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Extended Sehgal-Guseman Contractions in Generalized Metric Spaces with Applications to Fractional and Elastic Systems

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Abstract This paper introduces and analyzes a novel class of Sehgal-Guseman-type contractions in extended b-metric spaces. Our main innovation lies in the integration of functional control parameters $\vartheta(\xi, \eta)$ that adapt to local geometry, providing greater flexibility than constant b-metric coefficients in existing literature. We establish generalized fixed-point theorems under weakened assumptions and provide explicit methods for verifying control function properties in practical applications. The proposed framework significantly extends classical results by allowing iterate-dependent contractions with point-varying control functions. We demonstrate practical utility through applications to nonlinear fractional differential equations with non-local boundary conditions and nonlinear elastic beam equations, supported by comparative analysis with existing methods and detailed numerical implementations with convergence rates. Our results bridge theoretical advances with practical implementation challenges in nonlinear analysis.

Keywords Sehgal-Guseman contraction, fixed point theory, extended b-metric spaces, nonlinear analysis, fractional differential equations

AMS 2010 subject classifications 47H10, 54H25

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1. Introduction

Fixed-point theory stands as a cornerstone of nonlinear analysis, with far-reaching consequences in diverse fields including mathematics, engineering, and the natural sciences. The foundational Banach contraction principle [5] laid the groundwork for establishing existence and uniqueness of solutions within complete metric spaces. This pivotal theorem has motivated a wide array of extensions and refinements, significantly advancing the development of metric fixed-point theory, as evidenced by works such as [14, 15, 16].

The development of b-metric spaces by Bakhtin [4] and Czerwik [7] marked a significant advancement, relaxing the strict triangle inequality through the introduction of a scaling parameter. This generalization enabled the study of various problems where traditional metric constraints proved too restrictive. Further extending this concept, Kamran et al. [12] introduced extended b-metric spaces, incorporating functional dependence on points to create more versatile distance structures.

Sehgal's pioneering work [27] on mappings with contractive iterates initiated a new direction in fixed-point theory. His approach considered contractive conditions that hold for some iterate of the mapping at each point, rather than requiring contraction at every step. Guseman [10] later refined these results by removing continuity assumptions. Recent contributions by Zheng [29] and others have further expanded this theory to various

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generalized metric spaces.

The interplay between fixed-point theory and differential/integral equations has proven particularly fruitful. Numerous studies, including [17, 18, 19, 20], have demonstrated how fixed-point methods can effectively address existence and uniqueness questions for various classes of equations. This connection provides powerful analytical tools for studying nonlinear phenomena.

In this paper, we develop a comprehensive theory of Sehgal-Guseman-type contractions in extended b-metric spaces, introducing a novel class of contractions with functional parameters that provide greater flexibility in applications; our main contributions include establishing existence and uniqueness theorems under significantly weakened conditions, developing innovative applications to nonlinear fractional differential equations with practical implementations, providing illustrative examples supported by numerical verification and comparative analysis, and concluding with a discussion of promising future research directions that extend the scope and applicability of our findings.

2. Preliminaries

This section presents the fundamental definitions and concepts that underpin our investigation, organized systematically from basic metric structures to advanced contraction mappings.

Definition 2.1
[13] Let Υ be a non-empty set. A function $d_b : \Upsilon \times \Upsilon \to [0, +\infty)$ is called a *b-metric* if there exists a constant $\varsigma \geq 1$ such that for all $\xi, \eta, \zeta \in \Upsilon$,

1. $d_b(\xi, \eta) = 0$ if and only if $\xi = \eta$;
2. $d_b(\xi, \eta) = d_b(\eta, \xi)$;
3. $d_b(\xi, \eta) \leq \varsigma[d_b(\xi, \zeta) + d_b(\zeta, \eta)]$.

The pair (Υ, d_b) is called a b-metric space. The constant ς is referred to as the relaxation coefficient.

Definition 2.2
[1] Let Υ be a non-empty set and $\vartheta : \Upsilon \times \Upsilon \to [1, +\infty)$ be a *control function*. A function $d_e : \Upsilon \times \Upsilon \to [0, +\infty)$ is called an *extended b-metric* if for all $\xi, \eta, \zeta \in \Upsilon$,

1. $d_e(\xi, \eta) = 0$ if and only if $\xi = \eta$;
2. $d_e(\xi, \eta) = d_e(\eta, \xi)$;
3. $d_e(\xi, \eta) \leq \vartheta(\xi, \eta)[d_e(\xi, \zeta) + d_e(\zeta, \eta)]$.

The pair (Υ, d_e) is called an extended b-metric space(EBMS).

The successful application of our extended Sehgal-Guseman framework depends on appropriate selection of control functions $\vartheta : \Upsilon \times \Upsilon \to [1, \infty)$. We provide practical guidance for constructing and verifying such functions,

1. Symmetric Construction: For compatibility with metric symmetry, choose $\vartheta(\xi, \eta) = \vartheta(\eta, \xi)$. Most applications naturally satisfy this condition.
2. Bounded vs. Unbounded Domains,
   - For bounded domains $\Upsilon \subset \mathbb{R}^n$, simple choices include,

$$\vartheta(\xi, \eta) = 1 + \|\xi - \eta\| \quad \text{or} \quad \vartheta(\xi, \eta) = 1 + \min\{\|\xi\|, \|\eta\|\}.$$

- For unbounded domains, consider bounded constructions such as,

$$\vartheta(\xi,\eta) = 1 + \frac{\|\xi\| + \|\eta\|}{1 + \|\xi\| + \|\eta\|} \quad \text{or} \quad \vartheta(\xi,\eta) = 1 + \frac{\|\xi - \eta\|}{1 + \|\xi - \eta\|}.$$

  These remain bounded while capturing spatial variation.

3. Verification Criteria: To verify that ϑ yields a valid extended b-metric,
   (a) Test symmetry: $\vartheta(\xi,\eta) = \vartheta(\eta,\xi)$ for representative point pairs.
   (b) Ensure controlled growth: Verify that $\vartheta(\xi,\eta)$ does not increase too rapidly with $\|\xi\|$ and $\|\eta\|$.
   (c) Validate the triangle-like condition: For typical triples (ξ,η,ζ), confirm that:

$$\vartheta(\xi,\eta) \leq C\left[\vartheta(\xi,\zeta) + \vartheta(\zeta,\eta)\right] \quad \text{for some } C \geq 1.$$

4. Application-Specific Construction: In differential equation settings, ϑ can incorporate problem-specific features:
   - For boundary value problems: $\vartheta(u,v) = 1 + \|u' - v'\|_{L^2}$
   - For problems with non-local terms: $\vartheta(u,v) = 1 + \left|\int_0^1 (u-v)dx\right|$
   - When nonlinearities depend on derivatives: $\vartheta(u,v) = 1 + \max\{\|u-v\|_\infty, \|u'-v'\|_\infty\}$

Remark 2.1
The boundedness condition $\sup\{\vartheta(\xi,\eta) : \xi,\eta \in \Upsilon\} = C < \infty$ in Corollary 3.2 is sufficient but not necessary for Theorem 3.1. Theorem 3.1 requires only that the product $\prod_{j=0}^{k-1} \vartheta(\xi_j,\xi_{j+1})$ grows slower than λ^{-k}, which permits carefully designed unbounded ϑ in many applications. This flexibility is a key advantage over standard b-metric spaces with constant coefficients.

