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Dynamic Modeling and Multi-Objective Optimization of Takaful Insurance System

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Abstract This paper proposes a novel dynamical systems approach to model and optimize Takaful insurance operations, contributing to the growing body of research in Islamic finance. To the best of our knowledge, this is the first study to formalize the interactions between the three core components of Takaful—participants, claims, and the mutual fund—within a continuous-time dynamical framework. The model integrates key operational parameters such as enrollment and attrition rates, claim frequency, contribution levels, and profit-and-loss sharing mechanisms. We first establish the mathematical well-posedness of the system, proving existence, uniqueness, positivity, and boundedness of solutions, followed by a stability analysis of equilibrium points supported by numerical simulations. Building on this foundation, we formulate a multi-objective optimization problem to address the strategic goals of Takaful operators: maximizing participant retention, minimizing claim incidence, and ensuring fund stability. The problem consists in determining the optimal values for the attrition rate, claim occurrence rate, and average contribution, subject to realistic operational constraints. We solve this problem using an integrated NSGA-II and entropy weighting approach, enabling robust trade-off analysis between conflicting objectives. The proposed methodology offers practitioners a quantitative decision-support tool for enhancing membership strategies and risk management while maintaining financial sustainability in accordance with Sharia-compliant principles.

Keywords Takaful Insurance, Dynamical System, Stability analysis, Multi-objective optimization, Genetic Algorithm, NSGA-II, Pareto frontier, Entropy weight method (EWM).

AMS 2010 subject classifications 91G80, 37N40, 34D20, 90C29, 58E17, 68T20

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1. Introduction

Islamic finance, based on Sharia principles, represents an ethical and sustainable alternative to the traditional financial system. It is based on fundamental values such as the prohibition of riba (interest), gharar (excessive uncertainty) and haram (illicit) activities, while emphasizing justice, transparency and social responsibility. This is reflected in the use of specific financial instruments, such as mourabaha (margin selling), moudaraba (profit sharing) and ijara (Islamic leasing), which encourage the financing of the real economy and the equitable sharing of risks. Since its emergence in the 1970s, Islamic finance has experienced spectacular growth. According to last Islamic Financial Services Board (IFSB) report , the global Islamic financial services industry (IFSI) recorded strong growth in 2024, with total assets reaching 3.88 trillion, representing a 14.9% year-on-year increase compared to 2023. The Gulf Cooperation Council (GCC) region remains the largest hub, accounting for 53.1% of global IFSI assets. The East Asia and Pacific (EAP) region follows with 21.9%, led by Malaysia and Indonesia. The Middle East and North Africa (MENA, excluding GCC) contributes 16.9% [1].

Among the numerous variety of Islamic financial products, Takaful insurance stands out as a Sharia-compliant risk-sharing mechanism, primarily based on mutual useful resource among individuals. Each member makes

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voluntary contributions—dealt with as charitable donations (tabarru)—into to a commonplace fund (takaful fund) whose purpose is to compensate contributors dealing with claims. The entity accountable for administering Takaful, whether or not performing as agent (wakil) or capital manager (moudarib), does not own the fund, but handiest manages it. Claims are paid without delay from the mutual fund, emphasizing the device's non-earnings and cooperative nature. While Takaful and traditional insurance do possess a common purpose—risk mitigation and financial protection—their structural, legal, and ethical foundations are fundamentally different. Conventional coverage is predicated on danger-switch contracts with profit motives, while Takaful embodies principles of cohesion (ta'awun), transparency, and religious compliance (halal). Here, the legitimacy of the process is as vital as its monetary efficiency, reflecting a unique equilibrium between ethical finance and operational viability [2].

Confronted by the rapid growth and increasing complexity of Takaful insurance, several quantitative techniques have been implemented to model economic and financial processes, while ensuring compliance with Shariah principles. Unlike conventional insurance, Takaful relies on specific mechanisms, such as solidarity between participants, the exclusion of riba, and the redistribution of surpluses, which implies adapted models for pricing and risk management.

Several studies illustrate the dynamic of Takaful. Kouach et al. [4] have developed a Retakaful model for the Moroccan market, combining the excess of loss model with machine learning algorithms. Another contribution by El Attar and El Hachloufi [5] proposes a contribution optimization method for Murabaha contracts. In motor insurance, Kouach et al. [6] demonstrate the effectiveness of machine learning-based approaches against traditional methods. From a spatial point of view, the SGLMM model, used by El Msiyah and El Hachloufi [7], then validated on a large scale by Hajipour et al.[8], enables geographic dependencies to be better taken into account.

Furthermore, Saputra et al. [9] compare pricing practices between conventional and Takaful life insurance, highlighting the importance of religious constraints. Billah [10] stresses the actuary's key role in ensuring compliance with Islamic standards in Malaysia. Finally, Zhang and Walton [11] propose a hybrid approach, combining generalized linear models and Gaussian processes, adapted to the requirements of Takaful.

In the context of investment-linked insurance products, Saputra et al. [12] develop a dynamic model for valuing premiums on fund-backed family policies, based on an adjusted version of the Black-Scholes model that respects Sharia principles. Meanwhile, Bensed and Fasly [13] provide an overview of the Takaful market in Morocco, highlighting the regulatory and structural bottlenecks that are holding back its growth.

Regarding the coverage of extreme risks, Essadik and Achchab [14] point out that the Pareto distribution offers a better representation of rare claims in a Takaful framework. On the contractual front, Ali [15] traces the evolution of the main forms of management (Mudharabah, Wakalah, or hybrids) and argues for the adoption of the Musharakah Ta'awuniyah model, deemed more balanced. Hassan [3] pursues this line of thought, highlighting the doctrinal diversity of Takaful models and the challenge of international harmonization. The question of retrocession is also explored by Mansor and Noordin [16], who show its strategic use by Retakaful operators, particularly within integrated groups. For their part, Hassan et al. [17] recall the theoretical foundations of the Takaful system - notably tabarru', solidarity, as well as the prohibition of gharar and maysir - and propose a classification of the most widespread operational models. Ultimately, Kamal et al. [18] identify several levers influencing operators' financial performance, including company size, solvency, investment returns and claims experience.

From a theoretical perspective, Kwon [19] reinterprets the Islamic foundations of insurance, highlighting the discrepancies between principles and their application. From a historical perspective, Salman et al. [20] trace the evolution of modern Takaful and its main products, while highlighting persistent regulatory challenges. Empirically, Lahoucine [21] demonstrates, through an ARDL model, the positive long-term effect of family Takaful assets on Malaysian manufacturing output. Furthermore, Rahman and Aziz [25] propose to rethink the basis of Takaful by replacing tabarru' with the concept of ta'awun, deemed more consistent with the values of equity and cooperation. In short, these contributions illustrate a cross-disciplinary effort between theology, economics and actuarial science, in favor of a Takaful model that is better adapted to contemporary contexts and more faithful to the ethical principles of Islamic finance.

