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Estimation of Reliability Based on Rayleigh Distribution

Ayman A Hazaymeh [1,*], Anwar Bataihah [1], Naser Odat [1], Tariq Qawasmeh [2], Ra'ft Abdelrahim [3], Abdulgabar Adam Hassan [3], Faisal Al-Sharqi[4,5]

[1]*Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan*
[2]*Department of Mathematics, Faculty of Arts and Sciences, Amman Arab University, Amman, Jordan*
[3]*Department of Mathematics, College of Art and Sciences, Jouf University, Saudi Arabia*
[4]*Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, Anbar, Iraq*
[5]*College of Pharmacy, National University of Science and Technology, DhiQar, Iraq*

Abstract The estimation of reliability $R = P(X < Y)$ for independent Rayleigh-distributed random variables X and Y is examined in this study. This estimation is essential for reliability analysis and stress-strength models in engineering. Three estimators are derived and compared: a Bayesian estimator utilizing conjugate Gamma priors, the method of moments (Mom) estimate, and the maximum likelihood estimator (MLE). The normality of the MLE and the creation of confidence intervals using the Fisher information matrix are two of its acknowledged asymptotic characteristics. A thorough simulation analysis measures bias, mean squared error (MSE), and confidence interval coverage to assess how well these estimators perform across a range of sample sizes and parameter combinations. Our findings show that, especially for small to moderate sample sizes, the MLE consistently performs better in terms of MSE than the Mom and Bayesian estimators. Despite its flexibility, the Bayesian technique exhibits sensitivity to previous specifications when there is a considerable difference between the scale parameters θ_1 and θ_2. With applications in quality control and reliability engineering, the study offers useful recommendations for choosing estimators depending on sample size and parameter configurations. Numerical examples are provided to demonstrate the suggested approaches, and their expansions to more intricate systems are explored.

Keywords Rayleigh Distribution, maximum likelihood, Bayesian estimator, Moment method, Reliability.

AMS 2010 subject classifications 62D05, 62D99

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1. Introduction

A fundamental challenge in reliability theory and engineering statistics is the estimation of reliability parameters $R = P(X < Y)$, where X and Y are independent random variables that indicate strength and stress, respectively. This stress-strength model is essential for evaluating failure risk, system dependability, and product durability since it measures the likelihood that a component with strength X can sustain a stress Y applied to it. The estimation of R becomes practically significant and analytically tractable when both X and Y follow Rayleigh distributions, which are frequently used to model failure rates and lifespan in engineering systems. Numerous authors have made contributions to the estimation of dependability in a range of distributional situations, such as Pareto, normal, and exponential models. The situation where both variables exhibit Rayleigh distributions, however, has been the subject of comparatively fewer studies. This study contributes to the literature by generating and comparing estimators of R under the assumption that X and Y are independently and identically distributed according to the Rayleigh distribution. In particular, the study proposes Bayesian estimators under various prior assumptions, the

*Correspondence to: Ayman A Hazaymeh (Email: aymanha@jadara.edu.jo). Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan

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method of moments estimator (Mom), and the maximum likelihood estimator (MLE). Additionally, confidence intervals for R are built using both classical and computational methods, and the asymptotic aspects of the estimators are investigated.

Simulation experiments are conducted for different sample sizes and parameter configurations in order to verify and compare the performance of the suggested estimators. These simulations assess coverage probabilities of confidence intervals, mean squared error (MSE), and estimator bias. According to the results, the Bayesian approach provides more flexibility and resilience, particularly when including prior information, even though the MLE performs well in a variety of circumstances. The findings have implications for reliability analysis in engineering applications where Rayleigh-distributed variables are a natural fit.

The maximum likelihood estimate when x and y have a bivariate exponential distribution was studied in [1]. [2], [3], [4], [5], [6], [7],[8] and , [9] have all looked at the estimation of a comparable problem for the multivariate normal distribution. These studies all looked at the estimation of when x and y are regularly distributed. Assuming that x and y are independent exponential random variables, [10] have investigated the interval estimation of reliability parameters and hypothesis testing in strength models with two parameter exponential distributions. [11] extended the Pareto distribution using the Epanechnikov kernel technique. [17] investigated the Volterra integral equation's solution stability using a random kernel. Examine a new method for a modified Midzuno scheme in [13]. [14] looked into the statistically convergent sequences. [11] estimated dependability using the Pareto distribution. In [15], the ranked set selection for simple linear regression was examined. In the future, we hope to include the fuzzy soft set into our work as the [19, 20]

[16] Rank set sampling in a modified ratio estimator.

