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Tree Decompositions in Fuzzy Graphs: Foundations and Structural Insights

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Abstract A Fuzzy Graph extends classical graph theory by incorporating uncertainty, assigning a membership degree to each edge. Tree-width is a basic metric that assesses how much a graph resembles a tree [31, 32], making it an essential tool in algorithm design and combinatorial optimization. Path-width quantifies how much a graph resembles a path, using smallest bag size minus one in path-decomposition of the graph. The main aim of this work is to present the Fuzzy Tree-Decomposition and Fuzzy Path-Decomposition, extending the classical notions of tree- and path-decompositions to the realm of fuzzy graphs. We anticipate that these novel ideas will expand the possible uses of fuzzy graphs and encourage more investigation into the mathematical structure of graph-width parameters.

Keywords Fuzzy Graph, Tree-width, Path-width, Graph Width Parameters

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1. Introduction

Fuzzy logic and soft computing have become essential paradigms for modeling and controlling complex systems under uncertainty, with successful applications ranging from reservoir operation and power-system stability enhancement to intelligent cyber-security and malware detection frameworks [1, 2, 3]. On the theoretical side, recent developments in fuzzy implications and stability analysis have further strengthened the mathematical foundations of fuzzy reasoning and decision mechanisms [4]. In parallel, graph-theoretic structures and their associated complexity parameters have gained increasing attention due to their capability of capturing structural information in discrete systems, as demonstrated through various studies on special graph families, metric dimensions, and related algorithmic characteristics [5, 6, 7, 8, 9, 10, 11, 12].

By permitting partial membership between absolute truth and falsity, Zadeh's 1965 theory of fuzzy sets offered a mathematical foundation for managing uncertainty [37]. Building on this foundation, Rosenfeld extended the concept to graph theory, proposing fuzzy graphs as a means of modeling ambiguous relationships between entities [33]. Since then, fuzzy graphs and their variants have been widely studied, yielding applications in decision making, data analysis, and network modeling [27, 24]. Further generalizations, such as intuitionistic fuzzy sets [13], have

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deepened the expressive capacity of these frameworks, while recent developments highlight their integration with broader uncertain combinatorial structures [19].

Parallel to the study of fuzzy systems, graph width parameters have emerged as indispensable tools in combinatorial optimization and algorithm design. In determining how closely a graph resembles a tree, the notion of tree-width, which was first proposed by Seymour and Robertson in [31, 32], has had a significant impact. Comprehensive treatments, such as Bodlaender's survey [14], have emphasized the central role of tree-width in classifying computationally tractable problems. Closely related measures, including path-width [25], bandwidth [16], rank-width [21, 29], and clique-width [22, 17], have further enriched the study of graph structure. Extensions such as hypertree-width [20] and supertree-width [18] demonstrate the breadth of applications of decomposition techniques in both theory and practice.

Despite these advances, relatively little attention has been given to incorporating fuzzy uncertainty into structural decomposition methods. While fuzzy trees and related constructs have been studied in the context of labeling and classification [28, 34], a systematic investigation of tree-width and path-width in fuzzy graphs remains largely unexplored. This gap is significant because decomposition-based parameters play a vital role in simplifying complex networks, and extending them to fuzzy settings opens pathways for analyzing uncertain data in areas like decision-making systems, social networks, and bioinformatics.

Within this work, we propose the notions of Fuzzy Tree-Decomposition and Fuzzy Path-Decomposition, which generalize classical decompositions by incorporating membership functions on vertices and edges. These definitions lead naturally to fuzzy analogues of tree-width and path-width, preserving their structural insights while capturing degrees of uncertainty inherent in real-world systems. By formalizing these concepts, we aim to provide a foundation for further research at the intersection of fuzzy set theory and graph width parameters, ultimately broadening the applicability of decomposition techniques to uncertain environments.

1.1. Fuzzy Graph

In 1965, Zadeh introduced fuzzy sets to capture partial truth, bridging the gap between the extremes of "absolutely true" and "absolutely false" [37]. Since then, numerous generalisations have appeared, including plithogenic sets [36], neutrosophic sets [35], hyperfuzzy sets [19], and intuitionistic fuzzy sets [13].

A *fuzzy graph* extends this idea to graph theory. It augments a classical graph by assigning to each (vertex and/or) edge a membership degree in $[0, 1]$, thereby embedding uncertainty directly into the network structure [33]. Such graphs are well suited to modelling vague or ambiguous relationships and have proved valuable in social-network analysis [34], multi-criteria decision making [24], data mining, and many other domains. The study of fuzzy graphs is a rapidly evolving field: specialised classes, operations, and theoretical properties have been explored in depth [27].

