e_2 \in \mathcal{A},\ u \in \mathcal{U}$ be arbitrary.
Since $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_2$ if and only if $e_1 \rightharpoonup e_2 \in (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})$ and $e_2 \rightharpoonup e_1 \in (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})$,

$$\begin{array}{lrcll} & e_1\mathcal{F} & = & e_2\mathcal{F} & \\ \Leftrightarrow & \mathcal{C}^{e_1\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \text{and} \\ & \mathcal{C}^{e_2\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \\ \Leftrightarrow & e_1 \rightharpoonup e_2 & \in & (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A}) & \text{and} \\ & e_2 \rightharpoonup e_1 & \in & (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A}) & \\ \Leftrightarrow & e_1 & \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} & e_2 & \end{array}$$

Lemma 3.5
If $(\mathcal{F},\mathcal{A})$ be a softmultiset filter over $(\mathcal{U},\mathcal{E})$, then for any $e \in [e_2]_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$,
$\mathcal{C}^{e}_{(e_1\mathcal{F},\mathcal{A})}(u) = \min\{\mathcal{C}^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\}$ where $[a]_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} = \{b \in \mathcal{A}\ :\ a \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} b\}$.

Proof

$$\begin{array}{lrcll} & \text{Let}\quad e & \in & [e_2]_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} & \text{be arbitrary.} \\ \Leftrightarrow & e & \equiv & _{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}\ e_2 & \\ \Leftrightarrow & e \rightharpoonup e_2 & \in & (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A}) & \text{and} \\ & e_2 \rightharpoonup e & \in & (\mathcal{F}_{\mathcal{F}(1)},\mathcal{A}) & \\ \Leftrightarrow & \mathcal{C}^{e\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \text{and} \\ & \mathcal{C}^{e_2\rightharpoonup e}_{(\mathcal{F},\mathcal{A})}(u) & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \text{byDefinition} \\ & \text{Now,}\quad e & \leqslant & ((e \rightharpoonup e_1) \rightharpoonup e_1) & \text{by } (r_2) \\ \Leftrightarrow & e_2 \rightharpoonup e & \leqslant & e_2 \rightharpoonup ((e \rightharpoonup e_1) \rightharpoonup e_1) & \text{by } (r_5) \\ & & = & (e \rightharpoonup e_1) \rightharpoonup (e_2 \rightharpoonup e_1) & \text{by } (r_3) \\ \Rightarrow & \mathcal{C}^{e_2\rightharpoonup e}_{(\mathcal{F},\mathcal{A})}(u) & \leqslant & \mathcal{C}^{(e\rightharpoonup e_1)\rightharpoonup(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u) & \text{by } (\mathbb{S}_1) \end{array}$$

Since $\mathcal{C}^{e_2 \rightharpoonup e}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u)$, it follows that $\mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{(e \rightharpoonup e_1) \rightharpoonup (e_2 \rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)$ which is not possible.
Therefore,

$$\begin{aligned} \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) &= \mathcal{C}^{(e \rightharpoonup e_1) \rightharpoonup (e_2 \rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u) \\ \mathcal{C}^{e \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) &\leqslant \mathcal{C}^{e_2 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) \end{aligned}$$

Analogously, $\mathcal{C}^{e_2 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) \leqslant \mathcal{C}^{e \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u)$ follows.
i.e., $\mathcal{C}^{e_2 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{e \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u)$
Similarly, $\mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{e_1 \rightharpoonup e}_{(\mathcal{F},\mathcal{A})}(u)$ can be obtained.

$$\begin{aligned} \mathcal{C}^{e}_{(e_1 \mathcal{F},\mathcal{A})}(u) &= \min\{\mathcal{C}^{(e_1 \rightharpoonup e)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e \rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\} \\ &= \min\{\mathcal{C}^{(e_1 \rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_2 \rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\} \end{aligned}$$

Corollary 3.6
$\mathcal{C}^{e_2}_{(e_1 \mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u)$ for all $e_2 \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$

Proof Since $(\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A}) = [1]_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ and thus $e_1 \equiv_{(F_{F(1)},A)} 1$

$$\begin{aligned} \mathcal{C}^{e_2}_{(e_1 \mathcal{F},\mathcal{A})}(u) &= \min\{\mathcal{C}^{(e_1 \rightharpoonup 1)}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(1 \rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\} \\ &= \min\{\mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u)\} \\ &= \mathcal{C}^{e_1}_{(\mathcal{F},\mathcal{A})}(u) \end{aligned}$$

Lemma 3.7
If $(\mathcal{F}, \mathcal{A})$ is a softmultiset filter over $(\mathcal{U}, \mathcal{E})$ then $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is a congruence relation