When X and Y are independent random variables, the problem of predicting the probability that one random variable will surpass the other, or $(X < Y)$, has continuously drawn interest. The parameter R represents the reliability parameter. In the context of classical stress-strength reliability, the question arises if the random strength (X) of a component is greater than the stress (Y) to which it is subjected; if $X \leq Y$, the component fails or the system that employs the component may malfunction.

2. Problem Formulation

In this paper we consider the problem of estimation of the reliability $R(\theta_1, \theta_2) = P(X < Y)$, based on $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim}$ Rayleigh Distribution, where X is the strength with probability density function:

$$f(x) = 2\theta_1 x e^{-\theta_1 x^2}, \quad x \geq 0$$

Since $X \sim \text{Ray}(\theta_1)$ and $Y \sim \text{Ray}(\theta_2)$ where X and Y are independent and identically distributed, then:

$$R(\theta_1, \theta_2) = P(X < Y) = \iint_0^y 4\theta_1\theta_2 xy e^{-\theta_1 x^2} e^{-\theta_2 y^2}\, dx\, dy$$

$$R(\theta_1, \theta_2) = \frac{\theta_1}{\theta_1 + \theta_2} \tag{3}$$

3. Estimation Methods of $R = P(X < Y)$

3.1. Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a Rayleigh Distribution with population parameter θ_1, and $y_1, y_2, \ldots, y_n$ be a random sample of size n from a Rayleigh Distribution with population parameter θ_2. Then the likelihood function is given by:

$$L(\theta_1,\theta_2) = 4^n\theta_1^n\theta_2^n\prod_{i=1}^{n} x_i\prod_{i=1}^{n} y_i e^{-\theta_1\sum x_i^2}e^{-\theta_2\sum y_i^2} \tag{1}$$

By taking the natural logarithm to both sides, equation (1) becomes:

$$\ln L(\theta_1,\theta_2) = n\ln(4) + n\ln(\theta_1) + n\ln(\theta_2) + \sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(y_i) - \theta_1\sum x_i^2 - \theta_2\sum y_i^2 \tag{2}$$

By deriving $\ln L(\theta_1,\theta_2)$ with respect to θ_1 and θ_2 and equating the results to zero, we get:

$$\frac{\partial \ln L}{\partial\theta_1} = \frac{n}{\theta_1} - \sum x_i^2 = 0 \tag{3}$$

$$\frac{\partial \ln L}{\partial\theta_2} = \frac{n}{\theta_2} - \sum y_i^2 = 0 \tag{4}$$

The MLEs for the unknown parameters θ_1, θ_2 are given by:

$$\hat{\theta_1} = \frac{n}{\sum x_i^2} \tag{5}$$

$$\hat{\theta_2} = \frac{n}{\sum y_i^2} \tag{6}$$

By substituting equations (5) and (6) in equation (3) we get:

$$\hat{R}_{\text{mle}} = \frac{\frac{n}{\sum x_i^2}}{\frac{n}{\sum x_i^2} + \frac{n}{\sum y_i^2}} \tag{7}$$

3.2. Method of Moments

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a Rayleigh Distribution with population parameter θ_1 with probability density function:

$$f(x) = 2\theta_1 x e^{-\theta_1 x^2}, \quad x \geq 0, \theta_1 > 0$$

And let $y_1, y_2, \ldots, y_n$ be a random sample of size n from a Rayleigh Distribution with population parameter θ_2 with probability density function:

$$f(y) = 2\theta_2 y e^{-\theta_2 y^2}, \quad y \geq 0, \theta_2 > 0$$

The expected value of X and Y are:

$$E(X) = \sqrt{\frac{\pi}{4\theta_1}} \tag{8}$$

$$E(Y) = \sqrt{\frac{\pi}{4\theta_2}} \tag{9}$$

By equating the sample mean with the corresponding population mean, we get:

$$\sqrt{\frac{\pi}{4\theta_1}} = \bar{x} \tag{10}$$

$$\sqrt{\frac{\pi}{4\theta_2}} = \bar{y} \tag{11}$$

This implies that:

$$\hat{\theta}_1 = \frac{\pi}{4\bar{x}^2} \tag{12}$$

$$\hat{\theta}_2 = \frac{\pi}{4\bar{y}^2} \tag{13}$$

By substituting equations (15) and (16) in equation (3) we get:

$$\hat{R}_{\text{mom}} = \frac{\frac{\pi}{4\bar{x}^2}}{\frac{\pi}{4\bar{x}^2} + \frac{\pi}{4\bar{y}^2}} \tag{14}$$