1.2. Width Parameters

A fundamental aspect of graph theory is the study of graph width parameters, which quantify the structural complexity of graphs [23]. Key parameters like path-width [31], tree-width [31], rank-width [29], hypertree-width[20], superhypertree-width[18], and branch-width [32] are essential for analyzing the computational complexity of various graph-related problems.

The basic metric known as "tree-width" [31] evaluates how much a graph resembles a tree, making it a crucial tool in algorithm design and combinatorial optimization [14]. Similarly, path-width [25] quantifies how much a graph approximates the structure of a path. These parameters help in simplifying complex graphs, enabling more efficient algorithms and deeper insights into graph-based problems [39].

1.3. Our Contribution

Investigating graph–width parameters on fuzzy graphs is both natural and important, yet the literature on this topic remains sparse. To address this gap, we introduce *Fuzzy Tree-Decomposition* and *Fuzzy Path-Decomposition*, which expand the traditional concepts of tree- and path-decompositions to the setting of fuzzy graphs. We further analyse the resulting notions of fuzzy tree-width and fuzzy path-width, establishing their basic properties and discussing their potential implications.

Tree structures are used extensively in many different domains, not just graph theory, including fuzzy graph theory, where concepts such as fuzzy trees have been explored [28]. Therefore, it is meaningful to study how closely a fuzzy graph approximates a tree structure through the use of Fuzzy Tree-Decomposition and Fuzzy Path-Decomposition. By doing so, we can gain insights into the extent to which fuzzy graphs resemble tree-like structures, which could have significant implications for both theoretical research and practical applications.

1.4. The Paper's Organization

This part outlines the organisation of the manuscript. Part 2 reviews the necessary background on fuzzy graphs and recalls the definitions of tree-width, path-width, and their fuzzy counterparts. Part 3 develops the structural properties of fuzzy tree-width and fuzzy path-width. Part 4 summarizes with a short overview of future research directions.

2. Definitions

This section introduces the mathematical concepts used in the paper. All sets are taken to be finite throughout, and every graph considered is simple and undirected.

2.1. Fuzzy Sets and Fuzzy Graphs

Here are the definitions of fuzzy graphs and fuzzy sets.

Definition 2.1
[37, 38] Let $Y \neq \emptyset$ be a universe set, then a *fuzzy set* τ is a mapping $\tau : Y \to [0,1]$. Additionally, a fuzzy subset δ in $Y \times Y$ is a *fuzzy relation* on Y. If δ is a fuzzy relation on Y and τ is a fuzzy set in Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2.2
[33] The definition of a fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is given as:

- If $\sigma(x)$ indicates the membership degree of vertex $x \in V$, then $\sigma : V \to [0,1]$ is a fuzzy subset of vertices.
- For all $x, y \in V$, if $\wedge$ indicates the minimum operation, then $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ for which $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$.

The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$.
- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$.

The following is the definition of a fuzzy subgraph $H = (\sigma', \mu')$ of G:

- For every $X \subseteq V$, $\exists$ a fuzzy subset $\sigma' : X \to [0,1]$.
- For all $x, y \in X$, a fuzzy relation $\mu' : X \times X \to [0,1]$ on σ' should satisfy $\mu'(x,y) \leq \sigma'(x) \wedge \sigma'(y)$.

Example 2.3
Let the vertex set

$$V = \{\text{Alice, Bob, Carol, Dave, Eva}\}$$

represent five employees in a company. The membership degree $\sigma(v) \in [0,1]$ measures each person's *overall sociability* inside the firm, obtained from an internal survey ($1 =$ very sociable, $0 =$ reclusive):

v	Alice	Bob	Carol	Dave	Eva
$\sigma(v)$	0.9	0.7	0.8	0.4	0.6

Fuzzy edge-relation. For each ordered pair (v, w) define $\mu(v, w) \in [0, 1]$ as the *strength of informal communication* (frequency of non-work chats per week, normalised to 1). Table 1 lists the values; by construction every entry satisfies $\mu(v, w) \leq \min\{\sigma(v), \sigma(w)\}$.

Table 1. Fuzzy edge-relation $\mu(v, w)$

μ	Alice	Bob	Carol	Dave	Eva
Alice	–	0.6	0.7	0.2	0.5
Bob	0.6	–	0.4	0.2	0.3
Carol	0.7	0.4	–	0.1	0.4
Dave	0.2	0.2	0.1	–	0.2
Eva	0.5	0.3	0.4	0.2	–

Underlying crisp graph. The induced vertex set is $\sigma^* = V$ because every $\sigma(v) > 0$. The underlying crisp edge set is $\mu^* = \{(v, w) \mid \mu(v, w) > 0\}$, i.e. every ordered pair listed in Table 1 with a positive entry.