Proof Let $e_1 \in \mathcal{A},\ u \in \mathcal{U}$ be arbitrary

$$\begin{aligned} e_1 \rightharpoonup e_1 &= 1 && \text{by } (r_1) \\ \Rightarrow \mathcal{C}^{e_1 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) &= \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) && \text{by } \mathbb{S}_3 \\ \Rightarrow e_1 \rightharpoonup e_1 &\in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A}) \\ \Rightarrow e_1 &\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_1 \end{aligned}$$

i.e., $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is reflexive.
Let $e_1, e_2 \in \mathcal{A},\ u \in \mathcal{U}$ be arbitrary

$$\begin{aligned} e_1 &\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_2 \\ \Leftrightarrow \mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) &= \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) = \mathcal{C}^{e_2 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) \\ \Leftrightarrow e_2 &\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_1 \end{aligned}$$

i.e., $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is symmetric.
Let $e_1, e_2, e_3 \in \mathcal{A}$ be arbitrary.
Suppose $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_2$ and $e_2 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_3$.
So, $e_1 \rightharpoonup e_2, e_2 \rightharpoonup e_1, e_2 \rightharpoonup e_3, e_3 \rightharpoonup e_2 \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$
Then for any $u \in \mathcal{U}$, it follows that

$\mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_2 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_2 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_3 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u)$

$$\begin{array}{llll}
 & e_1 \rightharpoonup e_2 & \leqslant & (e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3) \quad \text{by}(r_5) \\
\Rightarrow & \mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) & \leqslant & \mathcal{C}^{(e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u) \quad \text{by } \mathbb{S}_1 \\
\Rightarrow & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \leqslant & \mathcal{C}^{(e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u) \\
\Rightarrow & \min\{\mathcal{C}^{e_2 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{(e_2 \rightharpoonup e_3) \rightharpoonup (e_1 \rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u)\} & \leqslant & \mathcal{C}^{e_1 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u) \\
\Rightarrow & \min\{\mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u)\} & \leqslant & \mathcal{C}^{e_1 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u) \\
\Rightarrow & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) & \leqslant & \mathcal{C}^{e_1 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u) \\
\Rightarrow & e_1 \rightharpoonup e_3 & \in & (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})
\end{array}$$

Similarly, $e_3 \rightharpoonup e_1 \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$ can be derived in the same way and thus $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_3$
i.e., $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is transitive.
Let $e_1, e_2, e_3, e_4 \in \mathcal{A}$ be arbitrary.
Suppose $e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_3$ and $e_2 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_4$
Then $e_1 \rightharpoonup e_3, e_3 \rightharpoonup e_1, e_2 \rightharpoonup e_4, e_4 \rightharpoonup e_2 \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$
It follows for any $u \in \mathcal{U},\ (\mathcal{F}, \mathcal{A})$ satisfies the following :

$$\mathcal{C}^{e_1 \rightharpoonup e_3}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_3 \rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_2 \rightharpoonup e_4}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u), \mathcal{C}^{e_4 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u) \geqslant \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) \tag{1}$$

$$\begin{array}{llll}
\text{Then} & (e_1 \curlywedge e_3) \rightharpoonup (e_2 \curlywedge e_4) & \geqslant & (e_1 \rightharpoonup e_2) \odot (e_3 \rightharpoonup e_4) \quad \text{by}(r_{11}) \\
\Rightarrow & \mathcal{C}^{(e_1 \curlywedge e_3) \rightharpoonup (e_2 \curlywedge e_4)}_{(\mathcal{F},\mathcal{A})}(u) & \geqslant & \mathcal{C}^{(e_1 \rightharpoonup e_2) \odot (e_3 \rightharpoonup e_4)}_{(\mathcal{F},\mathcal{A})}(u) \quad \text{by } \mathbb{S}_1 \\
 & & \geqslant & \min\{\mathcal{C}^{e_1 \rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{e_3 \rightharpoonup e_4}_{(\mathcal{F},\mathcal{A})}(u)\} \quad \text{by } \mathbb{S}_2 \\
 & & \geqslant & \min\{\mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u),\ \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u)\} \\
 & & = & \mathcal{C}^{1}_{(\mathcal{F},\mathcal{A})}(u) \\
\Rightarrow & (e_1 \curlywedge e_3) \rightharpoonup (e_2 \curlywedge e_4) & \in & (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})
\end{array}$$

Similarly $(e_2 \curlywedge e_4) \rightharpoonup (e_1 \curlywedge e_3) \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$ can be showed
Thus $(e_1 \curlywedge e_3\}) \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} (e_2 \curlywedge e_4)$
Analogously it becomes upon using (r_{12}) that $(e_1 \curlyvee e_3) \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} (e_2 \curlyvee e_4)$
i.e., $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ satisfies the substitutional property for $'\curlywedge'$ and $'\curlyvee'$
Hence $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is a congruence relation.

