STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 1–10.
Published online in International Academic Press (www.IAPress.org)

Enhancing the Accuracy of Standard Normal Distribution Approximations

Omar M. Eidous [1], Loai M. Alzoubi [2], Ahmad A. Hanandeh [3,*]

[1]*Department of Statistics, Yarmouk University, Irbid (21163), Jordan*
[2]*Department of Mathematics, Al al-Bayt University, Mafraq (25113), Jordan*
[3]*Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia*

Abstract In this paper, we introduce a novel approximation for the standard normal distribution function, significantly improving its accuracy. Using the maximum absolute error (Max-AE) and mean absolute error (MAE) as metrics, our approximation achieves a Max-AE of 2.95×10^{-5}, outperforming most existing methods. Additionally, we present an approximation for the inverse normal distribution, showing its superiority over many current models. Numerical comparisons validate the efficiency of our methods, making them applicable in fields like statistical analysis, machine learning, and financial modeling.

Keywords Normal distribution, Approximations, Cumulative distribution function, Maximum absolute error, Mean absolute error

AMS 2010 subject classifications 62E17

DOI:10.19139/soic-2310-5070-3116

1. Introduction

The normal distribution, also known as the Gaussian distribution, is a key concept in statistics. It is a continuous probability distribution that is symmetric around its mean, creating a bell-shaped curve. Most data points tend to cluster around the mean, with frequencies decreasing as values move further away from it. This distribution is widely applicable in real-world scenarios, such as measurements in science, intelligence testing, and error analysis. The normal distribution is crucial because of its role in the Central Limit Theorem, which states that the sum of many independent random variables will approximate a normal distribution. It is also essential in statistical analyses, including hypothesis testing and confidence intervals. With its well-defined properties, such as mean and standard deviation, the normal distribution is foundational in understanding data patterns and is used across various fields like economics, psychology, and medicine.

The cumulative distribution function (cdf) of the normal distribution is widely used in many scientific fields, particularly in probability and statistics. For the standard normal distribution, which has a mean ($\mu = 0$) and standard deviation ($\sigma = 1$), the cdf provides the probability that a random variable will take a value less than or equal to a specific value z. This standard normal distribution is denoted as $Z \sim N(0, 1)$, and its probability density function (pdf), $\varphi(z)$, is given by:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty.$$

*Correspondence to: Ahmad A. Hanandeh (Email: hanandeh@iu.edu.sa).

ISSN 2310-5070 (online) ISSN 2311-004X (print)

The cdf, $\Phi(z)$, is the integral of the pdf from $-\infty$ to z, representing the cumulative probability up to that point:

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt.$$

While the cdf does not have a simple closed-form solution, it can be approximated numerically or referenced from statistical tables. This function is essential for determining the likelihood of certain outcomes in standardized data, making it crucial for fields like finance, engineering, and social sciences.

Dealing with the integral in the formula for $\Phi(z)$ is not straightforward because no closed-form expression exists for it. Since the exact value of this integral cannot be determined algebraically, various methods have been developed to approximate or bound it. Some researchers approach this by finding bounds for the integral (as discussed in [8, 7] and [1], while others focus on deriving approximations for $\Phi(z)$ (as suggested by [10]). Numerous approximations for $\Phi(z)$ have been proposed over the years, such as those by [30, 13, 15, 21] and [11]). [10] compiled 45 different approximations and introduced nine new ones. These approximations have become essential tools for practical computations involving the standard normal distribution.

In addition to the various approximations for $\Phi(z)$, two important functions that have been extensively studied in the literature are the error function (erf(z)) and the Q-function ($Q(z)$), both of which are closely related to the standard normal cdf. The error function, denoted as erf(z), is defined by the equation

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} \exp(-t^2)dt = 2\Phi(\sqrt{2}z) - 1,$$

and it provides a way to express the cumulative probability for a normal distribution in a more manageable form. The error function is often used in various scientific and engineering disciplines, particularly when dealing with problems that involve Gaussian-like distributions. It allows for the transformation of the standard normal distribution into a more directly computable form, offering a significant advantage when calculating probabilities over ranges of values or solving differential equations.

The Q-function, defined as

$$Q(z) = \int_{z}^{\infty} \varphi(t)dt = 1 - \Phi(z),$$

is another crucial function related to the standard normal distribution, particularly useful in fields like communications, signal processing, and reliability engineering. The Q-function represents the tail probability of the normal distribution, providing the probability that a normally distributed random variable exceeds a given value z. This is particularly important in error rate analysis, where it quantifies the likelihood of a signal falling outside a certain threshold, such as in Gaussian noise environments.