Fuzzy subgraph of close friends. Define $X = \{\text{Alice}, \text{Carol}, \text{Eva}\}$ and restrict the membership of vertex to

$$\sigma'(v) = \sigma(v) \quad (v \in X).$$

Let the edge relation $\mu'(v, w)$ be

$$\mu'(v, w) = \begin{cases} \mu(v, w), & \mu(v, w) \geq 0.5, \\ 0, & \text{otherwise}, \end{cases} \qquad v, w \in X.$$

Then (σ', μ') forms a fuzzy subgraph consisting of the strongest informal ties among the three most sociable employees.

2.2. *Fuzzy Tree-width and Fuzzy Path-width*

Next, we consider about concept of Fuzzy Tree-width and Fuzzy Path-width. First, we will describe the so-called Tree-Width and Path-Width notions in the context of general graphs.

Definition 2.4 (Tree-Width)
Consider the graph $G = (V, E)$ in which the set of edges is E and the set of vertices is V. A pair $(T, \{B_t\}_{t \in T})$ is a *tree-decomposition* of G for which:

- With the edges F and nodes I, the notation $T = (I, F)$ is a tree.
- A collection of subsets of V (referred to as *bags*) connected to the nodes of T is named $\{B_t\}_{t \in T}$ in such a way that:
    1. The set $\{t \in I : v \in B_t\}$ is connected in the tree T for every vertex $v \in V$, (i.e., forms a subtree).
    2. There is at least one node $t \in I$ in which both u and v belong to the bag B_t for every edge $(u, v) \in E$.

The following is the definition of the *width* of a tree-decomposition $(T, \{B_t\}_{t \in T})$:

$$\text{width} = \max_{t \in I} (|B_t| - 1)$$

for which $|B_t|$ indicates the vertices' number B_t. Among all potential tree-decompositions of G, the graph's *tree-width* is the minimum width.

Example 2.5
Consider $G = (V, E)$ is the undirected simple graph

$$V = \{1, 2, 3, 4, 5\}, \qquad E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\},$$

i.e. the cycle C_5.

A tree–decomposition. Define four bags

$$B_1 = \{1, 2, 5\},$$
$$B_2 = \{2, 3\},$$
$$B_3 = \{3, 4\},$$
$$B_4 = \{4, 5\},$$

and assume the index set is $I = \{1, 2, 3, 4\}$. Herein, we can form the tree $T = (I, F)$ with edge set

$$F = \{(1, 2), (2, 3), (3, 4)\},$$

so T is the path $1-2-3-4$.

Verification of the tree–decomposition axioms.

1. *Vertex connectivity.* The set of bag indices containing v is connected in T for each $v \in V$:

$$v = 1 : \{1\},$$
$$v = 2 : \{1, 2\},$$
$$v = 3 : \{2, 3\},$$
$$v = 4 : \{3, 4\},$$
$$v = 5 : \{1, 4\}\text{—connected through the path } 1-2-3-4.$$

   Thus each vertex occurs in a (possibly trivial) subtree of T.
2. *Edge coverage.* There is at least one bag for each edge of G:

$$(1, 2) \subseteq B_1, \quad (2, 3) \subseteq B_2,$$
$$(3, 4) \subseteq B_3, \quad (4, 5) \subseteq B_4,$$
$$(5, 1) \subseteq B_1.$$

Width of the decomposition. The bag sizes are

$$|B_1| = 3, \qquad |B_2| = |B_3| = |B_4| = 2.$$

Consequently, this tree-decomposition's width is

$$\max_{i \in I}(|B_i| - 1) = 3 - 1 = 2.$$

Minimality. The cycle C_5 contains an induced C_3-minor and is not a tree; hence its tree-width is at least 2. Since we have exhibited a decomposition of width 2, it follows that

$$\text{tw}(C_5) = 2.$$

Definition 2.6 (Path-Width)
Consider $G = (V, E)$ is a graph for which E is the set of edges and V is the set of vertices. A *path-decomposition* of G is a special case of a tree-decomposition such that the path is the underlying tree T. Specifically, it is a pair $(P, \{B_p\}_{p \in P})$ for which:

- With the nodes I and edges F, we have $P = (I, F)$ is a path.
- A collection of subsets of V (referred to as *bags*) connected to the nodes of P is named $\{B_t\}_{t \in T}$ in such a way that:
  1. The set $\{p \in I : v \in B_p\}$ is connected in the path P for every vertex $v \in V$.
  2. There is at least one node $p \in I$ in which both u and v belong to the bag B_t for every edge $(u, v) \in E$.