Lemma 3.8
Let $(\mathcal{F}, \mathcal{A})$ be a softmultiset filter over $(\mathcal{U}, \mathcal{E})$.
If for any $e_1, e_2, e_3, e_4 \in \mathcal{A},\ e_1\mathcal{F} = e_3\mathcal{F}$ and $e_2\mathcal{F} = e_4\mathcal{F}$, then(1) $(e_1 \curlywedge e_2)\mathcal{F} = (e_3 \curlywedge e_4)\mathcal{F}$,
(2) $(e_1 \curlyvee e_2)\mathcal{F} = (e_3 \curlyvee e_4)\mathcal{F}$
(3) $(e_1 \odot e_2)\mathcal{F} = (e_3 \odot e_4)\mathcal{F}$
(4) $(e_1 \rightharpoonup e_2)\mathcal{F} = (e_3 \rightharpoonup e_4)\mathcal{F}$

Proof
Let $(\mathcal{F}, \mathcal{A})$ be a softmultiset filter over $(\mathcal{U}, \mathcal{E})$.
Assume $e_1\mathcal{F} = e_3\mathcal{F}$ and $e_2\mathcal{F} = e_4\mathcal{F}$ for any $e_1, e_2, e_3, e_4 \in \mathcal{A}$
$\Rightarrow e_1 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_3$ and $e_2 \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} e_4$ using Corollary 3.4
$\Rightarrow (e_1 \odot e_2) \equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})} (e_3 \odot e_4)$ since $\equiv_{(\mathcal{F}_{\mathcal{F}(1)},\mathcal{A})}$ is a congruence relation
$\Rightarrow (e_1 \odot e_2)\mathcal{F} = (e_3 \odot e_4)\mathcal{F}$
Analogously, (1),(2) and (4) can be obtained in the same manner.

Definition 3.9
Let $(\mathcal{F}, \mathcal{A})$ be a softmultiset filter over $(\mathcal{U}, \mathcal{E})$.

Let $\mathcal{A}/\mathcal{F}$ denote the set of all cosets of $(\mathcal{F},\mathcal{A})$. i.e., $\mathcal{A}/\mathcal{F} = \{e_1\mathcal{F} : e_1\mathcal{F}$ is a coset of $(\mathcal{F},\mathcal{A}),\ e_1 \in \mathcal{A}\}$.
For any $e_1\mathcal{F}, e_2\mathcal{F} \in \mathcal{A}/\mathcal{F}$ we define : $(e_1 \curlywedge e_2)\mathcal{F} = e_1\mathcal{F} \sqcap e_2\mathcal{F}$
$(e_1 \curlyvee e_2)\mathcal{F} = e_1\mathcal{F} \sqcup e_2\mathcal{F},\ (e_1 \odot e_2)\mathcal{F} = e_1\mathcal{F} * e_1\mathcal{F},\ (e_1 \rightharpoonup e_2)\mathcal{F} = e_1\mathcal{F} \diagdown e_1\mathcal{F}$

Theorem 3.10
If $(\mathcal{F},\mathcal{A})$ is a softmultiset filter over $(\mathcal{U},\mathcal{E})$. Then $\mathcal{A}/\mathcal{F} = (\mathcal{A}/\mathcal{F}, \sqcap, \sqcup, *, \diagdown, 0\mathcal{F}, 1\mathcal{F})$
is a residuated lattice

Proof
Clearly, the operations on $\mathcal{A}/\mathcal{F}$ become well-defined by the above Lemma 3.8 and Definition 3.9. It is to be shown first that $\mathcal{A}/(\mathcal{F},\mathcal{A})$ forms a bounded lattice.
Let $e_1\mathcal{F}, e_2\mathcal{F} \in \mathcal{A}/\mathcal{F}$ be arbitrary.
$\Rightarrow e_1, e_2 \in \mathcal{A}$
$\Rightarrow (e_1 \curlywedge e_2) \in \mathcal{A}$ [since A is a lattice]
$\Rightarrow (e_1 \curlywedge e_2)\mathcal{F} \in \mathcal{A}/\mathcal{F}$
$\Rightarrow e_1\mathcal{F} \sqcap e_2\mathcal{F} \in \mathcal{A}/\mathcal{F}$
Similarly $e_1\mathcal{F} \sqcup e_2\mathcal{F} \in \mathcal{A}/\mathcal{F}$ exists for every $e_1, e_2 \in \mathcal{A}$ as A is a lattice.
Analogously, it can be shown $0F$ is the greatest lower bound of $(\mathcal{A}/\mathcal{F}, \sqcap, \sqcup)$ as 0 is the greatest lower bound of A and $1F$ is the least upper bound of $(\mathcal{A}/\mathcal{F}, \sqcap, \sqcup)$ as 1 is the least upper bound of A. Thus $(\mathcal{A}/\mathcal{F}, \sqcap, \sqcup, 0\mathcal{F}, 1\mathcal{F})$ is a bounded lattice.
Next, it is to be shown that $\mathcal{A}/(\mathcal{F},\mathcal{A})$ forms a commutative monoid.
Let $e_1\mathcal{F}, e_2\mathcal{F}, e_3\mathcal{F} \in \mathcal{A}/\mathcal{F}$ be arbitrary. Then it follows from Definition 3.9 that