Both the error function and the Q-function provide essential tools for approximating and understanding the behavior of the normal distribution, especially when dealing with tail probabilities and cumulative values. Their widespread use across diverse fields like engineering, finance, and physics underscores their importance in the precise calculation and prediction of outcomes based on the normal distribution. The results related to the three functions $\Phi(z)$, erf(z) and $Q(z)$ are widely used and have applications in various fields such as communication systems (see, [27]), statistical computations, mathematical models in biology, mathematical physics and diffusion theory (see, [2, 19]).

The most popular criteria to measure the accuracy of an approximation $\widehat{\Phi}(z)$ of $\Phi(z)$ at a specific value of z are the maximum absolute error (Max-AE) and the mean absolute error (MAE). They are defined as follows ([17]),

$$Max - AE(\widehat{\Phi}(z)) = \max_{z} |\widehat{\Phi}(z) - \Phi(z)|,$$

$$MAE(\widehat{\Phi}(z)) = \frac{\sum_z |\widehat{\Phi}(z) - \Phi(z)|}{\#\text{ of } z}.$$

For any number of particular selection of values of z the value of $MAE(\widehat{\Phi}(z))$ can be figured out. For illustration, if z takes the values between 0 and 5 with step 0.001, then we use 5001 values to calculate $MAE(\widehat{\Phi}(z))$. These values are 0, 0.001, ..., 4.999, 5.

This paper aims to introduce more accurate approximations of $\Phi(z)$, evaluated through robust error metrics—Max-AE and MAE—to ensure computational efficiency without compromising accuracy. The rest of the paper is as follows: Section 2 reviews existing approximations for $\Phi(z)$, setting the stage for the improvements made in later sections. In Section 3, we introduce a new approximation for $\Phi(z)$, denoted $\Phi_{11}(z)$, which significantly reduces the Max-AE from 1.79×10^{-4} to 2.95×10^{-5}. Section 4 proposes a new approximation for the inverse of $\Phi(z)$, $\hat{z}_3$, which outperforms previous methods. Section 5 compares the accuracy of the proposed methods with existing approximations, demonstrating their superiority in terms of both Max-AE and MAE. Section 6 discusses applications of these new approximations in fields such as financial engineering, machine learning, and statistical quality control, showing how they enhance computational efficiency and precision. Finally, Section 7 concludes the paper, summarizing the key contributions and suggesting directions for future work to further improve the accuracy and applicability of these approximations.

2. Some existing approximations for $\Phi(z)$

Many approximations for $\Phi(z)$ have been suggested in the literature. Here, we listed some of these approximations as follows (for more approximation formulas, see [10]):

- [22] presented the approximation

$$\Phi_1(z) = 0.5\left[1 + \tanh\left(\frac{2z\left(1 + \frac{0.089430}{2}z^2\right)}{\sqrt{2\pi}}\right)\right], \quad z \in \mathbb{R}.$$

  The Max-AE of $\Phi_1(z)$ is 1.79×10^{-4}.
- [18] suggested another approximation to improve the accuracy of Page's approximation [22]. Their formula is,

$$\Phi_2(z) = 0.5\left[1 + \tanh\left(\frac{39z}{2\sqrt{2\pi}} - \frac{111}{2}\arctan\left(\frac{35z}{111\sqrt{2\pi}}\right)\right)\right], \quad z \in \mathbb{R},$$

  with Max-AE 6.13×10^{-5}.
- Another approximation of $\Phi(z)$ that used the function $\tanh$ is suggested by [30], which is,

$$\Phi_3(z) = \begin{cases} 0.5 - 1.136H_1 + 2.47H_2 - 3.013H_3, & 0 \le z \le 3.36 \\ 1, & z > 3.36 \end{cases},$$

  where $H_1 = \tanh(-0.2695z)$, $H_2 = \tanh(0.5416z)$ and $H_3 = \tanh(0.4134z)$. The Max-AE of $\Phi_3(z)$ equals 1.25×10^{-3}.
- [6] proposed the following approximation to improve the Tocher's approximation [26], which is given by,

$$\Phi_4(z) = \frac{1}{1 + e^{-y}},$$

  where $y = 1.526z(1 + 0.1034z)$. The Max-AE of $\Phi_4(z)$ is 2.09×10^{-3}. We notice here that the Tocher's approximation of $\Phi(z)$ is $\frac{1}{1+e^{-2\sqrt{2/\pi}z}}$ with Max-AE 1.77×10^{-2}.
- [28] approximated $\Phi(z)$ by,

$$\Phi_5(z) = \frac{1}{1 + e^{-y}},$$

  where $y = \sqrt{8/\pi}z + \sqrt{2/\pi}(4 - \pi)z^3/3\pi$. The Max-AE of $\Phi_5(z)$ is 3.1×10^{-4}, which is more accurate than $\Phi_4(z)$.