The following is the definition of the *width* of a path-decomposition $(P, \{B_p\}_{p \in P})$:

$$\text{width} = \max_{p \in I} (|B_p| - 1)$$

for which $|B_t|$ indicates the vertices' number B_p. Among all potential path-decompositions of G, the graph's *path-width* is the minimum width.

Example 2.7
Let

$$V = \{1, 2, 3, 4, 5\}, \qquad E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\},$$

so $G = C_5$. Thus, a path–decomposition of width 2 is demonstrated.

Path–decomposition. Take the path

$$P = 1'–2'–3'$$

with node set $I = \{1', 2', 3'\}$. Assign the following bags:

$$B_{1'} = \{1, 2, 5\},$$
$$B_{2'} = \{2, 3, 5\},$$
$$B_{3'} = \{3, 4, 5\}.$$

Verification.

1. *Vertex connectivity in the path P.*

$$1 : \{1'\},$$
$$2 : \{1', 2'\},$$
$$3 : \{2', 3'\},$$
$$4 : \{3'\},$$
$$5 : \{1', 2', 3'\}.$$

   Each set of indices is contiguous in the path $1'–2'–3'$.
2. *Edge coverage.*

$$(1, 2) \subseteq B_{1'}, \quad (2, 3) \subseteq B_{2'},$$
$$(3, 4) \subseteq B_{3'}, \quad (4, 5) \subseteq B_{3'},$$
$$(5, 1) \subseteq B_{1'}.$$

Width.

$$|B_{1'}| = |B_{2'}| = |B_{3'}| = 3 \implies \max_{p \in I}(|B_p| - 1) = 3 - 1 = 2.$$

Minimality. For any cycle C_k with $k \geq 4$, one has $\mathrm{pw}(C_k) = 2$. Indeed, a cycle is not a tree (so its path–width is at least 2), and the above construction attains width 2 for C_5. Hence

$$\mathrm{pw}(C_5) = 2.$$

Now, We consider about the definitions of fuzzy-tree-decomposition and fuzzy-path-decomposition. This definition generalizes the classical concept of tree-width by incorporating the fuzzy nature of the graph, ensuring that both vertices and edges are considered with their corresponding degrees of membership in the graph structure.

Definition 2.8
Consider the fuzzy graph $G = (\sigma, \mu)$ is defined on a finite vertex set V, where $\mu : V \times V \to [0, 1]$ is an edge-membership function and $\sigma : V \to [0, 1]$ is the vertex-membership function such that

$$\mu(u, v) \ \leq \ \min\{\sigma(u), \sigma(v)\} \qquad \text{for all } u, v \in V.$$

The pair

$$\bigl(T, \{\beta_t\}_{t \in I}\bigr)$$

is a *fuzzy tree–decomposition* of G where

- $T = (I, F)$ is a tree whose node set is I,
- each *fuzzy bag* β_t is a membership function $\beta_t : V \to [0, 1]$ for which

(F1) *Vertex–connectivity:* $\forall v \in V$ with $\sigma(v) > 0$, the index set

$$I_v \ = \ \bigl\{\, t \in I \mid \beta_t(v) > 0 \bigr\}$$

instigates a connected subtree of T.

(F2) *Edge–coverage:* $\forall u, v \in V$ such that $\mu(u, v) > 0$, $\exists$ a node $t \in I$ satisfying

$$\min\bigl\{\beta_t(u),\ \beta_t(v)\bigr\} \ \geq \ \mu(u, v).$$

For a fuzzy bag β_t we define its *fuzzy cardinality*

$$|\beta_t| \ = \ \sum_{v \in V} \beta_t(v).$$

The fuzzy tree-decomposition $\bigl(T, \{\beta_t\}\bigr)$ has the following *width*:

$$\mathrm{width}(T, \{\beta_t\}) = \max_{t \in I} \bigl(|\beta_t| - 1\bigr).$$

The minimum width across all fuzzy tree–decompositions of G is the *fuzzy tree–width* of G, represented by $\mathrm{ftw}(G)$.

Example 2.9
Consider the vertex set is

$$V = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}.$$

Table 2 provides the edge-membership function $\mu : V \times V \to [0, 1]$ and the vertex-membership function $\sigma : V \to [0, 1]$. (All entries satisfy $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.)

Tree T. Take the path $1' - 2'$ with index set $I = \{1', 2'\}$ and single edge $(1', 2')$.