$$\begin{array}{rcl}
\Rightarrow e_1, e_2, e_3 & \in & \mathcal{A} \\
\Rightarrow e_1\mathcal{F} * (e_2\mathcal{F} * e_3\mathcal{F}) & = & e_1\mathcal{F} * (e_2 \odot e_3)\mathcal{F} \\
& = & (e_1 \odot (e_2 \odot e_3))\mathcal{F} \\
& = & ((e_1 \odot e_2) \odot e_3)\mathcal{F} \\
& = & ((e_1 \odot e_2)\mathcal{F} * e_3\mathcal{F}) \\
& = & ((e_1\mathcal{F} * e_2\mathcal{F}) * e_3\mathcal{F})
\end{array}$$

i.e., $e_1\mathcal{F} * (e_2\mathcal{F} * e_3\mathcal{F}) = (e_1\mathcal{F} * e_2\mathcal{F}) * e_3\mathcal{F},\ \forall\ e_1\mathcal{F}, e_2\mathcal{F}, e_3\mathcal{F} \in \mathcal{A}/\mathcal{F}$
Similarly, $e_1\mathcal{F} * e_2\mathcal{F} = e_2\mathcal{F} * e_1\mathcal{F},\ \forall\ e_1\mathcal{F}, e_2\mathcal{F} \in \mathcal{A}/\mathcal{F}$ holds and $e\mathcal{F} * 1\mathcal{F} = e\mathcal{F},\ \forall\ e\mathcal{F} \in \mathcal{A}/\mathcal{F}$ holds. Clearly $1\mathcal{F}$ is the identity element of $(\mathcal{A}/\mathcal{F}, *)$. Thus $(\mathcal{A}/\mathcal{F}, *, 1\mathcal{F})$ is a commutative monoid.
It is to be shown next that $(*, \diagdown)$ is an adjoint pair in $\mathcal{A}/\mathcal{F}$.
Define the lattice order relation $\sqsubseteq$ on $\mathcal{A}/\mathcal{F}$ as $e_1\mathcal{F} \sqsubseteq e_2\mathcal{F}$ if and only if $e_1\mathcal{F} \sqcup e_2\mathcal{F} = e_2\mathcal{F}$ and $e_1\mathcal{F} \sqcap e_2\mathcal{F} = e_1\mathcal{F}$.
Let $e_1\mathcal{F}, e_2\mathcal{F}, e_3\mathcal{F} \in \mathcal{A}/(\mathcal{F},\mathcal{A})$ be arbitrary.

$$\begin{array}{rccl}
& e_1\mathcal{F} * e_2\mathcal{F} & \sqsubseteq & e_3\mathcal{F} \\
\Leftrightarrow & (e_1 \odot e_2)\mathcal{F} & \sqsubseteq & e_3\mathcal{F} \\
\Leftrightarrow & C_{(\mathcal{F},\mathcal{A})}^{(e_1 \odot e_2) \rightharpoonup e_3}(u) & \leqslant & C_{(\mathcal{F},\mathcal{A})}^{1}(u) \quad \forall\ u \in \mathcal{U} \\
\Leftrightarrow & C_{(\mathcal{F},\mathcal{A})}^{e_1 \rightharpoonup (e_2 \rightharpoonup e_3)}(u) & \leqslant & C_{(\mathcal{F},\mathcal{A})}^{1}(u) \quad \forall\ u \in \mathcal{U} \\
\Leftrightarrow & C_{(\mathcal{F},\mathcal{A})}^{e_1}(u) & \leqslant & C_{(\mathcal{F},\mathcal{A})}^{e_2 \rightharpoonup e_3}(u) \quad \forall\ u \in \mathcal{U} \\
\Leftrightarrow & e_1\mathcal{F} & \sqsubseteq & (e_2 \rightharpoonup e_3)\mathcal{F} \\
\Leftrightarrow & e_1\mathcal{F} & \sqsubseteq & e_2\mathcal{F} \diagdown e_3\mathcal{F}
\end{array}$$

i.e., $e_1\mathcal{F} * e_2\mathcal{F} \sqsubseteq e_3\mathcal{F} \Leftrightarrow e_1\mathcal{F} \sqsubseteq e_2\mathcal{F} \diagdown e_3\mathcal{F}$
Therefore, $(*, \diagdown)$ is an adjoint pair in $\mathcal{A}/\mathcal{F}$. Hence $(\mathcal{A}/\mathcal{F}, \sqsubseteq, \sqcap, \sqcup, *, \diagdown, 0\mathcal{F}, 1\mathcal{F})$ is a residuated lattice