- [29] also derived another approximation, which is more accurate than Tocher's one and $\Phi_5(z)$. Their approximation is,

$$\Phi_6(z) = \frac{1}{1+e^{-y}},$$

where $y = \sqrt{\pi}(0.9z + 0.0418198z^3 - 0.0004406z^5)$. The Max-AE of $\Phi_6(z)$ is 4.4×10^{-5}, which is more accurate than $\Phi_5(z)$.
- Some corrections are introduced by [5] to improve the approximation of [28]. They suggested the following approximation.

$$\Phi_7(z) = \frac{1}{1+e^{-1.5976z-0.07056z^3}},$$

which is the same as $\Phi_5(z)$ with some corrections on the coefficients of z and z^3. The Max-AE of $\Phi_7(z)$ is 1.42×10^{-4} less than that of $\Phi_5(z)$.
- [4] employed the approximation,

$$\Phi_8(z) = \frac{1}{1+e^{-y}},$$

where $y = \frac{1}{2}(-0.506445 + 10.4467\tanh(1.3448 + 0.3264z) + 9.8475\tanh(-1.3519 + 0.3376z) + 1.5976z + 0.070565992z^3)$. The Max-AE of $\Phi_8(z)$ is 2.40×10^{-5} less than that of $\Phi_6(z)$.
- [15] derived the following approximation for $\Phi(z)$,

$$\Phi_9(z) = 0.5\left(1 + \sqrt{1 - e^{-\frac{81}{130}z^2}}\right)$$

The Max-AE is 1.6×10^{-4}.
- [12] suggested the following approximation for $\Phi(z)$,

$$\Phi_{10}(z) = \frac{1}{1+e^{-az}}, \quad z \geq 0$$

where $a = 1.59635 + 0.072154z^2 - 0.000175z^5$. Its Max-AE is 4.95×10^{-5}.

The main objective of this paper is to propose a new approximation for $\Phi(z)$ and thus approximate erf(z) and $Q(z)$. In particular, the proposed approximation improves the approximation $\Phi_1(z)$. The Max-AE of $\Phi_1(z)$ reduces from 1.79×10^{-4} to 2.95×10^{-5} for the proposed approximation. Moreover, another new approximation is proposed for the inverse of $\Phi(z)$.

3. Proposed approximation of $\Phi(z)$

To evaluate the accuracy of our proposed approximations, we benchmark against reference values computed using established scientific libraries set to high precision with about 16 digits as outlined in [20]. While this practice is standard in the literature and typically serves as a reliable proxy for the "exact" value of the normal cdf or its quantile, it should be noted that such high-precision reference values may not always be readily available and can be computationally intensive to obtain.

To improve the accuracy of $\Phi_1(z)$ that derived by [22], we propose modifying the formulation as follows,

$$\Phi_{11}(z) = 0.5(1 + \tanh(az)), \quad z \geq 0,$$

Now, we find the values of a for some specific values of $z \geq 0$ that minimize the absolute difference between $\Phi_{11}(z)$ and the exact $\Phi(z)$. The best regression of the form $a = a_0 + a_1z + a_2z^2 + \ldots + a_5z^5$ is then obtained by using the specific values of z and the resulting values of a, which gives,

$$a = 0.7977 + 0.0369z^2 - 0.0004z^4 + 0.00003z^5$$

To justify the polynomial structure of the parameter a, consider the exact relationship required for the approximation to be an equality. If we set $\Phi(z) = 0.5(1 + \tanh(a(z) \cdot z))$, we can solve for the exact function $a^*(z)$ as:

$$a^*(z) = \frac{1}{z}\text{arctanh}(2\Phi(z) - 1)$$

The function $a^*(z)$ represents the ideal parameter variation required to yield zero error. However, $a^*(z)$ involves the evaluation of $\Phi(z)$ itself, which is computationally expensive. The goal of our approximation is to replace this complex transcendental function $a^*(z)$ with a computationally efficient polynomial $P_n(z)$ of degree n. Our objective is to find the set of coefficients for $P_n(z)$ that minimizes the maximum absolute error (Max-AE) over the domain $z \geq 0$. This corresponds to minimizing the Chebyshev norm (L_∞ norm):

$$\min_{P_n} ||\Phi_{11}(z|P_n) - \Phi(z)||_\infty$$

This framework aligns with the Minimax Approximation Theory. By performing the regression analysis described in the previous paragraph, we numerically approximate the optimal Minimax polynomial. The theoretical validity of this approach is supported by the error behavior shown in Figure 1. The error function $\psi(z)$ exhibits the "equioscillation property," where the error alternates signs and reaches comparable maximum magnitudes at several points in the domain (approximately $0.33, 1.06, 1.86, 2.91$). This behavior is a hallmark of near-optimal Minimax approximations, indicating that the derived polynomial efficiently utilizes its degrees of freedom to distribute the error uniformly across the range of interest.