Dynamic models, drawing upon complex systems theory and population dynamics [22, 23, 24], constitute a powerful analytical framework for modeling the temporal evolution of financial assets, portfolio dynamics, and fund flows within uncertain economic environments. These mathematical approaches offer remarkable capabilities for simulating nonlinear growth trajectories, optimizing investment strategies across different time horizons, and characterizing complex financial interactions among system stakeholders. Within this context, and to address a significant gap in the literature concerning dynamic model applications in Islamic finance, we propose an innovative approach specifically dedicated to Takaful fund management. More precisely, we develop a comprehensive mathematical model that captures the dynamic interactions between the three fundamental components of the Takaful system: participants, claims, and the mutual fund. This research has two primary objectives: (1) to establish a modeling framework enabling precise quantitative understanding of the Takaful system's temporal evolution, and (2) to significantly enhance Takaful operators' operational performance by providing a quantitative decision-support tool. The latter is achieved through the formulation and resolution of a multi-objective optimization problem that reconciles conflicting priorities: maximizing participant numbers, minimizing claim impacts, and ensuring the Takaful fund's financial stability.

Although continuous-time models are used for insurance and pension funds, to our knowledge, no study has examined the Takaful system using a cooperative structure, a tripartite dynamic involving participants, claims, and funds, and multi-objective optimization. The specific characteristics of Takaful in terms of governance, solidarity, and religious compliance require mathematical formulations that differ from those used in traditional actuarial models.

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation of the dynamic model and establishes its fundamental properties, including the existence, uniqueness, positivity, and boundedness of solutions. Section 3 focuses on the system's stability analysis, featuring theoretical results on local equilibrium stability complemented by numerical simulations. Section 4 addresses the multi-objective optimization problem, providing an overview of genetic algorithms before detailing the problem formulation and analyzing the obtained Pareto front results. Finally, Section 5 concludes the study by summarizing key findings and outlining future research directions.

2. Mathematical model

2.1. Model formulation

In this section, we present a new nonlinear dynamic model describing the evolution of a Takaful insurance system. The model incorporates three coupled variables:

- $P(t)$: Number of participants at time t.
- $C(t)$: Number of claims at time t.
- $F(t)$: Collective fund at time t.

The proposed mathematical model is governed by the following ordinary differential equations:

$$\frac{dP}{dt} = \alpha P\left(1 - \frac{P}{K}\right) - \beta P. \tag{1}$$

$$\frac{dC}{dt} = \gamma P - \delta C. \tag{2}$$

$$\frac{dF}{dt} = \eta P - \theta CF - \mu_1 F - \mu_2 F. \tag{3}$$

The initial conditions for the model are:

$$P(0) = P_0, \quad C(0) = C_0, \quad F(0) = F_0. \tag{4}$$

A detailed description of the variables' dynamics follows.

In Equation. (1), the participant dynamics follow a modified logistic growth. The term $\alpha P(1 - P/K)$ represents recruitment with saturation effects, where α is the natural recruitment rate and K is the system's maximum capacity. The term $-\beta P$ accounts for participant attrition at a constant rate β.

In Equation. (2), the claims dynamics comprise two principal components: (i) the claim generation term γP, where γ denotes the claim occurrence rate, and (ii) the claim resolution term $-\delta C$ with resolution rate δ.

The Takaful fund F, governed by Equation. (3), exhibits dynamic growth through participant contributions at an average premium η. The fund depletion is modeled through three distinct channels: (i) claim compensations via the nonlinear term θCF with compensation rate θ, (ii) operational costs $\mu_1 F$ (fund management fees at rate μ_1), and (iii) surplus distribution $\mu_2 F$ (participant profit-sharing at rate μ_2).

The bilinear interaction term θCF introduces essential nonlinear coupling between risk exposure (C) and capital pool (F), establishing the fundamental risk-capital feedback mechanism characteristic of Takaful systems. This nonlinearity captures the critical interdependence between claim liabilities and financial sustainability in Islamic insurance models.

Our model formally operationalizes the core principles of Takaful through its mathematical structure. The tabarru' (donation) principle is captured by the contribution rate parameter η, representing participants' voluntary, irrevocable payments to the mutual fund. Risk mutualization emerges through the nonlinear term θCF, which intrinsically links compensation amounts to both claim volume (C) and available capital (F), thereby implementing conditional solidarity while preventing gharar through transparent, capacity-based payouts. The mudarabah principle appears in the $\mu_1 F$ term, quantifying the fund manager's (mudarib) compensation as a fixed proportion of assets under management, while $\mu_2 F$ operationalizes profit-sharing by distributing surplus to participants. The model's complete dependence on observable state variables (P, C, F) ensures Shariah-compliant transparency, excluding speculative external parameters.

The parameters of the model are non-negative and are summarized in Table 1.

Table 1. Model parameters and their descriptions.

Parameter	Description
α	Participant growth rate
K	System maximum capacity
β	Attrition rate
γ	Claim occurrence rate
δ	Claim settlement rate
η	Contribution rate per participant
θ	Compensation rate per claim
μ_1	Management fee rate
μ_2	Surplus distribution rate

It is important to note that the model presented in this study is a simplified version designed to analyze the fundamental mechanisms of the Takaful system. Certain operational dynamics, such as investment income, heterogeneity in risk profiles, and the impact of exogenous shocks, have been excluded to preserve the system's analytical traceability. These elements are natural extensions of the proposed framework and will be presented in more detail in future work.

To ensure rigorous analysis and clear results, we made structural assumptions. We assume model parameters remain constant over time, and do not include investment returns, participant heterogeneity, or macroeconomic shocks. These assumptions focus on the fundamental mechanisms of takaful mutualism and set a reference point for future work.

2.2. Model proprieties

Prior to engaging in quantitative analysis, it is essential to validate the model's suitability for long-term simulations by ensuring that its solutions remain within an economically meaningful domain. To this end, this section is devoted to the theoretical study of three fundamental properties: (i) solution positivity, (ii) boundedness, and (iii) existence and uniqueness of solutions. The positivity property is formalized in the following theorem:

Theorem 1
Consider the Takaful system described by (1)-(3). The non-negative cone $\mathbb{R}^3_+$ is positively invariant under the dynamics of the system. In other words, for any initial condition $(P_0, C_0, F_0) \in \mathbb{R}^3_+$, the corresponding solution $(P(t), C(t), F(t))$ remain in $\mathbb{R}^3_+$ for all $t \geq 0$.

Proof
Considering the bounding planes of the non-negative cone $\mathbb{R}^3_+$, we have:

$$\left.\frac{dP}{dt}\right|_{P=0} = 0,$$
$$\left.\frac{dC}{dt}\right|_{C=0} = \gamma P \geq 0,$$
$$\left.\frac{dF}{dt}\right|_{F=0} = \eta P \geq 0.$$

The vector field is: Tangent to the $P = 0$ plane and directed inward on $C = 0$ and $F = 0$ planes when $P > 0$. So, the non-negative cone $\mathbb{R}^3_+$ is positively invariant. We therefore conclude that for any initial condition $(P_0, C_0, F_0) \in \mathbb{R}^3_+$, the solution $(P(t), C(t), F(t))$ satisfies:

$$P(t) \geq 0, \; C(t) \geq 0, \; F(t) \geq 0 \quad \text{for all} \quad t \geq 0.$$

□

Theorem 2
The solutions of system (1)-(3) are bounded.

Proof

- Boundedness of $P(t)$.