Theorem 3.11
Let $(\mathcal{F},\mathcal{A})$ be a softmultiset filter over $(\mathcal{U},\mathcal{E})$. Define $\varphi : \mathcal{A} \to \mathcal{A}/(\mathcal{F},\mathcal{A})$
by $\varphi(e_1) = e_1\mathcal{F}$. Then
(1) φ is an onto homomorphism
(2) $\ker\varphi = (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$
(3) $\mathcal{A}/(\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$ is isomorphic to $\mathcal{A}/(\mathcal{F},\mathcal{A})$

Proof
(1) Clearly φ is one-one and onto.

$$\begin{aligned}\varphi(e_1 \curlywedge e_2) &= (e_1 \curlywedge e_2)\mathcal{F} \\ &= e_1\mathcal{F} \sqcap e_2\mathcal{F} \\ &= \varphi(e_1) \sqcap \varphi(e_2)\end{aligned}$$

i.e., φ preserves '$\curlywedge$'. Similarly, it can be shown that φ preserves '$\curlyvee, \odot$ and $\rightharpoonup$'.
(2) Let $e \in \ker\varphi$ be arbitrary.
$\Leftrightarrow \varphi(e) = 1\mathcal{F} \Leftrightarrow e\mathcal{F} = 1\mathcal{F} \Leftrightarrow e \equiv_{(\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})} 1 \Leftrightarrow e \in (\mathcal{F}_{\mathcal{F}(1)}, \mathcal{A})$ and hence $\ker\varphi = \mathcal{F}_{\mathcal{F}(1)}$
(3) It immediately follows from (1) and (2)

In this section, we have examined the structural characteristics of cosets of soft multiset filters through a sequence of propositions and intermediate results. The study culminates in two pivotal outcomes: first, the set of all cosets of a soft multiset filter is shown to form a residuated lattice; and second, a mapping defined from this set to its corresponding parameter structure is established as an onto homomorphism whose kernel induces an isomorphism between the associated quotient structures. These foundational results emphasize the algebraic depth of coset structures and naturally motivate the introduction of relational frameworks, leading to the study of soft multiset equivalence and congruence relations in the subsequent section.

4. Softmultiset equivalence relation and softmultiset congruence relation

Extending the structural framework developed in the previous section, this part focuses on the relational aspects of soft multiset theory. We introduce the concepts of soft multiset equivalence and congruence relations along with their fundamental properties and axioms.

Definition 4.1
A soft multiset relation $(\xi, \mathcal{A})$ over $(\mathcal{U}, \mathcal{E})$ is said to be softmultiset equivalence relation on $\mathcal{A} \times \mathcal{A}$ if it satisfies the following :
$\xi(e_1, e_1) = \sup\{\xi(e_2, e_3) : e_2, e_3 \in \mathcal{A}\}$ (reflexive)
$\xi(e_1, e_2) = \xi(e_2, e_1)$ (symmetric)
$\xi(e_1, e_2) \succeq \xi(e_1, e_3) \curlywedge \xi(e_2, e_3)\}$ (transitive))

Example 4.2
Let $A = \{e_1, e_2, e_3\}$ be a finite residuated lattice of parameters subset of E with the order $e_1 \preceq e_2 \preceq e_3$. We define the following operations over A:
Monoid operation $\odot : A \times A \to A$

$\odot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_3	e_3
e_3	e_3	e_3	e_3

Residuum operation $\rightharpoonup : A \times A \to A$ by $e_i \rightharpoonup e_j = \begin{cases} e_3 & \text{if } e_i \preceq e_j, \\ e_1 & \text{otherwise.} \end{cases}$

Also,

$\rightharpoonup$	e_1	e_2	e_3
e_1	e_3	e_3	e_3
e_2	e_1	e_3	e_3
e_3	e_1	e_1	e_3

Define softmultiset relation $\xi : A \times A \to A$ by

ξ	e_1	e_2	e_3
e_1	e_3	e_2	e_1
e_2	e_2	e_3	e_2
e_3	e_1	e_2	e_3

It is clear that ξ is a softmultiset equivalence relation on $A \times A$

Lemma 4.3

If ξ is a softmultiset equivalence relation on $\mathcal{A} \times \mathcal{A}$, then ξ satisfies the following :
(1) $\xi(1,1) = \xi(e_i, e_i), \ \forall \ e_1 \in \mathcal{A}$
(2) $\xi(1,1) \succeq \xi(e_i, e_j), \ \forall \ e_1, e_2 \in \mathcal{A}$