Therefore, our proposed approximation of $\Phi(z)$ is,

$$\Phi_{11}(z) = 0.5(1 + \tanh(az)), \quad z \geq 0.$$

To determine the Max-AE of $\Phi_{11}(z)$, set $\psi(z) = \Phi_{11}(z) - \Phi(z)$ then the first derivative of $\psi(z)$ with respect to z is,

$$\psi'(z) = \Phi'_{11}(z) - \varphi(z) = \frac{a}{2}\text{Sech}^2(az) - \frac{1}{\sqrt{2\pi}}e^{-z^2/2}.$$

Using Mathematica, Version 11, we found four roots for $\psi'(z)$, which are (approximately) 0.333137, 1.06219, 1.85916 and 2.91009. Calculating the values of $\psi(z)$ at these roots, we obtain $\psi(0.333137) = 0.0000196$, $\psi(1.06219) = -0.0000264$, $\psi(1.85916) = 0.0000191$ and $\psi(2.91009) = -0.0000295$. It is clear that the Max-AE of $\Phi_{11}(z)$ is equal to 2.95×10^{-5}, which occurs at $z \cong 2.91$.

Figure 1 below displays the difference between $\Phi_{11}(z)$ and $\Phi(z)$. That is, it displays the graphs of a function $\psi(z) = \Phi_{11}(z) - \Phi(z)$ for $0 \leq z \leq 8$.

4. Approximation for inverse of $\Phi(z)$

The inverse of the cdf of standard normal, $p = \Phi(z), \quad z \geq 0$ is $z = \Phi^{-1}(p)$. We are concerned in approximating the value of z, such that $p(Z \leq z) = p$ given the value of $p(0.5 \leq p < 1)$. A simple approximation for z is derived by [23], it is given by,

$$\hat{z}_1 = \frac{p^{0.135} - (1-p)^{0.135}}{0.1975}, \quad p \geq 0.5.$$

Another approximation is derived by [24] for given $p \geq 0.5$,

$$\hat{z}_2 = -5.531\left(\left(\frac{1-p}{p}\right)^{0.1193} - 1\right).$$

In this paper, we have suggested the following approximation for z,

$$\hat{z}_3 = \begin{cases} \frac{1}{b}\tanh^{-1}\left(\frac{p}{0.5} - 1\right), & 0.5 < p < 0.99 \\ \frac{1}{c}\tanh^{-1}\left(\frac{p}{0.5} - 1\right), & 0.99 \leq p < 1 \end{cases}$$

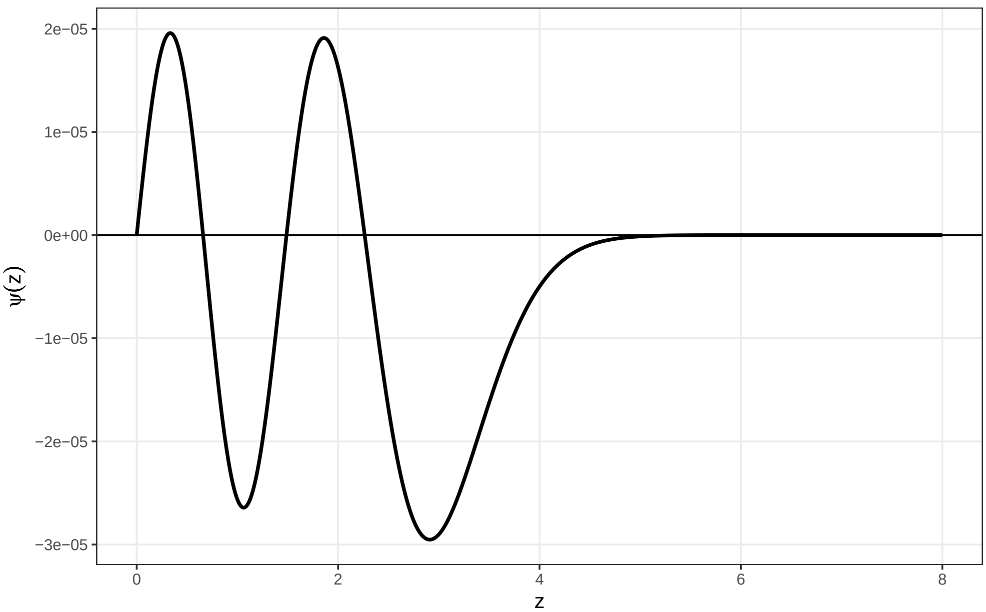

Figure 1. The difference between $\Phi_{11}(z)$ and $\Phi(z)$, i.e. $\psi(z) = \Phi_{11}(z) - \Phi(z)$.

where, $b = 0.609 + 1.38p^2 - 4.237p^4 + 3.24p^5$ and $c = -86733.11 + 292895.15p^2 - 445092.947p^4 + 238932.108p^5$.