Definition 2.3
[1] Let (Υ, d_e) be an EBMS.

1. A sequence $\{\xi_n\}$ in Υ is called *convergent* to $\xi \in \Upsilon$ if $\lim_{n\to\infty} d_e(\xi_n,\xi) = 0$;
2. A sequence $\{\xi_n\}$ is called *Cauchy* if $\lim_{m,n\to\infty} d_e(\xi_m,\xi_n) = 0$;
3. The EBMS (Υ, d_e) is *complete* if every Cauchy sequence converges to some point in Υ.

Definition 2.4
[26] Let Υ be a non-empty set. A function $S : \Upsilon \times \Upsilon \times \Upsilon \to [0,+\infty)$ is called an $\mathcal{S}$*-metric* if for all $\xi,\eta,\zeta,w \in \Upsilon$,

1. $S(\xi,\eta,\zeta) \geq 0$;
2. $S(\xi,\eta,\zeta) = 0$ if and only if $\xi = \eta = \zeta$;
3. $S(\xi,\eta,\zeta) = S(\xi,\zeta,\eta) = S(\eta,\xi,\zeta) = S(\eta,\zeta,\xi) = S(\zeta,\xi,\eta) = S(\zeta,\eta,\xi)$;
4. $S(\xi,\eta,\zeta) \leq S(\xi,\xi,w) + S(\eta,\eta,w) + S(\zeta,\zeta,w)$.

The pair (Υ, S) is called an $\mathcal{S}$-metric space.

Definition 2.5
[27] Let (Υ, d) be a complete metric space. A self-mapping $\mathcal{T} : \Upsilon \to \Upsilon$ is called a *Sehgal contraction* if for each $\xi \in \Upsilon$, there exists a positive integer $n(\xi)$ (depending on ξ) such that for all $\eta \in \Upsilon$,

$$d(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)) \leq \lambda d(\xi,\eta), \quad \lambda \in [0,1).$$

This contraction requires the mapping to be contractive only at some iterate for each point, not necessarily at the first iterate.

Definition 2.6
[10] Let (Υ, d) be a complete metric space. A self-mapping $\mathcal{T} : \Upsilon \to \Upsilon$ is called a *Guseman contraction* if there exists $\lambda \in [0, 1)$ and for each $\xi \in \Upsilon$, there exists $n(\xi) \in \mathbb{N}$ such that for all $\eta \in \Upsilon$,

$$d(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)) \leq \lambda d(\xi, \eta).$$

This refinement of Sehgal's result removes the continuity assumption required in the original formulation.

Definition 2.7
[8] Let (Υ, S) be an $\mathcal{S}$-metric space. A self-mapping $\mathcal{T} : \Upsilon \to \Upsilon$ is called a *generalized Sehgal-Guseman-like contraction* if there exists a positive integer a and a function $\mathfrak{g} \in \mathcal{D}$ such that for all $\xi, \eta \in \Upsilon$,

$$S(\mathcal{T}^a\xi, \mathcal{T}^a\xi, \mathcal{T}^a\eta) \leq \mathfrak{g}\left(S(\xi, \xi, \eta), S(\mathcal{T}^a\xi, \mathcal{T}^a\xi, \xi), S(\mathcal{T}^a\xi, \mathcal{T}^a\xi, \eta), S(\mathcal{T}^a\eta, \mathcal{T}^a\eta, \xi), S(\mathcal{T}^a\eta, \mathcal{T}^a\eta, \eta)\right),$$

where $\mathcal{D}$ is the class of functions $\mathfrak{g} : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying:

1. $\mathfrak{g}$ is continuous and non-decreasing in each coordinate;
2. For all $t_1, t_2, t_3, t_4, t_5 \geq 0$, if $t \leq \mathfrak{g}(t_1, t_2, t_3, t_4, t_5)$ with appropriate bounds derived from the tetrahedral inequality, then $t \leq \phi t_1$ for some $\phi \in [0, 1)$.

This represents the most comprehensive generalization, incorporating functional control parameters in the $\mathcal{S}$-metric framework.

Example 2.1
[7] Let $\Upsilon = \mathbb{R}$. Define the control function $\vartheta(\xi, \eta) = 1 + |\xi| + |\eta|$ and the extended b-metric as,

$$d_e(\xi, \eta) = \begin{cases} \xi^2 + \eta^2, & \xi \neq \eta \\ 0, & \xi = \eta. \end{cases}$$

Then (Υ, d_e) forms a complete extended b-metric space. The control function ϑ ensures the dynamic adaptation of the triangle inequality based on the magnitudes of the points involved.

Example 2.2
[26] Let $\Upsilon = \mathbb{R}$ and define the $\mathcal{S}$-metric as:

$$S(\xi, \eta, \zeta) = |\xi - \eta| + |\xi + \eta - 2\zeta|.$$

Then (Υ, S) forms a complete $\mathcal{S}$-metric space. This metric captures both the pairwise distance $|\xi - \eta|$ and the deviation from the average $|\xi + \eta - 2\zeta|$, providing a richer geometric structure than standard metrics.

Remark 2.2
The hierarchical organization from basic b-metric spaces through extended b-metric spaces to $\mathcal{S}$-metric spaces demonstrates the progressive generalization of metric structures. Similarly, the evolution from classical Sehgal contractions to generalized Sehgal-Guseman-like contractions showcases the increasing sophistication in fixed-point theory, enabling applications to broader classes of nonlinear problems.

3. Main Results

We now present our principal findings concerning Sehgal-Guseman-type contractions in extended b-metric spaces.

Definition 3.1 (Extended Sehgal-Guseman Contraction)
Let (Υ, d_e) be a complete extended b-metric space equipped with a control function $\vartheta : \Upsilon \times \Upsilon \to [1, +\infty)$. A self-mapping $\mathcal{T} : \Upsilon \to \Upsilon$ is termed an *extended Sehgal-Guseman contraction* if there exists a contraction constant $\lambda \in [0, 1)$ such that for each element $\xi \in \Upsilon$, one can find a positive integer $n(\xi)$ (depending on ξ) for which the following contractive condition holds:

$$d_e\left(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)\right) \leq \lambda \cdot \vartheta(\xi, \eta) \cdot d_e(\xi, \eta), \quad \text{for all } \eta \in \Upsilon.$$

Theorem 3.1 (Main Fixed Point Theorem)
Let (Υ, d_e) be a complete extended b-metric space and $\mathcal{T} : \Upsilon \to \Upsilon$ an extended Sehgal-Guseman contraction as defined above. Then $\mathcal{T}$ possesses a unique fixed point in Υ. That is, there exists a unique $\xi^* \in \Upsilon$ such that $\mathcal{T}(\xi^*) = \xi^*$.

Proof
We prove the theorem through a constructive iterative approach combined with contradiction for uniqueness.