3.3. Bayes Estimation of R

The Bayesian methodology has a number of benefits over the conventional frequentist approach. Bayes' theorem offers a systematic framework for updating our beliefs regarding the parameters in light of the observed data (see Bolstad [19]). This subsection investigates the stress-strength reliability of the Rayleigh distribution using conjugate Bayesian analysis.

Assuming independent Gamma priors for the unknown parameters θ_1 and θ_2 with hyperparameters (a_i, b_i), $i = 1, 2$, the gamma density function is given by:

$$g(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \quad x \geq 0, \quad a, b > 0 \tag{15}$$

Then, the joint posterior density function of θ_1 and θ_2 is:

$$\pi(\theta_1, \theta_2) = \frac{L(\theta_1, \theta_2 \mid \boldsymbol{x}) g(\theta_1) g(\theta_2)}{\iint_0^\infty L(\theta_1, \theta_2 \mid \boldsymbol{x}) g(\theta_1) g(\theta_2)\, d\theta_1 d\theta_2} \tag{16}$$

$$= \frac{4^n \theta_1^n \theta_2^n \prod_{i=1}^n x_i \prod_{i=1}^n y_i e^{-\theta_1 \sum x_i^2} e^{-\theta_2 \sum y_i^2} \frac{b_1^{a_1}}{\Gamma(a_1)} \theta_1^{a_1 - 1} e^{-b_1 \theta_1} \frac{b_2^{a_2}}{\Gamma(a_2)} \theta_2^{a_2 - 1} e^{-b_2 \theta_2}}{\iint_0^\infty 4^n \theta_1^n \theta_2^n \prod_{i=1}^n x_i \prod_{i=1}^n y_i e^{-\theta_1 \sum x_i^2} e^{-\theta_2 \sum y_i^2} \frac{b_1^{a_1}}{\Gamma(a_1)} \theta_1^{a_1 - 1} e^{-b_1 \theta_1} \frac{b_2^{a_2}}{\Gamma(a_2)} \theta_2^{a_2 - 1} e^{-b_2 \theta_2}\, d\theta_1 d\theta_2} \tag{17}$$

$$= \frac{\theta_1^{n+a_1-1} e^{-\theta_1 (\sum x_i^2 + b_1)} \theta_2^{n+a_2-1} e^{-\theta_2 (\sum y_i^2 + b_2)}}{\iint_0^\infty \theta_1^n \theta_2^n e^{-\theta_1 \sum x_i^2} e^{-\theta_2 \sum y_i^2} \theta_1^{a_1 - 1} e^{-b_1 \theta_1} \theta_2^{a_2 - 1} e^{-b_2 \theta_2}\, d\theta_1 d\theta_2} \tag{18}$$

$$= \frac{\theta_1^{n+a_1-1} e^{-\theta_1 (\sum x_i^2 + b_1)} \theta_2^{n+a_2-1} e^{-\theta_2 (\sum y_i^2 + b_2)}}{\int \theta_1^{n+a_1-1} e^{-\theta_1 (\sum x_i^2 + b_1)}\, d\theta_1 \int \theta_2^{n+a_2-1} e^{-\theta_2 (\sum y_i^2 + b_2)}\, d\theta_2} \tag{19}$$