Proof
(1) It can easily be deduced from the reflexivity of ξ
(2) Let $e_i, e_j \in \mathcal{A}$ be arbitrary. Then $\xi(1,1) = \xi(e_i, e_j) = \sup\{\xi(e_j, e_k) : e_j, e_k \in \mathcal{A}\} \succeq \xi(e_i, e_j)$

Definition 4.4
A softmultiset equivalence relation ξ on $\mathcal{A} \times \mathcal{A}$ is called softmultiset congruence relation on $\mathcal{A} \times \mathcal{A}$ if for any $e_1, e_2, e_3, e_4 \in \mathcal{A}$ it satisfies the following compatibility conditions :
(1) $\xi([e_1 \curlywedge e_2], \ [e_3 \curlywedge e_4]) \succeq \xi(e_1, e_3) \ \curlywedge \ \xi(e_2, e_4)$
(2) $\xi([e_1 \curlyvee e_2], \ [e_3 \curlyvee e_4]) \succeq \xi(e_1, e_3) \ \curlywedge \ \xi(e_2, e_4)$
(3) $\xi(e_1 \odot e_2, \ e_3 \odot e_4) \succeq \xi(e_1, e_3) \ \curlywedge \ \xi(e_2, e_4)$
(4) $\xi(e_1 \rightharpoonup e_2, \ e_3 \rightharpoonup e_4) \succeq \xi(e_1, e_3) \ \curlywedge \ \xi(e_2, e_4)$

Example 4.5
The previously defined Example 4.2 satisfies the compatibility conditions with respect to $\curlywedge, \curlyvee, \odot$ and $\rightharpoonup$ and hence ξ forms a softmultiset congruence relation on $A \times A$.

Theorem 4.6
Let ξ be a softmultiset equivalence relation on $\mathcal{A} \times \mathcal{A}$. Then ξ is a softmultiset congruence relation on $\mathcal{A} \times \mathcal{A}$ if and only if it satisfies the following :
(1) $\xi([e_1 \curlywedge e_3], [e_2 \curlywedge e_3]) \succeq \xi(e_1, e_2)$
(2) $\xi([e_1 \curlyvee e_3], [e_2 \curlyvee e_3]) \succeq \xi(e_1, e_2)$
(3) $\xi(e_1 \odot e_3, \ e_2 \odot e_3) \succeq \xi(e_1, e_2)$
(4) $\xi(e_1 \rightharpoonup e_3, \ e_2 \rightharpoonup e_3) \succeq \xi(e_1, e_2)$ and $\xi(e_3 \rightharpoonup e_1, \ e_3 \rightharpoonup e_2) \succeq \xi(e_1, e_2)$

Proof
Suppose ξ is a softmultiset congruence relation on $\mathcal{A} \times \mathcal{A}$. Then for any $e_1, e_2, e_3 \in \mathcal{A}$, we have

$$\begin{aligned} \xi([e_1 \curlywedge e_3], [e_2 \curlywedge e_3]) &\geqslant \xi(e_1, e_2) \ \curlywedge \ \xi(e_3, e_3) \\ &= \xi(e_1, e_2) \ \curlywedge \ \xi(1,1) \\ &= \xi(e_1, e_2) \end{aligned}$$

Similarly, (2),(3) and (4) can be derived
Conversely, ξ is a softmultiset equivalence relation on $\mathcal{A} \times \mathcal{A}$ which satisfies (1),(2),(3) and (4). Then for any $e_1, e_2, e_3 \in \mathcal{A}$, we have

$$\begin{aligned} \xi(e_1 \curlywedge e_2 \ , \ e_3 \curlywedge e_4\}) &\geqslant \xi(e_1 \curlywedge e_2 \ , \ e_2 \curlywedge e_3) \ \curlywedge \ \xi(e_2 \curlywedge e_3 \ , \ e_3 \curlywedge e_4) \\ &= \xi(e_1 \curlywedge e_3) \ \curlywedge \ \xi(e_2 \curlywedge e_4) \end{aligned}$$

Similarly, other conditions of Definition 4.4 can be derived
Hence ξ is a softmultiset congruence relation on $\mathcal{A} \times \mathcal{A}$