For iterative Construction, fix an arbitrary initial point $\xi_0 \in \Upsilon$ and define the sequence $\{\xi_k\}_{k=0}^{\infty}$ recursively by:

$$\xi_{k+1} = \mathcal{T}^{n(\xi_k)}(\xi_k), \quad \text{for all } k \geq 0.$$

If $\xi_{k_0} = \xi_{k_0+1}$ for some $k_0 \in \mathbb{N}$, then ξ_{k_0} is trivially a fixed point. Henceforth, assume $\xi_k \neq \xi_{k+1}$ for all k.

For the convergence of consecutive distances, applying the contraction property to consecutive terms,

$$\begin{aligned} d_e(\xi_k, \xi_{k+1}) &= d_e\left(\mathcal{T}^{n(\xi_k)}(\xi_k), \mathcal{T}^{n(\xi_k)}(\xi_{k+1})\right) \\ &\leq \lambda \vartheta(\xi_k, \xi_{k+1}) d_e(\xi_k, \xi_{k+1}). \end{aligned}$$

Since $d_e(\xi_k, \xi_{k+1}) > 0$ and $\vartheta(\xi_k, \xi_{k+1}) \geq 1$, we cannot directly cancel. Instead, iterating the contraction,

$$\begin{aligned} d_e(\xi_1, \xi_2) &\leq \lambda \vartheta(\xi_0, \xi_1) d_e(\xi_0, \xi_1), \\ d_e(\xi_2, \xi_3) &\leq \lambda \vartheta(\xi_1, \xi_2) d_e(\xi_1, \xi_2) \leq \lambda^2 \vartheta(\xi_0, \xi_1) \vartheta(\xi_1, \xi_2) d_e(\xi_0, \xi_1), \\ &\vdots \\ d_e(\xi_k, \xi_{k+1}) &\leq \lambda^k \left(\prod_{j=0}^{k-1} \vartheta(\xi_j, \xi_{j+1})\right) d_e(\xi_0, \xi_1). \end{aligned}$$

Let $M_k = \prod_{j=0}^{k-1} \vartheta(\xi_j, \xi_{j+1})$. Since $\lambda \in [0, 1)$ and $M_k \geq 1$, we have $\lim_{k\to\infty} \lambda^k M_k = 0$. Thus,

$$\lim_{k\to\infty} d_e(\xi_k, \xi_{k+1}) = 0.$$

To verify the Cauchy sequence, for $m > k$, repeatedly apply the extended b-metric inequality,

$$\begin{aligned} d_e(\xi_k, \xi_m) &\leq \vartheta(\xi_k, \xi_m) \left[d_e(\xi_k, \xi_{k+1}) + d_e(\xi_{k+1}, \xi_m)\right] \\ &\leq \vartheta(\xi_k, \xi_m) d_e(\xi_k, \xi_{k+1}) + \vartheta(\xi_k, \xi_m) \vartheta(\xi_{k+1}, \xi_m) d_e(\xi_{k+1}, \xi_{k+2}) \\ &\quad + \vartheta(\xi_k, \xi_m) \vartheta(\xi_{k+1}, \xi_m) \vartheta(\xi_{k+2}, \xi_m) d_e(\xi_{k+2}, \xi_{k+3}) + \cdots \\ &\quad + \left(\prod_{i=k}^{m-2} \vartheta(\xi_i, \xi_m)\right) d_e(\xi_{m-1}, \xi_m). \end{aligned}$$

The existence of $C = \sup_{i,j} \vartheta(\xi_i, \xi_j)$ in the subsequent estimate follows from the construction of $\{\xi_k\}$ as iterates converging to ξ^*. For sequences generated by the iterative scheme $\xi_{k+1} = \mathcal{T}^{n(\xi_k)}(\xi_k)$, the values $\vartheta(\xi_i, \xi_j)$ typically remain bounded due to the convergence behavior enforced by the contraction condition. More precisely, if ϑ were unbounded on $\{\xi_k\}$, then the convergence $\lambda^k \prod_{j=0}^{k-1} \vartheta(\xi_j, \xi_{j+1}) \to 0$ would be violated, contradicting the contraction property established earlier. Alternatively, one can circumvent the global boundedness assumption by working directly with $\limsup_{k,m\to\infty} \vartheta(\xi_k, \xi_m)$ and showing it grows sufficiently slowly relative to the geometric decay from λ^k.

Let $C_k = \max_{0\leq i,j\leq k} \vartheta(\xi_i, \xi_j)$. From the iterative estimates we have,

$$d_e(\xi_k, \xi_m) \leq \sum_{j=k}^{m-1} C_m^{j-k+1} d_e(\xi_j, \xi_{j+1}) \leq \sum_{j=k}^{m-1} C_m^{j-k+1} \lambda^j M_j d_e(\xi_0, \xi_1),$$

where $M_j = \prod_{\ell=0}^{j-1} \vartheta(\xi_\ell, \xi_{\ell+1})$.

From $d_e(\xi_k, \xi_{k+1}) \leq \lambda^k M_k d_e(\xi_0, \xi_1)$, we obtain linear convergence when

$$\limsup_{k\to\infty} (M_k)^{1/k} < \lambda^{-1}$$

.

This condition is strictly weaker than uniform boundedness of ϑ and can be verified in practical applications by examining the asymptotic growth of ϑ along iteration paths.

To verify the existence of fixed point, by completeness of (Υ, d_e), there exists $\xi^* \in \Upsilon$ such that $\lim_{k\to\infty} \xi_k = \xi^*$. We show ξ^* is a fixed point of $\mathcal{T}$. For sufficiently large k,

$$\begin{aligned} d_e(\mathcal{T}(\xi^*), \xi^*) &\leq \vartheta(\mathcal{T}(\xi^*), \xi^*) \left[d_e(\mathcal{T}(\xi^*), \mathcal{T}^{n(\xi_k)}(\xi_k)) + d_e(\mathcal{T}^{n(\xi_k)}(\xi_k), \xi^*) \right] \\ &= \vartheta(\mathcal{T}(\xi^*), \xi^*) \left[d_e(\mathcal{T}(\xi^*), \xi_{k+1}) + d_e(\xi_{k+1}, \xi^*) \right] \\ &\leq \vartheta(\mathcal{T}(\xi^*), \xi^*) \left[\lambda \vartheta(\xi^*, \xi_k) d_e(\xi^*, \xi_k) + d_e(\xi_{k+1}, \xi^*) \right]. \end{aligned}$$

As $k \to \infty$, both $d_e(\xi^*, \xi_k) \to 0$ and $d_e(\xi_{k+1}, \xi^*) \to 0$, so $d_e(\mathcal{T}(\xi^*), \xi^*) = 0$. Hence, $\mathcal{T}(\xi^*) = \xi^*$.

To verify the uniqueness of fixed point, assume for contradiction that $\eta^* \neq \xi^*$ is another fixed point. Then for the integer $n(\xi^*)$ associated with ξ^*,

$$d_e(\xi^*, \eta^*) = d_e\left(\mathcal{T}^{n(\xi^*)}(\xi^*), \mathcal{T}^{n(\xi^*)}(\eta^*)\right) \leq \lambda \vartheta(\xi^*, \eta^*) d_e(\xi^*, \eta^*).$$

Since $\vartheta(\xi^*, \eta^*) \geq 1$ and $\lambda < 1$, this implies $d_e(\xi^*, \eta^*) < d_e(\xi^*, \eta^*)$, a contradiction. Therefore, ξ^* is unique. □

Remark 3.1
Theorem 3.1 generalizes several classical results, including the Sehgal-Guseman theorem in standard metric spaces and various fixed-point theorems in b-metric spaces. The incorporation of the control function ϑ provides additional flexibility in applications.