  For the Equation. (1), the non-negativity conditions $\beta \geq 0$ and $P(t) \geq 0$ (established by the positivity theorem (1)) yield the differential inequality:

  $$\dot{P} \leq \alpha P \left(1 - \frac{P}{K}\right).$$

  Consider the reference logistic equation with identical initial condition:

  $$\dot{w} = \alpha w \left(1 - \frac{w}{K}\right), \quad w(0) = P(0) = w_0,$$

  whose exact solution takes the form:

  $$w(t) = \frac{K}{1 + \left(\frac{K}{w_0} - 1\right) e^{-\alpha t}}.$$

  This logistic solution satisfies:
  $$w(t) \leq \max(P(0), K) = B.$$
  Applying the comparison principle for ordinary differential equations to the inequality $\dot{P} \leq \dot{w}$, we obtain:

  $$P(t) \leq w(t) \quad \forall t \geq 0.$$

  The boundedness result $P(t) \leq B$ for all $t \geq 0$ follows immediately from this inequality and the properties of $w(t)$.

- Boundedness of $C(t)$.
  From Equation. (2), we have:
$$\dot{C} + \delta C = \gamma P.$$
  The solution is given by:
$$C(t) = C(0)e^{-\delta t} + \int_0^t \gamma P(s) e^{-\delta(t-s)}\, ds.$$
  Since $0 \leq P(s) \leq B$ for all $s \geq 0$, we obtain:
$$\begin{aligned} C(t) &= C(0)e^{-\delta t} + \int_0^t \gamma P(s) e^{-\delta(t-s)}\, ds \\ &\leq C(0)e^{-\delta t} + \gamma B \int_0^t e^{-\delta(t-s)}\, ds \\ &\leq C(0)e^{-\delta t} + \frac{\gamma B(1 - e^{-\delta t})}{\delta}. \end{aligned}$$
  Further, as $\delta \geq 0$ and $e^{-\delta t} \in (0, 1]$ for all $t \geq 0$, it follows that:
$$C(t) \leq C(0) + \frac{\gamma B}{\delta}, \text{ for all } t \geq 0.$$
- Boundedness of $F(t)$.
  From Equation (3), under the non-negativity conditions $\theta \geq 0$ and $F(t) \geq 0$, we derive the differential inequality:
$$\dot{F} = \eta P - \theta C F - (\mu_1 + \mu_2)F \leq \eta P - (\mu_1 + \mu_2)F.$$
  Therefore,
$$F(t) \leq F(0)e^{-(\mu_1+\mu_2)t} + \int_0^t \eta P(s) e^{-(\mu_1+\mu_2)(t-s)}\, ds.$$
  Since $0 \leq P(s) \leq B$ for all $s \geq 0$, we obtain:
$$F(t) \leq F(0)e^{-(\mu_1+\mu_2)t} + \frac{\eta B(1 - e^{-(\mu_1+\mu_2)t})}{\mu_1 + \mu_2}.$$
  From which we deduce:
$$F(t) \leq F(0) + \frac{\eta B}{\mu_1 + \mu_2}, \text{ for all } t \geq 0.$$

□

Theorem 3
The Cauchy problem associated with the system of Equations (1)–(3) and the initial conditions given in (4) admits a unique solution.

Proof
Let $X(t) = (P(t), C(t), F(t))$. The Cauchy problem (1)-(4) is expressed in vector form as follows:
$$\dot{X}(t) = f(X(t)) \text{ and } X_0 = (P_0, C_0, F_0),$$
where
$$f(X) = \begin{bmatrix} \alpha P \left(1 - \dfrac{P}{K}\right) - \beta P \\ \gamma P - \delta C \\ \eta P - \theta C F - (\mu_1 + \mu_2)F \end{bmatrix} = \begin{bmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \end{bmatrix}.$$

The functions f_1, f_2, and f_3 are continuous on $\mathbb{R}^3$. To apply the Cauchy–Lipschitz theorem, it remains to show that f is locally Lipschitz continuous with respect to X.

Consider two vectors $X_1 = (P_1, C_1, F_1)$ and $X_2 = (P_2, C_2, F_2)$. Then we have:

$$f(X_1) - f(X_2) = \begin{bmatrix} (\alpha-\beta)(P_1-P_2) - \dfrac{\alpha}{K}\left(P_1^2 - P_2^2\right) \\ \gamma(P_1-P_2) - \delta(C_1-C_2) \\ \eta(P_1-P_2) - \theta(C_1F_1 - C_2F_2) - (\mu_1+\mu_2)(F_1-F_2) \end{bmatrix}.$$

Using $\|\cdot\|_1$ norm on $\mathbb{R}^3$ and the boundedness of solutions, we obtain:

$$\begin{aligned}
\|f(X_1) - f(X_2)\|_1 = & \left|(\alpha-\beta)(P_1-P_2) - \frac{\alpha}{K}\left(P_1^2 - P_2^2\right)\right| \\
& + |\gamma(P_1-P_2) - \delta(C_1-C_2)| \\
& + |\eta(P_1-P_2) - \theta(C_1F_1 - C_2F_2) - (\mu_1+\mu_2)(F_1-F_2)| \\
\leq & |(\alpha-\beta)(P_1-P_2)| + \left|\frac{\alpha}{K}(P_1-P_2)(P_1+P_2)\right| \\
& + |\gamma(P_1-P_2)| + |\delta(C_1-C_2)| + |\eta(P_1-P_2)| \\
& + |\theta(C_1-C_2)| + |(\mu_1+\mu_2)(F_1-F_2)| \\
\leq & (\alpha+\beta+2\frac{\alpha B}{K}+\gamma+\eta)|P_1-P_2| \\
& + \delta|C_1-C_2| + \theta|C_1(F_1-F_2) + (C_1-C_2)F_2| + (\mu_1+\mu_2)|F_1-F_2| \\
\leq & (\alpha+\beta+2\frac{\alpha B}{K}+\gamma+\eta)|P_1-P_2| + \delta|C_1-C_2| + \theta\times\left(C(0)+\frac{\gamma B}{\delta}\right)|F_1-F_2| \\
& + \theta\times\left(F(0)+\frac{\eta B}{\mu_1+\mu_2}\right)|C_1-C_2| + (\mu_1+\mu_2)|F_1-F_2| \\
\leq & (\alpha+\beta+2\frac{\alpha B}{K}+\gamma+\eta)|P_1-P_2| + \left(\delta+\theta\left(F(0)+\frac{\eta B}{\mu_1+\mu_2}\right)\right)|C_1-C_2| \\
& + \left(\mu_1+\mu_2+\theta\left(C(0)+\frac{\gamma B}{\delta}\right)\right)|F_1-F_2| \\
\leq & M\|X_1-X_2\|,
\end{aligned}$$

where

$$M = \max\left(\alpha+\beta+2\frac{\alpha B}{K}+\gamma+\eta,\ \delta+\theta\left(F(0)+\frac{\eta B}{\mu_1+\mu_2}\right),\ \mu_1+\mu_2+\theta\left(C(0)+\frac{\gamma B}{\delta}\right)\right).$$

Thus, the Cauchy–Lipschitz theorem guarantees the existence and uniqueness of a solution to our problem. □

3. Stability analysis

To better understand the dynamics of the proposed model, we examine its behavior around the steady states. This analysis is essential for identifying the necessary conditions for equilibrium stability. By setting Equations (1)-(3) equal to zero, it can be shown that the system admits two equilibrium points:

- A trivial equilibrium:

$$E_0 = (0,\ 0,\ 0).$$

- A non-trivial equilibrium:

$$E_1 = (P^*,\ C^*,\ F^*) = \left(\frac{K(\alpha-\beta)}{\alpha}, \frac{\gamma K(\alpha-\beta)}{\delta\alpha}, \frac{\delta\eta K(\alpha-\beta)}{\theta\gamma K(\alpha-\beta)+\delta\alpha(\mu_1+\mu_2)}\right).$$

3.1. Local stability resulats

Theorem 4
The trivial equilibrium point E_0 is locally asymptotically stable if $\alpha < \beta$ and unstable if $\alpha > \beta$.