Figure 2 displays the plot of the difference between $\hat{z}_3$ and $z = \Phi^{-1}(p)$.

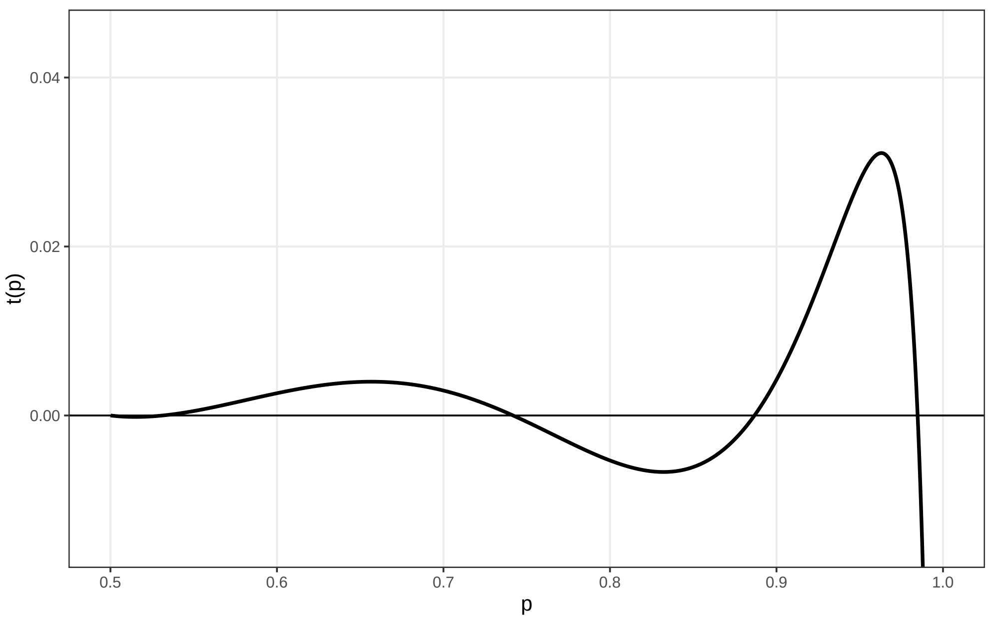

Figure 2. The difference between $\hat{z}_3$ and the invertible function $z = \Phi^{-1}(p)$, i.e. $t(p) = \hat{z}_3 - z$.

5. Comparisons

The MAE value for all approximations of $\Phi(z)$ is calculated based on the values of z from 0 to 5 with step 0.001 (i.e. $z = 0(0.001)5$) to ensure the precision of these approximations. Table 1 include the results of MAE and the $Max - AE$ to simplify the comparison between the approximations.

Table 1. The Max-AE and the MAE for the different existing approximations $\Phi_1(z)$ to $\Phi_{10}(z)$ and the proposed approximation $\Phi_{11}(z)$.

Approximation	MAE	$Max - AE$
$\Phi_1(z)$	7.65×10^{-5}	1.79×10^{-4}
$\Phi_2(z)$	1.61×10^{-5}	6.13×10^{-5}
$\Phi_3(z)$	5.68×10^{-4}	1.25×10^{-3}
$\Phi_4(z)$	9.78×10^{-4}	2.09×10^{-3}
$\Phi_5(z)$	9.99×10^{-5}	3.10×10^{-4}
$\Phi_6(z)$	1.69×10^{-5}	4.40×10^{-5}
$\Phi_7(z)$	6.88×10^{-5}	1.42×10^{-4}
$\Phi_8(z)$	7.26×10^{-6}	2.40×10^{-5}
$\Phi_9(z)$	1.6×10^{-4}	6.66×10^{-4}
$\Phi_{10}(z)$	1.59×10^{-5}	4.95×10^{-5}
$\Phi_{11}(z)$	1.29×10^{-5}	2.95×10^{-5}

The proposed approximation $\Phi_{11}(z)$ is quite accurate compared to the other approximations based on the two criteria: $Max - AE$ and MAE. However, there is still room for improvement by adopting the same technique used in this paper and increasing the number of polynomial terms that arose in the quantity a of $\Phi_{11}(z)$.

To explore the accuracy of the proposed approximation $\hat{z}_3$ of the inverse of normal distribution function, a number of values for z are selected from 0 to 4.4 with step 0.4 (i.e. = 0 (0.4) 4.4). For each value of z the corresponding value of $p = \Phi(z)$ is computed. Then $\hat{z}_3$ is computed at each value of p. For the sake of comparison, the results of the three approximations $\hat{z}_1$, $\hat{z}_2$ and $\hat{z}_3$ are included in Table 2. We can also compare the values of the different approximations with the exact values of z (first column).