Definition 4.7

Let $(\mathcal{F},\mathcal{A})$ be a soft multiset filter over $(\mathcal{U},\mathcal{E})$. Consider a softmultiset relation $\xi_{\mathcal{F}}$ on $\mathcal{A}\times\mathcal{A}$ induced by $(\mathcal{F},\mathcal{A})$ is denoted by $\xi_{\mathcal{F}}(e_1,e_2)$ and is defined as follows :
$C^{(e_1,e_2)}_{(\xi_{\mathcal{F}},\mathcal{A})}(x)=\min\{C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(x),\ C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(x)\}$ for any $x\in\mathcal{U},\ e_1,e_2\in\mathcal{A}$

Theorem 4.8

Let $(\mathcal{F},\mathcal{A})$ be a soft multiset filter over $(\mathcal{U},\mathcal{E})$. Then the soft multiset relation $\xi_{\mathcal{F}}$ on $\mathcal{A}\times\mathcal{A}$ induced by $(\mathcal{F},\mathcal{A})$ is a soft multiset congruence relation over$(\mathcal{U},\mathcal{E})$.

Proof

(i) Let $u\in\mathcal{U}$ and $e_1\in\mathcal{A}$ be arbitrary. Then it follows from Definition 4.7 that

$$\begin{aligned}
C^{(1,1)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) &= \min\{C^{(1\rightharpoonup 1)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(1\rightharpoonup 1)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= \min\{C^{1}_{(\mathcal{F},\mathcal{A})}(u),\ C^{1}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= \min\{C^{(e_1\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_1\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= C^{(e_1,e_1)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u)\\
\text{i.e., }\ \xi_{\mathcal{F}}(1,1) &= \xi_{\mathcal{F}}(e_1,e_1)
\end{aligned}$$

(ii) Let $u\in\mathcal{U}$ and $e_1,e_2\in\mathcal{A}$ be arbitrary. Then

$$\begin{aligned}
C^{(e_1,e_2)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) &= \min\{C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= \min\{C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= C^{(e_2,e_1)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u)\\
\text{i.e., }\ \xi_{\mathcal{F}}(e_1,e_2) &= \xi_{\mathcal{F}}(e_2,e_1)
\end{aligned}$$

(iii) Let $u\in\mathcal{U}$ and $e_1,e_2,e_3\in\mathcal{A}$ be arbitrary.

$$\begin{aligned}
\min\{C^{(e_1,e_2)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u),\ C^{(e_2,e_3)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u)\} &= \min\{\min\{C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\},\ \min\{C^{(e_2\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_3\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\}\}\\
&= \{\min\{C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_3\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&= \min\{\min\{C^{(e_1\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u)\},\ \min\{C^{(e_3\rightharpoonup e_2)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\}\}\\
&\leqslant \min\{C^{(e_1\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_3\rightharpoonup e_1)}_{(\mathcal{F},\mathcal{A})}(u)\} \qquad \text{by } \mathbb{S}_7\\
&= C^{(e_1,e_3)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) \qquad \text{by Definition 4.7}\\
\text{i.e., }\ \xi_{\mathcal{F}}(e_1,e_3)\curlywedge\xi_{\mathcal{F}}(e_2,e_3) &\leqslant \xi_{\mathcal{F}}(e_1,e_2)
\end{aligned}$$

(iv) Let $u\in\mathcal{U}$ and $e_1,e_2,e_3\in\mathcal{A}$ be arbitrary. Then it follows from $(r_1),(r_{11})$ that

$$\begin{aligned}
C^{(e_1\curlywedge e_3\ ,\ e_2\curlywedge e_3)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) &= \min\{C^{(e_1\curlywedge e_3)\rightharpoonup(e_2\curlywedge e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\curlywedge e_3)\rightharpoonup(e_1\curlywedge e_3)}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&\geqslant \min\{C^{(e_1\rightharpoonup e_2)\odot(e_3\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)\odot(e_3\rightharpoonup e_3)}_{(\mathcal{F},\mathcal{A})}(u)\} \qquad \text{by } (r_{11}) \text{ and } \mathbb{S}_1\\
&= \min\{C^{(e_1\rightharpoonup e_2)\odot 1}_{(\mathcal{F},\mathcal{A})}(u),\ C^{(e_2\rightharpoonup e_1)\odot 1}_{(\mathcal{F},\mathcal{A})}(u)\} \qquad \text{by } (r_1)\\
&= \min\{C^{e_1\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u),\ C^{e_2\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u)\} \qquad \text{by} (r_0)\\
&= C^{(e_1,e_2)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) \qquad \text{by Definition 4.7}
\end{aligned}$$

i.e., $\xi_{\mathcal{F}}(e_1\curlywedge e_3\ ,\ e_2\curlywedge e_3)\geqslant\xi_{\mathcal{F}}(e_1,e_2)$

(v) It can be shown in the similar way

(vi) Let $u\in\mathcal{U}$ and $e_1,e_2,e_3\in\mathcal{A}$ be arbitrary.