Proof
The Jacobian matrix at the point (P, C, F) associated with system (1)-(3) is given by:

$$J(P,C,F) = \begin{pmatrix} \alpha\left(1 - \dfrac{2P}{K}\right) - \beta & 0 & 0 \\ \gamma & -\delta & 0 \\ \eta & -\theta F & -\theta C - \mu_1 - \mu_2 \end{pmatrix}.$$

Evaluating the matrix J at the point E_0, we obtain:

$$J(E_0) = \begin{pmatrix} \alpha - \beta & 0 & 0 \\ \gamma & -\delta & 0 \\ \eta & 0 & -\mu_1 - \mu_2 \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = \alpha - \beta, \ \lambda_2 = -\delta, \text{ and } \lambda_3 = -\mu_1 - \mu_2.$$

Since λ_2 and λ_3 are always negative for positive parameter values, the local asymptotic stability of E_0 depends on the sign of λ_1. Specifically, E_0 is locally asymptotically stable if $\alpha < \beta$ and unstable if $\alpha > \beta$. □

Theorem 5
The non-trivial equilibrium point E_1 is locally asymptotically stable when $\alpha > \beta$.

Proof
For the non-trivial equilibrium point E_1, the Jacobien matrix is given by:

$$J(E_1) = \begin{pmatrix} \beta - \alpha & 0 & 0 \\ \gamma & -\delta & 0 \\ \eta & -\dfrac{\theta\eta\delta K(\alpha-\beta)}{\theta\gamma K(\alpha-\beta) + \alpha\delta(\mu_1+\mu_2)} & -\dfrac{\theta\gamma K(\alpha-\beta)}{\delta\alpha} - \mu_1 - \mu_2 \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = \beta - \alpha, \ \lambda_2 = -\delta, \text{ and } \lambda_3 = -\frac{\theta\gamma K(\alpha-\beta)}{\delta\alpha} - \mu_1 - \mu_2.$$

The eigenvalue λ_2 is negative since δ is positive. Consequently, the local stability of the equilibrium point E_1 depends on the signs of λ_1 and λ_3. These eigenvalues are both negative when $\alpha > \beta$, which implies that E_1 is locally asymptotically stable in this case. □

3.2. Numerical simulation

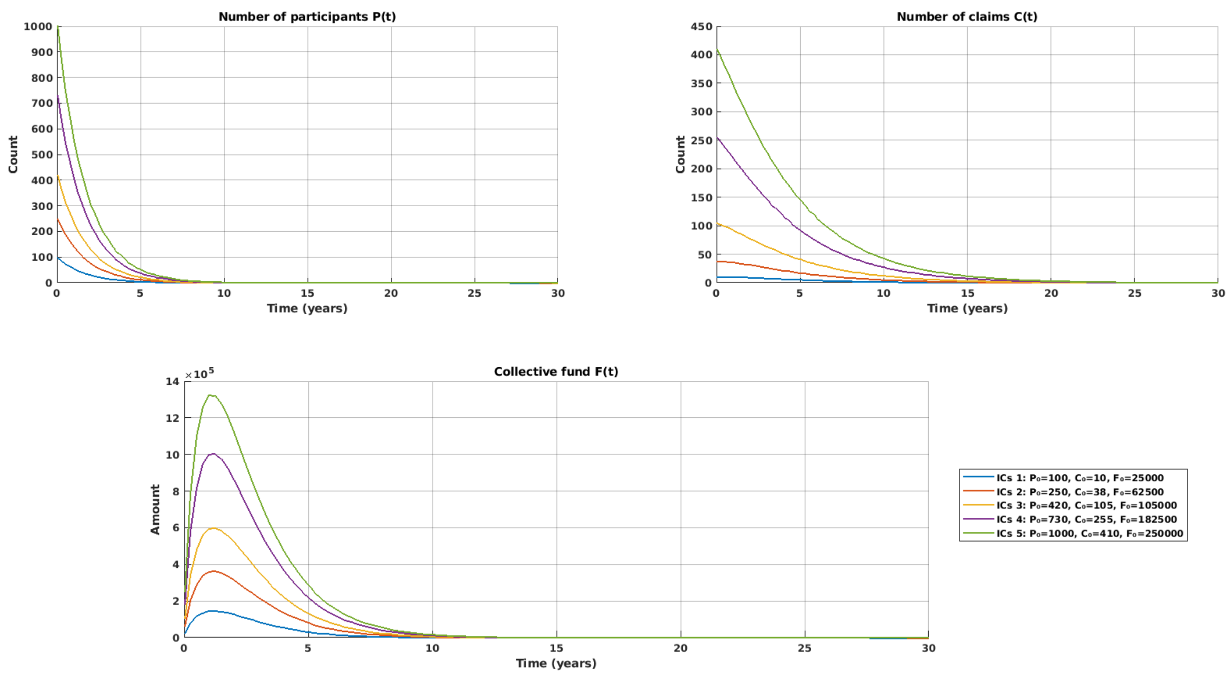
We simulate the dynamic evolution of the three variables in our Takaful model (Equations (1)-(3)) over a 30-year horizon in two cases: Case (1), where initial conditions are set near the trivial equilibrium E_0, with $\alpha < \beta$; and Case (2), where the initial values are chosen near the non-trivial equilibrium E_1, with $\alpha > \beta$. The remaining model parameters are provided in Table 2. Simulations are conducted using MATLAB's `ode45` solver.

Figure 1 illustrates the local dynamics around the trivial equilibrium $E_0 = (0, 0, 0)$, which corresponds to a complete collapse of the system—no participants, no claims, and no capital in the collective fund. Assuming $\alpha < \beta$, multiple sets of initial conditions (P_0, C_0, F_0) are tested, as shown in the legend. The results reveal a

Table 2. Model parameters values.

Parameter	Description	Value
K	System maximum capacity	6×10^6
γ	Claim occurrence rate	0.039
δ	Claim settlement rate	0.25
η	Contribution rate per participant	3,000 MAD
θ	Compensation rate per claim	5.263×10^{-4}
μ_1	Management fee rate	0.2
μ_2	Surplus distribution rate	0.8

Figure 1. Stabilty of the trivial equilibrium E_0, with $\alpha = 0.083$ and $\beta = 0.672$.



systematic convergence of all trajectories toward the zero state, regardless of the initial values: (1) The number of participants $P(t)$ declines rapidly to zero, with a sharper decrease observed for higher P_0, though all trajectories vanish within approximately 15 years. (2) The number of claims $C(t)$ follows a similar pattern, with a slightly slower, yet unavoidable decline. (3) The collective fund $F(t)$, while sometimes exhibiting a transient growth phase (notably in scenarios with higher initial capital), also eventually converges to zero. This initial rise can be attributed to system inertia before demographic decline and diminishing contributions dominate.

From an actuarial and financial perspective, this simulation highlights a structural risk of extinction for a fund that is either undercapitalized or initiated with an insufficient participant base. A low initial number of participants or a recruitment rate that fails to exceed the attrition rate cannot generate the necessary cash flows to maintain the system. Participant dropouts and reduced benefit flows inevitably lead to the system's collapse.