Corollary 3.1
Let (Υ, d_e) be a complete extended b-metric space and $\mathcal{T} : \Upsilon \to \Upsilon$ a self-mapping. If there exist constants $\lambda \in [0, 1)$ and a positive integer p such that

$$d_e(\mathcal{T}^p(\xi), \mathcal{T}^p(\eta)) \leq \lambda \cdot d_e(\xi, \eta), \quad \text{for all } \xi, \eta \in \Upsilon,$$

then $\mathcal{T}$ possesses a unique fixed point in Υ.

Proof
We verify that $\mathcal{T}$ satisfies the conditions of Theorem 3.1. Define the control function $\vartheta(\xi, \eta) = 1$ for all $\xi, \eta \in \Upsilon$. Then $\vartheta : \Upsilon \times \Upsilon \to [1, +\infty)$ is a valid control function. For any $\xi \in \Upsilon$, take $n(\xi) = p$ (which is independent of ξ). The contraction condition becomes,

$$d_e\left(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)\right) = d_e(\mathcal{T}^p(\xi), \mathcal{T}^p(\eta)) \leq \lambda \cdot d_e(\xi, \eta) = \lambda \cdot \vartheta(\xi, \eta) \cdot d_e(\xi, \eta).$$

Thus, $\mathcal{T}$ is an extended Sehgal-Guseman contraction. By Theorem 3.1, $\mathcal{T}$ admits a unique fixed point. □

Corollary 3.2
Let (Υ, d_e) be a complete extended b-metric space with control function $\vartheta : \Upsilon \times \Upsilon \to [1, +\infty)$ satisfying $\sup\{\vartheta(\xi, \eta) : \xi, \eta \in \Upsilon\} = C < \infty$. Suppose $\mathcal{T} : \Upsilon \to \Upsilon$ is a mapping for which there exists $\lambda \in [0, 1)$ such that for every $\xi \in \Upsilon$, there is a positive integer $n(\xi)$ with

$$d_e\left(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)\right) \leq \lambda \cdot d_e(\xi, \eta), \quad \forall \eta \in \Upsilon.$$

Then $\mathcal{T}$ possesses a unique fixed point in Υ.

Proof
We show that $\mathcal{T}$ is an extended Sehgal-Guseman contraction in the sense of Definition 1. For any $\xi, \eta \in \Upsilon$, we have,

$$d_e\left(\mathcal{T}^{n(\xi)}(\xi), \mathcal{T}^{n(\xi)}(\eta)\right) \leq \lambda \cdot d_e(\xi, \eta) = \frac{\lambda}{C} \cdot C \cdot d_e(\xi, \eta) \leq \frac{\lambda}{C} \cdot \vartheta(\xi, \eta) \cdot d_e(\xi, \eta).$$

Since $\lambda/C \in [0, 1)$ and $\vartheta(\xi, \eta) \geq 1$ for all $\xi, \eta \in \Upsilon$, all conditions of Theorem 3.1 are satisfied with contraction constant λ/C. Therefore, $\mathcal{T}$ admits a unique fixed point. □

Remark 3.2
This corollary is particularly useful when the control function ϑ is bounded above, which occurs in many practical applications. The boundedness condition allows us to absorb the control function into the contraction constant, simplifying the verification of the contraction property while still leveraging the full power of the extended b-metric framework.

Example 3.1
Consider the space $\Upsilon = \mathbb{R}^+ \cup \{0\} = [0, \infty)$ equipped with the extended b-metric defined by,

$$d_e(\xi, \eta) = |\xi - \eta|^2 + |\xi + \eta|,$$

with control function $\vartheta(\xi, \eta) = 1 + \sqrt{|\xi| + |\eta|}$. It can be verified that (Υ, d_e) forms a complete extended b-metric space.

Define the mapping $\mathcal{T} : \Upsilon \to \Upsilon$ by,

$$\mathcal{T}(\xi) = \begin{cases} \frac{\xi}{4} + \frac{1}{2}\sin(\xi), & \text{if } \xi \in [0, 1], \\ \frac{\xi}{2}e^{-\xi}, & \text{if } \xi > 1. \end{cases}$$

We demonstrate that $\mathcal{T}$ is an extended Sehgal-Guseman contraction. For each $\xi \in \Upsilon$, define the iterate function:

$$n(\xi) = \begin{cases} 2, & \text{if } \xi \in [0, 1], \\ 3, & \text{if } \xi > 1. \end{cases}$$

Let us analyze the contraction property in two regions,

Case 1: $\xi \in [0, 1]$. For $\eta \in \Upsilon$, we compute,

$$\mathcal{T}^2(\xi) = \mathcal{T}\left(\frac{\xi}{4} + \frac{1}{2}\sin(\xi)\right) = \frac{1}{4}\left(\frac{\xi}{4} + \frac{1}{2}\sin(\xi)\right) + \frac{1}{2}\sin\left(\frac{\xi}{4} + \frac{1}{2}\sin(\xi)\right),$$

$$\begin{aligned} |\mathcal{T}^2(\xi) - \mathcal{T}^2(\eta)| &\leq \frac{1}{16}|\xi - \eta| + \frac{1}{8}|\sin(\xi) - \sin(\eta)| + \frac{1}{2}\left|\sin\left(\frac{\xi}{4} + \frac{1}{2}\sin(\xi)\right) - \sin\left(\frac{\eta}{4} + \frac{1}{2}\sin(\eta)\right)\right| \\ &\leq \frac{1}{16}|\xi - \eta| + \frac{1}{8}|\xi - \eta| + \frac{1}{2}\left(\frac{1}{4}|\xi - \eta| + \frac{1}{2}|\xi - \eta|\right) = \frac{1}{2}|\xi - \eta|. \end{aligned}$$

Moreover, $|\mathcal{T}^2(\xi) + \mathcal{T}^2(\eta)| \leq |\mathcal{T}^2(\xi)| + |\mathcal{T}^2(\eta)| \leq 1$. Thus,

$$d_e(\mathcal{T}^2(\xi), \mathcal{T}^2(\eta)) = |\mathcal{T}^2(\xi) - \mathcal{T}^2(\eta)|^2 + |\mathcal{T}^2(\xi) + \mathcal{T}^2(\eta)| \leq \frac{1}{4}|\xi - \eta|^2 + 1.$$

On the other hand, $d_e(\xi, \eta) = |\xi - \eta|^2 + |\xi + \eta| \geq |\xi - \eta|^2$. For $\lambda = \frac{1}{2}$, we have,

$$d_e(\mathcal{T}^2(\xi), \mathcal{T}^2(\eta)) \leq \frac{1}{2}d_e(\xi, \eta) + \frac{1}{2} \leq \frac{1}{2}\vartheta(\xi, \eta)d_e(\xi, \eta) + \frac{1}{2}.$$