Therefore, the joint posterior distribution factorizes as:

$$\pi(\theta_1, \theta_2) = \frac{\theta_1^{n+a_1-1}}{\Gamma(a_1)} e^{-\theta_1 (\sum x_i^2 + b_1)} \cdot \frac{\theta_2^{n+a_2-1}}{\Gamma(a_2)} e^{-\theta_2 (\sum y_i^2 + b_2)} = \pi(\theta_1) \pi(\theta_2) \tag{20}$$

where $\theta_1 \mid \boldsymbol{x} \sim \text{Gamma}(a_1 + n, b_1 + \sum x_i^2)$ and $\theta_2 \mid \boldsymbol{y} \sim \text{Gamma}(a_2 + n, b_2 + \sum y_i^2)$.
The Bayesian estimators of θ_1 and θ_2 are:

$$\hat{\theta}_1 = E(\theta_1 \mid \boldsymbol{x}) = \frac{a_1 + n}{b_1 + \sum x_i^2} \tag{21}$$

$$\hat{\theta}_2 = E(\theta_2 \mid \boldsymbol{x}) = \frac{a_2 + n}{b_2 + \sum y_i^2} \tag{22}$$

And the variances of θ_1 and θ_2 are:

$$V(\theta_1 \mid \boldsymbol{x}) = \frac{a_1 + n}{(b_1 + \sum x_i^2)^2} \tag{23}$$

$$V(\theta_2 \mid \boldsymbol{y}) = \frac{a_2 + n}{(b_2 + \sum y_i^2)^2} \tag{24}$$

For $a_i = b_i = 0$, the non-informative prior gives $\theta_1 \mid \boldsymbol{x} \sim \text{Gamma}(n, \sum x_i^2)$ and $\theta_2 \mid \boldsymbol{y} \sim \text{Gamma}(n, \sum y_i^2)$. Then:

$$\hat{\theta}_1 = E(\theta_1 \mid \boldsymbol{x}) = \frac{n}{\sum x_i^2}, \tag{25}$$

$$\hat{\theta}_2 = E(\theta_2 \mid \boldsymbol{x}) = \frac{n}{\sum y_i^2} \tag{26}$$

This matches the MLE of θ_1 and θ_2. Then the Bayesian estimator for R is:

$$\hat{R}_{\text{Bayes}} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} \tag{27}$$

4. Asymptotic Distribution of R-hat and Different Confidence Intervals

This section derives the asymptotic distribution of the Maximum Likelihood Estimator (MLE) of the reliability parameter R. The approach involves first establishing the asymptotic distribution of the parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and then applying the delta method to the function $R(\theta_1, \theta_2)$.

4.1. Asymptotic Distribution of $\hat{\boldsymbol{\theta}}$

Let $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)$ be the MLE of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. From standard asymptotic theory of maximum likelihood estimation, $\hat{\boldsymbol{\theta}}$ is consistent and asymptotically normal under regularity conditions.

The Fisher Information Matrix, $I(\boldsymbol{\theta})$, for the parameter vector $\boldsymbol{\theta}$ based on samples of size n from both X and Y is given by the negative expectation of the second derivatives of the log-likelihood function.

The log-likelihood function, as derived in Equation (5), is:

$$\ln L(\theta_1, \theta_2) = n \ln(4) + n \ln(\theta_1) + n \ln(\theta_2) + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} \ln(y_i) - \theta_1 \sum x_i^2 - \theta_2 \sum y_i^2 \tag{28}$$

The second derivatives are:

$$\frac{\partial^2 \ln L}{\partial \theta_1^2} = -\frac{n}{\theta_1^2}, \tag{29}$$

$$\frac{\partial^2 \ln L}{\partial \theta_2^2} = -\frac{n}{\theta_2^2}, \tag{30}$$

$$\frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1} = 0 \tag{31}$$

Taking the negative expectation, the Fisher Information Matrix is:

$$I(\boldsymbol{\theta}) = -\begin{bmatrix} E\left(\frac{\partial^2 \ln L}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2}\right) & E\left(\frac{\partial^2 \ln L}{\partial \theta_2^2}\right) \end{bmatrix} = \begin{bmatrix} \frac{n}{\theta_1^2} & 0 \\ 0 & \frac{n}{\theta_2^2} \end{bmatrix} \tag{32}$$

Thus, the inverse Fisher Information Matrix is:

$$I^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\theta_1^2}{n} & 0 \\ 0 & \frac{\theta_2^2}{n} \end{bmatrix} \tag{33}$$