Figure 2. Stabilty of the non-trivial equilibrium E_1, with $\alpha = 0.771$ and $\beta = 0.076$.

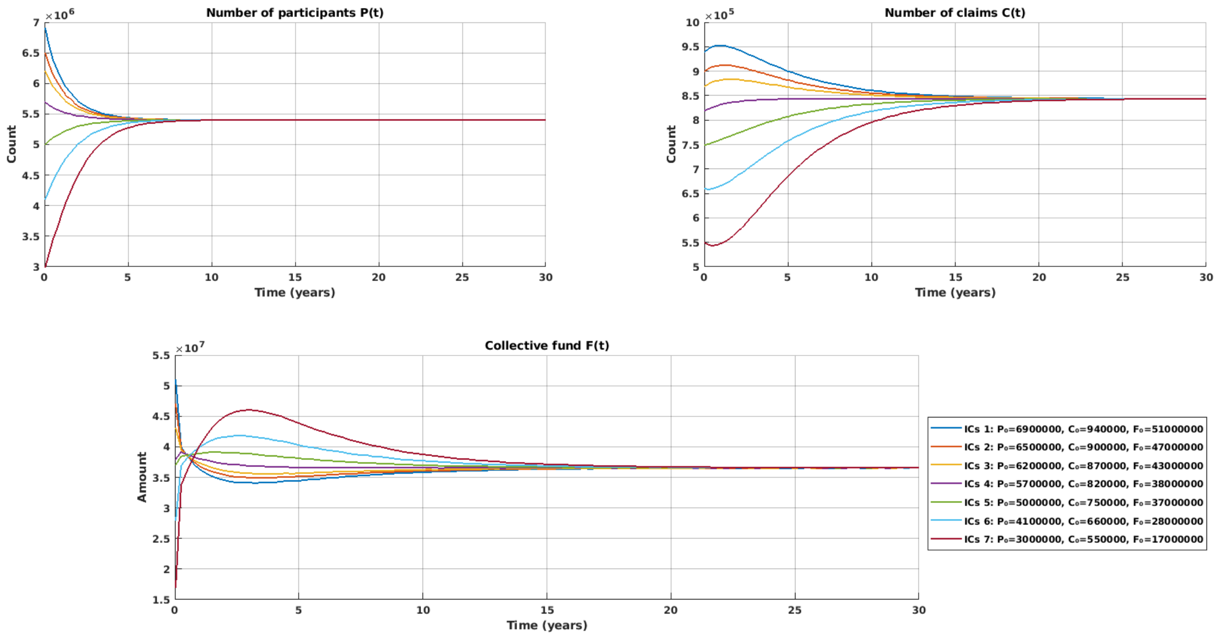


Figure 2 shows the system's evolution when $\alpha > \beta$. Each curve represents a different set of initial conditions, as indicated in the legend. All trajectories systematically converge to stable values associated with the non-trivial equilibrium, regardless of the starting point. Specifically, the number of participants converges to around 5.4 million, claims to approximately 845,000, and the collective fund to nearly 36.6 million monetary units. This behavior suggests the existence of an attractive long-term equilibrium.

Actuarially and financially, such behavior demonstrates the system's capacity to absorb initial shocks, whether in participant demographics, the volume of claims, or the initial capital. This robustness implies the presence of built-in adjustment mechanisms, possibly reflecting endogenous contribution regulation or automatic compensation between inflows and liabilities.

The collective fund trajectory $F(t)$ deserves special attention. Under certain initial conditions—particularly those with low starting capital—the fund exhibits a transient phase of growth followed by convergence. Other scenarios begin with a decline that gradually stabilizes. This diversity in transient behavior reveals short-term sensitivity to initial conditions, but also confirms the system's long-term resilience, which is critical for financial sustainability.

In conclusion, these numerical simulations highlight the foundational principles of sustainability in the Takaful system, from both demographic and financial perspectives. They support the earlier theoretical stability results for both the trivial and non-trivial equilibria. Ultimately, they underscore that the long-term viability of the Takaful system hinges on maintaining a recruitment rate that consistently exceeds the rate of disengagement—a condition essential for ensuring demographic stability and financial equilibrium, even in the presence of shocks or temporary imbalances.

The initial parameters and conditions used in the numerical simulations are hypothetical and were selected from realistic and acceptable ranges commonly used in actuarial literature. Due to the lack of publicly available operational data on Takaful systems, these simulations are designed primarily to illustrate the model's dynamic behavior and qualitatively analyze the stability of equilibria, rather than to provide calibrated results for a real operator.

4. Multi-Objective optimisation problem

In this section, we present a practical implementation of our model through a constrained multi-objective optimization problem, solved using genetic algorithms (GA). The choice of GA is justified by several key advantages: they require no gradient computation, inherently handle constraints (enabling efficient exploration of the feasible solution space), and most importantly, promote global search capabilities. This latter property allows them to escape local optima and converge toward near-globally optimal solutions.

Our objective is to determine optimal values for several key parameters in our model, specifically: the contribution rate η, claim rate γ, and attrition rate β. These parameters were selected for their substantial impact on the financial stability and sustainability of the fund. The attrition rate β directly influences participant retention - a crucial factor in Takaful systems where member persistence ensures stable contribution flows. The claim rate γ determines the claim dynamics and significantly affects financial reserves, requiring careful balancing between risk coverage and economic viability. Finally, the contribution rate η plays a pivotal role in both the fund's profitability and its attractiveness to participants, presenting a delicate trade-off between competitive pricing and liability coverage.

4.1. An overview of Genetic Algorithms

Genetic algorithms (GA), introduced by Holland in the 1970s, are metaheuristics that use an iterative and stochastic search strategy to find optimal solutions, mimicking Darwin's "survival of the fittest" evolutionary principle [26]. Inspired by natural evolutionary mechanisms - particularly selection, crossover, and mutation - they work with a population of candidate solutions (individuals or chromosomes), encoded in binary or real-valued form, that iteratively evolve toward optimal solutions. The process involves evaluating individuals through a fitness function, followed by selecting the best specimens for reproduction. Crossover operators (typically with high probability) and mutation operators (with low probability) generate a new population, respectively promoting intensification around promising solutions and diversification to avoid local optima. Selection methods (like roulette wheel) guide this evolution, which stops when predefined criteria (such as a maximum number of iterations) are met [27, 28].

Genetic algorithms differ from classical optimization methods, such as differential or enumerative approaches, through several fundamental characteristics. Unlike these methods that work with a single solution point, GA utilize a population of individuals, enabling parallel and robust exploration of the search space. Moreover, they only require evaluation of the objective function, without auxiliary information (like derivatives), making them domain-independent methods particularly suited to complex or ill-defined problems. Their probabilistic rather than deterministic transition mechanism enhances flexibility. Their population-based nature offers two major advantages: (1) implicit parallelism, allowing efficient implementation on distributed architectures to reduce computation time, and (2) informational synergy through crossover operators, facilitating solution exchange between distinct regions of the search space. These properties give GA enhanced effectiveness for optimizing multivariate and nonlinear functions, often outperforming traditional methods without requiring problem-specific knowledge [29].