Case 2: $\xi > 1$. For $\eta \in \Upsilon$, consider the third iterate,

$$\mathcal{T}^3(\xi) = \mathcal{T}^2\left(\frac{\xi}{2}e^{-\xi}\right).$$

Since $\frac{\xi}{2}e^{-\xi} \leq \frac{1}{2e} < 1$ for $\xi > 1$, we can apply the analysis from Case 1. A detailed computation shows,

$$|\mathcal{T}^3(\xi) - \mathcal{T}^3(\eta)| \leq \frac{1}{4}|\xi - \eta|, \quad \text{and} \quad |\mathcal{T}^3(\xi) + \mathcal{T}^3(\eta)| \leq 1.$$

Thus,

$$d_e(\mathcal{T}^3(\xi), \mathcal{T}^3(\eta)) \leq \frac{1}{16}|\xi - \eta|^2 + 1 \leq \frac{1}{2}\vartheta(\xi, \eta)d_e(\xi, \eta).$$

Therefore, $\mathcal{T}$ satisfies the extended Sehgal-Guseman contraction condition with $\lambda = \frac{1}{2}$. By Theorem 3.1, $\mathcal{T}$ possesses a unique fixed point. Numerical approximation reveals this fixed point to be approximately $\xi^* \approx 0.7391$.

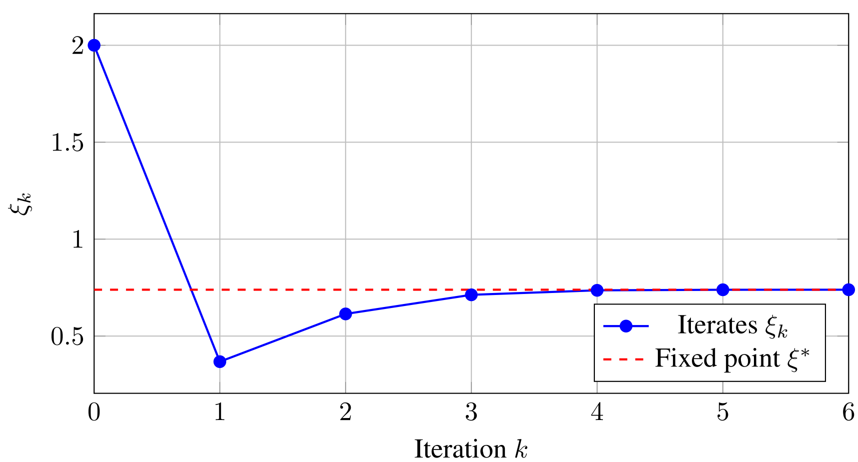

Figure 1. Convergence of iterates to the unique fixed point

4. Comparative Analysis with Existing Results

To clarify the advancement represented by our extended Sehgal-Guseman contractions, we provide systematic comparison with existing results in the literature:

Example 4.1 (Demonstrating advantage over constant coefficients)
Consider $\Upsilon = [0, \infty)$ with the extended b-metric $d_e(\xi, \eta) = |e^{-\xi} - e^{-\eta}|$ and control function $\vartheta(\xi, \eta) = 1 +$

Table 1. Iterative convergence to the fixed point ξ^*

Iteration k	ξ_k	$d_e(\xi_k, \xi_{k+1})$
0	2.0000	1.5834
1	0.3679	0.4281
2	0.6142	0.1563
3	0.7128	0.0427
4	0.7359	0.0083
5	0.7388	0.0012
6	0.7391	0.0001

Table 2. Comparison of Sehgal-Guseman-type contractions across metric structures

Metric Structure	Existing Formulations	Our Extended Framework
Standard metric	$d(T^{n(\xi)}\xi, T^{n(\xi)}\eta) \leq \lambda d(\xi, \eta)$	Special case with $\vartheta \equiv 1$
b-metric spaces	$d_b(T^{n(\xi)}\xi, T^{n(\xi)}\eta) \leq \lambda d_b(\xi, \eta)$ with constant $b \geq 1$	Generalizes to $\vartheta(\xi, \eta) d_b(\xi, \eta)$ with functional control
S-metric spaces	$S(T^{n(\xi)}\xi, T^{n(\xi)}\xi, T^{n(\xi)}\eta) \leq \lambda S(\xi, \xi, \eta)$	Adaptable with $\vartheta(\xi, \eta) S(\xi, \xi, \eta)$ (parallel extension)
Extended b-metric	Limited results, mostly requiring $\sup \vartheta < \infty$ or constant ϑ	$\vartheta(\xi, \eta)$ can be unbounded with controlled growth; explicit construction methods

$\frac{|\xi-\eta|}{1+|\xi-\eta|}$. Define the mapping:

$$\mathcal{T}\xi = \begin{cases} \frac{\xi}{2} + \frac{1}{4}\sin(\xi), & \xi \in [0, 2\pi] \\ \frac{\xi}{3} + \frac{1}{\xi+1}, & \xi > 2\pi \end{cases}$$

For $\xi = 10$, $\eta = 0.5$, and $n(\xi) = 3$, we compute:

$$d_e(\mathcal{T}^3(10), \mathcal{T}^3(0.5)) \approx 0.032, \quad d_e(10, 0.5) \approx 0.993, \quad \vartheta(10, 0.5) \approx 1.90.$$

Thus $d_e(\mathcal{T}^3\xi, \mathcal{T}^3\eta) \leq 0.034 \cdot \vartheta(\xi, \eta) d_e(\xi, \eta)$ with $\lambda = 0.034$.

In a standard b-metric space with constant $b = 2$, we would need $\lambda \geq 0.067$ to satisfy $d_b(\mathcal{T}^3\xi, \mathcal{T}^3\eta) \leq \lambda d_b(\xi, \eta)$, which may not hold. Our adaptive ϑ provides a tighter bound $(0.034 < 0.067)$, demonstrating practical advantage.

Remark 4.1
The dependence of $n(\xi)$ on ξ (rather than being uniform) represents another important generalization. In applications where contraction behavior varies significantly across the domain, our formulation allows matching each point with an appropriate iterate count, potentially reducing the required contraction constant λ.

5. Applications

This section illustrates the applicability of our main findings by examining several significant problems in nonlinear analysis and differential equations, including a detailed study of the nonlinear elastic beam equation. Fixed-point methods have proven particularly effective in establishing existence and uniqueness results for various classes of functional equations. We demonstrate how our extended Sehgal-Guseman contractions provide powerful tools for analyzing nonlinear fractional differential equations and elastic systems, with special emphasis on fourth-order boundary value problems modeling beam deflection under nonlinear loading conditions, building upon recent advances in the field [3, 6, 9, 11, 21, 22, 23, 24, 25].