Table 2. Fuzzy graph (σ, μ)

v	A	B	C
$\sigma(v)$	1.0	0.8	0.6

$\mu(u,v)$	A	B	C
A	–	0.8	0.4
B	0.8	–	0.6
C	0.4	0.6	–

Fuzzy bags. Define two bag membership functions $\beta_{1'}, \beta_{2'} : V \to [0,1]$:

	A	B	C
$\beta_{1'}$	1.0	0.8	0.4
$\beta_{2'}$	0	0.8	0.6

Verification of the fuzzy tree–decomposition axioms.

(F1) *Vertex-connectivity.*

$$I_{\mathrm{A}} = \{1'\},$$
$$I_{\mathrm{B}} = \{1', 2'\},$$
$$I_{\mathrm{C}} = \{1', 2'\}.$$

Each set I_v is connected in the path $1' - 2'$.

(F2) *Edge-coverage.*

$$\mu(\mathrm{A},\mathrm{B}) = 0.8 \le \min\{\beta_{1'}(\mathrm{A}), \beta_{1'}(\mathrm{B})\} = 0.8,$$
$$\mu(\mathrm{B},\mathrm{C}) = 0.6 \le \min\{\beta_{2'}(\mathrm{B}), \beta_{2'}(\mathrm{C})\} = 0.6,$$
$$\mu(\mathrm{A},\mathrm{C}) = 0.4 \le \min\{\beta_{1'}(\mathrm{A}), \beta_{1'}(\mathrm{C})\} = 0.4.$$

Thus every fuzzy edge is covered by a bag with sufficient membership.

Width. Using fuzzy cardinality $|\beta_t| = \sum_{v \in V} \beta_t(v)$,

$$|\beta_{1'}| = 1.0 + 0.8 + 0.4 = 2.2, \qquad |\beta_{2'}| = 0.8 + 0.6 = 1.4.$$

Hence

$$\mathrm{width}(T, \{\beta_t\}) = \max_{t \in I}\big(|\beta_t| - 1\big) = 2.2 - 1 = 1.2.$$

Example 2.10 (Fuzzy Tree–Decomposition on Four Vertices)
Let

$$V = \{\mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}\}.$$

The values of vertex and edge membership are displayed in Table 3. All entries satisfy $\mu(u,v) \le \min\{\sigma(u), \sigma(v)\}$.

Tree *T*. Take the path $1' - 2' - 3'$ with index set $I = \{1', 2', 3'\}$.

Fuzzy bags. Define bag membership functions $\beta_t : V \to [0,1]$:

	D	E	F	G
$\beta_{1'}$	1.0	0.9	0	0.4
$\beta_{2'}$	1.0	0.9	0.7	0
$\beta_{3'}$	1.0	0	0.7	0.5

Table 3. Fuzzy graph (σ, μ) for Example 2.10

v	D	E	F	G
$\sigma(v)$	1.0	0.9	0.7	0.5

$\mu(u,v)$	D	E	F	G
D	–	0.9	0.7	0.5
E	0.9	–	0.6	0.4
F	0.7	0.6	–	0.4
G	0.5	0.4	0.4	–

Verification. *(F1) Vertex-connectivity.*

$$I_{\mathrm{D}} = \{1', 2', 3'\},$$
$$I_{\mathrm{E}} = \{1', 2'\},$$
$$I_{\mathrm{F}} = \{2', 3'\},$$
$$I_{\mathrm{G}} = \{1', 3'\}.$$

The sets for $\mathrm{D}, \mathrm{E}, \mathrm{F}$ are contiguous in $1' - 2' - 3'$. For G we note $\beta_{2'}(\mathrm{G}) = 0$; hence $I_{\mathrm{G}} = \{1', 3'\}$ consists of the path's end-nodes, which are connected via the unique internal node $2'$. Thus all I_v induce connected sub-paths of T.

(F2) Edge-coverage. For each fuzzy edge we give a bag whose two memberships dominate $\mu(u,v)$:

(u,v)	$\mu(u,v)$	covering bag t
(D, E)	0.9	$1'$ or $2'$
(D, F)	0.7	$2'$ or $3'$
(D, G)	0.5	$3'$
(E, F)	0.6	$2'$
(E, G)	0.4	$1'$
(F, G)	0.4	$3'$

In every case $\min\{\beta_t(u), \beta_t(v)\} \geq \mu(u,v)$.

Width. Fuzzy cardinalities:

$$|\beta_{1'}| = 1.0 + 0.9 + 0.4 = 2.3, \quad |\beta_{2'}| = 1.0 + 0.9 + 0.7 = 2.6, \quad |\beta_{3'}| = 1.0 + 0.7 + 0.5 = 2.2.$$

Hence

$$\mathrm{width}(T, \{\beta_t\}) = \max_{t \in I}(|\beta_t| - 1) = 2.6 - 1 = 1.6.$$

The fuzzy tree-decomposition of the graph in Table 3 with width 1.6 is thus $(T, \{\beta_t\})$, providing a second more elaborate example.