In finance, genetic algorithms have seen growing development thanks to their ability to solve complex optimization and forecasting problems in dynamic, nonlinear environments. Specific GA applications for commodity price forecasting have been demonstrated by Drachal and Pawlowski [45]. Furthermore, Huang et al. [30] used genetic algorithms to optimize variational mode decomposition (VMD) parameters, thereby improving short-term financial time series forecasting accuracy. Several studies have also explored hybrid approaches combining GA with other techniques like quadratic programming, which has proven effective for financial portfolio management. This approach optimizes asset allocation according to Markowitz's mean-variance model, particularly in complex scenarios where classical methods show limitations [32]. Finally, Thakkar and Chaudhari [31] systematically analyzed GA applications for stock market prediction, covering work from 2013 to 2022. Their study highlights the evolution and effectiveness of GA-based approaches in finance while examining their competitiveness and complementarity relative to other methods. Genetic algorithms also find extensive applications across various fields including economics, computer science, and engineering (see [33, 34, 35] for a detailed review).

4.2. Problem formulation and resolution

As introduced in this section, we consider a constrained multi-objective optimization problem, based on the dynamics of the system (1)-(3) and parametric constraints. The goal is to identify the optimal parameters β (attrition rate), γ (claim rate), and η (average contribution), which belong to the admissible domain:

$$\mathcal{D} = \left\{ (\beta, \gamma, \eta) \in \mathbb{R}^3 \,\middle|\, 0 \leq \beta \leq \alpha,\ 0 \leq \gamma \leq 1,\ \eta_{\min} \leq \eta \leq \eta_{\max} \right\}, \tag{5}$$

and which achieve the following three objectives:

- **Objective 1 :** Maximize the cumulative number of participants, in order to maintain a large base of loyal members over the time horizon $[0, T]$:

$$J_1(\beta, \gamma, \eta) = \int_0^T P(t)\, dt. \tag{6}$$

- **Objective 2 :** Minimize the cumulative number of claims, thereby reducing the system's expenses:

$$J_2(\beta, \gamma, \eta) = \int_0^T C(t)\, dt. \tag{7}$$

- **Objective 3 :** Maximize the final amount of the fund, ensuring its financial robustness:

$$J_3(\beta, \gamma, \eta) = F(T). \tag{8}$$

The multi-objective optimization problem is then formulated as the search for the optimal vector:

$$(\beta^*, \gamma^*, \eta^*) = \arg \max_{(\beta,\gamma,\eta)\in\mathcal{D}} \left[J_1(\beta, \gamma, \eta),\ -J_2(\beta, \gamma, \eta),\ J_3(\beta, \gamma, \eta) \right], \tag{9}$$

where J_1, J_2, and J_3 are constrained by the system dynamics (1)-(3).

To solve our multi-objective optimization problem, we adopted an approach based on the NSGA-II genetic algorithm (Non-dominated Sorting Genetic Algorithm II) [36, 37, 38]. This methodological choice proves particularly suitable for efficiently approximating the Pareto front, enabling identification of optimal trade-offs between three critical objectives: participant retention, claims management, and fund financial stability. We propose a Matlab implementation (see Algorithm 1) that begins with an initialization phase, where system constants are defined and the feasible domain of decision variables $\mathcal{D}$ is established. Subsequently, genetic process parameters are configured, including population size, maximum number of generations, as well as crossover and mutation rates. An initial population is randomly generated within the feasible solution space.

At each generation, all individuals are evaluated. This evaluation involves simulating the dynamic system (1)-(3) using the `ode45` numerical solver over the time interval $[0, T]$. The resulting trajectories $(P(t), C(t), F(t))$ are extracted, objective functions (6)-(8) are computed, then reformulated as a multi-objective minimization problem $\min[-J_1, J_2, -J_3]$. Population evolution is managed through standard genetic algorithm steps: selection, crossover, mutation, and elitism. These operators generate a new population while preserving the most promising solutions. The Pareto front is updated at each iteration. The process continues until meeting a stopping criterion, which depends either on Pareto front stagnation over a predefined number of generations or on reaching the maximum allowed generations. Upon algorithm completion, the final Pareto front is returned, representing the set of Pareto-optimal solutions. Table 3 presents the genetic algorithm configuration parameters. The simulations are performed using the following fixed parameters:

- Constant values: $\alpha = 0.771$, K, δ, θ, μ_1, and μ_2 (as specified in Table 2).
- Decision variable bounds: $0 \leq \beta \leq \alpha,\ 0 \leq \gamma \leq 1$, and $\eta \in [\eta_{\min}, \eta_{\max}] = [1\,000, 10\,000]$ MAD.
- Time horizon: $T = 20$ years.
- Initial conditions: $P(0) = 100\,000$, $C(0) = 4\,000$, and $F(0) = 1\,000\,000$ MAD.

Algorithm 1 Multi-Objective Optimization using Genetic Algorithm

Parameter Initialization
  Define system constants
  Set decision variable bounds
Genetic Algorithm Configuration
  Set population size, number of generations
  Set crossover and mutation rates
  Generate initial population
while not stagnated **and** generation $<$ MaxGen **do**
  Evaluate Population
  for each individual in population **do**
    Simulate system using `ode45` over $[0, T]$
    Extract state trajectories
    Compute objective functions
    Convert objectives to minimization form
  end for
  Evolve Population
   Perform selection
   Apply crossover
   Apply mutation
   Apply elitism
   Extract updated Pareto front
  if Pareto front stagnant for N generations **or** generation $=$ MaxGen **then**
    break
  end if
end while
Return final Pareto front

Table 3. Genetic algorithm configuration parameters.

Parameter	Value
Population size	50
Maximum generations	100
Termination tolerance (Fitness limit)	1×10^{-6}
Crossover fraction	0.8
Mutation function	Adaptive feasible
Crossover function	Intermediate
Selection function	Tournament
Elitism ratio	50%

Having presented our approach for solving the multi-objective optimization problem and detailed the algorithmic parameters employed, we now examine the obtained Pareto-optimal solutions through their graphical representations (Figures 3–6). This visual analysis enables identification of the essential trade-offs between the different objectives. We subsequently apply a rigorous multicriteria selection methodology to determine the solution offering the optimal balance between the three considered objectives.

Figure 3 illustrates a clear trade-off between the number of participants and the number of claims. As J_1 (Members) increases, J_2 (Claims) tends to rise as well. This suggests that high customer retention—driven by more lenient underwriting policies or less stringent client selection—could weaken claims control and lead to a significant increase in claims.

Figure 3. Pareto frontier (J_1, J_2).

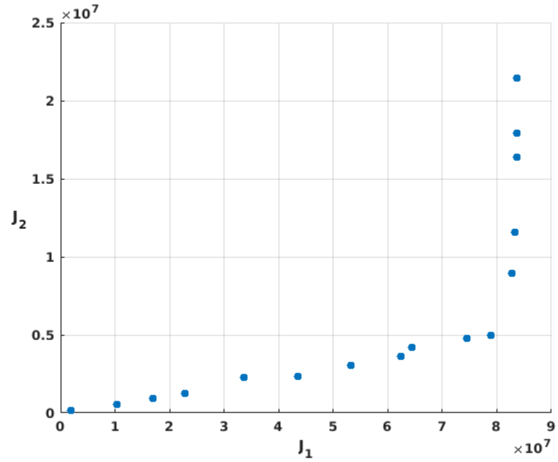


Figure 4. Pareto frontier (J_1, J_3).