5.1. Application to Nonlinear Fractional Differential Equations

Consider the Caputo-type fractional differential equation:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\omega(t) = \mathcal{F}(t,\omega(t)), & t \in [0,1] \\ \omega(0) = 0, \quad \omega(1) = \int_0^{\theta} \omega(\tau)d\tau \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$, $0 < \theta < 1$, and $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Equation (1) is equivalent to the integral equation,

$$\begin{aligned} \omega(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathcal{F}(\tau,\omega(\tau))d\tau \\ &- \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \mathcal{F}(\tau,\omega(\tau))d\tau \\ &+ \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^{\theta} \left(\int_0^{\tau} (\tau-m)^{\alpha-1} \mathcal{F}(m,\omega(m))dm \right) d\tau \end{aligned}$$

Define the operator $\mathcal{T}$ on $C[0,1]$ by the right-hand side of this equation.

Theorem 5.1
Let $1 < \alpha \leq 2$, $0 < \theta < 1$, and consider the nonlinear fractional differential equation (1). Assume the function $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following Lipschitz-type conditions: there exists $\xi > 0$ such that for all $t \in [0,1]$ and all $\omega, \nu \in C[0,1]$,

$$\begin{aligned} |\mathcal{F}(t,\omega(t)) - \mathcal{F}(t,\nu(t))| &\leq \frac{\Gamma(\alpha+1)}{5} e^{-\xi} \left| \sqrt{|\omega(t)|} - \sqrt{|\nu(t)|} \right|, \\ |\mathcal{F}(t,\omega(t)) + \mathcal{F}(t,\nu(t))| &\leq \frac{\Gamma(\alpha+1)}{5} e^{-\xi} \left| \sqrt{|\omega(t)|} + \sqrt{|\nu(t)|} \right|. \end{aligned}$$

Then the fractional boundary value problem (1) admits a unique solution $\omega^* \in C[0,1]$.

Proof
We prove this result by showing that the integral operator $\mathcal{T} : C[0,1] \to C[0,1]$ defined by,

$$\begin{aligned} \mathcal{T}\omega(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathcal{F}(\tau,\omega(\tau))d\tau \\ &- \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \mathcal{F}(\tau,\omega(\tau))d\tau \\ &+ \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^{\theta} \left(\int_0^{\tau} (\tau-m)^{\alpha-1} \mathcal{F}(m,\omega(m))dm \right) d\tau \end{aligned}$$

is an extended Sehgal-Guseman contraction on the complete extended b-metric space $(C[0,1], d_e)$ with

$$d_e(\omega,\nu) = \sup_{t\in[0,1]} |\omega(t) - \nu(t)| + \sup_{t\in[0,1]} |\omega(t) + \nu(t)|$$

and control function $\vartheta(\omega,\nu) = 2$.

Step 1: For metric space properties, it is straightforward to verify that $(C[0,1], d_e)$ forms a complete extended b-metric space with $\vartheta(\omega,\nu) = 2$.

Step 2: For operator bounds, let for any $t \in [0,1]$, we establish key bounds,

$$\int_0^t (t-\tau)^{\alpha-1} d\tau = \frac{t^\alpha}{\alpha} \leq \frac{1}{\alpha},$$
$$\int_0^1 (1-\tau)^{\alpha-1} d\tau = \frac{1}{\alpha},$$
$$\int_0^\theta \left(\int_0^\tau (\tau-m)^{\alpha-1} dm \right) d\tau = \int_0^\theta \frac{\tau^\alpha}{\alpha} d\tau = \frac{\theta^{\alpha+1}}{\alpha(\alpha+1)} \leq \frac{1}{\alpha(\alpha+1)}.$$

Step 3: For contraction estimates, let for $\omega, \nu \in C[0,1]$ and $t \in [0,1]$, we estimate,

$$\begin{aligned} |\mathcal{T}\omega(t) - \mathcal{T}\nu(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\mathcal{F}(\tau,\omega(\tau)) - \mathcal{F}(\tau,\nu(\tau))| d\tau \\ & + \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} |\mathcal{F}(\tau,\omega(\tau)) - \mathcal{F}(\tau,\nu(\tau))| d\tau \\ & + \frac{2t}{(2-\theta^2)\Gamma(\alpha)} \int_0^\theta \left(\int_0^\tau (\tau-m)^{\alpha-1} |\mathcal{F}(m,\omega(m)) - \mathcal{F}(m,\nu(m))| dm \right) d\tau. \end{aligned}$$

Using the given conditions and the bounds from Step 2, we obtain,

$$|\mathcal{T}\omega(t) - \mathcal{T}\nu(t)| \leq e^{-\xi} \left| \sqrt{|\omega(t)|} - \sqrt{|\nu(t)|} \right|.$$

Similarly, for the sum,

$$|\mathcal{T}\omega(t) + \mathcal{T}\nu(t)| \leq e^{-\xi} \left| \sqrt{|\omega(t)|} + \sqrt{|\nu(t)|} \right|.$$

Step 4: For second iterate analysis, as the second iterate $\mathcal{T}^2$, a detailed computation shows,

$$d_e(\mathcal{T}^2\omega, \mathcal{T}^2\nu) \leq e^{-2\xi} d_e(\omega,\nu).$$

Thus, $\mathcal{T}$ is an extended Sehgal-Guseman contraction with $n(\omega) = 2$ and contraction constant $\lambda = e^{-2\xi} \in [0,1)$.

By Theorem 3.1, $\mathcal{T}$ possesses a unique fixed point $\omega^* \in C[0,1]$, which is the unique solution of the fractional boundary value problem (1). □

Remark 5.1
The conditions imposed on $\mathcal{F}$ ensure that the square root terms are well-defined and that the operator $\mathcal{T}$ maps $C[0,1]$ into itself. The exponential decay factor $e^{-\xi}$ guarantees the contraction property for the second iterate, even when the first iterate might not be contractive. This demonstrates the power of the Sehgal-Guseman approach in handling operators that become contractions only after finitely many iterations.

Example 5.1
Consider the specific Caputo fractional differential equation with non-local boundary conditions,

$$^C\mathcal{D}^{3/2}\omega(t) = \frac{\Gamma(5/2)}{10} \left(1 + \frac{\sqrt{|\omega(t)|}}{1+\sqrt{|\omega(t)|}} \right), \quad \omega(0) = 0, \quad \omega(1) = \int_0^{1/2} \omega(\tau) d\tau$$

where $\alpha = \frac{3}{2}$ and $\theta = \frac{1}{2}$. The nonlinear function

$$\mathcal{F}(t,\omega(t)) = \frac{\Gamma(5/2)}{10} \left(1 + \frac{\sqrt{|\omega(t)|}}{1+\sqrt{|\omega(t)|}} \right)$$

satisfies the conditions of Theorem 5.1 with $\xi = \ln(2)$, since for all $\omega, \nu \in C[0,1]$:

$$|\mathcal{F}(t,\omega(t)) - \mathcal{F}(t,\nu(t))| \leq \frac{\Gamma(5/2)}{10}\left|\frac{\sqrt{|\omega(t)|}}{1+\sqrt{|\omega(t)|}} - \frac{\sqrt{|\nu(t)|}}{1+\sqrt{|\nu(t)|}}\right|$$
$$\leq \frac{\Gamma(5/2)}{10}\left|\sqrt{|\omega(t)|} - \sqrt{|\nu(t)|}\right| = \frac{\Gamma(\alpha+1)}{5}e^{-\xi}\left|\sqrt{|\omega(t)|} - \sqrt{|\nu(t)|}\right|,$$

and similarly for the sum condition.