Table 2. The exact values of z from 0 to 4.4 (first column) and the corresponding exact values of $\Phi(z)$ (second column). The other columns give the values of different approximations of z.

z	$p = \Phi(z)$	$\hat{z}_1$	$\hat{z}_2$	$\hat{z}_3$
0.0	0.5000	0.000	0.000	0.000
0.4	0.6554	0.3976	0.4084	0.3960
0.8	0.7881	0.7969	0.8024	0.8044
1.2	0.8849	1.1989	1.1948	1.2005
1.6	0.9452	1.6038	1.5932	1.5744
2.0	0.9773	2.0093	1.9993	1.9783
2.4	0.9918	2.4105	2.4097	2.3874
2.8	0.9974	2.7999	2.8168	2.8015
3.2	0.9993	3.1686	3.2109	3.1608
3.6	0.9998	3.5084	3.5826	3.6787
4.0	0.99997	3.8130	3.9239	4.3221
4.4	0.99999	4.0783	4.2293	5.0504

The differences between the three approximations of z, $\hat{z}_i, i = 1, 2, 3$, and the exact value of z are included in Table 3 to make the comparison easier.

Tables 2 and 3 show that the proposed approximation is a good competitor to $\hat{z}_1$ and $\hat{z}_2$ in particular for small values of $\Phi(z)$. All results and graphs are obtained by using Mathematica, Version 12.

Table 3. The exact values of z from 0 to 4.4 (first column) and the corresponding exact values of $\Phi(z)$ (second column). The third column up to fifth column give the values of the differences between $\hat{z}_i, i = 1, 2, 3$ and exact z.

z	$p = \Phi(z)$	$\hat{z}_1$	$\hat{z}_2$	$\hat{z}_3$
0.0	0.5000	0.	0.	0.
0.4	0.6554	-0.00238	0.00839	-0.00399
0.8	0.7881	-0.00313	0.00237	0.00435
1.2	0.8849	-0.00107	-0.00521	0.00049
1.6	0.9452	0.00380	-0.00685	-0.02564
2.0	0.9773	0.00932	-0.00070	-0.02174
2.4	0.9918	0.01051	0.00972	-0.01525
2.8	0.9974	-0.00015	0.01681	0.00145
3.2	0.9993	-0.03141	0.01094	-0.03925
3.6	0.9998	-0.09157	-0.01738	0.07866
4.0	0.99997	-0.18706	-0.07607	0.32207
4.4	0.99999	-0.32173	-0.17066	0.65041

6. Application

The application of the newly proposed approximations for the cdf of the standard normal distribution and its inverse extends across multiple domains where precision and computational efficiency are crucial. These approximations, distinguished by their high accuracy coupled with relative simplicity, enhance both theoretical and practical implementations requiring normal probabilistic calculations.

6.1. Numerical Experiment: Option Pricing Sensitivity

To illustrate the practical impact of the proposed approximation, we consider the Black-Scholes model for pricing European call options [3]. The price C of a call option is given by:

$$C = S\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$

where S is the spot price, K is the strike price, and $\Phi(\cdot)$ is the standard normal cumulative distribution function. The accuracy of the calculated price C is directly proportional to the accuracy of the approximation used for $\Phi(d_1)$ and $\Phi(d_2)$. Consider a scenario where the input parameters result in $d_1 = 2.91$. As shown in Section 3, this value corresponds to the region where approximation errors are typically maximized for this class of functions. • Asset Value: Let us assume a notional portfolio value of $S = \$1,000,000$. • Standard Method Error: Using the approximation by [22] (Φ_1), the maximum absolute error is approximately 1.79×10^{-4}. This results in a pricing error of roughly:

$$\text{Error}_{\Phi_1} \approx \$1,000,000 \times 1.79 \times 10^{-4} = \$179.00$$

Using the proposed approximation (Φ_{11}), the maximum absolute error is reduced to 2.95×10^{-5}. The corresponding pricing error is:

$$\text{Error}_{\Phi_{11}} \approx \$1,000,000 \times 2.95 \times 10^{-5} = \$29.50$$

In high-frequency trading or large-scale risk management contexts, where millions of options are re-valued continuously, this reduction in systematic bias—from 179 down to 29.50 per million dollars of exposure—represents a significant improvement in valuation integrity without incurring the computational overhead of algorithmic high-precision libraries.

6.2. Applications in Financial Engineering

The standard normal distribution underlies many cornerstone models in financial engineering, notably the Black-Scholes option pricing formula, where both the normal cdf and inverse cdf ("probit" function) are essential.

Efficient and accurate approximations significantly reduce computational burden in real-time pricing environments such as high-frequency trading, where latency is critical. Moreover, risk metrics such as Value at Risk (VaR) and Conditional Value at Risk (CVaR) fundamentally depend on tail probabilities of financial return distributions. Improved approximations of the normal cdf enable more precise tail quantile estimation, thereby enhancing portfolio risk evaluation and regulatory compliance in risk management frameworks [14]. Given the numerical challenges often faced in such applications, the accuracy levels of the proposed approximations, reaching Max-AEs on the order of 10^{-5}, substantially improve computational efficiency without sacrificing precision.