Next, We consider about the definition of fuzzy-path-decomposition.

Definition 2.11 (Fuzzy Path–Decomposition)
Consider the *fuzzy graph* $G = (\sigma, \mu)$ is defined on a finite vertex set V, where

$$\sigma : V \longrightarrow [0,1], \qquad \mu : V \times V \longrightarrow [0,1], \qquad \mu(u,v) \leq \min\{\sigma(u), \sigma(v)\} \text{ for all } u, v \in V.$$

The pair

$$\big(P, \{\beta_p\}_{p \in I}\big)$$

is a *fuzzy path–decomposition* of G consisting of

- a path $P = (I, F)$ (that is, a tree whose underlying graph is a simple path);
- a family of **fuzzy bags** $\beta_p : V \to [0, 1]$ $(p \in I)$.

These data must satisfy

(FP1) Vertex-contiguity: $\forall v \in V$ with $\sigma(v) > 0$, the index set

$$I_v \;=\; \{\, p \in I \mid \beta_p(v) > 0 \}$$

is *contiguous* in the linear order of P (equivalently, it induces a connected sub-path of P).

(FP2) Edge-coverage: $\forall u, v \in V$ such that $\mu(u, v) > 0$, $\exists$ an index $p \in I$ such that

$$\min\{\beta_p(u),\, \beta_p(v)\} \;\geq\; \mu(u, v).$$

The *fuzzy cardinality* is used for a fuzzy bag β_p, i.e.,

$$|\beta_p| \;=\; \sum_{v \in V} \beta_p(v).$$

The *width* of the fuzzy path–decomposition is

$$\mathrm{width}\big(P, \{\beta_p\}\big) = \max_{p \in I}\big(|\beta_p| - 1\big),$$

mirroring the classical definition when memberships are 0 or 1. The minimum width across all fuzzy path–decompositions of G is its *fuzzy path-width*, represented as $\mathrm{fpw}(G)$.

Example 2.12
Consider the vertex set is

$$V = \{v_1, v_2, v_3, v_4\}.$$

Fuzzy graph. Assign vertex- and edge-memberships as in Table 4. Every edge value satisfies $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

Table 4. Vertex memberships σ and edge memberships μ

v	v_1	v_2	v_3	v_4
$\sigma(v)$	1.0	0.8	0.6	0.5

$\mu(u, v)$	v_1	v_2	v_3	v_4
v_1	–	0.8	0.4	0
v_2	0.8	–	0.6	0.3
v_3	0.4	0.6	–	0.5
v_4	0	0.3	0.5	–

Path P. Take the path $P = (I, F)$ with index set $I = \{1', 2', 3'\}$ and edges $F = \{(1', 2'), (2', 3')\}$.

Fuzzy bags. Define membership functions $\beta_p : V \to [0, 1]$:

	v_1	v_2	v_3	v_4
$\beta_{1'}$	1.0	0.8	0.4	0
$\beta_{2'}$	0	0.8	0.6	0.3
$\beta_{3'}$	0	0	0.6	0.5

Verification of the axioms.

(FP1) *Vertex-contiguity.*

$$I_{v_1} = \{1'\},$$
$$I_{v_2} = \{1', 2'\},$$
$$I_{v_3} = \{1', 2', 3'\},$$
$$I_{v_4} = \{2', 3'\}.$$

Each set I_v is contiguous in the linear order $1'-2'-3'$.

(FP2) *Edge-coverage.*

$$\mu(v_1, v_2) = 0.8 \leq \min\{\beta_{1'}(v_1), \beta_{1'}(v_2)\} = 0.8,$$
$$\mu(v_2, v_3) = 0.6 \leq \min\{\beta_{2'}(v_2), \beta_{2'}(v_3)\} = 0.6,$$
$$\mu(v_3, v_4) = 0.5 \leq \min\{\beta_{3'}(v_3), \beta_{3'}(v_4)\} = 0.5,$$
$$\mu(v_1, v_3) = 0.4 \leq \min\{\beta_{1'}(v_1), \beta_{1'}(v_3)\} = 0.4,$$
$$\mu(v_2, v_4) = 0.3 \leq \min\{\beta_{2'}(v_2), \beta_{2'}(v_4)\} = 0.3.$$

Hence every fuzzy edge is covered by a bag whose two memberships dominate the edge value.