$$\begin{aligned}
C^{((e_1\odot e_3),\ (e_2\odot e_3))}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) &= \min\{C^{((e_1\odot e_3)\rightharpoonup(e_2\odot e_3))}_{(\mathcal{F},\mathcal{A})}(u),\ C^{((e_2\odot e_3)\rightharpoonup(e_1\odot e_3))}_{(\mathcal{F},\mathcal{A})}(u)\}\\
&\geqslant \min\{C^{e_1\rightharpoonup e_2}_{(\mathcal{F},\mathcal{A})}(u),\ C^{e_2\rightharpoonup e_1}_{(\mathcal{F},\mathcal{A})}(u)\} \qquad \text{by } \mathbb{S}_8\\
&= C^{(e_1,e_2)}_{(\xi_{\mathcal{F}},\mathcal{A})}(u) \qquad \text{by Definition 4.7}\\
\text{i.e., }\ \xi_{\mathcal{F}}((e_1\odot e_3),\ (e_2\odot e_3)) &\geqslant \xi_{\mathcal{F}}(e_1,e_2)
\end{aligned}$$

(vii) Let $u \in \mathcal{U}$ and $e_1, e_2, e_3 \in \mathcal{A}$ be arbitrary.

$$\begin{aligned}
& \min\{\mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_1 \rightharpoonup e_3,\ e_2 \rightharpoonup e_3)}(u),\ \mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_3 \rightharpoonup e_1,\ e_3 \rightharpoonup e_2)}(u)\} \\
= \ & \min\{\min\{\mathcal{C}_{(\mathcal{F},\mathcal{A})}^{(e_1 \rightharpoonup e_3)\rightharpoonup(e_2 \rightharpoonup e_3)}(u),\ \mathcal{C}_{(\mathcal{F},\mathcal{A})}^{(e_2 \rightharpoonup e_3)\rightharpoonup(e_1 \rightharpoonup e_3)}(u)\},\ \min\{\mathcal{C}_{(\mathcal{F},\mathcal{A})}^{(e_3 \rightharpoonup e_1)\rightharpoonup(e_3 \rightharpoonup e_2)}(u),\ \mathcal{C}_{(\mathcal{F},\mathcal{A})}^{(e_3 \rightharpoonup e_2)\rightharpoonup(e_3 \rightharpoonup e_1)}(u)\}\} \\
\geqslant \ & \min\{\min\{\mathcal{C}_{(\mathcal{F},\mathcal{A})}^{e_2 \rightharpoonup e_1}(u),\ \mathcal{C}_{(\mathcal{F},\mathcal{A})}^{e_1 \rightharpoonup e_2}(u)\},\ \min\{\mathcal{C}_{(\mathcal{F},\mathcal{A})}^{e_1 \rightharpoonup e_2}(u),\ \mathcal{C}_{(\mathcal{F},\mathcal{A})}^{e_2 \rightharpoonup e_1}(u)\}\} \quad \text{by } \mathbb{S}_9 \text{ and } \mathbb{S}_{10} \\
= \ & \min\{\mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_1,e_2)}(u),\ \mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_2,e_1)}(u)\} \\
= \ & \min\{\mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_1,e_2)}(u),\ \mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_1,e_2)}(u)\} \quad \text{by (ii)} \\
= \ & \mathcal{C}_{(\xi_{\mathcal{F}},\mathcal{A})}^{(e_1,e_2)}(u)
\end{aligned}$$

i.e., $\xi_{\mathcal{F}}(e_1 \rightharpoonup e_3,\ e_2 \rightharpoonup e_3) \curlywedge \xi_{\mathcal{F}}(e_3 \rightharpoonup e_1,\ e_3 \rightharpoonup e_2) \geqslant \xi_{\mathcal{F}}(e_1, e_2)$
It is clear from (i),(ii),(iii),(iv) and (v) that $\xi_{\mathcal{F}}$ is a softmultiset congruence relation on $\mathcal{A} \times \mathcal{A}$

Note 4.9
Observed from Definition 3.1,4.7 and Theorem 4.8 that Any coset of a soft multiset filter of a residuated lattice is a softmultiset congruence relation on the residuated lattice.

5. Conclusion

This study has developed a comprehensive framework for analyzing cosets and congruence relations in residuated lattices using the soft multiset approach, thereby unifying algebraic precision with soft computational flexibility. The established interrelations among filters, equivalence, and congruence relations enrich the theoretical foundation of soft algebraic systems. The proposed results extend traditional lattice concepts to uncertain and graded environments. In future work, this framework can be expanded to soft multiset modules, homomorphisms, and quotient structures, offering broader applications in fuzzy inference, decision theory, and soft computing models. Further exploration of algorithmic implementations and real-world data-driven validations is also envisioned.

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