6.3. Applications in Machine Learning and Data Science

Probabilistic models in machine learning extensively utilize the Gaussian distribution. Algorithms such as Gaussian Naive Bayes classifiers and Gaussian Processes frequently compute $\Phi(x)$, the standard normal cdf, in likelihood evaluations or kernel computations. Employing the new approximations accelerates inference especially in large-scale datasets or resource-constrained environments (e.g., embedded systems). Furthermore, approximate inference techniques like Expectation Propagation or Variational Inference, which often require repeated evaluation of normal cdfs and their inverses, benefit from the reduced computational expense and faster convergence made possible by these approximations [25]. Importantly, inverse cdf approximations are critical in inverse transform sampling to generate Gaussian random variables efficiently within simulation-based models, including reinforcement learning and probabilistic generative frameworks.

6.4. Applications in Statistical Quality Control

Statistical quality control (SQC) utilizes the normal distribution extensively to design control charts and compute process capability indices (Cp, Cpk). These indices require evaluation of cumulative probabilities to measure how a process distribution fits within specification limits. The use of fast and accurate normal cdf approximations facilitates real-time process monitoring and anomaly detection in manufacturing lines by enabling swift calculation of control limits and capability indices. This is especially relevant in contexts involving truncated normal distributions where approximations have also been developed [16]. The proposed approximations, by virtue of their ease of implementation and high fidelity, enable robust deployment in automated quality control systems, enhancing responsiveness and decision-making.

7. Conclusions and Further direction

In this paper, we have proposed novel approximations for the cumulative distribution function $\Phi(z)$, the error function erf(z), and the Q-function $Q(z)$. Our proposed approximation for $\Phi(z)$ significantly improves upon the existing approximation $\Phi_1(z)$, reducing the Max-AE from 1.79×10^{-4} to 2.95×10^{-5}, thus enhancing the precision of normal distribution calculations. Furthermore, we introduced an accurate approximation for the inverse of $\Phi(z)$, which demonstrates superior performance in evaluating standard normal quantiles. The proposed approximations were benchmarked using Max-AE and MAE metrics, and results show that the new approximations are highly accurate across a wide range of values, outperforming many established methods. These improvements are particularly beneficial in fields requiring computational efficiency and high accuracy, such as financial engineering, machine learning, and SQC. The proposed methods provide a significant advancement in the accuracy and efficiency of normal distribution function approximations, offering a solid foundation for further developments in statistical computations and applications requiring normal distribution evaluations. Future work could focus on extending these approximations for broader applications and exploring potential optimizations for even better accuracy at extreme values of z.

REFERENCES

1. E. Ananbeh, and O. Eidous, *New simple bounds for standard normal distribution function*, Communications in Statistics - Simulation and Computation, vol. 54, no. 7, pp. 2762–2769, 2025, 10.1080/03610918.2024.2326596.