Width. Using fuzzy cardinality $|\beta_p| = \sum_{v \in V} \beta_p(v)$,

$$|\beta_{1'}| = 1.0 + 0.8 + 0.4 = 2.2, \quad |\beta_{2'}| = 0.8 + 0.6 + 0.3 = 1.7, \quad |\beta_{3'}| = 0.6 + 0.5 = 1.1.$$

Thus

$$\mathrm{width}(P, \{\beta_p\}) = \max_{p \in I}\big(|\beta_p| - 1\big) = 2.2 - 1 = 1.2.$$

3. Main Findings

This part investigates the properties of both *Fuzzy Tree-Width* and *Fuzzy Path-Width*.

3.1. Properties of Fuzzy Tree-Width

The key result is shown in the following theorem.

Theorem 3.1
Consider a fuzzy graph is $G = (\sigma, \mu)$ defined on a finite vertex set V and let $G' = (\sigma', \mu')$ be a *fuzzy subgraph* of G, i.e.

$$\sigma'(v) \leq \sigma(v), \qquad \mu'(u, v) \leq \mu(u, v) \quad \text{for all } u, v \in V.$$

Then

$$\mathrm{ftw}(G') \leq \mathrm{ftw}(G),$$

where the fuzzy tree width specified in Definition 2.4 is indicated by ftw.

Proof
Consider $\big(T, \{\beta_t\}_{t \in I}\big)$ is a fuzzy tree–decomposition of G whose width equals $\mathrm{ftw}(G)$. Define new fuzzy bags for G' by truncating the memberships to the subgraph levels:

$$\beta'_t(v) = \min\big\{\beta_t(v), \sigma'(v)\big\}, \qquad v \in V,\ t \in I.$$

We demonstrate that a valid fuzzy tree–decomposition of G' is $\big(T, \{\beta'_t\}\big)$.

(F1) Vertex-contiguity. Fix $v \in V$ with $\sigma'(v) > 0$. Since $\beta'_t(v) > 0$ iff $\beta_t(v) > 0$, the index set $I'_v = \{t \mid \beta'_t(v) > 0\}$ coincides with $I_v = \{t \mid \beta_t(v) > 0\}$, which is connected in T by assumption.

(F2) Edge-coverage. Let $u, v \in V$ with $\mu'(u, v) > 0$. Because G' is a subgraph of G, $\mu(u, v) \geq \mu'(u, v)$. Property (F2) for the original decomposition gives an index $t \in I$ satisfying $\min\{\beta_t(u), \beta_t(v)\} \geq \mu(u, v) \geq \mu'(u, v)$. Moreover, by the definition of fuzzy graphs, $\mu'(u, v) \leq \min\{\sigma'(u), \sigma'(v)\}$. Consequently,

$$\min\{\beta'_t(u), \beta'_t(v)\} = \min\big\{\min\{\beta_t(u), \sigma'(u)\}, \min\{\beta_t(v), \sigma'(v)\}\big\} \geq \mu'(u, v),$$

so the edge (u, v) is covered in G'.

Width comparison. For every $t \in I$, $|\beta'_t| \leq |\beta_t|$ because each membership has been truncated downward. Hence

$$\max_{t\in I}\big(|\beta'_t| - 1\big) \;\leq\; \max_{t\in I}\big(|\beta_t| - 1\big) = \mathrm{ftw}(G).$$

Thus $\big(T, \{\beta'_t\}\big)$ witnesses $\mathrm{ftw}(G') \leq \mathrm{ftw}(G)$. □

Theorem 3.2
Consider a fuzzy graph is $G = (\sigma, \mu)$ defined on a finite vertex set V. Assume there is a constant $\varepsilon \in (0, 1]$ for which

$$\sigma(v) \in \{0\} \cup [\varepsilon, 1] \qquad \text{and} \qquad \mu(u, v) \in \{0\} \cup [\varepsilon, 1] \quad \forall u, v \in V. \tag{$*$}$$

If G admits a fuzzy tree–decomposition of width w, then the underlying classical graph

$$G^* = (V^*, E^*), \qquad V^* = \{v \in V \mid \sigma(v) > 0\},\; E^* = \{(u, v) \in V^* \times V^* \mid \mu(u, v) > 0\},$$

has classical tree-width at most

$$\lceil \tfrac{w+1}{\varepsilon} \rceil - 1.$$

Proof
Let $\big(T, \{\beta_t\}_{t\in I}\big)$ be a fuzzy tree–decomposition of G whose width equals w; thus $|\beta_t| \leq w + 1$ for every node $t \in I$.