Therefore, the asymptotic distribution of the MLE $\hat{\boldsymbol{\theta}}$ is:

$$\sqrt{n}\begin{pmatrix} \hat{\theta}_1 - \theta_1 \\ \hat{\theta}_2 - \theta_2 \end{pmatrix} \xrightarrow{d} N\left(\mathbf{0}, I^{-1}(\boldsymbol{\theta})\right) \tag{34}$$

4.2. Asymptotic Distribution of $\hat{R}_{\text{mle}}$ using the Delta Method

The reliability parameter is a function of $\boldsymbol{\theta}$: $R = g(\theta_1, \theta_2) = \frac{\theta_1}{\theta_1+\theta_2}$. We apply the delta method to find the asymptotic distribution of $\hat{R}_{\text{mle}} = g(\hat{\theta}_1, \hat{\theta}_2)$.

The gradient vector of R with respect to $\boldsymbol{\theta}$, denoted ∇R, is:

$$\nabla R = \begin{bmatrix} \frac{\partial R}{\partial \theta_1} \\ \frac{\partial R}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \frac{\theta_2}{(\theta_1+\theta_2)^2} \\ -\frac{\theta_1}{(\theta_1+\theta_2)^2} \end{bmatrix} \tag{35}$$

The asymptotic variance of $\hat{R}_{\text{mle}}$ is given by:

$$V = (\nabla R)^\top I^{-1}(\boldsymbol{\theta}) \nabla R \tag{36}$$

Substituting the expressions for $I^{-1}(\boldsymbol{\theta})$ and ∇R:

$$V = \begin{bmatrix} \frac{\theta_2}{(\theta_1+\theta_2)^2} & -\frac{\theta_1}{(\theta_1+\theta_2)^2} \end{bmatrix} \begin{bmatrix} \frac{\theta_1^2}{n} & 0 \\ 0 & \frac{\theta_2^2}{n} \end{bmatrix} \begin{bmatrix} \frac{\theta_2}{(\theta_1+\theta_2)^2} \\ -\frac{\theta_1}{(\theta_1+\theta_2)^2} \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} \frac{\theta_2}{(\theta_1+\theta_2)^2} & -\frac{\theta_1}{(\theta_1+\theta_2)^2} \end{bmatrix} \begin{bmatrix} \frac{\theta_1^2\theta_2}{n(\theta_1+\theta_2)^2} \\ \frac{\theta_1\theta_2^2}{n(\theta_1+\theta_2)^2} \end{bmatrix} \tag{38}$$

$$= \frac{\theta_1^2\theta_2^2}{n(\theta_1+\theta_2)^4} + \frac{\theta_1^2\theta_2^2}{n(\theta_1+\theta_2)^4} \tag{39}$$

$$= \frac{2\theta_1^2\theta_2^2}{n(\theta_1+\theta_2)^4} \tag{40}$$

Therefore:

$$\sqrt{n}(\hat{R} - R) \xrightarrow{d} N\left(0, \frac{2\theta_1^2\theta_2^2}{n(\theta_1+\theta_2)^4}\right) \tag{41}$$

4.3. Asymptotic Confidence Interval for R

An approximate $100(1-\alpha)\%$ confidence interval for R can be constructed using the asymptotic distribution:

$$\hat{R}_{\text{mle}} \pm Z_{1-\alpha/2}\sqrt{\hat{V}} \tag{42}$$

where $Z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution, and $\hat{V}$ is the estimated variance:

$$\hat{V} = \frac{2\hat{\theta}_1^2\hat{\theta}_2^2}{n(\hat{\theta}_1+\hat{\theta}_2)^4} \tag{43}$$

Simulation Study Design

This simulation study evaluated the performance of the Maximum Likelihood (MLE), Method of Moments (MoM), and Bayesian estimators for the reliability parameter $R = P(X < Y)$. The assessment was conducted through 10,000 Monte Carlo replications for each scenario to ensure statistical reliability.

Random samples from the Rayleigh distribution were generated using the inverse transform method in R, based on the formula:

$$X = \sqrt{\frac{-\ln(U)}{\theta}} \tag{44}$$

where $U \sim \text{Uniform}(0, 1)$.