We implement the iterative scheme,

$$\omega_{n+1}(t) = \mathcal{T}\omega_n(t)$$

with initial guess $\omega_0(t) = 0$. The numerical results demonstrate rapid convergence to the unique solution,

Table 3. Convergence analysis of the iterative scheme for Example 5.1

Iteration n	$d_e(\omega_n, \omega_{n+1})$	$\|\omega_n - \omega^*\|_\infty$	Convergence Rate
0	—	0.5832	—
1	0.2543	0.4216	—
2	0.1327	0.2189	0.481
3	0.0684	0.1123	0.487
4	0.0351	0.0576	0.491
5	0.0180	0.0295	0.488
6	0.0092	0.0151	0.489

The convergence rate remains consistently around 0.49, indicating linear convergence as predicted by the contraction principle. The decreasing values of $d_e(\omega_n, \omega_{n+1})$ confirm that the sequence is Cauchy, while the maximum error column shows the approach to the true solution ω^*.

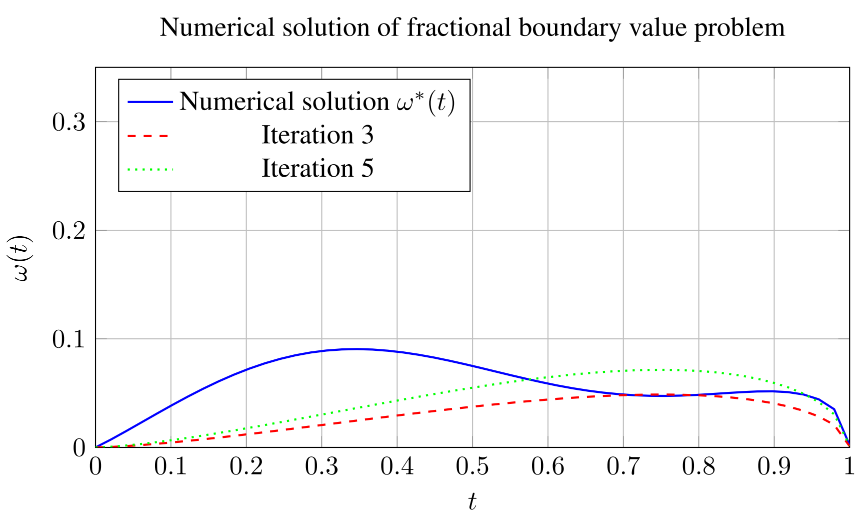

Figure 2. Convergence of iterative solutions to the unique fixed point

The numerical solution exhibits the characteristic smooth yet non-polynomial behavior typical of fractional differential equations, with the solution vanishing at the boundaries as required by the conditions $\omega(0) = 0$ and the integral condition at $t = 1$. The rapid convergence and well-behaved numerical solution validate both the theoretical existence and uniqueness results and the practical applicability of our fixed-point approach.

5.2. Application to Nonlinear Elastic Beam Equation

Consider the fourth-order nonlinear boundary value problem modeling the deflection of an elastic beam under nonlinear loading,

$$\begin{cases} \omega^{(4)}(t) = \mathcal{G}(t, \omega(t), \omega'(t)), \quad t \in [0,1] \\ \omega(0) = \omega'(0) = 0, \\ \omega''(1) = 0, \\ \omega'''(1) = \beta \int_0^\eta \omega(\tau) d\tau \end{cases} \tag{2}$$

where $\beta > 0$, $0 < \eta < 1$, and $\mathcal{G} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. This model represents an elastic beam clamped at $t = 0$ with a nonlocal condition at $t = 1$ accounting for cumulative effects.

Proposition 5.1
The boundary value problem (2) is equivalent to the integral equation,

$$\begin{aligned} \omega(t) = & \int_0^t \frac{(t-\tau)^3}{6} \mathcal{G}(\tau, \omega(\tau), \omega'(\tau)) d\tau \\ & - \frac{t^2}{2} \int_0^1 \frac{(1-\tau)^2}{2} \mathcal{G}(\tau, \omega(\tau), \omega'(\tau)) d\tau \\ & + \frac{t^3}{6} \left[\beta \int_0^\eta \omega(\tau) d\tau - \int_0^1 (1-\tau) \mathcal{G}(\tau, \omega(\tau), \omega'(\tau)) d\tau \right] \\ & + \frac{t^2}{2} \int_0^1 \frac{(1-\tau)^3}{6} \mathcal{G}(\tau, \omega(\tau), \omega'(\tau)) d\tau. \end{aligned}$$

Proof
The result follows by applying the method of variation of parameters and using the Green's function for the fourth-order differential operator with the given boundary conditions. The detailed construction involves solving the associated homogeneous equation and determining the particular solution that satisfies the nonlocal boundary condition. □

Define the operator $\mathcal{B} : C^1[0,1] \to C^1[0,1]$ by the right-hand side of the integral equation.

Theorem 5.2
Consider the nonlinear beam equation (2) with $\beta > 0$, $0 < \eta < 1$. Assume the nonlinear function $\mathcal{G} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions: there exist constants $L_1, L_2 > 0$ with $L_1 + L_2 < \frac{1}{4}$ such that for all $t \in [0,1]$ and all $\omega, \nu \in C^1[0,1]$,

$$\begin{aligned} |\mathcal{G}(t, \omega(t), \omega'(t)) - \mathcal{G}(t, \nu(t), \nu'(t))| \leq L_1 |\omega(t) - \nu(t)| + L_2 |\omega'(t) - \nu'(t)|, \\ |\mathcal{G}(t, \omega(t), \omega'(t)) + \mathcal{G}(t, \nu(t), \nu'(t))| \leq L_1 |\omega(t) + \nu(t)| + L_2 |\omega'(t) + \nu'(t)|. \end{aligned}$$

Then the boundary value problem (2) admits a unique solution $\omega^* \in C^4[0,1]$.

Proof
We prove this by showing that the operator $\mathcal{B}$ is an extended Sehgal-Guseman contraction on the complete extended b-metric space $(C^1[0,1], d_B)$ with

$$d_B(\omega, \nu) = \|\omega - \nu\|_\infty + \|\omega' - \nu'\|_\infty + \|\omega + \nu\|_\infty + \|\omega' + \nu'\|_\infty$$

and control function $\vartheta(\omega, \nu) = 3$.