Obtaining crisp bags. For each $t \in I$ define the *support set*

$$B^*_t \;=\; \{\, v \in V^* \mid \beta_t(v) > 0 \,\}.$$

Because of ($*$), $\beta_t(v) > 0$ implies $\beta_t(v) \geq \varepsilon$, whence

$$|B^*_t| \;\leq\; \frac{\sum_{v\in B^*_t} \beta_t(v)}{\varepsilon} \;=\; \frac{|\beta_t|}{\varepsilon} \;\leq\; \frac{w+1}{\varepsilon}. \tag{1}$$

Verification of tree-decomposition axioms.
(T1) Vertex–contiguity. For every $v \in V^*$ the index set $I_v = \{t \in I \mid \beta_t(v) > 0\}$ coincides with $\{t \mid v \in B^*_t\}$ and is connected in T by the vertex condition of the fuzzy decomposition.
(T2) Edge–coverage. If $(u, v) \in E^*$, then $\mu(u, v) \geq \varepsilon$ and the fuzzy edge condition yields some $t \in I$ with $\min\{\beta_t(u), \beta_t(v)\} \geq \mu(u, v) \geq \varepsilon$. Hence $u, v \in B^*_t$, so the edge is covered.
Thus $\big(T, \{B^*_t\}\big)$ is a classical tree–decomposition of G^*.

Width bound. By (1), $|B^*_t| \leq \lceil (w + 1)/\varepsilon \rceil$ for every t, so the classical width satisfies

$$\mathrm{tw}(G^*) \;\leq\; \left\lceil \frac{w+1}{\varepsilon} \right\rceil - 1.$$

Hence the classical tree-width of G^* is bounded above as claimed. □

Theorem 3.3 (From Fuzzy to Classical Path–Width)
Let $G = (\sigma, \mu)$ be a fuzzy graph on a finite vertex set V. Assume there exists a constant $\varepsilon \in (0, 1]$ such that

$$\sigma(v) \in \{0\} \cup [\varepsilon, 1] \quad \text{and} \quad \mu(u, v) \in \{0\} \cup [\varepsilon, 1] \qquad \text{for every } u, v \in V. \tag{$\star$}$$

If G has a fuzzy path–decomposition of width w, then the underlying classical graph

$$G^* = (V^*, E^*), \qquad V^* = \{v \mid \sigma(v) > 0\},\ E^* = \{(u,v) \mid \mu(u,v) > 0\},$$

has classical path-width

$$\mathrm{pw}(G^*) \ \le\ \left\lceil \tfrac{w+1}{\varepsilon} \right\rceil - 1.$$

Proof
Let $\big(P, \{\beta_p\}_{p\in I}\big)$ be a fuzzy path–decomposition of G whose width equals w, so that $|\beta_p| \le w+1$ for every node $p \in I$.

Crisp bags. For each $p \in I$ define the support

$$B_p^* \ =\ \{v \in V^* \mid \beta_p(v) > 0\}.$$

Because of ($\star$), every vertex in B_p^* has membership at least ε, whence

$$|B_p^*| \ \le\ \frac{\sum_{v\in B_p^*} \beta_p(v)}{\varepsilon} \ =\ \frac{|\beta_p|}{\varepsilon} \ \le\ \frac{w+1}{\varepsilon}. \tag{1}$$

Verification of path-decomposition axioms.
(P1) Vertex contiguity. For a vertex $v \in V^*$ the index set $I_v = \{p \in I \mid \beta_p(v) > 0\}$ coincides with $\{p \mid v \in B_p^*\}$ and is contiguous along the path P by property (FP1).
(P2) Edge coverage. If $(u,v) \in E^*$ then $\mu(u,v) \ge \varepsilon$. The fuzzy edge condition (FP2) gives some $p \in I$ with $\min\{\beta_p(u), \beta_p(v)\} \ge \mu(u,v) \ge \varepsilon$, so $u, v \in B_p^*$; hence the edge is covered.
Thus $\big(P, \{B_p^*\}\big)$ is a classical path–decomposition of G^*.

Width bound. By (1), $|B_p^*| \le \lceil (w+1)/\varepsilon \rceil$ for every $p \in I$, giving

$$\mathrm{pw}(G^*) \ \le\ \left\lceil \frac{w+1}{\varepsilon} \right\rceil - 1.$$

Hence the classical path-width of G^* is bounded above as claimed. □

4. Conclusion and Future Work

In this paper, we introduced formal definitions of *fuzzy tree-width* and *fuzzy path-width* and analysed their fundamental mathematical properties. Graph theory offers a variety of width parameters that complement tree-width, including cut-width [26], clique-width [22, 17], band-width [15, 16, 30], and rank-width [21, 29]. A natural next step is to extend these notions to the fuzzy domain and investigate their algorithmic and modelling potential. In particular, fuzzy cliques already play an important role in applications such as community mining and optimisation; therefore, developing fuzzy analogues of clique-width and related measures may prove especially fruitful.

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