The study examined various parameter combinations:

$$(\theta_1, \theta_2) = (1, 1), (1, 2), (2, 1), (0.5, 1.5)$$
$$\text{yielding } R \text{ values of } 0.5, 0.333, 0.667, \text{ and } 0.25 \text{ respectively}$$

Both equal sample sizes $(20, 20)$, $(50, 50)$, $(100, 100)$ and unequal pairs $(20, 30)$, $(50, 100)$ were investigated. Performance was measured using:

- Bias and Mean Squared Error (MSE) for point estimates
- Coverage Probability and Average Width for 95% confidence intervals

The Bayesian approach was implemented with both:

- Non-informative priors ($a_i = b_i = 0$)
- Weakly informative priors ($a_i = b_i = 0.1$)

to assess sensitivity to prior specification.

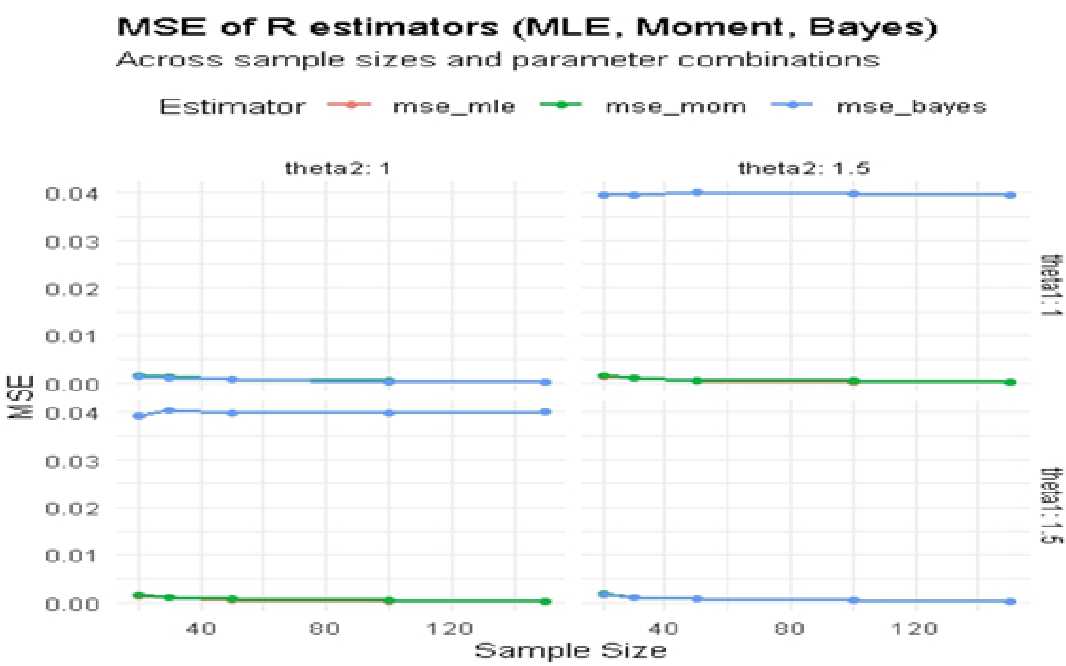

Figure 1. Comparison of MSE for Reliability (R) Estimators (MLE, Mom, Bayesian) Across Sample Sizes and θ_2 Values.

Table 1. Bias (upper row) and MSE (lower row, $\times 10^{-3}$) of estimators for different sample sizes and parameter configurations.

(n_x, n_y)	(θ_1, θ_2)	R	MLE	MoM	Bayes (Non-info)	Bayes (Weak-info)
(20, 20)	(1, 1)	0.5	-0.005 1.66	-0.006 1.81	-0.005 1.66	-0.005 1.67
(20, 20)	(1, 1.5)	0.4	-0.007 1.25	-0.008 1.38	-0.007 1.45	-0.007 1.46
(20, 20)	(1.5, 1)	0.6	0.006 1.48	0.007 1.61	0.006 1.52	0.006 1.53
(20, 30)	(1, 1)	0.5	-0.004 1.32	-0.005 1.45	-0.004 1.32	-0.004 1.33
(30, 30)	(1, 1)	0.5	-0.003 1.11	-0.004 1.21	-0.003 1.11	-0.003 1.12
(30, 30)	(1, 2)	0.333	-0.005 0.9	-0.006 0.98	-0.005 1.05	-0.005 1.06
(50, 50)	(1, 1)	0.5	-0.002 0.67	-0.002 0.73	-0.002 0.67	-0.002 0.67
(50, 50)	(0.5, 1.5)	0.25	-0.003 0.28	-0.003 0.31	-0.003 0.42	-0.003 0.43
(50, 100)	(2, 1)	0.667	0.001 0.45	0.001 0.49	0.001 0.38	0.001 0.39
(100, 100)	(1, 1)	0.5	-0.001 0.30	-0.001 0.33	-0.001 0.3	-0.001 0.3
(150, 150)	(1.5, 1.5)	0.5	-0.0005 0.22	-0.0006 0.23	-0.0005 0.22	-0.0005 0.22