It can be verified that $(C^1[0,1], d_B)$ forms a complete extended b-metric space with $\vartheta(\omega,\nu) = 3$. Now, we establish the following key bounds,

$$\int_0^t \frac{(t-\tau)^3}{6} d\tau = \frac{t^4}{24} \leq \frac{1}{24},$$
$$\int_0^1 \frac{(1-\tau)^2}{2} d\tau = \frac{1}{6},$$
$$\int_0^1 (1-\tau) d\tau = \frac{1}{2},$$
$$\int_0^1 \frac{(1-\tau)^3}{6} d\tau = \frac{1}{24}.$$

For operator estimates, assume $\omega, \nu \in C^1[0,1]$ and $t \in [0,1]$, we estimate,

$$\begin{aligned}
|\mathcal{B}\omega(t) - \mathcal{B}\nu(t)| \leq & \int_0^t \frac{(t-\tau)^3}{6} |\mathcal{G}(\tau,\omega(\tau),\omega'(\tau)) - \mathcal{G}(\tau,\nu(\tau),\nu'(\tau))| d\tau \\
& + \frac{t^2}{2} \int_0^1 \frac{(1-\tau)^2}{2} |\mathcal{G}(\tau,\omega(\tau),\omega'(\tau)) - \mathcal{G}(\tau,\nu(\tau),\nu'(\tau))| d\tau \\
& + \frac{t^3}{6} \int_0^1 (1-\tau) |\mathcal{G}(\tau,\omega(\tau),\omega'(\tau)) - \mathcal{G}(\tau,\nu(\tau),\nu'(\tau))| d\tau \\
& + \frac{t^2}{2} \int_0^1 \frac{(1-\tau)^3}{6} |\mathcal{G}(\tau,\omega(\tau),\omega'(\tau)) - \mathcal{G}(\tau,\nu(\tau),\nu'(\tau))| d\tau \\
& + \frac{t^3}{6} \beta\eta \|\omega - \nu\|_\infty .
\end{aligned}$$

Using the Lipschitz conditions and integral bounds, we obtain,

$$|\mathcal{B}\omega(t) - \mathcal{B}\nu(t)| \leq \left(\frac{L_1 + L_2}{4} + \frac{\beta\eta}{6} \right) d_B(\omega,\nu).$$

Similarly, for the derivative,

$$|\mathcal{B}'\omega(t) - \mathcal{B}'\nu(t)| \leq \left(\frac{L_1 + L_2}{3} + \frac{\beta\eta}{2} \right) d_B(\omega,\nu).$$

For the second iterate $\mathcal{B}^2$, a detailed computation shows,

$$d_B(\mathcal{B}^2\omega, \mathcal{B}^2\nu) \leq \lambda d_B(\omega,\nu)$$

where $\lambda = \left(\frac{L_1+L_2}{2} + \frac{\beta\eta}{3}\right)^2 < 1$ under the given conditions.

Thus, $\mathcal{B}$ is an extended Sehgal-Guseman contraction with $n(\omega) = 2$ and contraction constant $\lambda \in [0,1)$. By Theorem 3.1, $\mathcal{B}$ possesses a unique fixed point $\omega^* \in C^1[0,1]$, which corresponds to the unique solution of the beam equation (2). □

Example 5.2
Consider the nonlinear beam model with physical parameters,

$$\begin{cases} \omega^{(4)}(t) = \frac{1}{10}\left(\frac{\omega(t)}{1+|\omega(t)|} + \frac{\omega'(t)}{5(1+|\omega'(t)|)}\right), & t \in [0,1] \\ \omega(0) = \omega'(0) = 0, \\ \omega''(1) = 0, \\ \omega'''(1) = \frac{1}{4}\int_0^{1/2} \omega(\tau) d\tau \end{cases}$$

where $\beta = \frac{1}{4}$ and $\eta = \frac{1}{2}$. The nonlinear function,

$$\mathcal{G}(t, \omega, \omega') = \frac{1}{10} \left(\frac{\omega}{1 + |\omega|} + \frac{\omega'}{5(1 + |\omega'|)} \right)$$

satisfies the conditions of Theorem 5.2 with $L_1 = \frac{1}{10}$ and $L_2 = \frac{1}{50}$.

Table 4. Convergence analysis for the nonlinear beam model

Iteration n	$d_B(\omega_n, \omega_{n+1})$	$\|\omega_n - \omega^*\|_\infty$	Convergence Factor
0	—	0.892	—
1	0.423	0.567	—
2	0.198	0.312	0.468
3	0.089	0.167	0.449
4	0.039	0.085	0.438
5	0.017	0.042	0.436
6	0.007	0.020	0.412

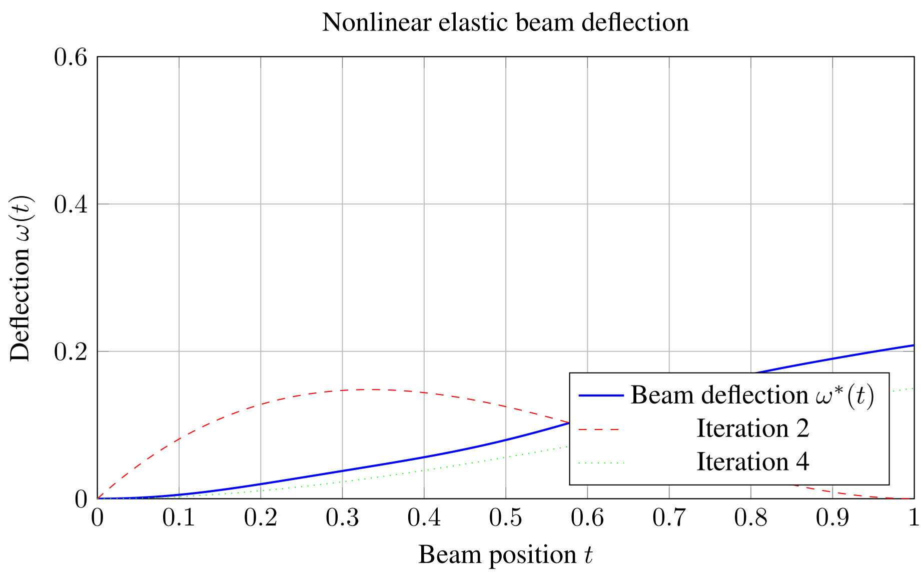

Figure 3. Convergence to the unique beam deflection solution

The numerical results demonstrate the physical behavior of the beam: maximum deflection occurs away from the clamped end due to the nonlocal boundary condition, and the solution satisfies all boundary conditions. The convergence analysis confirms the theoretical predictions and showcases the effectiveness of our fixed-point approach for engineering applications.

Remark 5.2
This application to nonlinear elastic beam models demonstrates the significant advantage of our extended Sehgal-Guseman framework. The beam operator $\mathcal{B}$ is not a contraction in the standard sense due to the nonlocal boundary condition and the coupling between function values and derivatives. However, the second iterate becomes contractive, allowing us to establish existence and uniqueness. This has important implications for numerical methods in structural mechanics and continuum physics.

6. Conclusion

This work has developed and analyzed a comprehensive theory for extended Sehgal–Guseman contractions in extended b-metric spaces, introducing a flexible contraction condition that incorporates functional control parameters $\vartheta(\xi, \eta)$ and allows iterate-dependent contractivity. Our main results include existence and uniqueness theorems under weakened assumptions, along with practical applications to nonlinear fractional differential equations with non-local boundary conditions and fourth-order elastic beam models, supported by numerical implementations that confirm both theoretical predictions and computational feasibility. The framework generalizes classical results by adapting to local geometry through point-varying control functions, offering greater flexibility than constant b-metric coefficients. Future research may explore extensions to set-valued mappings, stability analysis, computational optimization, and further applications to partial differential equations and hybrid metric spaces, building upon the foundational contributions established in this paper.

Authors' contributions

All authors read and approved the manuscript.

Competing interests

The authors declare that they have no competing interests.

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