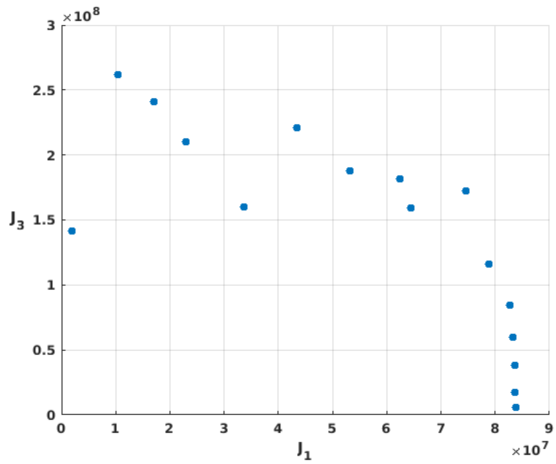


Figure 4 reveals an inverse, nonlinear relationship between the cumulative number of participants and the fund balance: beyond a certain threshold, increasing the number of participants leads to a reduction in the fund. This trend suggests that while customer retention efforts effectively expand the client base, they become financially counterproductive past a certain point. Two primary mechanisms explain this phenomenon: first, each additional membership increases risk exposure (through potential claims), progressively burdening the fund's resources; second, this erosion of reserves indicates a structural imbalance, reflecting either inadequate premium pricing relative to the covered risks, or abnormally high claim frequency or severity.

Figure 5 shows a strong inverse relationship between the number of claims and the fund balance: as claims increase, the fund decreases proportionally. This indicates that the fund is highly sensitive to claim volume, underscoring the need for stringent claims management policies. By minimizing improper payouts and optimizing claims processing, such measures would enhance the financial stability of the Takaful system.

3D visualization (Figure 6) highlights the complex interactions between the three objectives. The data points form a characteristic convex Pareto front surface in the space, confirming the absence of a single optimal solution - each point represents a specific trade-off between objectives. The analysis reveals that no solution can simultaneously optimize all three axes, demonstrating the need for strategic compromises: any action to improve

Figure 5. Pareto frontier (J_2, J_3).

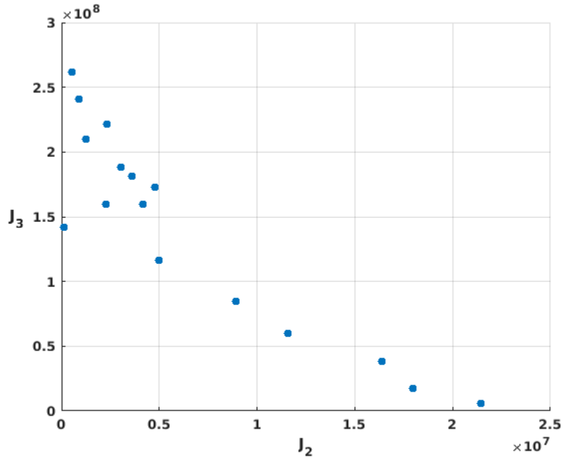
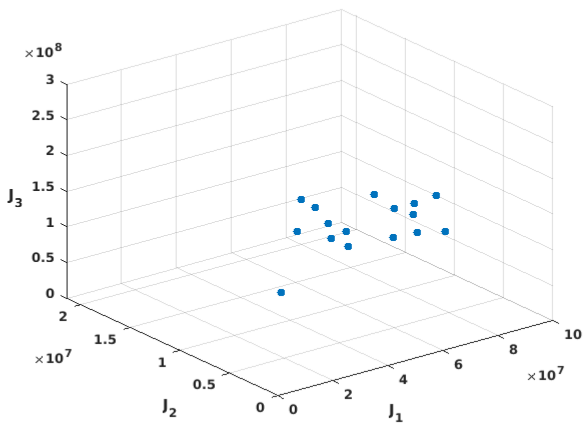


Figure 6. Pareto frontier (J_1, J_2, J_3).



one objective (such as reducing claims through enhanced controls) necessarily impacts other dimensions, either by negatively affecting member retention (as overly strict controls may degrade user experience) or by increasing operational costs. A particularly notable antagonistic relationship emerges, where joint increases in J_1 and J_2 systematically lead to decreases in J_3, confirming the inherent tensions in this multi-criteria optimization.

Although all solutions on the Pareto front are mathematically optimal in the sense of efficiency, their practical relevance varies depending on the operational priorities of the application context. Selecting the final solution therefore requires the integration of additional decision criteria to identify the one that achieves the best trade-off among the key objectives: participant retention, claim control, and financial stability of the fund.

To this end, we employ the entropy weight method (EWM) for assigning criterion weights based on Shannon's entropy theory [39, 40, 41]. This objective approach, widely used in multi-criteria decision analysis (MCDA), quantifies the relative importance of each criterion based on the intrinsic information contained in the data, without relying on subjective judgments. Entropy measures the dispersion of solution performances across each criterion: a criterion with low entropy (i.e., low variability) provides little discriminative information and is assigned a lower weight, whereas a criterion with high entropy (highly discriminative) receives a higher weight [42, 43].

Using the solutions from the Pareto front (Table 4), and by applying the entropy method [42, 44] we obtain the weighting coefficients ω_j $(j = 1, 2, 3)$ associated with the three evaluation criteria J_1, J_2, and J_3 (Table 5). The results show that criterion J_2—to be minimized—has the highest weight (67.8%), highlighting its critical importance in the decision process. In contrast, J_1 and J_3 have comparable weights of 16.9% and 15.3%, respectively, indicating a similar relative importance in the overall assessment.

Table 4. Solutions on the Pareto front.

Solution	β	γ	η	J_1	J_2	J_3
1	0.0304	0.0872	1000.22	83733092	21432842	5653058.21
2	0.0308	0.0730	2575.28	83672890	17945651	17369602.03
3	0.0311	0.0668	5139.38	83628129	16393339	37924190.88
4	0.0336	0.0473	5742.82	83274188	11565567	59796435.71
5	0.0376	0.0367	6306.50	82683189	8916353	84542329.18
6	0.0642	0.0215	5064.70	78838024	4965660	116031557.17
7	0.0940	0.0219	7663.50	74543306	4768634	172597936.51
8	0.1644	0.0225	7218.52	64440155	4191895	159346926.46
9	0.1785	0.0200	7298.14	62427509	3605580	181331869.78
10	0.1993	0.0159	4987.94	41593247	1664099	170404147.26
11	0.2440	0.0200	7494.39	53160920	3035958	188041095.47
12	0.3136	0.0191	8257.04	43445831	2328706	221210906.19
13	0.3857	0.0245	7406.97	33632718	2265241	159641402.51
14	0.4552	0.0180	5780.47	9193169	355486	238435946.50
15	0.4687	0.0202	7634.57	22912273	1249592	209824759.87
16	0.5184	0.0198	8308.02	17040244	909896	240829233.19
17	0.5855	0.0189	8383.41	10335734	538691	261668252.30
18	0.7673	0.0200	7768.89	1844496	135781	141870430.93

Table 5. Criterion weights determined using the entropy method.