2. G. Bercu, *New refinements for the error function with applications in diffusion theory*, Symmetry, vol. 12, 2020, 10.3390/sym12122017.
3. F. Black, and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, vol. 81, no. 3, pp. 637–654, 1973
4. N. K. Boiroju, and K. R. Rao, *Logistic approximations to standard normal distribution function*, Assam Statistical Review (ASR), vol. 28, no. 1, pp. 27–40, 2014.
5. S. R. Bowling, M. T. Khasawneh, S. Kaewkuekool, and B. R. Cho, *A logistic approximation to the cumulative normal distribution*, Journal of Industrial Engineering and Management, vol. 2, no. 1, pp. 114–127, 2009, 10.3926/jiem.2009.v2n1.p114-127.
6. D. R. Divgi, *Approximations for three statistical functions*, Center for Naval Analyses, 1990.
7. O. Eidous, *Lower and upper bounds for Q-function and its related functions*, Communications in Statistics - Simulation and Computation, pp. 1–8, 2025, 10.1080/03610918.2025.2515188.
8. O. Eidous, *New tightness lower and upper bounds for the standard normal distribution function and related functions*, Mathematical Methods in the Applied Sciences, vol. 46, no. 14, pp. 15011–15019, 2023, 10.1002/mma.9358.
9. O. Eidous, and J. Abu-Hawwas, *An accurate approximation for the standard normal distribution function*, Journal of Information and Optimization Science, vol. 42, no. 1, pp. 17–27, 2021, 10.1080/02522667.2019.1661642.
10. O. Eidous, and R. Abu Shareefa, *New approximations for standard normal distribution function*, Communications in Statistics - Simulation and Computation, vol. 49, no. 6, pp. 1357–1374, 2020, 10.1080/03610926.2018.1563166.
11. O. Eidous, and M. Al-Rawwash, *Approximations for standard normal distribution function and its invertible*, Journal of Algorithms & Computational Technology, vol. 19, pp. 1–5, 2025, 10.1177/17483026251322100.
12. O. Eidous, and E. Ananbeh, *Approximations for cumulative distribution function of standard normal*, Journal of Statistics and Management Systems, vol. 25, no. 3, 2022, 10.1080/09720510.2021.1892278.
13. M. Edous, and O. Eidous, *A simple approximation for normal distribution function*, Mathematics and Statistics, vol. 6, no. 4, pp. 47–49, 2018, 10.13189/ms.2018.060401.
14. I. Groparu-Cojocaru, and L. G. Doray, *Inference for the generalized normal Laplace distribution*, Communications in Statistics - Simulation and Computation, vol. 42, no. 9, pp. 1989–1997, 2013, 10.1080/03610918.2012.687064.
15. A. Hanandeh, and O. Eidous, *A new one-term approximation to the standard normal distribution*, Pakistan Journal of Statistics and Operation Research, vol. 17, no. 2, 2021, 10.18187/pjsor.v17i2.3556.
16. M. M. Hamasha, *Practitioner advice: approximation of the cumulative density of left-sided truncated normal distribution using logistic function and its implementation in Microsoft Excel*, Quality Engineering, vol. 29, no. 2, pp. 322–328, 2017, 10.1080/08982112.2016.1196373.
17. A. Hanandeh, and O. M. Eidous, *Some improvements for existing simple Approximations of the Normal Distribution Function*, Pakistan Journal of Statistics and Operation Research, vol. 18, no. 3, pp. 555–559, 2022, 10.18187/pjsor.v18i3.4007.
18. H. V. Leal, R. C. Sheissa, V. F. Nino, A. S. Reyes, and J. S. Orea, *High accurate simple approximation of normal distribution integral*, Mathematical Problems in Engineering, Article ID 124029, 22 pages, 2012, 10.1155/2012/124029.
19. J. Lipoth, Y. Tereda, S. Papalexiou, and R. Spiteri, *A new very simply explicitly invertible approximation for the standard normal cumulative distribution function*, AIMS Mathematics, vol. 7, no. 7, pp. 11635–11646, 2022, 10.3934/math.2022648.
20. G. Marsaglia, *Evaluating the normal distribution*, Journal of Statistical Software, vol. 11, pp. 1–11, 2004, 10.18637/jss.v011.i04.
21. M. Obeidat, R. Al-Jamal, and A. A. Hanandeh, *New highly accurate improvements for single-term approximations of the standard normal distribution function*, Pakistan Journal of Statistics and Operation Research, vol. 21, no. 2, pp. 119–128, 2025, 10.18187/pjsor.v21i2.4949.
22. E. Page, *Approximations to the cumulative normal function and its inverse for use on a pocket calculator*, Applied Statistics, vol. 26, pp. 75–76, 1977, 10.2307/2346872.
23. B. Schmeiser, *Approximations to the inverse cumulative normal function for use on hand calculators*, Applied Statistics, vol. 28, pp. 175–176, 1979, 10.2307/2346737.
24. H. Shore, *Simple Approximations for the Inverse Cumulative Function, the Density Function and the Loss Integral of the Normal Distribution*, Journal of the Royal Statistical Society. Series C (Applied Statistics), vol. 31, no. 2, pp. 108–114, 1982, 10.2307/2347972.
25. H. Shore, *Accurate RMM-based approximations for the CDF of the normal distribution*, Communications in Statistics—Theory and Methods, vol. 34, no. 3, pp. 507–513, 2005, 10.1081/STA-200052102.
26. K. D. Tocher, *The art of simulation*, English Universities Press, London, 1963.
27. I. Trigui, E. Agbogla, M. Benjillali, W. Ajib, and W. Zhu, *Bit error rate analysis for reconfigurable intelligent surfaces with phase errors*, IEEE Communications Letters, vol. 25, no. 7, pp. 2176–2180, 2021, 10.1109/LCOMM.2021.3071433.
28. J. D. Vedder, *An invertible approximation to the normal distribution function*, Computational Statistics & Data Analysis, vol. 16, pp. 119–123, 1993, 10.1016/0167-9473(93)90248-R.
29. G. R. Waissi, and D. F. Rossin, *A sigmoid approximation to the standard normal integral*, Applied Mathematics and Computation, vol. 77, pp. 91–95, 1996, 10.1016/0096-3003(95)00190-5.
30. R. Yerukala, N. K. Boiroju, and M. K. Reddy, *An approximation to the CDF of standard normal distribution*, International Journal of Mathematical Archive, vol. 2, pp. 1077–1079, 2011, 10.13140/RG.2.1.4599.7607.