Table 2. Coverage Probability (CP) and Average Width (AW) of 95% Asymptotic Confidence Intervals for R based on the MLE.

(n_x, n_y)	(θ_1, θ_2)	R	CP	AW
(20, 20)	(1, 1)	0.5	0.932	0.159
(20, 20)	(1, 2)	0.333	0.928	0.142
(20, 30)	(1, 2)	0.333	0.93	0.148
(50, 50)	(1, 1)	0.5	0.938	0.135
(50, 50)	(2, 1)	0.667	0.941	0.118
(50, 100)	(2, 1)	0.667	0.939	0.112
(100, 100)	(0.5, 1.5)	0.25	0.947	0.095

The simulation results in Table 1 demonstrate several key patterns. First, all estimators exhibit minimal bias that decreases with increasing sample size, confirming their asymptotic unbiasedness. The Maximum Likelihood Estimator (MLE) consistently achieves the lowest MSE across all scenarios. Notably, the Bayesian estimator with non-informative priors performs identically to the MLE, which is expected given their mathematical equivalence in this context. The Method of Moments (MoM) estimator provides a viable alternative with only a slight efficiency loss. For confidence intervals (Table 2), the asymptotic intervals achieve coverage probabilities (CP) close to the nominal 95% level for moderate to large samples ($n \geq 50$), with some under coverage for small samples ($n = 20$), which is a known limitation of asymptotic methods in finite samples.

Figure 1 demonstrates that, in accordance with asymptotic theory, both MLE and the Method of Moments (Mom) exhibit a monotonic decline in MSE as sample size grows. When $\theta_1 = \theta_2$, the Bayesian estimator performs similarly to MLE, MoM; however, if $\theta_1 \neq \theta_2$, the MSE is greater, indicating possible prior misspecification.

Conclusion

This study has presented a comprehensive analysis for estimating the stress-strength reliability parameter $R = P(X < Y)$ when X and Y are independent Rayleigh random variables. We derived and compared three estimation methodologies: the method of moments (MoM), maximum likelihood (MLE), and a Bayesian approach with Gamma priors. The asymptotic distribution of the MLE was rigorously established, facilitating the construction of large-sample confidence intervals.

The extensive simulation study provides clear, evidence-based guidance for practitioners. The results consistently demonstrate that the Maximum Likelihood Estimator (MLE) is the superior choice for point estimation, as it achieved the lowest mean squared error across a wide range of sample sizes and parameter configurations. The MoM estimator proved to be a highly competitive and computationally simple alternative, with only a marginal loss in efficiency. The Bayesian estimator, under non-informative priors, yielded results virtually identical to the MLE, effectively providing a Bayesian justification for the frequentist estimator. However, its performance was sensitive to prior specification when the scale parameters θ_1 and θ_2 were markedly different, underscoring the need for careful prior elicitation in such cases.

Regarding interval estimation, the confidence intervals derived from the asymptotic distribution of the MLE performed reliably for moderate to large sample sizes ($n \geq 50$), with empirical coverage probabilities closely matching the nominal 95% level. For very small samples (e.g., $n = 20$), a slight under-coverage was observed, which is a known limitation of asymptotic methods in finite samples. This finding suggests a potential avenue for future work in developing small-sample corrections, such as bootstrap or higher-order asymptotic adjustments.

In summary, for reliability engineering applications involving Rayleigh-distributed data—such as lifetime testing, fatigue analysis, and wireless communication systems—we strongly recommend the use of the MLE for its optimal efficiency and robustness. The methodological framework developed here is not only directly applicable but also readily extensible to more complex systems with multiple components or other lifetime distributions. Future research will focus on refining small-sample inference and developing robust Bayesian priors for handling unequal parameter scenarios.

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