Criterion	**Weight** ω_j
J_1	16.9%
J_2	67.8%
J_3	15.3%

Once the weights are determined, we normalize the criteria using the min-max method to ensure comparability. Since J_2 is a cost-type criterion (to be minimized), it is inverted in the global score calculation to maintain directional consistency with the benefit-type criteria. The weighted global score is computed as follows:

$$\text{Score} = \omega_1 \cdot J_{1,\text{norm}} + \omega_2 \cdot (1 - J_{2,\text{norm}}) + \omega_3 \cdot J_{3,\text{norm}}$$

The final ranking (Table 6) enables the identification of the most balanced solution according to the derived priority weights across the three criteria J_1, J_2, and J_3, where J_1 and J_3 are to be maximized while J_2 is to be minimized. It reveals that Solution 17 $(\boldsymbol{\beta = 0.5855, \gamma = 0.0189, \eta = 8,383.41})$ emerges as the most balanced option due to its optimal combination of all three criteria, as evidenced by its highest overall score of **0.836**. This superior score confirms that this solution presents the best possible compromise: it achieves the second-lowest J_2 value $(538,691)$, surpassed only by Solution 18 which suffers from an unacceptably low J_1 $(1,844,496)$. Simultaneously, it delivers the absolute highest J_3 performance $(261,668,252.30$ MAD$)$ across all solutions, while maintaining a respectable J_1 level $(10,335,734)$ that, although not the maximum available, remains well within acceptable. When compared to alternative solutions, Solution 17's balanced superiority becomes even more apparent. While Solution 14 does offer a marginally better J_2 $(355, 486)$, it does so at the cost of significantly inferior J_1 $(9, 193, 169)$ and J_3 $(238, 435, 946.50$ MAD) performance. Similarly, Solution 16 provides a higher J_1 $(17, 040, 244)$ but is penalized by substantially worse J_2 $(909, 896)$ and J_3 $(240, 829, 233.19$ MAD) values. This comparative analysis clearly demonstrates that Solution 17 represents the optimal equilibrium point, offering the best compromise between actuarial performance, participant attractiveness, and financial robustness.

Table 6. Ranking of solutions based on the weighted score.

Solution	β	γ	η	$J_{1,norm}$	$J_{2,norm}$	$J_{3,norm}$	**Score**	**Rank**
17	0.5855	0.0189	8383.41	0.104	0.019	1.000	0.836	1
16	0.5184	0.0198	8308.02	0.186	0.036	0.919	0.826	2
14	0.4552	0.0180	5780.47	0.090	0.010	0.909	0.826	3
12	0.3136	0.0191	8257.04	0.508	0.103	0.842	0.823	4
10	0.1993	0.0159	4987.94	0.485	0.072	0.644	0.810	5
15	0.4687	0.0202	7634.57	0.257	0.052	0.797	0.808	6
11	0.2440	0.0200	7494.39	0.627	0.136	0.712	0.801	7
9	0.1785	0.0200	7298.14	0.740	0.163	0.686	0.798	8
7	0.0940	0.0219	7663.50	0.888	0.218	0.652	0.780	9
8	0.1644	0.0225	7218.52	0.764	0.190	0.600	0.770	10
13	0.3857	0.0245	7406.97	0.388	0.100	0.601	0.768	11
18	0.7673	0.0200	7768.89	0.000	0.000	0.532	0.759	12
6	0.0642	0.0215	5064.70	0.940	0.227	0.431	0.749	13
5	0.0376	0.0367	6306.50	0.987	0.412	0.308	0.613	14
4	0.0336	0.0473	5742.82	0.994	0.537	0.211	0.514	15
3	0.0311	0.0668	5139.38	0.999	0.763	0.126	0.349	16
2	0.0308	0.0730	2575.28	0.999	0.836	0.046	0.287	17
1	0.0304	0.0872	1000.22	1.000	1.000	0.000	0.169	18

Multi-objective optimization is performed using hypothetical parameter values chosen from admissible domains to systematically examine how improving one Takaful system objective may require a sacrifice in another. The optimal solutions obtained should be interpreted as illustrative results highlighting different trends and possible trade-offs, rather than as recommendations that are directly applicable at the operational level. However, the framework remains flexible and can be calibrated and empirically validated once real data becomes available.

5. Conclusion and future work

This study makes a significant contribution to the emerging literature on mathematical modeling in Islamic finance, particularly in the field of Takaful insurance. To the best of our knowledge, it represents the first work to establish a formal framework based on dynamical systems for modeling the complex interactions among the three fundamental components of Takaful: participants, claims, and the mutual fund. Our approach enables a detailed analysis of the system's dynamic behavior by incorporating key parameters of Islamic insurance, including participant enrollment rate, claim frequency and cost, contribution rate, operational expenses, and profit-and-loss sharing ratios.

After establishing the fundamental mathematical properties of the model—namely, the existence, uniqueness, positivity, and boundedness of solutions—we conducted an in-depth stability analysis of the equilibrium points by deriving conditions for local stability, supported and validated through numerical simulations. In a second phase, we applied the model to a concrete case study by formulating a multi-objective optimization problem that captures the strategic priorities of Takaful operators: participant retention, mitigation of claim impacts, and stabilization of the Takaful fund.

More specifically, we considered a constrained vector optimization problem grounded in the dynamical system (equations (1)–(3)) governing the time evolution of the three key variables—Participants, Claims, and Fund—alongside a set of parametric constraints reflecting real-world operational conditions in the Takaful market. The central objective is to determine the optimal triplet of control parameters: β (participant attrition rate), γ (claim occurrence rate), and η (average participant contribution), such that the cumulative number of participants over the time horizon is maximized, the cumulative number of claims is minimized, and the final fund value is maximized simultaneously.

Solving this problem combines, in an innovative way, the NSGA-II genetic algorithm—used for efficient exploration of the solution space—with the entropy method for objective weighting of criteria, enabling the identification of optimal trade-offs among the three conflicting objectives. This methodology provides Takaful practitioners with robust quantitative tools to optimize their membership acquisition and risk management strategies, while ensuring the long-term financial sustainability of their operations in accordance with Islamic finance principles.

In addition to its mathematical properties, the proposed system aligns with the conceptual framework of Takaful. The model's parameters reflect the fundamental principles of Islamic finance: the contribution rate η represents the amount paid under Tabarru', a voluntary donation intended for collective solidarity; the coefficient μ_2 reflects the surplus sharing mechanism associated with the Mudarabah contract; while the parameter θ, linked to the intensity of compensation, embodies the principle of Ta'awun (mutual aid). Similarly, the model's formulation ensures the absence of Gharar by imposing deterministic, transparent flows. Furthermore, multi-objective optimization does not aim to reduce legitimate compensation but rather to limit fraudulent or excessive claims within an ethical framework fully compliant with Sharia law.

The optimized parameters can be interpreted in practical terms. For example, a decrease in the attrition rate β reflects concrete measures to strengthen customer loyalty or improve customer service. The γ loss ratio may reflect the impact of better risk management, prevention programs, or anti-fraud mechanisms. As for the η contribution level, it should be interpreted as a proposal resulting from the compromise reached on the Pareto frontier, achieved through an open discussion and consensus-building process between the operator and the members before any decision is made, in accordance with Takaful governance.

This research opens several promising avenues for future work. A natural extension would involve incorporating more complex market dynamics, such as macroeconomic effects and exogenous shocks that may impact the stability of the Takaful fund. Developing a stochastic framework could further improve the model by better accounting for the inherent uncertainty in financial flows and participant behavior. Another interesting direction would be to adapt the model to different types of Takaful products (e.g., family, health) by incorporating their specific contractual features. Finally, building a decision-support interface integrating these optimization tools could significantly facilitate their adoption by industry practitioners. Such developments would enhance both the theoretical reach and practical relevance of the model for the Islamic insurance industry.

Declarations

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

All authors contributed to the study conception and design, to the analysis of the results and to the writing of the manuscript.

Data Availability Statement

All data underlying the results are available as part of the article, and no additional source data are required.

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